Topological, Automata-Theoretic and Logical Characterization of Finitary Languages

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Abstract The class of ω-regular languages provide a robust specification language in verification. Every ω-regular condition can be decomposed into a safety part and a liveness part. The liveness part ensures that something good happens "eventually." Two main strengths of the classical, infinite-limit formulation of liveness are robustness (independence from the granularity of transitions) and simplicity (abstraction of complicated time bounds). However, the classical liveness formulation suffers from the drawback that the time until something good happens may be unbounded. A stronger formulation of liveness, so-called *finitary* liveness, overcomes this drawback, while still retaining robustness and simplicity. Finitary liveness requires that there exists an unknown, fixed bound b such that something good happens within b transitions. In this work we consider the finitary parity and Streett (fairness) conditions. We present the topological, automata-theoretic and logical characterization of finitary languages defined by finitary parity and Streett conditions. We (a) show that the finitary parity and Streett languages are Σ_2-complete; (b) present a complete characterization of the expressive power of various classes of automata with finitary and infinitary conditions (in particular we show that non-deterministic finitary parity and Streett automata cannot be determinized to deterministic finitary parity or Streett automata); and (c) show that the languages defined by non-deterministic finitary parity automata exactly characterize the star-free fragment of ωB-regular languages.

1 Introduction

Classical ω-regular languages: strengths and weakness. The class of ω-regular languages provide a robust language for specification for solving control and verification problems (see, e.g., [PR89,RW87]). Every ω-regular specification can be decomposed into a safety part and a liveness part [AS85]. The safety part ensures that the component will not do anything "bad" (such as violate an invariant) within any finite number of transitions. The liveness part ensures that the component will do something "good" (such as proceed, or respond, or terminate) in the long-run. Liveness can be violated only in the limit, by infinite sequences of transitions, as no bound is stipulated on when the "good" thing must happen. This infinitary, classical formulation of liveness has both strengths and weaknesses. A main strength is robustness, in particular, independence from the chosen granularity of transitions. Another main strength is simplicity, allowing liveness to serve as an abstraction for complicated safety conditions. For example, a component may always respond in a number of transitions that depends, in some complicated manner, on the exact size of the stimulus. Yet for correctness, we may be interested only that the component will respond "eventually." However, these strengths also point to a weakness of the classical definition of liveness: it can be satisfied by components that in practice are quite unsatisfactory because no bound can be put on their response time.

Stronger notion of liveness. For the weakness of the infinitary formulation of liveness, alternative and stronger formulations of liveness have been proposed. One of these is *finitary* liveness [AH94,DJP03]: finitary liveness does not insist on a response within a known bound b (i.e., every stimulus is followed by a response within b transitions), but on response within some unknown bound (i.e., there exists b such that every stimulus is followed by a response within b transitions). Note that in the finitary case, the bound b may be arbitrarily large, but the response time must not grow forever from one stimulus to the next. In this way, finitary liveness still maintains the robustness (independence of step granularity) and simplicity (abstraction of complicated safety) of traditional liveness, while removing unsatisfactory implementations.

Finitary parity and Streett conditions. The classical infinitary notion of fairness is given by the Streett condition: a Streett condition consists of a set of d pairs of requests and corresponding responses (grants) and the condition requires that every request that appears infinitely often must be responded infinitely often. The finitary Streett condition requires that there is a bound b such that in the limit every request is responded within b steps. The classical infinitary parity condition consists of a priority function and the condition requires that the minimum priority visited infinitely often is even. The finitary parity condition requires that there is a bound b such that in the limit after every odd priority a lower even priority is visited within b steps.

Characterization of infinitary parity and Streett automata. There are several robust language-theoretic characterization of the languages expressible by automata with infinitary liveness (Büchi), parity and Streett conditions. Some of the important characterizations are as follows: (a) *Topological characterization:* it is known that deterministic automata with Büchi conditions are Π_2-complete, whereas non-deterministic Büchi and both deterministic and non-deterministic parity and Streett automata lie in the boolean closure of Σ_2 and Π_2 [MP92]; (b) *Automata theoretic characterization:* non-deterministic automata with Büchi conditions have the same expressive power as deterministic and non-deterministic parity and Streett automata [Cho74,Saf92]; and (c) *Logical characterization:* the class of languages expressed by deterministic parity (that is equivalent to non-deterministic Büchi, parity and Streett automata) is equivalent to the class of ω-regular languages and is also characterized by the *monadic second-order logic* (MSOL) (see the handbook [Tho97] for details).

Our results. For finitary Büchi, parity and Streett automata the topological, automata-theoretic, and logical characterization were all missing. In this work we present all the three characterizations. Our main results are as follows.

1. *Topological characterization.* We show that the class of languages defined by finitary Büchi, parity and Streett conditions are Σ_2-complete, and thus present a precise topological characterization of finitary Büchi, parity and Streett languages.

2. *Automata-theoretic characterization.* We show that languages defined by finitary parity and Streett automata are incomparable in expressive power as compared to infinitary parity and Streett automata. We show that non-deterministic automata with finitary parity and Streett conditions have the same expressive power as non-deterministic automata with finitary Büchi conditions, and deterministic parity and Streett automata have the same expressive power and is strictly more expressive than deterministic finitary Büchi automata. However, in contrast to infinitary parity condition, for finitary parity condition, non-deterministic automata is strictly more expressive than the deterministic counterpart. We also present a precise characterization of the closure properties of finitary automata with respect to union, intersection and complementation.

3. *Logical characterization.* Since finitary automata are incomparable in expressive power as compared to ω-regular languages, the result also holds for MSOL. We consider the characterization of finitary automata through an extension of MSOL and ω-regular languages defined as $\mathrm{MSOL}\mathbb{A}$ and ωB-regular languages by [BC06]. We show that languages defined by non-deterministic finitary parity automata are exactly the star-free fragment of ωB-regular languages. It follows that in general $\mathrm{MSOL}\mathbb{A}$ and ωB-regular languages are strictly more expressive, and non-deterministic finitary parity automata exactly characterize the star-free fragment. Hence we obtain a precise logical characterization of the finitary languages.

2 Definitions

In this section we define languages, topology related to languages, then automata and languages described by automata with various acceptance conditions.

2.1 Languages, Cantor topology and Borel hierarchy

Languages. Let Σ be a finite set, we refer to Σ as the alphabet, and its elements as letters. A word w is a sequence of letters, which can be either finite or infinite. A word w will be described as a sequence

$w_0w_1 \ldots$ of letters, where $w_0, w_1, \cdots \in \Sigma$. Let Σ^* be the set of all finite words over Σ and Σ^ω the set of all infinite words over Σ. A language is a set of words, thus $L \subseteq \Sigma^*$ is a language over finite words and $L \subseteq \Sigma^\omega$ is a language over infinite words.

Cantor topology. The complexity of languages can be studied according to the topological definition. To present a topological definition on languages we first define *open* and *closed* sets. A language is open if it can be described as $W \cdot \Sigma^\omega$ where $W \subseteq \Sigma^*$. A closed set is a complement of an open set. Then we define the Cantor topology to obtain the topology over languages. It may be noted that the above topology defines the same topology as the one induced by the following distance over infinite words: $\mathrm{distword}(w, w') = \frac{1}{2^i}$, where i is the largest nonnegative integer such that $w_j = w'_j$ for all $0 \leq j < i$.

Borel hierarchy. We now define the Borel hierarchy of languages. Let Σ_1 denote the open sets, Π_1 denote the closed sets, and then inductively we have the following: Σ_{i+1} is obtained as countable union of Π_i sets; and Π_{i+1} is obtained as countable intersection of Σ_i sets. We note that the closed sets (languages in Π_1) correspond to *safety* properties. For $L \subseteq \Sigma^\omega$, let $\mathrm{pref}(L) \subseteq \Sigma^*$ be the set of finite prefixes of words in L: $u \in \Sigma^*$ belongs to $\mathrm{pref}(L)$ iff there exists $w \in L$ such that u is a finite prefix of w. Then the following property holds.

Proposition 1. *For all languages $L \subseteq \Sigma^\omega$, the following statements are equivalent: (a) L is closed; (b) for all infinite words w, if all finite prefixes of w are in $\mathrm{pref}(L)$, then $w \in L$.*

Topological reduction. The classes $\Sigma_1, \Pi_1, \Sigma_2, \Pi_2, \ldots$ are the levels of Borel hierarchy. Since they are closed under continuous preimage, we can define a notion of reduction: L reduces to L', denoted by $L \preceq L'$, if there exists a continuous function $f : \Sigma^\omega \to \Sigma^\omega$ such $L = f^-(L')$, where $f^-(L')$ is the preimage of L' by f. This defines the notion of Wadge reduction [Wad84]. A language is hard with respect to a class if all languages of this class reduce to it. If it additionally belongs to this class, then it is complete.

Classical languages. We now consider several classical notion of languages. For an infinite word w, let $\mathrm{Inf}(w) \subseteq \Sigma$ denote the set of letters that appear infinitely often in w. The class of reachability, safety, Büchi and coBüchi languages are defined as follows. Let $F \subseteq \Sigma$:

$$\mathrm{Reach}(F) = \{w \mid \exists i \in \mathbb{N}, w_i \in F\}; \qquad \mathrm{Safe}(F) = \Sigma^\omega \backslash \mathrm{Reach}(F) = \{w \mid \forall i \in \mathbb{N}, w_i \notin F\};$$

$$\mathrm{Büchi}(F) = \{w \mid \mathrm{Inf}(w) \cap F \neq \emptyset\}; \qquad \mathrm{CoBüchi}(F) = \Sigma^\omega \backslash \mathrm{Büchi}(F) = \{w \mid \mathrm{Inf}(w) \subseteq \Sigma \backslash F\}.$$

In other words, the reachability language $\mathrm{Reach}(F)$ requires that a letter in F appears at least once and the Büchi language $\mathrm{Büchi}(F)$ requires that some letter in F appears infinitely often. The $\mathrm{Safe}(F)$ and $\mathrm{CoBüchi}(F)$ are duals of $\mathrm{Reach}(F)$ and $\mathrm{Büchi}(F)$, respectively. The class of parity languages is defined as follows. Let $p : \Sigma \to \mathbb{N}$ be a priority function that maps letters to integer priorities. The parity languages are defined as follows:

$$\mathrm{Parity}(p) = \{w \mid \min(p(\mathrm{Inf}(w))) \text{ is even}\};$$

i.e., the parity condition accepts infinite words where the lowest priority infinitely visited is even. The parity conditions are self-dual. The class of Rabin and Streett languages are defined as follows. Let $(R, G) = (R_i, G_i)_{1 \leq i \leq d}$, where $R_i, G_i \subseteq \Sigma$ are request-grant pairs. Then we have

$$\mathrm{Streett}(R, G) = \{w \mid \forall i, 1 \leq i \leq d, \mathrm{Inf}(w) \cap R_i \neq \emptyset \Rightarrow \mathrm{Inf}(w) \cap G_i \neq \emptyset\};$$
$$\mathrm{Rabin}(R, G) = \{w \mid \exists i, 1 \leq i \leq d, \mathrm{Inf}(w) \cap R_i \neq \emptyset \wedge \mathrm{Inf}(w) \cap G_i = \emptyset\};$$

i.e., the Streett condition accepts infinite words w such that for all requests R_i, if R_i appears infinitely often in w, then the corresponding grant G_i also appears infinitely often in w. Rabin condition is the dual of Streett condition. Then we have the following theorem that presents the topological characterization of the classical languages.

Theorem 1 (Topological characterization of classical languages [MP92]). *The following assertions hold.*

– *For all $\emptyset \subset F \subset \Sigma$, we have (a)* $\mathrm{Reach}(F)$ *is* Σ_1-complete *and* $\mathrm{Safe}(F)$ *is* Π_1-complete; *and*
  (b) $\mathrm{B\ddot{u}chi}(F)$ *is* Π_2-complete *and* $\mathrm{CoB\ddot{u}chi}(F)$ *is* Σ_2-complete.
– *The parity, Streett and Rabin languages lie in the boolean closure of* Σ_2 *and* Π_2.

Finitary languages. Let $p : \Sigma \to \mathbb{N}$ a priority function. We define:

$$\mathrm{dist}_k(w,p) = \begin{cases} 0 & p(w_k) \text{ is even} \\ \inf\{k' - k \mid k' \geq k, p(w_{k'}) \text{ is even and } p(w_{k'}) < p(w_k)\} & p(w_k) \text{ is odd} \end{cases}$$

The finitary parity language $\mathrm{FinParity}(p)$ was defined as follows in [CHH09]:

$$\mathrm{FinParity}(p) = \{w \mid \limsup_k \mathrm{dist}_k(w,p) < \infty\},$$

i.e., the $\mathrm{FinParity}(p)$ requires that the supremum limit of the distance sequence is bounded. The definition for $\mathrm{FinStreett}(R,G)$ languages uses similar distance sequence as follows:

$$\mathrm{dist}_k^j(w,(R,G)) = \begin{cases} 0 & w_k \notin R_j \\ \inf\{k' - k \mid k' \geq k, w_{k'} \in G_j\} & w_k \in R_j \end{cases}$$

Then we have $\mathrm{dist}_k(w,(R,G)) = \max\{\mathrm{dist}_k^j(w,p) \mid 1 \leq j \leq d\}$, and the finitary Streett language $\mathrm{FinStreett}(R,G)$ was defined as follows in [CHH09]:

$$\mathrm{FinStreett}(R,G) = \{w \mid \limsup_k \mathrm{dist}_k(w,(R,G)) < \infty\},$$

i.e., similar to finitary parity languages $\mathrm{FinStreett}(R,G)$ requires the supremum limit of the distance sequence to be bounded.

2.2 Automata, ω-regular and finitary languages

In this section we consider automata with acceptance conditions and consider the class of languages defined by automata with various classes of acceptance conditions.

Definition 1. *An automaton is a tuple $\mathcal{A} = (Q, \Sigma, Q_0, \delta, Acc)$, where Q is a finite set of states, Σ is the finite input alphabet, $Q_0 \subseteq Q$ is the set of initial states, $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation and $Acc \subseteq Q^\omega$ is the acceptance condition.*

Deterministic and complete automata. We consider the special class of *deterministic* and *complete* automata. An automaton $\mathcal{A}$ is deterministic if (a) $|Q_0| = 1$, i.e., there is a single initial state; (b) for every letter and every state there is at most one transition, i.e., for all $q \in Q$, for all $\sigma \in \Sigma$ we have $|\{q' \mid (q, \sigma, q') \in \delta\}| \leq 1$. Deterministic automata will be described as $(Q, \Sigma, q_0, \delta, Acc)$, where $\delta : Q \times \Sigma \to Q$ is a function. If for all $q \in Q$ and for all $\sigma \in \Sigma$, there exists $q' \in Q$ such that $(q, \sigma, q') \in \delta$, then the corresponding automaton is complete. This is the case when the transition function is *total*.

Runs. A run $\rho = q_0 q_1 \ldots$ is a word over Q, where $q_0 \in Q_0$. The run ρ is accepting if it is infinite and $\rho \in Acc$. We will write $p \xrightarrow{a} q$ to denote $(p, a, q) \in \delta$. An infinite word $w = w_0 w_1 \ldots$ induces possibly several runs of $\mathcal{A}$: a word w induces a run $\rho = q_0 q_1 \ldots$ if we have $q_0 \in Q_0$ and

$$q_0 \xrightarrow{w_0} q_1 \xrightarrow{w_1} q_2 \ldots q_n \xrightarrow{w_n} q_{n+1} \ldots.$$

The language accepted by $\mathcal{A}$, denoted by $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^\omega$, is as follows:

$$\mathcal{L}(\mathcal{A}) = \{w \mid \text{there exists a run } \rho \text{ induced by } w \text{ such that } \rho \in Acc\}.$$

Note that for a deterministic automaton, every word w induces at most one run, whereas in a non-deterministic automaton a word may induce several possible runs.

Acceptance conditions. We will consider various acceptance conditions for automata obtained from the last section by considering Q as the alphabet. For $F \subseteq Q$, the conditions $\mathrm{Reach}(F)$, $\mathrm{Safe}(F)$, $\mathrm{B\ddot{u}chi}(F)$, $\mathrm{CoB\ddot{u}chi}(F)$, define reachability, safety, Büchi and coBüchi acceptance conditions, respectively. For $p : Q \to \mathbb{N}$, the conditions $\mathrm{Parity}(p)$ and $\mathrm{FinParity}(p)$ define parity and finitary parity acceptance conditions, respectively. For $(R, G) = (R_i, G_i)_{1 \leq i \leq d}$, where $R_i, G_i \subseteq Q$, the conditions $\mathrm{Streett}(R, G)$, $\mathrm{Rabin}(R, G)$, and $\mathrm{FinStreett}(R, G)$ define Streett, Rabin and finitary Streett acceptance conditions, respectively. The set of languages recognized by non-deterministic Büchi automata corresponds to the class of ω-regular languages [Büc62] and we will denote the class of ω-regular languages as $\mathbb{L}_\omega$.

Notation 1 *We use a standard notation to denote set of languages recognized by some class of automata. The first letter is either N or D, where N stands for "non-deterministic" and D stands for "deterministic". The last block of letters refers to the acceptance condition, for example, B stands for "Büchi", C stands for "CoBüchi", P stands for "parity" and S stands for "Streett". The acceptance condition may be prefixed by F for "finitary". For example, NP denotes non-deterministic parity automata, and DFS denotes deterministic finitary Streett automata. Hence we have the following combination:*

$$\left\{ \begin{array}{c} N \\ D \end{array} \right\} \cdot \left\{ \begin{array}{c} F \\ \varepsilon \end{array} \right\} \cdot \left\{ \begin{array}{c} B \\ C \\ P \\ S \end{array} \right\}$$

We now present the following theorem that summarizes the results of automata with classical languages, and the results of the theorem follows from [Büc62,Saf92,Cho74,GH82].

Theorem 2 (Automata-theoretic results for classical languages). *The following assertions hold:*

$$(1) \; \mathbb{L}_\omega = NB = NP = NS = DP = DS;$$
$$(2) \; DB \subset NB; \qquad (3) \; DC = NC \subset NB.$$

3 Topological Characterization of Finitary Languages

In this section we present the topological characterization of finitary Büchi, finitary parity and finitary Streett languages. We first present a definition and then use the definition for characterization of finitary languages.

Union of ω-regular and closed subset of a language. Given a language $L \subseteq \Sigma^\omega$, the language $\mathsf{UniCloOmg}(L) \subseteq \Sigma^\omega$ is the union of the languages M that are subset of L, ω-regular and closed, i.e., $\mathsf{UniCloOmg}(L) = \bigcup \{M \mid M \in \Pi_1, M \in \mathbb{L}_\omega, M \subseteq L\}$.

Proposition 2. *The following assertions hold: (a) the operator $\mathsf{UniCloOmg}$ is idempotent; i.e., for all languages L we have $\mathsf{UniCloOmg}(\mathsf{UniCloOmg}(L)) = \mathsf{UniCloOmg}(L)$; (b) the language $\mathsf{UniCloOmg}(L)$ is in Σ_2, i.e., for all languages L we have $\mathsf{UniCloOmg}(L) \in \Sigma_2$.*

Proof. We prove both the properties below.

1. By definition for all languages L' we have $\mathsf{UniCloOmg}(L') \subseteq L'$. Given a language L, let $L' = \mathsf{UniCloOmg}(L)$. Hence we have $\mathsf{UniCloOmg}(L') \subseteq L'$, i.e., $\mathsf{UniCloOmg}(\mathsf{UniCloOmg}(L)) \subseteq \mathsf{UniCloOmg}(L)$. We now show the other direction. For any language L' and $M \subseteq L'$, if $M \in \Pi_1$ and $M \in \mathbb{L}_\omega$, then $M \subseteq \mathsf{UniCloOmg}(L')$. Consider the language $L' = \mathsf{UniCloOmg}(L)$. For a language M such that $M \subseteq L$, $M \in \Pi_1$ and $M \in \mathbb{L}_\omega$, we have $M \subseteq \mathsf{UniCloOmg}(L)$, (i.e., $M \subseteq L'$) and hence $M \subseteq \mathsf{UniCloOmg}(L')$. Hence we have

$$L' = \bigcup \{M \mid M \in \Pi_1, M \in \mathbb{L}_\omega, M \subseteq L\} \subseteq \mathsf{UniCloOmg}(L').$$

The result follows.

2. Since $\mathbb{L}_\omega = NB$ (by Theorem 2), and the set of finite automata can be enumerated in sequence, it follows $\mathbb{L}_\omega$ is countable. It follows that for all languages L, the set $\mathsf{UniCloOmg}(L)$ is described as a countable union of closed sets. Hence $\mathsf{UniCloOmg}(L) \in \Sigma_2$.

The result follows. ∎

We now present a *pumping lemma* for regular languages, and will use it to present the topological characterization for finitary languages.

Lemma 1 (A pumping lemma). *Let M be a ω-regular language. There exists n_0 such that for all words $w \in M$, for all positions $j \geq n_0$, there exist $j \leq i_1 < i_2 \leq j + n_0$ such that for all $\ell \geq 0$ we have $w_0 w_1 w_2 \ldots w_{i_1-1} \cdot (w_{i_1} w_{i_1+1} \ldots w_{i_2-1})^\ell \cdot w_{i_2} w_{i_2+1} \ldots \in M$.*

Proof. Given M is a ω-regular language, let $\mathcal{A}$ be a complete and deterministic parity automata that recognizes M (such an automaton exists by Theorem 2), and let n_0 be the number of states of $\mathcal{A}$. Consider a word $w = w_0 w_1 w_2 \ldots$ such that $w \in M$, and let $\rho = q_0 q_1 q_2 \ldots$ be the unique run induced by w in $\mathcal{A}$. Consider a position j in w such that $j \geq n_0$. Then there exist $j \leq i_1 < i_2 \leq j + n_0$ such that $q_{i_1} = q_{i_2}$, this must happen as $\mathcal{A}$ has n_0 states. For $\ell \geq 0$, if we consider the word $w^\ell = w_0 w_1 w_2 \ldots w_{i_1-1} \cdot (w_{i_1} w_{i_1+1} \ldots w_{i_2-1})^\ell \cdot w_{i_2} w_{i_2+1} \ldots$, then the unique run induced by w^ℓ in $\mathcal{A}$ is $\rho^\ell = q_0 q_1 q_2 \ldots q_{i_1-1} \cdot (q_{i_1} q_{i_1+1} \ldots q_{i_2-1})^\ell \cdot q_{i_2} q_{i_2+1} \ldots$. Since the parity condition is independent of finite prefixes and the run ρ is accepted by $\mathcal{A}$, it follows that ρ^ℓ is accepted by $\mathcal{A}$. Since $\mathcal{A}$ recognizes M, it follows $w^\ell \in M$, and the result follows. ∎

We now present the main lemma of this section.

Lemma 2. *For all $(R, G) = (R_i, G_i)_{1 \leq i \leq d}$, where $R_i, G_i \subseteq \Sigma$, we have*

$$\mathsf{UniCloOmg}(\mathrm{Streett}(R, G)) = \mathrm{FinStreett}(R, G);$$

i.e., $\mathrm{FinStreett}(R, G)$ is obtained by applying the $\mathsf{UniCloOmg}$ operator to $\mathrm{Streett}(R, G)$.

Proof. We present the two directions of the proof.

1. We first show that $\mathsf{UniCloOmg}(\mathrm{Streett}(R, G)) \subseteq \mathrm{FinStreett}(R, G)$. Let $M \subseteq \mathrm{Streett}(R, G)$ such that M is closed and ω-regular. Let $w = w_0 w_1 \ldots \in M$, and assume towards contradiction, that $\limsup_k \mathrm{dist}_k(w, (R, G)) = \infty$. Hence for all $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $n \geq n_0$ and $\mathrm{dist}_n(w, (R, G)) \geq n_0$. Let $n_0 \in \mathbb{N}$ given by the pumping lemma on M, from above given n_0 we obtain j such that $j \geq n_0$ and $\mathrm{dist}_j(w, (R, G)) \geq n_0$. By the pumping lemma (Lemma 1), we obtain the witness $j \leq i_1 < i_2 \leq j + n_0$. Let $u = w_0 w_1 \ldots w_{i_1-1}$, $v = w_{i_1} w_{i_1+1} \ldots w_{i_2-1}$ and $w' = w_{i_2} w_{i_2+1} \ldots$. Since $w \in M$, by the pumping lemma for all $\ell \geq 0$ we have $u v^\ell w' \in M$. This entails that all finite prefixes of the infinite word $u v^\omega$ are in $\mathrm{pref}(M)$. Since M is closed, it follows to $u v^\omega \in M$. Since $\mathrm{dist}_j(w, (R, G)) \geq n_0$ it follows that there is some request i in position j, and there is no corresponding grant i for the next n_0 steps. Hence there is a position j' in v such that there is request i at j' and no corresponding grant in v, and thus it follows that the word $u v^\omega \notin \mathrm{Streett}(R, G)$. This contradicts that $M \subseteq \mathrm{Streett}(R, G)$. Hence it follows that $\mathsf{UniCloOmg}(\mathrm{Streett}(R, G)) \subseteq \mathrm{FinStreett}(R, G)$.

2. We now show the converse: $\mathsf{UniCloOmg}(\mathrm{Streett}(R, G)) \supseteq \mathrm{FinStreett}(R, G)$. We have

$$\mathrm{FinStreett}(R, G) = \{w \mid \limsup_k \mathrm{dist}_k(w, (R, G)) < \infty\} = \bigcup_{B \in \mathbb{N}} \{w \mid \limsup_k \mathrm{dist}_k(w, (R, G)) \leq B\}$$
$$= \bigcup_{B \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \{w \mid \forall k \geq n, \mathrm{dist}_k(w, (R, G)) \leq B\}$$

The language $\{w \mid \forall k \geq n, \mathrm{dist}_k(w, (R, G)) \leq B\}$ is closed, ω-regular, and included in $\mathrm{Streett}(R, G)$. It follows $\mathrm{FinStreett}(R, G) \subseteq \mathsf{UniCloOmg}(\mathrm{Streett}(R, G))$.

The result follows. ∎

Corollary 1. *For all $p : \Sigma \to \mathbb{N}$, we have* $\mathsf{UniCloOmg}(\mathrm{Parity}(p)) = \mathrm{FinParity}(p)$.

Proof. This follows from Lemma 2 and the fact that parity condition is a special case of Streett condition. ∎

Corollary 2. *For all $F \subseteq \Sigma$, we have* $\mathsf{UniCloOmg}(\mathrm{CoB\ddot{u}chi}(F)) = \mathrm{CoB\ddot{u}chi}(F)$.

Proof. We show that $\mathrm{CoB\ddot{u}chi}(F)$ properties are stable under $\mathsf{UniCloOmg}$ operator. By Lemma 2 we have $\mathsf{UniCloOmg}(\mathrm{CoB\ddot{u}chi}(F))$ and finitary coBüchi languages coincide, and since finitary coBüchi and coBüchi languages coincide, the result follows. ∎

We now present a characterization for finitary Büchi that will be used in the sequel. For a set $F \subseteq \Sigma$, let $\mathrm{next}_k(w, F) = \inf\{k' - k \mid k' \geq k, w_{k'} \in F\}$.

Corollary 3. *For all $F \subseteq \Sigma$, we have* $\mathsf{UniCloOmg}(\mathrm{B\ddot{u}chi}(F)) = \{w \mid \limsup_k \mathrm{next}_k(w, F) < \infty\}$.

We now present the results for topological characterization of finitary Büchi, parity and Streett languages.

Theorem 3 (Topological characterization of finitary languages). *The following assertions hold:*

1. *For all $p : \Sigma \to \mathbb{N}$, we have* $\mathrm{FinParity}(p) \in \Sigma_2$.
2. *For all $(R, G) = (R_i, G_i)_{1 \leq i \leq d}$, we have* $\mathrm{FinStreett}(R, G) \in \Sigma_2$.
3. *For all $\emptyset \subset F \subset \Sigma$, we have that* $\mathsf{UniCloOmg}(\mathrm{B\ddot{u}chi}(F))$ *is* Σ_2-*complete.*
4. *There exists $p : \Sigma \to \mathbb{N}$ such that* $\mathrm{FinParity}(p)$ *is* Σ_2-*complete.*
5. *There exists $(R, G) = (R_i, G_i)_{1 \leq i \leq d}$ such that* $\mathrm{FinStreett}(R, G)$ *is* Σ_2-*complete.*

Proof. We prove all the cases below.

1. It follows from Corollary 1 and Proposition 2.(a) that $\mathsf{UniCloOmg}(\mathrm{FinParity}(p)) = \mathrm{FinParity}(p)$, and then it follows from Proposition 2.(b) that $\mathrm{FinParity}(p) \in \Sigma_2$.
2. As above it follows from Lemma 2 and Proposition 2.
3. It follows from Proposition 2 that $\mathsf{UniCloOmg}(\mathrm{B\ddot{u}chi}(F)) \in \Sigma_2$. We have that $\mathrm{CoB\ddot{u}chi}(\Sigma \setminus F)$ is Σ_2-complete from Theorem 1. We now present a topological reduction to show that $\mathrm{CoB\ddot{u}chi}(\Sigma \setminus F) \preceq \mathsf{UniCloOmg}(\mathrm{B\ddot{u}chi}(F))$. Recall that $w \in \mathrm{CoB\ddot{u}chi}(\Sigma \setminus F)$ iff $\mathrm{Inf}(w) \subseteq F$. Let $b : \Sigma^\omega \to \Sigma^\omega$ be the stuttering function defined as follows:

$$
\begin{array}{lllllll}
w & = w_0 & w_1 & \ldots & w_n & & \ldots \\
b(w) & = w_0 & \underbrace{w_1 w_1}_{2} & \ldots & \underbrace{w_n w_n \ldots w_n}_{2^n} & & \ldots
\end{array}
$$

   The function b is continuous, since $\mathrm{distword}(b(w), b(w')) \leq \mathrm{distword}(w, w')$. It remains to show the following:
$$
\mathrm{Inf}(w) \subseteq F \ \text{ iff } \ \exists B \in \mathbb{N}, \exists n \in \mathbb{N}, \forall k \geq n, \mathrm{next}_k(b(w), F) \leq B.
$$

   Left to right direction: assume that from the position n of w, letters belong to F. Then from the position $2^n - 1$, letters of $b(w)$ belong to F, then $\mathrm{next}_k(b(w), F) = 0$ for $k \geq 2^n - 1$.
   Right to left direction: let B and n be integers such that for all $k \geq n$ we have $\mathrm{next}_k(b(w), F) \leq B$. Assume $2^{k-1} > B$ and $k \geq n$, then the letter in position $2^k - 1$ in $b(w)$ is repeated 2^{k-1} times, thus $\mathrm{next}_k(b(w), F)$ is either 0 or higher than 2^{k-1}. The latter is not possible since it must be less than B. It follows that the letter in position k in w belongs to F.
4. This follows from item 3 above and the fact that Büchi condition is a special case of parity condition.
5. This follows from item 3 above and the fact that Büchi condition is a special case of Streett condition.

The desired result follows. ∎

# 4 Automata-Theoretic Characterization of Finitary Languages

In this section we consider the automata-theoretic characterization of finitary languages. We compare the expressive power of various classes of automata with finitary acceptance conditions with respect to automata with classical ω-regular acceptance condition.

4.1 Comparison with classical languages

In this section we compare the expressive power of automata with finitary acceptance conditions as compared to automata with classical acceptance conditions. In the examples we will consider $\Sigma = \{a, b\}$.

Example 1 (DFB $\nsubseteq$ NB). Consider the finitary Büchi automaton $\mathcal{A}$ shown in Fig. 1 and the state labeled 0 is the accepting set F. The language of $\mathcal{A}$ is $L_B = \{(b^{j_0}a^{f(0)}) \cdot (b^{j_1}a^{f(1)}) \cdot (b^{j_2}a^{f(2)}) \ldots \mid f : \mathbb{N} \to \mathbb{N}, f$ bounded, $\forall i \in \mathbb{N}, j_i \in \mathbb{N}\}$. Indeed, 0-labeled state is visited while reading the letter b, and the 1-labeled state is visited while reading the letter a. An infinite word w is accepted iff the 0-labeled state is visited infinitely often, and there must be a bound between any two consecutive visits of the 0-labeled state. We now show that L_B is not ω-regular: assume towards contradiction that L_B is ω-regular. Then by
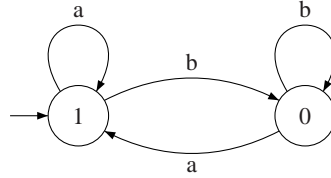

Figure 1. A finitary Büchi automaton $\mathcal{A}$

Theorem 2 there is a deterministic parity automaton $\mathcal{A}'$ that recognizes L_B, having N states. Without loss of generality we assume this automaton to be complete, and let the starting state be q_0. Since the word b^ω belongs to the language, the unique run on this word is accepting and can be decomposed as:

$$q_0 \xrightarrow{b^{n_0}} s_0 \xrightarrow{b^{p_0}} s_0 \xrightarrow{b^{p_0}} s_0 \ldots$$

where s_0 is the lowest priority state visited infinitely often (thus it has even priority), and $n_0 \leq N, 1 \leq p_0 \leq N$. Since the word $b^{n_0}a^\omega$ belongs to the language L_B, we can repeat the above construction. By induction, we define s_k and q_k as shown in the Figure 2: s_k is the lowest priority state visited infinitely often while
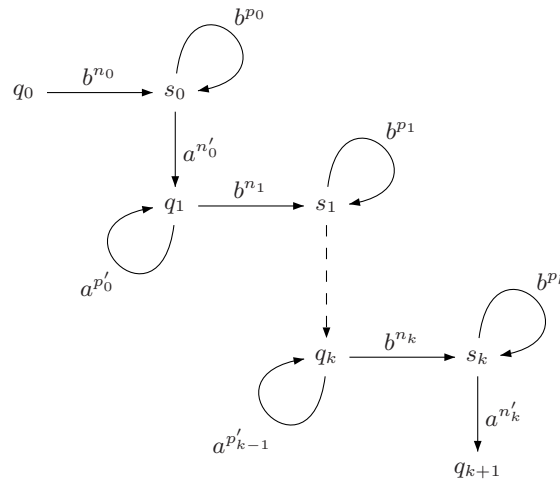
Figure 2. Inductive construction showing that $L_B \notin DP$.

reading $b^{n_0}a^{n'_0} \, b^{n_1}a^{n'_1} \ldots b^{n_{k-1}}a^{n'_{k-1}} \, b^{n_k} \, b^\omega$ (thus it has even priority), and $n_k \leq N, 1 \leq p_k \leq N$,

and similarly for q_k, reading $b^{n_0} a^{n'_0} \; b^{n_1} a^{n'_1} \ldots b^{n_{k-1}} a^{n'_{k-1}} \; a^\omega$. There exists $i < j$ such that $q_i = q_j$, and hence the infinite word $u \cdot (a^{p'_{i-1}} \, v) \cdot (a^{2p'_{i-1}} \, v) \ldots (a^{kp'_{i-1}} \, v) \ldots$, where $u = b^{n_0} a^{n'_0} \ldots a^{n'_{i-1}}$ and $v = b^{n_i} \ldots a^{n'_{j-1}}$, is accepted by $\mathcal{A}'$, and hence we have contradiction that $\mathcal{A}'$ recognizes L_B. ∎

We now show that there exist languages expressed by deterministic Büchi automata that cannot be expressed by non-deterministic finitary parity automata.

Example 2 (DB ⊈ NFP). Consider the language of infinitely many a's, i.e., $L_I = \{w \mid w$ has an infinite number of $a\}$. The language L_I is ω-regular and there is a deterministic Büchi automaton $\mathcal{A}'$ such that the language of $\mathcal{A}'$ is L_I. We now show that there is no non-deterministic finitary parity automata that recognizes L_I. Assume towards contradiction that $\mathcal{A}$ is a non-deterministic finitary parity automata recognizing L_I and let $\mathcal{A}$ have N states. Let us consider the infinite word $w = ab \; ab^2 \; ab^3 \; ab^4 \ldots ab^n \ldots \in L_I$. Since w must be accepted by $\mathcal{A}$, there must be an accepting run ρ, and we represent the accepting run as follows:

$$q_0 \xrightarrow{a} p_0 \xrightarrow{b} q_1 \xrightarrow{a} p_1 \xrightarrow{b^2} q_2 \ldots p_{n-1} \xrightarrow{b^n} q_n \xrightarrow{a} p_n \xrightarrow{b^{n+1}} q_{n+1} \ldots$$

and

$$p_{n-1} \xrightarrow{b} q_{n,1} \xrightarrow{b} q_{n,2} \ldots \xrightarrow{b} q_{n,n-1} \xrightarrow{b} q_{n,n} = q_n \ldots$$

The sequence satisfies that $\exists B \in \mathbb{N}, \exists n \in \mathbb{N}, \forall k \geq n$ we have $\text{dist}_k(\rho, p) \leq B$. Let c be the lowest priority infinitely visited, and c must be even. The state p_{k-1} is in position $\frac{k \cdot (k+1)}{2}$ in ρ. Let k be an integer such that (a) $\frac{k \cdot (k+1)}{2} \geq n$ and (b) $k \geq (N+1) \cdot B$. Let us consider the set of states $\{q_{k,1}, \ldots, q_{k,k}\}$. Since the distance function is bounded by B, the priority c appears at least once in each set of consecutively visited states of size B. Since $k \geq (N+1) \cdot B$, it appears at least $N+1$ times in $\{q_{k,1}, \ldots, q_{k,k}\}$. Since there is N states in $\mathcal{A}$, at least one state has been reached twice. We can thus iterate: the infinite word $w' = ab \; ab^2 \; ab^3 \; ab^4 \ldots b^{k-1} a \; b^\omega$, and the word w' is accepted by $\mathcal{A}$. However, $w' \notin L_I$ and hence we have a contradiction. ∎

Remark 1. From Example 2 we deduce the following result: *NFB* and *NFP* are not closed under complementation. The language $\{a, b\}^\omega \setminus L_I = \{w \mid w$ has a finite number of $a\} \in NFB$ (see Example 3 later); however, Example 2 shows that the complement is not expressible by non-deterministic finitary parity automata.

We summarize the results in the following theorem.

Theorem 4. *The following assertions hold: (a) DB ⊈ NFP and DFP ⊈ NB; (b) DB ⊈ NFB and DFB ⊈ NB.*

4.2 Deterministic finitary automata

In this subsection we consider deterministic automata with finitary acceptance conditions. Given a deterministic complete automaton $\mathcal{A}$ with accepting condition Acc, we will consider the language obtained by the finitary restriction of the acceptance condition. We first consider a function $C_\mathcal{A}$ as follows: $C_\mathcal{A} : \Sigma^\omega \to Q^\omega$ maps an infinite word w to the unique run ρ of $\mathcal{A}$ on w (there is a unique run since $\mathcal{A}$ is deterministic and complete). Then

$$\mathcal{L}(\mathcal{A}) = \{w \mid C_\mathcal{A}(w) \in Acc\} = C_\mathcal{A}^-(Acc).$$

We will focus on the following property: $C_\mathcal{A}^-(\mathsf{UniCloOmg}(Acc)) = \mathsf{UniCloOmg}(C_\mathcal{A}^-(Acc))$, which follows from the following lemma.

Lemma 3. *For all $\mathcal{A} = (Q, \Sigma, q_0, \delta, Acc)$ deterministic complete automaton, we have:*

1. *for all $A \subseteq Q^\omega$, A is closed $\Rightarrow C_\mathcal{A}^-(A)$ closed ($C_\mathcal{A}$ is continuous).*
2. *for all $L \subseteq \Sigma^\omega$, L is closed $\Rightarrow C_\mathcal{A}(L)$ closed ($C_\mathcal{A}$ is closed).*

3. for all $A \subseteq Q^\omega$, A is ω-regular $\Rightarrow C_{\mathcal{A}}^-(A)$ ω-regular.
4. for all $L \subseteq \Sigma^\omega$, L is ω-regular $\Rightarrow C_{\mathcal{A}}(L)$ ω-regular.

Proof. We prove all the cases below.

1. Let $A \subseteq Q^\omega$ such that A is closed. Let w be such that for all $n \in \mathbb{N}$ we have $w_0 \ldots w_n \in \mathrm{pref}(C_{\mathcal{A}}^-(A))$. We define the run $\rho = C_{\mathcal{A}}(w)$ and show that $\rho = q_0 q_1 \ldots \in A$. Since A is closed, we will show for all $n \in \mathbb{N}$ we have $q_0 \ldots q_n \in \mathrm{pref}(A)$. From the hypothesis we have $w_0 \ldots w_{n-1} \in \mathrm{pref}(C_{\mathcal{A}}^-(A))$, and then there exists an infinite word u such that $C_{\mathcal{A}}(w_0 \ldots w_{n-1}u) \in A$. Let $C_{\mathcal{A}}(w_0 \ldots w_{n-1}u) = q_0 q_1' \ldots q_n' \ldots$, then we have $q_0 \xrightarrow{w_0} q_1' \xrightarrow{w_1} q_2' \cdots \xrightarrow{w_{n-1}} q_n' \cdots$. Since $\mathcal{A}$ is deterministic, we get $q_i' = q_i$, and hence $q_0 \ldots q_n \in \mathrm{pref}(A)$.

2. Let $L \subseteq \Sigma^\omega$ such that L is closed. Let $\rho = q_0 q_1 \ldots$ such that for all $n \in \mathbb{N}$ we have $q_0 \ldots q_n \in \mathrm{pref}(C_{\mathcal{A}}(L))$. Then for all $n \in \mathbb{N}$, there exists a word $w_0 w_1 \ldots w_{n-1}$ such that $q_0 \xrightarrow{w_0} q_1 \xrightarrow{w_1} q_2 \ldots \xrightarrow{w_{n-1}} q_n$, and $w_0 w_1 \ldots w_{n-1} \in \mathrm{pref}(L)$. We define by induction on n an infinite nested sequence of finite words $w_0 w_1 \ldots w_n \in \mathrm{pref}(L)$. We denote by w the limit of this nested sequence of finite words. We have that $\rho = C_{\mathcal{A}}(w)$. Since L is closed, $w \in L$.

3. Let $A \subseteq Q^\omega$ such that A recognized by a Büchi automaton $\mathcal{B} = (Q_{\mathcal{B}}, Q, P_0, \tau, F)$. We define the Büchi automaton $\mathcal{C} = (Q \times Q_{\mathcal{B}}, \Sigma, \{q_0\} \times P_0, \gamma, Q_{\mathcal{B}} \times F)$, where $(q_1, p_1) \xrightarrow{\sigma} (q_2, p_2)$ iff $q_1 \xrightarrow{\sigma} q_2$ in $\mathcal{A}$ and $p_1 \xrightarrow{q_1} p_2$ in $\mathcal{B}$. We now show the correctness of our construction. Let $w = w_0 w_1 \ldots$ accepted by $\mathcal{C}$, then there exists an accepting run ρ, as follows:

$$(q_0, p_0) \xrightarrow{w_0} (q_1, p_1) \xrightarrow{w_1} (q_2, p_2) \ldots (q_n, p_n) \xrightarrow{w_n} (q_{n+1}, p_{n+1}) \ldots$$

where the second component visits F infinitely often. Hence:

$$q_0 \xrightarrow{w_0} q_1 \xrightarrow{w_1} q_2 \ldots q_n \xrightarrow{w_n} q_{n+1} \ldots \text{ in } \mathcal{A} \text{ and } p_0 \xrightarrow{q_0} p_1 \xrightarrow{q_1} p_2 \ldots p_n \xrightarrow{q_n} p_{n+1} \ldots \text{ in } \mathcal{B} \quad (\dagger)$$

Hence from ($\dagger$), we have $C_{\mathcal{A}}(w) = q_0 q_1 \cdots \in \mathcal{L}(\mathcal{B}) = A$, and it follows that $w \in C_{\mathcal{A}}^-(A)$. Conversely, let $w \in C_{\mathcal{A}}^-(A)$, then we have $\rho = C_{\mathcal{A}}(w) = q_0 q_1 \cdots \in A = \mathcal{L}(\mathcal{B})$. Then the above statement ($\dagger$) holds, which entails that w is accepted by $\mathcal{C}$. It follows that $\mathcal{C}$ recognizes $C_{\mathcal{A}}^-(A)$.

4. Let $L \subseteq \Sigma^\omega$ such that L is recognized by a Büchi automaton $\mathcal{B} = (Q_{\mathcal{B}}, \Sigma, P_0, \tau, F)$. We define the Büchi automaton $\mathcal{C} = (Q \times Q_{\mathcal{B}}, Q, \{q_0\} \times P_0, \gamma, Q \times F)$, where $(q, p_1) \xrightarrow{q} (q', p_2)$ iff there exists $\sigma \in \Sigma$, such that $q \xrightarrow{\sigma} q'$ in $\mathcal{A}$ and $p_1 \xrightarrow{\sigma} p_2$ in $\mathcal{B}$. A proof similar to above show that $\mathcal{C}$ recognizes $C_{\mathcal{A}}(L)$.

The desired result follows. ∎

Theorem 5. *For all deterministic complete automata $\mathcal{A} = (Q, \Sigma, q_0, \delta, Acc)$ recognizing a language L, the finitary restriction of this automaton $\mathsf{UniCloOmg}(\mathcal{A}) = (Q, \Sigma, q_0, \delta, \mathsf{UniCloOmg}(Acc))$ recognizes $\mathsf{UniCloOmg}(L)$.*

Proof. A word w is accepted by $\mathsf{UniCloOmg}(\mathcal{A})$ iff $w \in C_{\mathcal{A}}^-(\mathsf{UniCloOmg}(Acc)) = \mathsf{UniCloOmg}(C_{\mathcal{A}}^-(Acc)) = \mathsf{UniCloOmg}(L)$. ∎

Theorem 5 allows to extend all known results on deterministic classes to finitary deterministic classes, and we have the following corollary.

Corollary 4. *We have: (a) $DFP = DFS$; (b) $DFB \subset DFP$; (c) $DC \subset DFP$.*

We now show that non-deterministic finitary parity automata is more expressive than deterministic finitary parity automata.

Example 3 (DFP $\subset$ NFP). Consider the following language L_F of finitely many a's,

$$L_F = \{a, b\}^\omega \setminus L_I = \{w \mid w \text{ has a finite number of } a\} \in NFP.$$
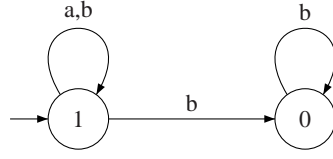
Figure 3. A *NFB* for L_F.

The language L_F is recognized by the non-deterministic finitary Büchi automata shown in Fig 3.

To show that deterministic finitary parity automata are strictly less expressive than non-deterministic finitary parity automata, i.e., $DFP \subset NFP$ we show $L_F \notin DFP$. Assume towards contradiction that there is a deterministic finitary parity automaton $\mathcal{A}$ with N states that recognizes L_F. Without loss of generality we assume this automaton to be complete, and let the starting state be q_0. Since the word b^ω belongs to the language, the unique run on this word is accepting and can be decomposed as:

$$q_0 \xrightarrow{b^{n_0}} s_0 \xrightarrow{b^{p_0}} s_0 \xrightarrow{b^{p_0}} s_0 \dots$$

where s_0 is the lowest priority state visited infinitely often (thus it has even priority), and $n_0, p_0 \leq N$. Let $s_0 \xrightarrow{a} r_0$. Since the word $b^{n_0} a\, b^\omega$ belongs to the language L_F, we can repeat the above construction. By induction, we define s_k and q_k as shown in the Figure 4: s_k is the lowest priority state visited infinitely
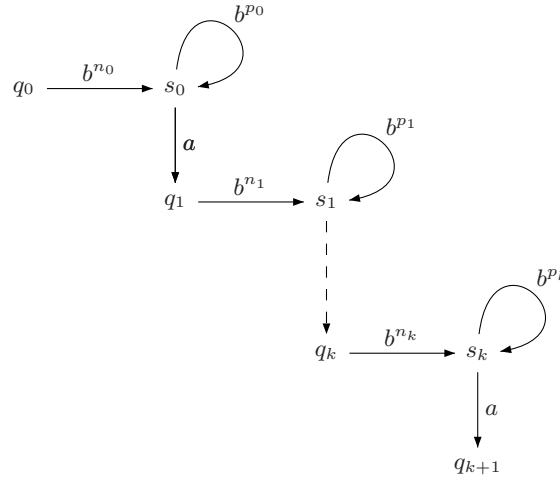
Figure 4. Inductive construction showing that $L_F \notin DFP$.

often while reading $b^{n_0} a\, b^{n_1} a \dots b^{n_{k-1}} a\, b^{n_k} b^\omega$ (thus it has even priority), and $n_k, p_k \leq N$.

There exists $i < j$ such that $q_i = q_j$, and hence the infinite word $u \cdot (b^{n_i+p_i} v) \cdot (b^{n_i+2p_i} v) \dots (b^{n_i+kp_i} v) \dots$ where $u = b^{n_0} a\, b^{n_1} a \dots b^{n_{i-1}} a$ and $v = ab^{n_i} \dots b^{n_{j-1}} a$, is accepted by $\mathcal{A}$.

$$q_0 \xrightarrow{u} q_i \xrightarrow{b^{n_i}} s_i \xrightarrow{b^{p_i}} s_i \xrightarrow{v} q_i \xrightarrow{b^{n_i}} s_i \xrightarrow{b^{2p_i}} \dots$$

Indeed, iterating on s_i's loop ensures that there is no bound between two consecutive visits of a state, for those which are not in this loop. In s_i's loop, s_i has the lowest priority, and it is even. There is a bound between two consecutive visits of s_i: the loop has less than N states, and the way from s_i by v to q_i and back to s_i has constant size $|v| + n_i$. Hence we have contradiction that $\mathcal{A}$ recognizes L_F. ∎

Theorem 6. *We have* $DFP \subset NFP$.

Remark 2. Observe that Theorem 5 does not hold for non-deterministic automata, since we have $DP = NP$ but $DFP \subset NFP$.

4.3 Non-deterministic finitary automata

We now show that non-deterministic finitary Streett automata can be reduced to non-deterministic finitary Büchi automata, and this would complete the picture of automata-theoretic characterization. We first show that non-deterministic finitary Büchi automata are closed under conjunction, and use it to show Theorem 7.

Lemma 4. *NFB is closed under conjunction.*

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_0^1, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_0^2, F_2)$ be two non-deterministic finitary Büchi automata. Without loss of generality we assume both $\mathcal{A}_1$ and $\mathcal{A}_2$ to be complete. We will define a construction similar to the synchronous product construction, where a switch between copies will happen while visiting F_1 or F_2. The finitary Büchi automaton is $\mathcal{A} = (Q_1 \times Q_2 \times \{1, 2\}, \Sigma, \delta, Q_0^1 \times Q_0^2 \times \{1\}, F_1 \times Q_2 \times \{2\} \cup Q_1 \times F_2 \times \{1\})$. We define the transition relation δ below:

$$\begin{aligned}
\delta = \ & \{((q_1, q_2, k), \sigma, (q_1', q_2', k)) \mid q_1' \notin F_1, q_2' \notin F_2, (q_1, \sigma, q_1') \in \delta_1, (q_2, \sigma, q_2') \in \delta_2, k \in \{1, 2\}\} \\
\cup\ & \{((q_1, q_2, 1), \sigma, (q_1', q_2', 2)) \mid q_1' \in F_1, (q_1, \sigma, q_1') \in \delta_1, (q_2, \sigma, q_2') \in \delta_2\} \\
\cup\ & \{((q_1, q_2, 2), \sigma, (q_1', q_2', 1)) \mid q_2' \in F_2, (q_1, \sigma, q_1') \in \delta_1, (q_2, \sigma, q_2') \in \delta_2\}
\end{aligned}$$

Intuitively, the transition function δ is as follows: the first component mimics the transition for automata $\mathcal{A}_1$, the second component mimics the transition for $\mathcal{A}_2$, and there is a switch for the third component from 1 to 2 visiting a state in F_1, and from 2 to 1 visiting a state in F_2.

We now prove the correctness of the construction. Consider a word w that is accepted by $\mathcal{A}_1$, and then there exists a bound B_1 and a run ρ_1 in $\mathcal{A}_1$ such that eventually, the number of steps between two visits to F_1 in ρ_1 is at most B_1; and similarly, there exists a bound B_2 and a run ρ_2 in $\mathcal{A}_2$ such that eventually the number of steps between two visits to F_2 in ρ_2 is at most B_2. It follows that in our construction there is a run ρ (that mimics the runs ρ_1 and ρ_2) in $\mathcal{A}$ such that eventually within $\max\{B_1, B_2\}$ steps a state in $F_1 \times Q_2 \times \{2\} \cup Q_1 \times F_2 \times \{1\}$ is visited in ρ. Hence w is accepted by $\mathcal{A}$. Conversely, consider a word w that is accepted by $\mathcal{A}$, and let ρ be a run and B be the bound such that eventually between two visits to the accepting states in ρ is separated by at most B steps. Let ρ_1 and ρ_2 be the decomposition of the run ρ in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. It follows that both in $\mathcal{A}_1$ and $\mathcal{A}_2$ the respective final states are eventually visited within at most $2 \cdot B$ steps in ρ_1 and ρ_2, respectively. It follows that w is accepted by both $\mathcal{A}_1$ and $\mathcal{A}_2$. Hence we have $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$. ∎

Theorem 7. *We have NFS $\subseteq$ NFP $\subseteq$ NFB.*

Proof. We will present a reduction of *NFS* to *NFB* and the result will follow. Since the Streett condition is a finite conjunction of conditions $\mathrm{Inf}(w) \cap R_i \neq \emptyset \Rightarrow \mathrm{Inf}(w) \cap G_i \neq \emptyset$, by Lemma 4 it suffices to handle the special case when $d = 1$. Hence we consider a non-deterministic Streett automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, (R, G))$ with $(R, G) = (R_1, G_1)$. Without loss of generality we assume $\mathcal{A}$ to be complete. We construct a non-deterministic Büchi automaton $\mathcal{A}' = (Q \times \{1, 2, 3\}, \Sigma, \delta', Q_0 \times \{1\}, Q \times \{2\})$, where the transition relation δ' is given as follows:

$$\begin{aligned}
\delta' = \ & \{(q, 1), \sigma, (q', j) \mid (q, \sigma, q') \in \delta, j \in \{1, 2\}\} \\
\cup\ & \{(q, 2), \sigma, (q', 2) \mid q' \notin R_1, (q, \sigma, q') \in \delta\} \\
\cup\ & \{(q, 2), \sigma, (q', 3) \mid q' \in R_1, (q, \sigma, q') \in \delta\} \\
\cup\ & \{(q, 3), \sigma, (q', 3) \mid q' \notin G_1, (q, \sigma, q') \in \delta\} \\
\cup\ & \{(q, 3), \sigma, (q', 2) \mid q' \in G_1, (q, \sigma, q') \in \delta\}
\end{aligned}$$

In other words, the state component mimics the transition of $\mathcal{A}$, and in the second component: (a) the automaton can choose to stay in component 1, or switch to 2; (b) there is a switch from 2 to 3 upon visiting a state in R_1; and (b) there is a switch from 3 to 2 upon visiting a state in G_1. Consider a word w accepted by $\mathcal{A}$ and an accepting run ρ in $\mathcal{A}$, and let B be the bound on the distance sequence. We show that w is accepted by $\mathcal{A}'$ by constructing an accepting run ρ' in $\mathcal{A}'$. We consider the following cases:

1. If infinitely many requests R_1 are visited in ρ, then in $\mathcal{A}'$ immediately switch to component 2, and then mimic the run ρ as a run ρ' in $\mathcal{A}'$. It follows that from some point j on every request is granted within B steps, and it follows that after position j, whenever the second component is 3, it becomes 2 within B steps. Hence w is accepted by $\mathcal{A}$.
2. If finitely many requests R_1 are visited in ρ, then after some point j, there are no more requests. The automaton $\mathcal{A}'$ mimics the run ρ by staying in the second component as 1 for j steps, and then switches to component 2. Then after j steps we always have the second component as 2, and hence the word is accepted.

Conversely, consider a word w accepted by $\mathcal{A}'$ and consider the accepting run ρ'. We mimic the run in $\mathcal{A}$. To accept the word w, the run ρ' must switch to the second component as 2, say after j steps. Then, from some point on whenever a state with second component 3 is visited, within some bound B steps a state with second component 2 is visited. Hence the run ρ is accepting in $\mathcal{A}$. Thus the languages of $\mathcal{A}$ and $\mathcal{A}'$ coincide, and the desired result follows. ∎

Corollary 5. *We have $DFB \subset DFP \subset NFB = NFP = NFS$.*

Our results establishing the precise automata-theoretic characterization of languages defined by automata with finitary acceptance condition is shown in Fig 5. In general NFP cannot be determinized to a DFP, however, for every language $L \in \mathbb{L}_\omega$ there is $\mathcal{A} \in DP$ such that $\mathcal{A}$ recognizes L, and hence the deterministic finitary parity automata $\mathsf{UniCloOmg}(\mathcal{A})$ recognizes $\mathsf{UniCloOmg}(L)$.

Corollary 6. *For every language $L \in \mathbb{L}_\omega$ there is a deterministic finitary parity automata $\mathcal{A}$ such that $\mathcal{A}$ recognizes $\mathsf{UniCloOmg}(L)$.*
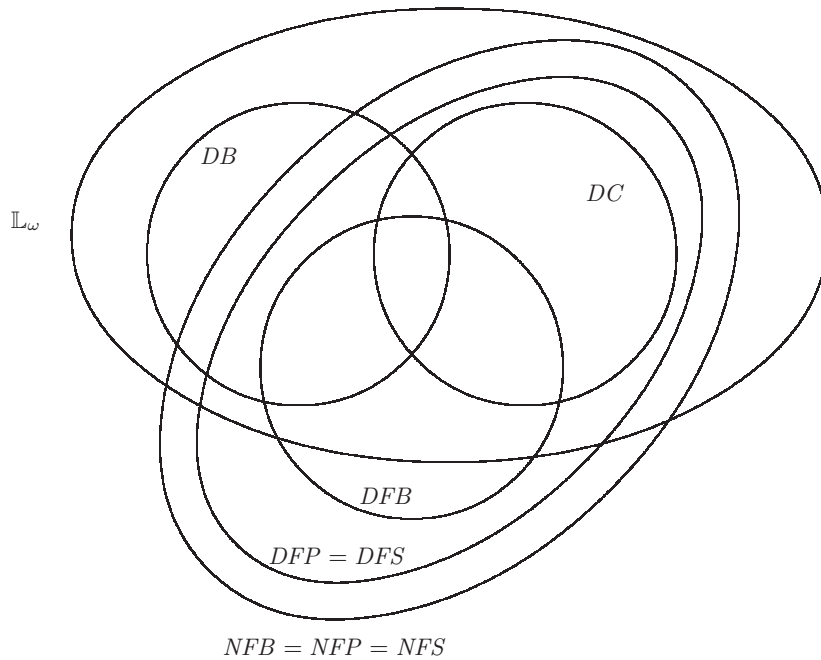
Figure 5. Automata-theoretic characterization

5   Logical Characterization of Finitary Languages

In this section we consider the logical characterization of finitary languages.

Closure properties. For a logical characterization of languages defined by automata with finitary acceptance conditions, we first study the closure properties of deterministic and non-deterministic automata with finitary acceptance conditions. We will consider DFP and NFP.

Theorem 8 (Closure properties). *The following closure properties hold:*

1. *DFP is closed under intersection.*
2. *DFP and NFP are not closed under complementation.*
3. *DFP is not closed under union.*
4. *NFP is closed under union and intersection.*

Proof. We prove all the cases below.

1. Intersection closure for DFP follows from Theorem 5 and from the observation that for all $L, L' \subseteq \Sigma^\omega$ we have $\mathsf{UniCloOmg}(L \cap L') = \mathsf{UniCloOmg}(L) \cap \mathsf{UniCloOmg}(L')$. The observation is proved as follows. Let $M \in \Pi_1 \cap \mathbb{L}_\omega$ and $M \subseteq L \cap L'$, then $M \subseteq \mathsf{UniCloOmg}(L) \cap \mathsf{UniCloOmg}(L')$, and hence $\mathsf{UniCloOmg}(L \cap L') \subseteq \mathsf{UniCloOmg}(L) \cap \mathsf{UniCloOmg}(L')$. Conversely, let $M_1 \subseteq \mathsf{UniCloOmg}(L)$ and $M_2 \subseteq \mathsf{UniCloOmg}(L')$, then $M_1 \cap M_2 \in \Pi_1 \cap \mathbb{L}_\omega$ and $M_1 \cap M_2 \subseteq L \cap L'$. Hence $M_1 \cap M_2 \subseteq \mathsf{UniCloOmg}(L \cap L')$, thus $\mathsf{UniCloOmg}(L) \cap \mathsf{UniCloOmg}(L') \subseteq \mathsf{UniCloOmg}(L \cap L')$.

2. It follows from Example 2 and Example 3 that there is a non-deterministic finitary parity automata that recognizes the language L_F, and the complementary language $\{a, b\}^\omega \setminus L_F = L_I$ is not recognized by a non-deterministic finitary parity automaton. It follows that NFP is not closed under complementation. The result for DFP is similar.

3. As for Example 1 we consider the languages $L_1 = \{(a^{j_0} b^{f(0)}) \cdot (a^{j_1} b^{f(1)}) \cdot (a^{j_2} b^{f(2)}) \ldots \mid f : \mathbb{N} \to \mathbb{N}, f \text{ bounded}, \forall i \in \mathbb{N}, j_i \in \mathbb{N}\}$ and $L_2 = \{(a^{f(0)} b^{j_0}) \cdot (a^{f(1)} b^{j_1}) \cdot (a^{f(2)} b^{j_2}) \ldots \mid f : \mathbb{N} \to \mathbb{N}, f \text{ bounded}, \forall i \in \mathbb{N}, j_i \in \mathbb{N}\}$, also described by the ωB-regular expressions $(a^* b^B)\omega$ and $(b^* a^B)\omega$, respectively. It follows from Example 1 that both L_1 and L_2 belong to DFP, and we show that $L_1 \cup L_2 \notin DFP$. The proof is the very similar to Example 1. Assume towards contradiction that $L_1 \cup L_2 \in DFP$, and let $\mathcal{A}$ be a deterministic complete finitary parity automaton that recognizes $L_1 \cup L_2$. Let $\mathcal{A}$ has N states, and let q_0 be the starting state. Since a^ω belongs to this language, its unique run on $\mathcal{A}$ is accepting: $q_0 \xrightarrow{a^{n_0}} s_0 \xrightarrow{a^{p_0}} s_0 \xrightarrow{a^{p_0}} \ldots$ where $n_0 \leq N$, $1 \leq p_0 \leq N$ and s_0 is the lowest priority visited infinitely often while reading a^ω. Then, $a^{n_0} b^\omega$ belongs to this language, its unique run on $\mathcal{A}$ is accepting: $q_0 \xrightarrow{a^{n_0}} s_0 \xrightarrow{b^{n'_0}} q_1 \xrightarrow{b^{p'_0}} q_1 \xrightarrow{b^{p'_0}} \ldots$ where $n'_0 \leq N$, $1 \leq p'_0 \leq N$ and q_1 is the lowest priority visited infinitely often while reading $a^{n_0} b^\omega$. Repeating this construction and by induction we have: $q_0 \xrightarrow{a^{n_0}} s_0 \xrightarrow{b^{n'_0}} q_1 \xrightarrow{a^{n_1}} s_1 \xrightarrow{b^{n'_1}} \ldots q_k \xrightarrow{a^{n_k}} s_k \xrightarrow{b^{n'_k}} q_{k+1}$ with $s_k \xrightarrow{a^{p_k}} s_k$ and $q_{k+1} \xrightarrow{b^{p'_k}} q_{k+1}$, where $n_k, n'_k \leq N$ and $1 \leq p_k, p'_k \leq N$; and s_k is the lowest priority visited infinitely often while reading $a^{n_0} b^{n'_0} \ldots a^{n_k} a^\omega$; and q_{k+1} is the lowest priority visited infinitely often while reading $a^{n_0} b^{n'_0} \ldots a^{n_k} b^{n'_k} b^\omega$. There must be $i < j$, such that $q_i = q_j$. Let $u = a^{n_0} b^{n'_0} \ldots b^{n'_{i-1}}$ and $v = b^{n'_i} \ldots b^{n'_{j-1}}$. The word

$$w^* = u \cdot (b^{p'_i} a^{n_i + p_i} v) \cdot (b^{2p'_i} a^{n_i + 2p_i} v) \ldots (b^{kp'_i} a^{n_i + kp_i} v) \ldots$$

is accepted by $\mathcal{A}$, but does not belong to $L_1 \cup L_2$. Hence we have a contradiction, and the result follows.

4. Union closure for NFP is obvious, intersection closure for NFP follows from Lemma 4, since $NFP = NFB$ (Corollary 5).

The result follows.   ∎

Comparison with ωB-regular expressions. We now study the expressive power of NFP as compared to ωB-regular expressions. The class of ωB-regular expressions was introduced in the work of [BC06] as an extension of ω-regular expressions. Regular expressions defines exactly regular languages over finite words, and has the following grammar:

$$L := \emptyset \mid \varepsilon \mid \sigma \mid L \cdot L \mid L^* \mid L + L; \quad \sigma \in \Sigma$$

In the above grammar, $\cdot$ stands for concatenation, $*$ for Kleene star and $+$ for union. Then ω-regular languages are finite union of $L \cdot L'^\omega$, where L and L' are regular languages of finite words. The class of ωB-regular languages, as defined in [BC06], is exactly described by finite union of $L \cdot M^\omega$, where L is a regular language over finite words and M is a B-regular language over infinite sequences of finite words. The grammar for B-regular languages is as follows:

$$M := \emptyset \mid \varepsilon \mid \sigma \mid M \cdot M \mid M^* \mid M^B \mid M + M; \quad \sigma \in \Sigma$$

The semantics of regular languages over infinite sequences of finite words will assign to a B-regular expression M, a language in $(\Sigma^*)^\omega$. The infinite sequence $\langle u_0, u_1, \ldots \rangle$ will be denoted by $\boldsymbol{u}$. The semantics is defined by structural induction as follows.

- $\emptyset$ is the empty language,
- ε is the language containing the single sequence $(\varepsilon, \varepsilon, \ldots)$,
- a is the language containing the single sequence $(a, a, \ldots)$,
- $M_1 \cdot M_2$ is the language $\{\langle u_0 \cdot v_0, u_1 \cdot v_1, \ldots \rangle \mid \boldsymbol{u} \in M_1, \boldsymbol{v} \in M_2\}$,
- M^* is the language $\{\langle u_1 \ldots u_{f(1)-1}, u_{f(1)} \ldots u_{f(2)-1}, \ldots \rangle \mid \boldsymbol{u} \in M, f : \mathbb{N} \to \mathbb{N}\}$,
- M^B is defined like M^* but we additionally require the values $f(i+1) - f(i)$ to be bounded,
- $M_1 + M_2$ is $\{\boldsymbol{w} \mid \boldsymbol{u} \in M_1, \boldsymbol{v} \in M_2, \forall i, w_i \in \{u_i, v_i\}\}$.

Finally, the ω operator on sequences with nonempty words on infinitely many coordinates: $\langle u_0, u_1, \ldots \rangle^\omega = u_0 u_1 \ldots$. This operation is naturally extended to languages of sequences by taking the ω power of every sequence in the language (ignoring those with nonempty words on finitely many coordinates). The class of ωB-regular languages is more expressive than NFP, and this is due to the $*$-operator. We will consider the following fragment of ωB-regular languages where we do not consider the $*$-operator for B-regular expressions (however, the $*$-operator is allowed for L, regular languages over finite words). We call this fragment the star-free fragment of ωB-regular languages. In the following two lemmas we show that star-free ωB-regular expressions express exactly $NFP = NFB$.

Lemma 5. *All languages in NFP can be described by a star-free ωB-regular expression.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, p)$ be a non-deterministic finitary parity automaton. Without loss of generality we assume $Q = \{1, \ldots, n\}$. Let $C = p(Q)$ be the set of priorities, $Q^{\text{even}} = \{q \in Q \mid p(q) \text{ even}\}$ the set of states with even priority and $Q^{\geq c} = \{q \in Q \mid p(q) \geq c\}$ the set of states with priority at least c. Let $L_{q,q'} = \{u \in \Sigma^* \mid q \xrightarrow{u} q'\}$ and $M_q^{\geq c} = \{\boldsymbol{u} \mid (|u_i|)_i \text{ is bounded and } \forall i, q \xrightarrow{u_i} q \text{ where all intermediate visited states have priority greater than } c\}$. Then

$$\mathcal{L}(\mathcal{A}) = \bigcup_{q_0 \in Q_0, q \in Q^{\text{even}}} L_{q_0,q} \cdot (M_q^{\geq p(q)})^\omega.$$

For all $q, q' \in Q$ we have $L_{q,q'} \subseteq \Sigma^*$ is regular. We now show that for all $q \in Q$ and $c \in C$ the language $M_q^{\geq c}$ is B-regular. We fix $c \in C$, and then for simplicity of notation abbreviate $M_q^{\geq c}$ to M_q. For all $0 \leq k \leq n$ and $q, q' \in Q$, let $M_{q,q'}^k = \{\boldsymbol{u} \mid (|u_i|)_i \text{ is bounded and } \forall i, q \xrightarrow{u_i} q' \text{ where all intermediate visited states are from } \{1, \ldots, k\} \text{ and have priority greater than } c\}$. We show by induction on $0 \leq k \leq n$ that for all $q, q' \in Q$ the language $M_{q,q'}^k$ is B-regular. The base case $k = 0$ follows from observation:

$$M_{q,q'}^0 = \begin{cases} a_1 + a_2 + \cdots + a_l & \text{if } q \neq q' \text{ and } (q, a, q') \in \delta \iff \exists i \in \{1, \ldots, l\}, a = a_i \\ \varepsilon + a_1 + a_2 + \cdots + a_l & \text{if } q = q' \text{ and } (q, a, q') \in \delta \iff \exists i \in \{1, \ldots, l\}, a = a_i \\ \emptyset & \text{otherwise} \end{cases}$$

The inductive case for $k > 0$ follows from observation:

$$M_{q,q'}^k = M_{q,k}^{k-1} \cdot (M_{k,k}^{k-1})^B \cdot M_{k,q'}^{k-1} + M_{q,q'}^{k-1}$$

Since $M_{q,q}^n = M_q$, we conclude that $\mathcal{L}(\mathcal{A})$ is described by a star-free ωB-regular expression. ∎

Lemma 6. *All languages described by a star-free ωB-regular expression is recognized by a non-deterministic finitary Büchi automaton.*

Proof. To prove this result, we will describe automata reading infinite sequences of finite words, and corresponding acceptance conditions. Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ a finitary Büchi automaton. While reading an infinite sequence $\boldsymbol{u}$ of finite words, $\mathcal{A}$ will accept if the following conditions are satisfied: (1) $\exists q_0 \in Q_0, q_1, q_2, \ldots \in F, \forall i \in \mathbb{N}$, we have $q_i \xrightarrow{u_i} q_{i+1}$ and (2) $(|u_n|)_n$ is bounded.

We show that for all M star-free B-regular expression, there exists a non-deterministic finitary Büchi automaton accepting M^B, language of infinite sequence of finite words, as described above. We proceed by induction on M.

- The cases $\emptyset, \varepsilon$ and $a \in \Sigma$ are easy.
- From M to M^B, the same automaton for M works for M^B as well, since B is idempotent.
- From M_1, M_2 to $M_1 + M_2$: this involves non-determinism. The automaton guesses for each finite word which word is used. Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_1^0, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_2^0, F_2)$ two non-deterministic finitary Büchi automata accepting M_1^B and M_2^B, respectively. For $k \in \{1, 2\}$ and $T \subseteq Q_k$, we define $\text{Final}(T) = \{q' \in F_k \mid \exists q \in T, \exists u \in \Sigma^*, q \xrightarrow{u}_{\mathcal{A}_k} q'\}$ to be the state of final states reachable from a state in T. We denote by Final^k the k-th iteration of Final, e.g., $\text{Final}^3(T) = \text{Final}(\text{Final}(\text{Final}(T)))$.

  We define a finitary Büchi automaton:

  $$\mathcal{A} = (\underbrace{(Q_1 \times 2^{Q_1}) \cup (Q_2 \times 2^{Q_1})}_{\text{computation states}} \cup \underbrace{2^{Q_1} \times 2^{Q_2}}_{\text{guess states}}, \Sigma, \delta, (Q_1^0, Q_2^0), F)$$

  where

  $$\begin{aligned}
  \delta = &\{((Q, Q'), \varepsilon, (q, \text{Final}(Q'))) \mid q \in Q\} && \text{(guess is 1)}\\
  \cup &\{((Q, Q'), \varepsilon, (q', \text{Final}(Q))) \mid q' \in Q'\} && \text{(guess is 2)}\\
  \cup &\{((q, T), \sigma, (q', T)) \mid (q, \sigma, q') \in \delta_1 \cup \delta_2\}\\
  \cup &\{((q_1, T), \varepsilon, (\{q_1\}, T)) \mid q_1 \in F_1\}\\
  \cup &\{((q_2, T), \varepsilon, (T, \{q_2\})) \mid q_2 \in F_2\}
  \end{aligned}$$

  There are two kinds of states. Computation states are (q, T) where $q \in Q_1$ and $T \subseteq Q_2$ (or symmetrically $q \in Q_2$ and $T \subseteq Q_1$), where q is the current state of the automaton that has been decided to use for the current finite word, and T is the set of final states of the other automaton that would have been reachable if one had chosen this automaton. Guess states are (Q, Q'), where Q is the set of states from $\mathcal{A}_1$ one can start reading the next word, and similarly for Q'.

  We now prove the correctness of our construction. Consider an infinite sequence $\boldsymbol{w}$ accepted by $\mathcal{A}$, and consider an accepting run ρ. There are three cases:
  1. either all guesses are 1;
  2. or all guesses are 2;
  3. else, both guesses happen.

  The first two cases are symmetric. In the first, we can easily see that $\boldsymbol{w}$ is accepted by $\mathcal{A}_1$, and similarly in the second $\boldsymbol{w}$ is accepted by $\mathcal{A}_2$.

  We now consider the third case. There are two symmetric subcases: either the first guess is 1, then

  $$\rho = (Q_1^0, Q_2^0) \cdot (q_1^0, \text{Final}(Q_2^0)) \ldots,$$

  with $q_1^0 \in Q_1^0$; or symmetrically the first guess is 2, then

  $$\rho = (Q_1^0, Q_2^0) \cdot (q_2^0, \text{Final}(Q_1^0)) \ldots,$$

with $q_2^0 \in Q_2^0$. We consider only the first subcase. Then

$$\rho = (Q_1^0, Q_2^0) \cdot (q_1^0, \mathrm{Final}(Q_2^0)) \ldots (q_1^1, \mathrm{Final}(Q_2^0)) \cdot (\{q_1^1\}, \mathrm{Final}(Q_2^0)) \ldots,$$

where u_0 is a finite prefix of $\boldsymbol{w}^\omega$ such that $q_1^0 \xrightarrow{u_0} q_1^1$ in $\mathcal{A}_1$ and $q_1^1 \in F_1$. We denote by ρ_0 the finite prefix of ρ up to $(q_1^1, \mathrm{Final}(Q_2^0))$. Let k be the first time when guess is 2: then

$$\rho = \rho_0 \cdot \rho_1 \cdot \rho_{k-1} \cdot (\{q_1^k\}, \mathrm{Final}^k(Q_2^0)) \cdot (q_2^0, \mathrm{Final}(\{q_k\})) \ldots,$$

where $q_2^0 \in \mathrm{Final}^k(Q_2^0)$ and for $1 \le i \le k-1$, we have

$$\rho_i = (\{q_1^i\}, \mathrm{Final}^i(Q_2^0)) \cdot (q_1^i, \mathrm{Final}^{i+1}(Q_2^0)) \ldots (q_1^{i+1}, \mathrm{Final}^{i+1}(Q_2^0)),$$

and u_i is a finite word such that $q_1^i \xrightarrow{u_i} q_1^{i+1}$ in $\mathcal{A}_1, q_1^{i+1} \in F_1$ and $u_0 u_1 \ldots u_{k-1}$ finite prefix of $\boldsymbol{w}^\omega$. Since $q_2^0 \in \mathrm{Final}^k(Q_2^0)$, there exists $v_0, v_1, \ldots, v_{k-1}$ finite words and $q_2^1, \ldots, q_2^k \in F_2$ such that: $q_2^0 \xrightarrow{v_0} q_2^1 \xrightarrow{v_1} \ldots \xrightarrow{v_{k-1}} q_2^k$. Then we can repeat this by induction, constructing $\boldsymbol{u} \in M_1^B$ and $\boldsymbol{v} \in M_2^B$, such that for all $i \in \mathbb{N}$, we have $w_i \in \{u_i, v_i\}$.

Conversely, let $\boldsymbol{u} \in M_1^B$ and $\boldsymbol{v} \in M_2^B$, and $\boldsymbol{w}$ such that $\forall i \in \mathbb{N}, w_i \in \{u_i, v_i\}$. Using $\mathcal{A}_1$ when $w_i = u_i$ and $\mathcal{A}_2$ otherwise, one can construct an accepting run for $\boldsymbol{w}$ and $\mathcal{A}$. Hence $\mathcal{A}$ recognizes $(M_1 + M_2)^B$.

– From M_1, M_2 to $M_1 \cdot M_2$: the automaton keeps tracks of pending states while reading the other word. Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_1^0, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_2^0, F_2)$ two non-deterministic finitary Büchi automata accepting M_1^B and M_2^B, respectively. Let $\mathcal{A} = ((Q_1 \times F_2) \cup (Q_2 \times F_1), \Sigma, \delta, Q_1^0 \times Q_2^0, F_1 \times F_2)$, where

$$\begin{aligned}
\delta = &\ \{((q,f), \sigma, (q',f)) \mid (q, \sigma, q') \in \delta_1, f \in F_2\} \\
&\cup \{((q,f), \sigma, (q',f)) \mid (q, \sigma, q') \in \delta_2, f \in F_1\} \\
&\cup \{((q_1,f), \varepsilon, (f, q_1)) \mid q_1 \in F_1\} \\
&\cup \{((q_2,f), \varepsilon, (f, q_2)) \mid q_2 \in F_2\}
\end{aligned}$$

Intuitively, the transition relation is as follows: either one is reading using $\mathcal{A}_1$ or $\mathcal{A}_2$. In both cases, the automaton remembers the last final state visited while reading in the other automaton in order to restore this state for the next word. Let $\boldsymbol{w}$ accepted by $\mathcal{A}$, an accepting run is as follows:

$$(q_1^0, q_2^0) \xrightarrow{w_0} (q_1^1, q_2^1) \xrightarrow{w_1} \ldots (q_1^i, q_2^i) \xrightarrow{w_i} (q_1^{i+1}, q_2^{i+1}) \ldots$$

where $(q_1^0, q_2^0) \in Q_1^0 \times Q_2^0$, for all $i \ge 1$, we have $(q_1^i, q_2^i) \in F_1 \times F_2$ and $(|w_n|)_n$ bounded. From the construction, for all $i \in \mathbb{N}$, we have $w_i = u_i^0 \cdot v_i^0 \cdot u_i^1 \cdot v_i^1 \ldots u_i^{k_i} \cdot v_i^{k_i}$, where

$$q_1^i = q_1^i(0) \xrightarrow{u_i^0} q_1^i(1) \xrightarrow{u_i^1} q_1^i(2) \ldots \xrightarrow{u_i^{k_i}} q_1^i(k_i + 1) = q_1^{i+1} \quad \text{in } \mathcal{A}_1$$

$$q_2^i = q_2^i(0) \xrightarrow{v_i^0} q_2^i(1) \xrightarrow{v_i^1} q_2^i(2) \ldots \xrightarrow{v_i^{k_i}} q_2^i(k_i + 1) = q_2^{i+1} \quad \text{in } \mathcal{A}_2$$

the states $(q_1^i(k), q_2^i(k))$ belong to $F_1 \times F_2$. We define $u_i = u_i^0 u_i^1 \ldots u_i^{k_i}$ and $v_i = v_i^0 v_i^1 \ldots v_i^{k_i}$. From the above follows that $\boldsymbol{u}$ and $\boldsymbol{v}$ are accepted by $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then $\boldsymbol{w} \in (M_1 \cdot M_2)^B$. Conversely, a sequence in $(M_1 \cdot M_2)^B$ is clearly accepted by $\mathcal{A}$. Hence $\mathcal{A}$ recognizes $(M_1 \cdot M_2)^B$.

We now prove that all star-free ωB-regular expressions are recognized by a non-deterministic finitary Büchi automaton. Since NFB are closed under finite union (Theorem 8), we only need to consider expressions $L \cdot M^\omega$, where $L \subseteq \Sigma^*$ is regular language of finite words and M star-free B-regular expression. The constructions above ensure that there exists $\mathcal{A}_M = (Q_M, \Sigma, \delta_M, Q_M^0, F_M)$, a non-deterministic finitary Büchi automaton that recognizes the language M^B of infinite sequences. Let $\mathcal{A}_L = (Q_L, \Sigma, \delta_L, Q_L^0, F_L)$ be a finite automaton over finite words that recognizes L. We construct a non-deterministic finitary Büchi automaton as follows: $\mathcal{A} = (Q_L \cup Q_M, \Sigma, \delta, Q_L^0, F_M)$ where $\delta = \delta_L \cup \delta_M \cup \{(q, \varepsilon, q') \mid q \in F_L, q' \in Q_M^0\}$. In other words, first $\mathcal{A}$ simulates $\mathcal{A}_L$, and when a finite prefix is recognized by $\mathcal{A}_L$, then $\mathcal{A}$ turns to $\mathcal{A}_M$ and simulates it.

We argue that $\mathcal{A}$ recognizes $L \cdot M^\omega$. Let w accepted by $\mathcal{A}$, and u the finite prefix read by $\mathcal{A}_L$, $w = u \cdot v$. From v infinite word, we define $\boldsymbol{v}$ an infinite sequence of finite words by sequencing v each time a final state (i.e., from F_L) is visited. The sequence $\boldsymbol{v}$ is accepted by $\mathcal{A}_M$, hence belongs to M^B, and since $\boldsymbol{v}^\omega = v$, we have $v \in (M^B)^\omega = M^\omega$, and finally $w \in L \cdot M^\omega$. Conversely, let $w = u \cdot \boldsymbol{v}^\omega$, where $u \in L$ and $\boldsymbol{v} \in M^B$. Let $q_0 \in Q_L^0, q \in F_L$ such that $q_0 \xrightarrow{u} q$. Let $q' \in Q_0, q_1, q_2, \ldots \in F_L$, such that for all $i \in \mathbb{N}$ we have $q_i \xrightarrow{v_i} q_{i+1}$. The *key*, yet simple observation is that for all M star-free B-regular expression, for all $\boldsymbol{v} \in M$, $(|v_n|)_n$ is bounded. This is straightforward by induction on M. Hence, from position $|u|$, the set F_L is visited infinitely many times, and there is a bound between two consecutive visits. Thus w is accepted by $\mathcal{A}$. ∎

The following theorem follows from Lemma 5 and Lemma 6.

Theorem 9. *NFP has exactly the same expressive power as star-free ωB-regular expressions.*

Monadic second-order logic. We now consider monadic second-order logic (MSOL). Terms are either 0 or a first-order variable i: $t := 0 \mid i$. They will stand for positions. Atomic formulas are of the form, where t, t' are terms and X second-order variable:

$$A := t = t' \mid S(t, t') \mid t < t' \mid t \in X \mid Q_a(t) \text{ for } a \in \Sigma$$

MSOL formulas are generated by the grammar:

$$\phi := A \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg\phi \mid \exists i, \phi \mid \forall i, \phi \mid \exists X, \phi \mid \forall X, \phi$$

Languages described by atomic formulas lie in *NFP*. We now consider closure properties of *NFP* under the logical constructors used in MSOL:

- The closure under conjunction and disjunction follows from Theorem 8.
- The failure of closure under negation follows from Theorem 8.
- The closure under existential quantification (both first and second-order) follows from non-determinacy.
- The closure under universal quantification (both first and second-order) fails: $L = \{w \mid w$ has an infinite number of $a\} \notin NFP$ as shown in Example 2, but it can be described using a universal quantifier and an existential one: $L = \{w \mid w \models \forall n, \exists k, k \geq n \wedge Q_a(k)\}$.

We already saw that *DFP* and *NFP* are not included in $\mathbb{L}_\omega$, thus MSOL is not expressive enough to describe *DFP* nor *NFP*, as MSOL describes exactly $\mathbb{L}_\omega$. MSOL$\mathbb{A}$ [BC06] is an extension of MSOL where we add the *above bounded quantifier* $\mathbb{A}$, whose semantics is:

$$\mathbb{A}X.\phi := \exists N \in \mathbb{N}, \forall X, |X| \geq N \Rightarrow \phi(X)$$

MSOL$\mathbb{A}$ is the set of formulas containing MSOL and closed under $\vee, \wedge, \forall, \exists$ and $\mathbb{A}$. MSOL$\mathbb{A}$ is equivalent in expressive power as ωB-regular expressions. Since *NFP* corresponds to the star-free fragment (Theorem 9), which is less expressive than ωB-regular expressions, it follows that MSOL$\mathbb{A}$ is strictly more expressive than *NFP*.

Theorem 10. MSOL$\mathbb{A}$ *is strictly more expressive than NFP*.

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