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Derivation of the Time Dependent Gross–Pitaevskii Equation in Two Dimensions

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Abstract: We present microscopic derivations of the defocusing two-dimensional cubic nonlinear Schrödinger equation and the Gross–Pitaevskii equation starting from an interacting N-particle system of bosons. We consider the interaction potential to be given either by $W_\beta(x) = N^{-1+2\beta} W(N^\beta x)$, for any $\beta > 0$, or to be given by $V_N(x) = e^{2N} V(e^N x)$, for some spherical symmetric, nonnegative and compactly supported $W, V \in L^\infty(\mathbb{R}^2, \mathbb{R})$. In both cases we prove the convergence of the reduced density corresponding to the exact time evolution to the projector onto the solution of the corresponding nonlinear Schrödinger equation in trace norm. For the latter potential V_N we show that it is crucial to take the microscopic structure of the condensate into account in order to obtain the correct dynamics.

Contents

1. Introduction

We are interested in the time evolution of bosonic quantum systems of N particles in two dimensions that interact with each other by a two-particle interaction potential. At a given time t, the state of the system is described by a wave function $\Psi_t \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$, where $L_s^2(\mathbb{R}^{2N}, \mathbb{C})$ denotes the Hilbert space of all $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ which are symmetric under permutations of the variables $x_1, \ldots, x_N \in \mathbb{R}^2$. The Hamiltonian of the system is given by

$$H_U = -\sum_{j=1}^{N} \Delta_j + \sum_{1 \le j < k \le N} U(x_j - x_k) + \sum_{j=1}^{N} A_t(x_j) \tag{1}$$

with $A_{\cdot} \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ being a time-dependent external potential and $U \colon \mathbb{R}^2 \to \mathbb{R}$ modeling the interaction between the particles. The time evolution of the system is described by the Schrödinger equation

$$i\partial_t \Psi_t = H_U \Psi_t \tag{2}$$

with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$. In general, even for small particle numbers N, it is not possible to solve the Schrödinger equation exactly or numerically. The time evolution of the system, however, can approximately be determined if one studies special classes of initial conditions and certain types of interaction potentials. In this paper, we are concerned with the dynamical evolution of a Bose–Einstein condensate. This state of matter appears if one cools bosons in an external trapping potential near absolute zero temperature such that almost all particles occupy the same quantum state (see e.g. [38] for a comprehensive discussion). After the trapping potential has been changed or completely switched off, the condensate is no longer in equilibrium and one would like to study its evolution in space.

    Mathematically, the appearance of a Bose–Einstein condensate is described by means of the one-particle reduced density matrix $\gamma_\Psi^{(1)}$ of the state Ψ. $\gamma_\Psi^{(1)}$ is a non-negative trace class operator on $L^2(\mathbb{R}^2, \mathbb{C})$ with an integral kernel given by

$$\gamma_\Psi^{(1)}(x, x') = \int_{\mathbb{R}^{2N-2}} \overline{\Psi(x, x_2, \ldots, x_N)} \Psi(x', x_2, \ldots, x_N) d^2x_2 \ldots d^2x_N.$$

A state Ψ is said to exhibit complete Bose–Einstein condensation, if there exists a one-particle wave function $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi\| = 1$ such that $\gamma_\Psi^{(1)} \to |\varphi\rangle\langle\varphi|$ in trace norm as $N \to \infty$.[1] Initially, we consider a complete condensed state Ψ_0 and then show

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   [1] We like to remark that it is well known that the convergence of $\gamma_\Psi^{(1)}$ to $|\varphi\rangle\langle\varphi|$ in trace norm is equivalent to the respective convergence in operator norm since $|\varphi\rangle\langle\varphi|$ is a rank-1-projection, see Remark 1.4. in [51]. For other indicators of condensation and their relation we refer to [41].

that $\gamma_{\Psi_t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t|$ as $N \to \infty$, where φ_t solves a nonlinear Schrödinger equation. This statement shows that the condensate is stable during the time evolution. Moreover, it proves that the time-evolution of the one-particle reduced density matrix which is given by the many-body Schrödinger equation can approximately be described by a much simpler nonlinear one-particle equation.

To state the exact form of the one-particle equation, we specify the potentials U we are interested in.

- For $\beta > 0$, we consider the so called nonlinear Schrödinger (NLS) scaling $U(x) = W_{\beta,N}(x) = N^{-1+2\beta} W(N^\beta x)$, for a compactly supported, spherically symmetric and nonnegative potential $W \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$.
  In the case of $\beta > 1/2$, such a scaling models strong but short range repulsive interactions. The origin of the scaling can heuristically be motivated by the fact that for a completely factorized wave function $\Psi = \varphi^{\otimes N}$ with $\varphi \in H^2(\mathbb{R}^2, \mathbb{C})$ the kinetic energy per particle[2] $\frac{1}{N}\langle\!\langle\Psi, \sum_{k=1}^N (-\Delta_k)\Psi\rangle\!\rangle = -\langle\varphi, \Delta\varphi\rangle = \mathcal{O}(1)$ is of the same order as the potential energy per particle $\frac{1}{N}\langle\!\langle\Psi, \sum_{1<j<k<N}^N W_\beta(x_j - x_k)\Psi\rangle\!\rangle = \mathcal{O}(1)$.
- We also consider exponentially scaled potentials $U(x) = V_N(x) = e^{2N}V(e^N x)$ with $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$ being spherically symmetric and nonnegative. This scaling will be denoted Gross–Pitaevskii scaling in the following.
  The motivation to consider an exponential scaling is similar to the Gross Pitaevskii scaling $V_N(x) = N^2 V(Nx)$ in three space dimensions. Namely, the kinetic and interaction energy are of the same order for a gas of fixed volume. This will be shown below, when discussing the scattering process of two particles, see (4). Furthermore, the interaction originates from a N-independent potential by rescaling space and time coordinates [see (7)]. Our results can be generalized to a wider class of N-dependent interactions covering most of the relevant cases discussed in the literature on two dimensional Bose gases [39].

For these scalings the condensate wave function φ_t satisfies the cubic nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\varphi_t = (-\Delta + A_t)\,\varphi_t + b_U|\varphi_t|^2\varphi_t =: h_{b_U}^{GP}\varphi_t \tag{3}$$

with initial datum φ_0. The precise definition of b_U will be given in Definition 2.1. At the moment however if suffices to note that for the potentials from above we have $b_{W_{\beta,N}} = N\|W_{\beta,N}\|_1 = \|W\|_1$ if $U = W_{\beta,N}$ and $b_{V_N} = 4\pi$ for $U = V_N$. In case that the coupling constant is given by $b_{V_N} = 4\pi$ Eq. (3) is also referred to as Gross–Pitaevskii equation.

We are going to explain on a heuristic level why the coupling constants differ in the NLS and Gross–Pitaevskii scaling. We first consider the exponential scaling and assume that the energy of the many-body state Ψ_t is comparable to the ground state energy of the system. In this case, the wave function develops a short scale correlation structure which prevents the particles from being too close to each other [39]. If we neglect for the moment all but two particle correlations, one may heuristically think of Ψ_t to be of Jastrow-type [38, p. 15 and p. 28], i.e. $\Psi_t(x_1, \ldots, x_N) \approx \prod_{i<j} F(x_i - x_j) \prod_{k=1}^N \varphi_t(x_k)$. The function F accounts for pair correlations between the particles at scales of order $\mathcal{O}(e^{-N})$. These correlations determine the time evolution of the condensate in a crucial

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[2] Throughout the paper we use the notation $\langle\!\langle\cdot,\cdot\rangle\!\rangle = \langle\cdot,\cdot\rangle_{L_s^2(\mathbb{R}^{2N},\mathbb{C})}$ while $\langle\cdot,\cdot\rangle$ always refers to the scalar product of $L^2(\mathbb{R}^2,\mathbb{C})$.

manner and must therefore explicitly be taken into account. Since V_N is a strong, short range potential, the interaction between the particles can in first order be described as a two-body scattering process. That is, the correlation function F should approximately be given by the zero energy scattering state $j_{N,R} \in C^1(\mathbb{R}^2, \mathbb{R})$ which is defined by

$$
\begin{cases}
\left(-\Delta_x + \frac{1}{2}e^{2N} V(e^N x)\right) j_{N,R}(x) = 0, \\
j_{N,R}(x) = 1 \text{ for } |x| = R
\end{cases}
\tag{4}
$$

for some $R \in (0, \infty)$ used to normalize $j_{N,R}$ via the second line of (4). Note, that it is a peculiarity of two dimensional scattering states that $\lim_{x \to \infty} |j_{N,R}(x)|$ does not exist for short range potentials and can not be used for normalization. A particle at location x then experiences the effective interaction

$$
\int_{\mathbb{R}^2} d^2 y N V_N(x-y) j_{N,R}(x-y)|\varphi_t(y)|^2 \approx |\varphi_t(x)|^2 \int_{\mathbb{R}^2} d^2 x N V_N(x) j_{N,R}(x),
$$

see e.g. [19] for a nice derivation. It will be shown in Sect. 5 that

$$
N \int_{\mathbb{R}^2} d^2 x V_N(x) j_{N,R}(x) = N \frac{4\pi}{\ln\left(\frac{R}{ae^{-N}}\right)},
$$

where a denotes the scattering length of the potential V. Since $\frac{4\pi}{\ln\left(\frac{R}{ae^{-N}}\right)} \approx \frac{4\pi}{N}$ holds for $a > 0$, the effective coupling b_{V_N} will be given by 4π. This shows that the scaling we used gives us a system where the kinetic energy and the interaction energy are of the same order.

Let us now turn to the NLS scaling and consider for $\beta > 0$ the scattering equation of the potential $W_{\beta,N}$

$$
\begin{cases}
\left(-\Delta_x + N^{-1+2\beta} W(N^\beta x)\right) F_{N,\beta,R}(x) = 0, \\
F_{N,\beta,R}(x) = 1 \text{ for } |x| = R.
\end{cases}
\tag{5}
$$

With $y = N^\beta x$, $\tilde{R} = N^\beta R$ and $G_{N,\beta,R} = F_{N,\beta,R}(N^{-\beta} \cdot)$, this can be written as

$$
\begin{cases}
\left(-\Delta_y + N^{-1} W(y)\right) G_{N,\beta,R}(y) = 0, \\
G_{N,\beta,R}(y) = 1 \text{ for } |y| = \tilde{R}.
\end{cases}
\tag{6}
$$

Due to the factor N^{-1}, the zero energy scattering state is almost constant for large N, $F_{N,\beta,R}(x) \approx 1 \; \forall \; |x| \leq R$. It can therefore be concluded that the microscopic structure has a negligible effect on the effective interaction on each particle which is approximated by[3]

$$
\int_{\mathbb{R}^2} d^2 y N W_{\beta,N}(x-y) F_{N,\beta,R}(x-y)|\varphi_t(y)|^2 \approx \int_{\mathbb{R}^2} d^2 y N W_{\beta,N}(x-y)|\varphi_t(y)|^2
$$
$$
\to \|W\|_1 |\varphi_t(x)|^2.
$$

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[3] Even if the effect of the microscopic structure is negligible in the interaction, one should note that Ψ_t is not close to a full product state $\prod_{k=1}^{N} \varphi_t(x_k)$ in norm. For certain types of interactions, it has rigorously been shown that Ψ_t can be approximated by a quasifree state satisfying a Bogoliubov-type dynamics. We refer to [5,12,23–25,31,35,43–46] for the precise statements.

This yields to the correct coupling in the effective equation (3) in the case of $U(x) = W_{\beta,N}(x)$.

Let us briefly compare the phenomenon of Bose–Einstein condensation in two and three dimensions. In three dimension the NLS scaling is defined by $N^{-1+3\beta}W(N^\beta x)$ only for $0 < \beta < 1$ while in the case of $\beta = 1$ the microscopic structure must be taken into account. This difference originates from the different form of the scattering state in two and three dimension, see Appendix C of [38]. In the case that the time evolution of Ψ_t is generated by H_{V_N} it is interesting to note that the effective evolution equation of φ_t does not depend on the scattering length a. Also this contrasts the three-dimensional case, where the correct mean field coupling is given by $8\pi a_{3D}$, a_{3D} denoting the scattering length of the potential in three dimensions. The universal coupling 4π in the case of a two-dimensional setup is known within the physical literature, see e.g. (30) and (A3) in [18] (note that $\hbar = 1$, $m = \frac{1}{2}$ in our choice of coordinates).

Actually, our dynamical result complements a more general theory describing the ground state properties of dilute, two-dimensional Bose gases. It was shown in [39] that for a gas with repulsive interaction $V \geq 0$, the ground state energy per particle is to leading order given by either the Gross–Pitaevskii energy functional with coupling parameter $8\pi/|\ln(\overline{\rho}a^2)|$ or a Thomas–Fermi type functional, depending on the diluteness of the gas, i.e. the mean-particle distance compared to the scattering length of the interaction. Here, $\overline{\rho}$ denotes the mean density of the gas and a is the scattering length which must decrease exponentially with N in the Gross–Pitaevskii limit [39, p. 20].

It should be pointed out that there has been some debate about the question whether two-dimensional Bose–Einstein condensation can be observed experimentally. This amounts to the question whether condensation takes place for temperatures $T > 0$. For an ideal, noninteracting gas in a box, the standard grand canonical computation for the critical temperature T_c of a Bose–Einstein condensate shows that there is no condensation for $T > 0$. For trapped, noninteracting bosons in a confining power-law potential, the findings in [3] however show that in that case $T_c > 0$ holds. Finally, it was proven in [37] that $\gamma_\Psi^{(1)}$ converges to $|\varphi\rangle\langle\varphi|$ in trace norm if Ψ is the ground state of H_{V_N} and φ is the ground state of the Gross–Pitaevskii energy functional, see (8). It was furthermore proven that one does not observe 100% condensation in the ground state of an interacting homogenous system. The emergence of 100% Bose–Einstein condensation as a ground state phenomenon thus highly depends on the particular physical system. Our approach is the following: Initially, we assume the convergence of $\gamma_{\Psi_0}^{(1)}$ to $|\varphi_0\rangle\langle\varphi_0|$. We then show the persistence of condensation for time scales of order one. Our assumption is thus in agreement with the findings in [37].

The rigorous derivation of effective evolution equations is well known in the literature, see e.g. [2,5,9–11,19–22,30,43,44,48–51] and references therein. For the two-dimensional case we consider, it has been proven, for $0 < \beta < 3/4$ and W nonnegative, that $\gamma_{\Psi_t}^{(1)}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ as $N \to \infty$ [27]. For $0 < \beta < 1/6$, it has been established in [14] that the reduced density matrices converge, assuming that the potential W is attractive, i.e. $W \leq 0$. This result was later extended to a larger class of scaling parameters β, under some assumptions on the negative part of the potential W [26,34]. In [45] a norm approximation to the two-dimensional focusing Schrödinger equation in the NLS scaling with $0 < \beta < 1$ was considered. Here, the evolution of the condensate is effectively described by the nonlinear Schrödinger equation while the evolution of the fluctuations around the condensate is governed by a quadratic Hamiltonian, resulting from Bogoliubov approximation. Another approach which relates more closely to the experimental setup is to consider a three-dimensional gas of bosons which is strongly

confined in one spatial dimension. Then, one obtains an effective two-dimensional system in the unconfined directions. We remark that in this dimensional reduction two limits appear, the length scale in the confined direction and the scaling of the interaction in the unconfined directions. A derivation of the two-dimensional Gross–Pitaevskii equation from the three-dimensional quantum many-body dynamics of strongly confined bosons was just recently given in [7]. Further results in this direction can be found in [4,6,8,15–17,28,29]. For known results regarding the ground state properties of dilute Bose gases, we refer to the monograph [38], which also summarizes the papers [37,39,40].

Our proof is based on [49], which covers the derivation of the time dependent Gross–Pitaevskii equation in three dimensions. In particular, the exponential scaling of the interaction forces us to adapt crucial ideas and refine many estimates. Additional difficulties arise amongst others from the logarithmic behaviour of the scattering state and the fact that $\|e^{2N}V(e^N\cdot)\|_{L^1(\mathbb{R}^2,\mathbb{C})} \sim 1$ while $\|N^{-1+3}V(N\cdot)\|_{L^1(\mathbb{R}^3,\mathbb{C})} \sim N^{-1}$ in the three-dimensional Gross–Pitaevskii regime.

We shortly discuss the physical relevance of the Gross–Pitaevskii scaling. It is possible to rescale space- and time-coordinates in such a way that in the new coordinates the interaction is *not* N-dependent. Choosing $y = e^N x$ and $\tau = e^{2N}t$ the Schrödinger equation reads

$$i\frac{d}{d\tau}\Psi_{e^{-2N}\tau} = \left(-\sum_{j=1}^{N}\Delta_{y_j} + \sum_{1\leq j<k\leq N}V(y_j - y_k) + \sum_{j=1}^{N}A_{e^{-2N}\tau}(e^{-N}y_j)\right)\Psi_{e^{-2N}\tau}.$$

(7)

The latter equation thus corresponds to an extremely dilute gas of bosons with density $\sim e^{-2N}$. In order to observe a nontrivial dynamics, this condensate is then monitored over time scales of order $\tau \sim e^{2N}$. Since the trapping potential is adjusted according to the density of the gas in the experiment, the N dependence of $A_{e^{-2N}\tau}(e^{-N}\cdot)$ is reasonable.

2. Main Result

Our main theorem consists of two parts, which consider potentials in the NLS and Gross–Pitaevskii scaling, respectively. For the proof of the theorem it is useful to enlarge the class of potentials in the NLS regime because it allows us in the derivation of the Gross–Pitaevskii equation to refer to various estimates that appear in first part of the proof.

Definition 2.1. (a) For $\beta > 0$, we define the following space of sequences $(W_{\beta,N})_{N\in\mathbb{N}}$.

$$\widetilde{\mathcal{W}}_\beta = \Big\{(W_{\beta,N})_{N\in\mathbb{N}}| \ W_{\beta,N} \in L_c^\infty(\mathbb{R}^2, \mathbb{R}), \exists C > 0 \text{ independent of}$$
$$N \text{ and } \beta\colon W_{\beta,N}(x) \geq 0 \ \forall x \in \mathbb{R}^2,$$
$$\|W_{\beta,N}\|_1 \leq CN^{-1}, \|W_{\beta,N}\| \leq CN^{-1+\beta}, \|W_{\beta,N}\|_\infty \leq CN^{-1+2\beta},$$
$$W_{\beta,N}(x) = 0 \ \forall|x| \geq CN^{-\beta}, \ W_{\beta,N} \text{ is spherically symmetric}\Big\}.$$

(b) For every $(W_{\beta,N})_{N\in\mathbb{N}} \in \widetilde{\mathcal{W}}_\beta$ we define the coupling parameter $b_{W_\beta} = \lim_{N\to\infty}N\|W_{\beta,N}\|_1$.

(c) Define the set of potentials $\mathcal{W}_\beta$ by

$$\mathcal{W}_\beta = \Big\{ (W_{\beta,N})_{N\in\mathbb{N}} \in \widetilde{\mathcal{W}}_\beta \mid \exists C > 0 \text{ independent of } N \text{ and } \beta:$$
$$\big| N \|W_{\beta,N}\|_1 - b_{W_\beta} \big| \leq C N^{-1} \ln(N) \Big\}.$$

To ease the notation, we often omit to display the dependence on N and denote both the sequence $(W_{\beta,N})_{N\in\mathbb{N}}$ and the element $W_{\beta,N}$ by W_β.

Remark 2.2. It should be noted that $N^{-1+2\beta} W(N^\beta x) \in \mathcal{W}_\beta$, if $W \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$ is nonnegative and spherically symmetric. In this case, $b_{W_\beta} = \|W\|_1$.

For notational convenience, it is in addition helpful to define a class of potentials with Gross–Pitaevskii scaling.

Definition 2.3. Define the set of sequences of potentials $(V_N)_{N\in\mathbb{N}}$ as

$$\mathcal{V}_N = \{(V_N)_{N\in\mathbb{N}} \mid \exists V \in L_c^\infty(\mathbb{R}^2, \mathbb{R}) \text{ not being identically zero}: V_N(x) = e^{2N} V(e^N x),$$
$$V(x) \geq 0 \; \forall x \in \mathbb{R}^2, \; V \text{ is spherically symmetric}\}.$$

With a slight abuse of notation we use V_N to denote the sequence $(V_N)_{N\in\mathbb{N}}$ and its Nth element.

For $U \in \{W_\beta, V_N\}$ and $A_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$, define the energy functional $\mathcal{E}_U: H^1(\mathbb{R}^{2N}, \mathbb{C}) \to \mathbb{R}$

$$\mathcal{E}_U(\Psi) = N^{-1} \langle\!\langle \Psi, H_U \Psi \rangle\!\rangle,$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the scalar product on $L^2(\mathbb{R}^{2N}, \mathbb{C})$. Furthermore, define the Gross–Pitaevskii energy functional $\mathcal{E}_{b_U}^{GP}: H^1(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$

$$\mathcal{E}_{b_U}^{GP}(\varphi) = \langle \nabla\varphi, \nabla\varphi \rangle + \langle \varphi, (A_t + \tfrac{1}{2} b_U |\varphi|^2)\varphi \rangle = \langle \varphi, (h_{b_U}^{GP} - \tfrac{1}{2} b_U |\varphi|^2)\varphi \rangle \quad (8)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(\mathbb{R}^2, \mathbb{C})$. Note that both $\mathcal{E}_U(\Psi)$ and $\mathcal{E}_{b_U}^{GP}(\varphi)$ depend on t, due to the time varying external potential A_t. For the sake of readability, we will not indicate this time dependence explicitly. Our main theorem is the following.

Theorem 2.4. *Let $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. Let $\varphi_0 \in L^2(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi_0\| = 1$. Let the external potential A_t satisfy $A_\cdot \in C^1(\mathbb{R}, L^\infty(\mathbb{R}^2, \mathbb{R}))$.*

(a) *Let $\beta > 0$, $W_\beta \in \mathcal{W}_\beta$ and let Ψ_t the unique solution to $i\partial_t \Psi_t = H_{W_\beta} \Psi_t$ with initial datum Ψ_0. Let φ_t the unique solution to $i\partial_t \varphi_t = h_{b_{W_\beta}}^{GP} \varphi_t$ with initial datum φ_0 and assume that $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C}) \; \forall t \in \mathbb{R}$. Let $\mathcal{E}_{W_\beta}(\Psi_0) \leq C$, where $C > 0$ is a constant independent of N. Then, for any $t > 0$ there exists a constant $0 < C_t < \infty$, which*

depends on t but not on N, such that

$$Tr\left|\gamma_{\Psi_t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \le e^{C_t}\left(\sqrt[4]{Tr\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|}\right.$$
$$\left. + \sqrt{\left|\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}_{bw_\beta}^{GP}(\varphi_0)\right|} + N^{-\gamma}\sqrt{\ln(N)}\right), \quad (9)$$

$$\left|\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}_{bw_\beta}^{GP}(\varphi_t)\right| \le e^{C_t}\left(\sqrt{Tr\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|}\right.$$
$$\left. + \left|\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}_{bw_\beta}^{GP}(\varphi_0)\right| + N^{-2\gamma}\ln(N)\right), \quad (10)$$

where $\gamma = \beta$ for $0 < \beta < 1/12$ and $\gamma = 1/20$ for $\beta \ge 1/12$.

(b) *Let $V_N \in \mathcal{V}_N$ and let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum Ψ_0. Let φ_t the unique solution to $i\partial_t\varphi_t = h_{4\pi}^{GP}\varphi_t$ with initial datum φ_0 and assume that $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C}) \; \forall t \in \mathbb{R}$. Let $\mathcal{E}_{V_N}(\Psi_0) \le C$, where $C > 0$ is a constant independent of N. Then, for any $t > 0$ there exists a constant $0 < C_t < \infty$, which depends on t but not on N, such that*

$$Tr\left|\gamma_{\Psi_t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \le e^{C_t}\left(\sqrt[4]{Tr\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|}\right.$$
$$\left. + \sqrt{\left|\mathcal{E}_{V_N}(\Psi_0) - \mathcal{E}_{4\pi}^{GP}(\varphi_0)\right|} + N^{-1/20}\right), \quad (11)$$

$$\left|\mathcal{E}_{V_N}(\Psi_t) - \mathcal{E}_{4\pi}^{GP}(\varphi_t)\right| \le e^{C_t}\left(\sqrt{Tr\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|}\right.$$
$$\left. + \left|\mathcal{E}_{V_N}(\Psi_0) - \mathcal{E}_{4\pi}^{GP}(\varphi_0)\right| + N^{-1/10}\right). \quad (12)$$

Remarks. (a) If one considers initial many-body states which exhibit condensation and whose energy per particle converges to the corresponding Gross–Pitaevskii energy, i.e.

$$\lim_{N\to\infty} Tr\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right| = 0 \quad \text{and} \quad \lim_{N\to\infty}\left|\mathcal{E}_U(\Psi_0) - \mathcal{E}_{bU}^{GP}(\varphi_0)\right| = 0$$
$$\text{with } U \in \{W_\beta, V_N\},$$

it follows from Theorem 2.4 that

$$\lim_{N\to\infty} Tr\left|\gamma_{\Psi_t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| = 0 \quad \text{and} \quad \lim_{N\to\infty}\left|\mathcal{E}_U(\Psi_t) - \mathcal{E}_{bU}^{GP}(\varphi_t)\right| = 0 \quad \text{for any } t > 0.$$

Our result consequently shows the stability of the condensate during the time evolution.

(b) It has been shown that in the limit $N \to \infty$ the energy-difference $\mathcal{E}_{V_N}(\Psi^{gs}) - \mathcal{E}_{4\pi}^{GP}(\varphi^{gs}) \to 0$, where Ψ^{gs} is the ground state of a trapped Bose gas and φ^{gs} the ground state of the respective Gross–Pitaevskii energy functional, see [39,40].

(c) The necessity to require $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ stems from the fact that the constant C_t in (9) and (11) depends on $\|\varphi_t\|_{H^3}$, see the discussion before Lemma 4.7. For regular enough external potentials A_t we expect the assumption $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ to follow from regularity assumptions on the initial datum φ_0. If $\varphi_0 \in \Sigma^3(\mathbb{R}^2, \mathbb{C}) = \{f \in L^2(\mathbb{R}^2, \mathbb{C}) | \sum_{\alpha+\beta \leq 3} \|x^\alpha \partial_x^\beta f\| < \infty\}$ holds, the bound $\|\varphi_t\|_{H^3} < \infty$ has been proven for external potentials which are at most quadratic in space, see [13] and Lemma 4.7. In particular, for $\varphi_0 \in \Sigma^3(\mathbb{R}^2, \mathbb{C})$, the bound $\|\varphi_t\|_{H^3} \leq C$ with $C > 0$ uniformly bounded in t holds if the external potential is not present, i.e. $A_t = 0$ [see [13] above (1.3.)].

(d) One can relax the conditions on the initial condition and only require $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$ using a standard density argument.

3. Organization of the Proof

The method we are applying to prove Theorem 2.4 was originally introduced in [50] and later generalized to derive various mean-field equations [1,8,30,32,33,42,46–49]. Our proof is primarily based on [49] which covers the three-dimensional counterpart of our system. The key idea of the method is to show the existence of Bose–Einstein condensation not in terms of reduced density matrices but to consider an equivalent measure of condensation. Heuristically speaking, we count for each time t the relative number of those particles which are not in the state of the condensate wave function φ_t. It is then possible to show that the rate of the particles which leave the condensate is small, if initially almost all particles were in the state φ_0. The counting of the particles will be performed with the help of a functional. In order to define it, we introduce the following operators.

Definition 3.1. Let $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi\| = 1$.

(a) For any $1 \leq j \leq N$ the projectors $p_j^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ and $q_j^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ are defined as

$$p_j^\varphi \Psi = \varphi(x_j) \int \varphi^*(\tilde{x}_j) \Psi(x_1, \ldots, \tilde{x}_j, \ldots, x_N) d^2\tilde{x}_j \quad \forall \, \Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$$

and $q_j^\varphi = 1 - p_j^\varphi$. We shall also use, with a slight abuse of notation, the bra-ket notation $p_j^\varphi = |\varphi(x_j)\rangle\langle\varphi(x_j)|$.

(b) For any $0 \leq k \leq N$ we define the set

$$\mathcal{S}_k = \left\{ \vec{s} = (s_1, s_2, \ldots, s_N) \in \{0, 1\}^N \; ; \; \sum_{j=1}^N s_j = k \right\}$$

and the orthogonal projector $P_k^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ as

$$P_k^\varphi = \sum_{\vec{s} \in \mathcal{S}_k} \prod_{j=1}^N \left(p_j^\varphi\right)^{1-s_j} \left(q_j^\varphi\right)^{s_j}.$$

For negative k and $k > N$ we set $P_k^\varphi = 0$.

(c) For any function $m: \mathbb{N}_0 \to \mathbb{R}_0^+$ we define the operator $\widehat{m}^\varphi: L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ as

$$\widehat{m}^\varphi = \sum_{j=0}^{N} m(j) P_j^\varphi. \tag{13}$$

We also need the shifted operators $\widehat{m}_d^\varphi: L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ given by

$$\widehat{m}_d^\varphi = \sum_{j=-d}^{N-d} m(j+d) P_j^\varphi \quad \text{with } d \in \mathbb{Z}.$$

Following a general strategy[4] we will define a functional $\alpha: L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ such that

(a) $\alpha(\Psi_0, \varphi_0) \to 0$ for suitably chosen initial data $(\Psi_0, \varphi_0) \in L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C})$.[5]
(b) If Ψ_t is a solution of (2) and φ_t a solution of (3), $\alpha(\Psi_t, \varphi_t)$ can be estimated by $\alpha(\Psi_0, \varphi_0) + \int_0^t ds\, C_s\big(\alpha(\Psi_s, \varphi_s) + \mathcal{O}(1)\big)$ for some time dependent constant C_s. Using a Grönwall type estimate, it then follows that $\alpha(\Psi_t, \varphi_t) \leq e^{2\int_0^t d\tau\, C_\tau}\big(\alpha(\Psi_0, \varphi_0) + \mathcal{O}(1)\big)$.
(c) $\alpha(\Psi_t, \varphi_t) \to 0$ implies the convergence of the one-particle reduced density matrix of Ψ_t to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm as well as the convergence of the energy per particle of the many-body system to the energy of the condensate wave function.

In [30,50] the mean field scaling $W_0(x) = N^{-1}W(x)$ and a condensate wave function which evolves according to the Hartree equation $i\partial_t \varphi_t = \big(-\Delta + A_t\big)\varphi_t + \big(W * |\varphi_t|^2\big)\varphi_t$ were considered in the three-dimensional setting. In these works it was shown that the persistence of condensation can be proven if one chooses

$$\alpha(\Psi_t, \varphi_t) = \Big\langle\!\!\Big\langle \Psi_t, \big(\widehat{n}^{\varphi_t}\big)^j \Psi_t \Big\rangle\!\!\Big\rangle,$$

where $n(k) = \sqrt{k/N}$, $j > 0$ and Ψ_t is a solution of (2) with $U = W_0$. The choice $j = 2$ corresponds to the functional $\big\langle\!\!\big\langle \Psi_t, \sum_{k=0}^{N} \frac{k}{N} P_k^{\varphi_t} \Psi_t \big\rangle\!\!\big\rangle$, whose action on Ψ_t can be viewed as "counting the relative number of particles which are not in the state φ_t". Other values of j or a different choice of $\widehat{m}^{\varphi_t}$ should be understood as a weighted measure of counting the number of particles which are not in the condensate state. We will therefore sometimes call m the weight function of the functional α.

In this work we are interest in interaction potentials which get peaked as $N \to \infty$. As explained in Sect. 6.1, it is then no longer possible to obtain a Grönwall estimate with the previous choice of the functional and we have to adjust it in accordance with the scaling of the interaction. The precise definition of the functional and the proof of Theorem 2.4 are given in Sect. 6. In the preceding chapters we introduce the necessary preliminaries.

The rest of the paper is organized as follows:

(a) In Sect. 4 we start by fixing the notation. Afterwards, we recall important properties of the operator $\widehat{m}$ and explain the required regularity conditions on the solutions of the nonlinear Schrödinger equation.

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[4] For an extensive introduction to the method we refer to [50].

[5] It should be noted that the requirement $\alpha(\Psi_0, \varphi_0) \to 0$ defines conditions on the initial states (Ψ_0, φ_0).

(b) In case of the exponential scaling, the interaction is so strong such that the many-body state develops a short scale correlation structure. This correlation structure affects the time evolution of the condensate and must therefore also be regarded in the definition of the functional. In Sect. 5, we explain the correlations structure in greater detail, provide certain estimates on the zero-energy scattering state and explain how the effective coupling parameter b_{V_N} can be inferred from the microscopic structure.

(c) In Sect. 6 we prove Theorem 2.4. We first consider the potential W_β and define a counting measure which allows us to establish a Grönwall estimate for all $\beta > 0$. We will explain in detail how one arrives at this Grönwall estimate. Afterwards, the counting measure is adjusted to the case V_N, taking the microscopic structure $j_{N,R}$ of the wave function into account. We then establish a Grönwall estimate and finally prove the second part of the main theorem.

(d) In order to improve the readability of the paper we only state the estimates which are needed for the proof of Theorem 2.4 in Sect. 6. Their derivation is provided afterwards in Sect. 7.

4. Preliminaries

We will first fix the notation we are going to employ during the rest of the paper.

Notation 4.1. (a) Throughout the paper hats $\widehat{\cdot}$ will always be used in the sense of Definition 3.1(c). The label n will always be used for the function $n(k) = \sqrt{k/N}$.

(b) For better readability, we will often omit the upper index φ on p_j, q_j, P_j and $\widehat{\cdot}$. It will be placed exclusively in formulas where the φ-dependence is crucial.

(c) The operator norm, defined for any linear operator $f\colon L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$, will be denoted by

$$\|f\|_{\mathrm{op}} = \sup_{\psi \in L^2(\mathbb{R}^{2N}, \mathbb{C}), \|\Psi\|=1} \|f\Psi\|.$$

(d) We will bound expressions which are uniformly bounded in N and t by some constant C. Constants appearing in a sequence of estimates will not be distinguished, i.e. in $X \le CY \le CZ$ the constants may differ.

(e) We will denote by $\mathcal{K}(\varphi_t, A_t)$ a generic polynomial with finite degree in $\|\varphi_t\|_\infty$, $\|\nabla\varphi_t\|_\infty$, $\|\nabla\varphi_t\|$, $\|\Delta\varphi_t\|$, $\|A_t\|_\infty$, $\int_0^t ds \|\dot{A}_s\|_\infty$ and $\|\dot{A}_t\|_\infty$. Note, in particular, that for a generic constant C the inequality $C \le \mathcal{K}(\varphi_t, A_t)$ holds. The exact form of $\mathcal{K}(\varphi_t, A_t)$ which appears in the final bounds can be reconstructed, collecting all contributions from the different estimates.

(f) We will denote for any multiplication operator $F\colon L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ the corresponding operator

$$\mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)}\colon L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$$

acting on the N-particle Hilbert space by $F(x_k)$. In particular, we will use, for any $\Psi, \Omega \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ the notation

$$\langle\!\langle \Omega, \mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)}\Psi \rangle\!\rangle = \langle\!\langle \Omega, F(x_k)\Psi \rangle\!\rangle.$$

In analogy, for any two-particle multiplication operator $K\colon L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2} \to L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2}$, we denote the operator acting on any $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ by multiplication in the variable x_i and x_j by $K(x_i, x_j)$. In particular, we denote

$$\langle\!\langle \Omega, K(x_i, x_j)\Psi \rangle\!\rangle = \int_{\mathbb{R}^{2N}} K(x_i, x_j)\Omega^*(x_1, \ldots, x_N)\Psi(x_1, \ldots, x_N)d^2x_1 \ldots d^2x_N.$$

Next, we prove some properties of the projectors p_j and q_j, which are defined in Definition 3.1.

Lemma 4.2. (a) *For any weights $m, r: \mathbb{N}_0 \to \mathbb{R}_0^+$ the commutation relations*

$$\widehat{mr} = \widehat{m}\widehat{r} = \widehat{r}\widehat{m}, \qquad \widehat{m}p_j = p_j\widehat{m}, \qquad \widehat{m}q_j = q_j\widehat{m}, \qquad \widehat{m}P_k = P_k\widehat{m}$$

hold.
(b) *Let $n: \mathbb{N}_0 \to \mathbb{R}_0^+$ be given by $n(k) = \sqrt{k/N}$. Then, the square of $\widehat{n}$ equals the relative particle number operator of particles not in the state φ, i.e.*

$$(\widehat{n})^2 = N^{-1} \sum_{j=1}^{N} q_j. \tag{14}$$

(c) *For any weight $m: \mathbb{N}_0 \to \mathbb{R}_0^+$ and any function $f \in L^\infty\left(\mathbb{R}^4, \mathbb{C}\right)$ and any $j, k = 0, 1, 2$*

$$\widehat{m}Q_j f(x_1, x_2)Q_k = Q_j f(x_1, x_2)\widehat{m}_{j-k}Q_k,$$

where $Q_0 = p_1 p_2$, $Q_1 \in \{p_1 q_2, q_1 p_2\}$ and $Q_2 = q_1 q_2$. Furthermore, for $j, k \in \{0, 1\}$ and $g \in L^\infty(\mathbb{R}^2, \mathbb{C})$ the relations

$$\widehat{m}\widetilde{Q}_j g(x_1)\widetilde{Q}_k = \widetilde{Q}_j g(x_1)\widehat{m}_{j-k}\widetilde{Q}_k \quad and \quad \widehat{m}\widetilde{Q}_j \nabla_1 \widetilde{Q}_k = \widetilde{Q}_j \nabla_1 \widehat{m}_{j-k}\widetilde{Q}_k$$

hold, where $\widetilde{Q}_0 = p_1$ and $\widetilde{Q}_1 = q_1$.
(d) *For any weight $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any functions $f \in L^\infty\left(\mathbb{R}^4, \mathbb{C}\right)$, $g \in L^\infty\left(\mathbb{R}^2, \mathbb{C}\right)$ the commutation relations*

$$[f(x_1, x_2), \widehat{m}] = [f(x_1, x_2), p_1 p_2(\widehat{m} - \widehat{m}_2) + (p_1 q_2 + q_1 p_2)(\widehat{m} - \widehat{m}_1)],$$
$$[g(x_1), \widehat{m}] = q_1 g(x_1)(\widehat{m} - \widehat{m}_1)p_1 - p_1(\widehat{m} - \widehat{m}_1)g(x_1)q_1$$

hold.
(e) *Let $f \in L^1\left(\mathbb{R}^2, \mathbb{C}\right)$, $g \in L^2\left(\mathbb{R}^2, \mathbb{C}\right)$. Then,*

$$\|p_j f(x_j - x_k)p_j\|_{op} \leq \|f\|_1 \|\varphi\|_\infty^2, \tag{15}$$
$$\|p_j g^*(x_j - x_k)\|_{op} = \|g(x_j - x_k)p_j\|_{op} \leq \|g\| \, \|\varphi\|_\infty, \tag{16}$$
$$\||\varphi(x_j)\rangle\langle\nabla_j\varphi(x_j)|g^*(x_j - x_k)\|_{op} = \|g(x_j - x_k)\nabla_j p_j\|_{op} \leq \|g\|\|\nabla\varphi\|_\infty. \tag{17}$$

Proof. (a) follows immediately from Definition 3.1, using that p_j and q_j are orthogonal projectors.
(b) Note that $\cup_{k=0}^{N} \mathcal{S}_k = \{0, 1\}^N$, so $1 = \sum_{k=0}^{N} P_k$. Using also $(q_j)^2 = q_j$ and $q_j p_j = 0$ we get

$$\sum_{j=1}^{N} q_j = \sum_{j=1}^{N} q_j \sum_{k=0}^{N} P_k = \sum_{k=0}^{N} \sum_{j=1}^{N} q_j P_k = \sum_{k=0}^{N} k P_k = N\widehat{n^2} = N\widehat{n}^2.$$

(c) Using the definitions above we have

$$\widehat{m} Q_j f(x_1, x_2) Q_k = \sum_{l=0}^{N} m(l) P_l Q_j f(x_1, x_2) Q_k.$$

The number of projectors q_j in $P_l Q_j$ in the coordinates $j = 3, \ldots, N$ is equal to $l - j$. The p_j and q_j with $j = 3, \ldots, N$ commute with $Q_j f(x_1, x_2) Q_k$. Thus $P_l Q_j f(x_1, x_2) Q_k = Q_j f(x_1, x_2) Q_k P_{l-j+k}$ and

$$\widehat{m} Q_j f(x_1, x_2) Q_k = \sum_{l=0}^{N} m(l) Q_j f(x_1, x_2) Q_k P_{l-j+k}$$

$$= \sum_{\tilde{l}=k-j}^{N+k-j} Q_j f(x_1, x_2) m(\tilde{l} + j - k) P_{\tilde{l}} Q_k = Q_j f(x_1, x_2) \widehat{m}_{j-k} Q_k.$$

Similarly one gets the second and third formula.

(d) First note that

$$[f(x_1, x_2), \widehat{m}] - [f(x_1, x_2), p_1 p_2 (\widehat{m} - \widehat{m}_2) + p_1 q_2 (\widehat{m} - \widehat{m}_1) + q_1 p_2 (\widehat{m} - \widehat{m}_1)]$$
$$= [f(x_1, x_2), q_1 q_2 \widehat{m}] + [f(x_1, x_2), p_1 p_2 \widehat{m}_2 + p_1 q_2 \widehat{m}_1 + q_1 p_2 \widehat{m}_1]. \tag{18}$$

We will show that the right hand side is zero. Multiplying the right hand side with $p_1 p_2$ from the left and using (c) one gets

$$p_1 p_2 f(x_1, x_2) q_1 q_2 \widehat{m} + p_1 p_2 f(x_1, x_2) p_1 p_2 \widehat{m}_2 - p_1 p_2 \widehat{m}_2 f(x_1, x_2)$$
$$+ p_1 p_2 f(x_1, x_2) p_1 q_2 \widehat{m}_1 + p_1 p_2 f(x_1, x_2) q_1 p_2 \widehat{m}_1$$
$$= p_1 p_2 \widehat{m}_2 f(x_1, x_2) q_1 q_2 + p_1 p_2 \widehat{m}_2 f(x_1, x_2) p_1 p_2 - p_1 p_2 \widehat{m}_2 f(x_1, x_2)$$
$$+ p_1 p_2 \widehat{m}_2 f(x_1, x_2) p_1 q_2 + p_1 p_2 \widehat{m}_2 f(x_1, x_2) q_1 p_2$$
$$= 0.$$

Multiplying (18) with $p_1 q_2$ from the left one gets

$$p_1 q_2 f(x_1, x_2) q_1 q_2 \widehat{m} + p_1 q_2 f(x_1, x_2) p_1 p_2 \widehat{m}_2 + p_1 q_2 f(x_1, x_2) p_1 q_2 \widehat{m}_1$$
$$+ p_1 q_2 f(x_1, x_2) q_1 p_2 \widehat{m}_1 - p_1 q_2 \widehat{m}_1 f(x_1, x_2).$$

Using (c) the latter is zero. Also multiplying with $q_1 p_2$ yields zero due to symmetry in interchanging x_1 with x_2. Multiplying (18) with $q_1 q_2$ from the left one gets

$$q_1 q_2 f(x_1, x_2) \widehat{m} q_1 q_2 - q_1 q_2 \widehat{m} f(x_1, x_2) + q_1 q_2 f(x_1, x_2) p_1 p_2 \widehat{m}_2$$
$$+ q_1 q_2 f(x_1, x_2) p_1 q_2 \widehat{m}_1 + q_1 q_2 f(x_1, x_2) q_1 p_2 \widehat{m}_1$$

which is again zero and so is (18).

By means of the identity $1 = p_1 + q_1$ one has

$$[g(x_1), \widehat{m}] = p_1 \big(g(x_1) \widehat{m} - \widehat{m} g(x_1) \big) p_1 + q_1 \big(g(x_1) \widehat{m} - \widehat{m} g(x_1) \big) q_1$$
$$+ q_1 \big(g(x_1) \widehat{m} - \widehat{m} g(x_1) \big) p_1 + p_1 \big(g(x_1) \widehat{m} - \widehat{m} g(x_1) \big) q_1.$$

The second relation from part (d) then follows from (a) and (c).

(e) To show (15), note that

$$p_j f(x_j - x_k) p_j = p_j (f * |\varphi|^2)(x_k). \tag{19}$$

It follows that

$$\|p_j f(x_j - x_k) p_j\|_{op} \leq \|f\|_1 \|\varphi\|_\infty^2.$$

For (16) we write

$$\begin{aligned}
\|g(x_j - x_k) p_j\|_{op}^2 &= \sup_{\|\Psi\|=1} \|g(x_j - x_k) p_j \Psi\|^2 \\
&= \sup_{\|\Psi\|=1} \langle\!\langle \Psi, p_j |g(x_j - x_k)|^2 p_j \Psi \rangle\!\rangle \\
&\leq \|p_j |g(x_j - x_k)|^2 p_j\|_{op}.
\end{aligned}$$

With (15) we get (16). For (17) we use

$$\begin{aligned}
\|g(x_j - x_k)\nabla_j p_j\|_{op}^2 &= \sup_{\|\Psi\|=1} \langle\!\langle \Psi, p_j(|g|^2 * |\nabla\varphi|^2)(x_k)\Psi \rangle\!\rangle \leq \||g|^2 * |\nabla\varphi|^2\|_\infty \\
&\leq \|g\|^2 \|\nabla\varphi\|_\infty^2.
\end{aligned}$$

The Lemma then follows from the fact that, for bounded operators A, $\|A\|_{op} = \|A^*\|_{op}$ holds, where A^* is the adjoint operator of A. □

Within our estimates we will encounter wave functions where some of the symmetry is broken (at this point the reader should exemplarily think of the wave function $V_\beta(x_1 - x_2)\Psi$ which is not symmetric under exchange of the variables x_1 and x_3). This leads to the following definition

Definition 4.3. For any finite set $\mathcal{M} \subset \{1, 2, \ldots, N\}$, define the space $\mathcal{H}_\mathcal{M} \subset L^2(\mathbb{R}^{2N}, \mathbb{C})$ as the set of functions which are symmetric in all variables in $\mathcal{M}$

$$\Psi \in \mathcal{H}_\mathcal{M} \Leftrightarrow \Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) = \Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)$$
for all $j, k \in \mathcal{M}$.

Based on the combinatorics of the p_j and q_j, we obtain the following

Lemma 4.4. *For any $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any finite set $\mathcal{M}_a \subset \{1, 2, \ldots, N\}$ with $1 \in \mathcal{M}_a$ and any finite set $\mathcal{M}_b \subset \{1, 2, \ldots, N\}$ with $1, 2 \in \mathcal{M}_b$*

$$\left\|\widehat{f} q_1 \Psi\right\|^2 \leq \frac{N}{|\mathcal{M}_a|} \|\widehat{f\widehat{n}}\Psi\|^2 \quad \text{for any } \Psi \in \mathcal{H}_{\mathcal{M}_a}, \tag{20}$$

$$\left\|\widehat{f} q_1 q_2 \Psi\right\|^2 \leq \frac{N^2}{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)} \|\widehat{f(\widehat{n})^2}\Psi\|^2 \quad \text{for any } \Psi \in \mathcal{H}_{\mathcal{M}_b}. \tag{21}$$

Proof. Let $\Psi \in \mathcal{H}_{\mathcal{M}_a}$ for some finite set $1 \in \mathcal{M}_a \subset \{1, 2, \ldots, N\}$. By Lemma 4.2 (b), (20) can be estimated as

$$\|\widehat{f}\widehat{n}\Psi\|^2 = \langle\!\langle \Psi, (\widehat{f})^2(\widehat{n})^2\Psi \rangle\!\rangle = N^{-1} \sum_{k=1}^{N} \langle\!\langle \Psi, (\widehat{f})^2 q_k \Psi \rangle\!\rangle$$

$$\geq N^{-1} \sum_{k\in\mathcal{M}_a} \langle\!\langle \Psi, (\widehat{f})^2 q_k \Psi \rangle\!\rangle = \frac{|\mathcal{M}_a|}{N} \langle\!\langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle\!\rangle$$

$$= \frac{|\mathcal{M}_a|}{N} \|\widehat{f}q_1\Psi\|^2.$$

Similarly, we obtain for $\Psi \in \mathcal{H}_{\mathcal{M}_b}$

$$\|\widehat{f}(\widehat{n})^2\Psi\|^2 = \langle\!\langle \Psi, (\widehat{f})^2(\widehat{n})^4\Psi \rangle\!\rangle \geq N^{-2} \sum_{j,k\in\mathcal{M}_b} \langle\!\langle \Psi, (\widehat{f})^2 q_j q_k \Psi \rangle\!\rangle$$

$$= \frac{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)}{N^2} \langle\!\langle \Psi, (\widehat{f})^2 q_1 q_2 \Psi \rangle\!\rangle + \frac{|\mathcal{M}_b|}{N^2} \langle\!\langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle\!\rangle$$

$$\geq \frac{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)}{N^2} \|\widehat{f}q_1 q_2 \Psi\|^2$$

which concludes the Lemma. $\square$

Corollary 4.5. *Let* $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$. *For any weight* $m \colon \mathbb{N}_0 \to \mathbb{R}_0^+$

$$\|\nabla_2 \widehat{m} q_2 \Psi\| \leq 2\|\widehat{m}\|_{op}\|\nabla_2 q_2 \Psi\|, \tag{22}$$

$$\|\nabla_2 \widehat{m} q_1 q_2 \Psi\| \leq C\|\widehat{mn}\|_{op}\|\nabla_2 q_2 \Psi\|. \tag{23}$$

Proof. Using $p_2 + q_2 = 1$ and triangle inequality,

$$\|\nabla_2 \widehat{m} q_2 \Psi\| \leq \|p_2 \nabla_2 \widehat{m} q_2 \Psi\| + \|q_2 \nabla_2 \widehat{m} q_2 \Psi\|, \tag{24}$$

$$\|\nabla_2 \widehat{m} q_1 q_2 \Psi\| \leq \|p_2 \nabla_2 \widehat{m} q_1 q_2 \Psi\| + \|q_2 \nabla_2 \widehat{m} q_1 q_2 \Psi\|. \tag{25}$$

With Lemma 4.2 (c) we get

$$(24) = \|\widehat{m}_1 p_2 \nabla_2 q_2 \Psi\| + \|\widehat{m} q_2 \nabla_2 q_2 \Psi\| \leq (\|\widehat{m}_1\|_{op} + \|\widehat{m}\|_{op})\|\nabla_2 q_2 \Psi\|.$$

Note that the wave function $p_2\nabla_2 q_2 \Psi$ is symmetric under the exchange of any two variables but x_2. Thus we can use Lemma 4.4 to get

$$(25) = \|q_1 \widehat{m}_1 p_2 \nabla_2 q_2 \Psi\| + \|q_1 \widehat{m} q_2 \nabla_2 q_2 \Psi\|$$

$$\leq \frac{N}{N-1}(\|\widehat{m}_1\widehat{n}\|_{op} + \|\widehat{mn}\|_{op})\|\nabla_2 q_2 \Psi\|.$$

Since $\sqrt{k} \leq \sqrt{k+1}$ for $k \geq 0$ it follows that the latter is bounded by

$$C(\|\widehat{m}_1\widehat{n}_1\|_{op} + \|\widehat{mn}\|_{op})\|\nabla_2 q_2 \Psi\|.$$

Using that $\|\widehat{r}\|_{op} = \sup_{0 \leq k \leq N}\{r(k)\} = \|\widehat{r}_d\|_{op}$ for any $d \in \mathbb{N}$ and any weight r, the Corollary follows. $\square$

Lemma 4.6. *Let $\Omega, \chi \in \mathcal{H}_\mathcal{M}$ for some $\mathcal{M}$, let $1 \notin \mathcal{M}$ and $2, 3 \in \mathcal{M}$. Let $O_{j,k}$ be an operator acting on the jth and k^{th} coordinate. Then*

$$|\langle\!\langle \Omega, O_{1,2}\chi \rangle\!\rangle| \leq \|\Omega\|^2 + \left|\langle\!\langle O_{1,2}\chi, O_{1,3}\chi \rangle\!\rangle\right| + (|\mathcal{M}|)^{-1}\|O_{1,2}\chi\|^2.$$

Proof. Using symmetry and Cauchy Schwarz

$$|\langle\!\langle \Omega, O_{1,2}\chi \rangle\!\rangle| = |\mathcal{M}|^{-1}|\langle\!\langle \Omega, \sum_{j\in\mathcal{M}} O_{1,j}\chi \rangle\!\rangle| \leq |\mathcal{M}|^{-1}\|\Omega\| \, \|\sum_{j\in\mathcal{M}} O_{1,j}\chi\|.$$

For the second factor we can write

$$\begin{aligned}
\|\sum_{j\in\mathcal{M}} O_{1,j}\chi\|^2 &= \langle\!\langle \sum_{j\in\mathcal{M}} O_{1,j}\chi, \sum_{k\in\mathcal{M}} O_{1,k}\chi \rangle\!\rangle \\
&\leq \sum_{j\in\mathcal{M}} |\langle\!\langle O_{1,j}\chi, O_{1,j}\chi \rangle\!\rangle| + |\sum_{j\neq k\in\mathcal{M}} \langle\!\langle O_{1,j}\chi, O_{1,k}\chi \rangle\!\rangle| \\
&\leq |\mathcal{M}||\langle\!\langle O_{1,2}\chi, O_{1,2}\chi \rangle\!\rangle| + |\mathcal{M}|(|\mathcal{M}| - 1)|\langle\!\langle O_{1,2}\chi, O_{1,3}\chi \rangle\!\rangle|.
\end{aligned}$$

Since $ab \leq 1/2a^2 + 1/2b^2$ and $(a + b)^2 \leq 2a^2 + 2b^2$ holds for any real numbers a and b, the Lemma follows. $\square$

In our estimates, we need the regularity conditions

$$\|\nabla\varphi_t\|_\infty < \infty, \qquad \|\varphi_t\|_\infty < \infty, \qquad \|\nabla\varphi_t\| < \infty, \qquad \|\Delta\varphi_t\| < \infty.$$

That is, we need $\varphi_t \in H^2(\mathbb{R}^2, \mathbb{C}) \cap W^{1,\infty}(\mathbb{R}^2, \mathbb{C})$. Then, $\|\Delta|\varphi_t|^2\|$, $\|\Delta|\varphi_t|^2\|_1$ and $\|\varphi_t^2\|$, which also appear in our estimates, can be bounded by

$$\Delta|\varphi_t|^2 = \varphi_t^*\Delta\varphi_t + \varphi_t\Delta\varphi_t^* + 2(\nabla\varphi_t^*) \cdot (\nabla\varphi_t)$$

$$\|\Delta|\varphi_t|^2\| \leq 2\|\Delta\varphi_t\|\|\varphi_t\|_\infty + 2\|\nabla\varphi_t\|\|\nabla\varphi_t\|_\infty$$

$$\|\Delta|\varphi_t|^2\|_1 \leq 4\|\Delta\varphi_t\|$$

$$\|\varphi_t^2\| \leq \|\varphi_t\|_\infty\|\varphi_t\|.$$

Recall the Sobolev embedding Theorem, which implies in particular $H^k(\mathbb{R}^2, \mathbb{C}) = W^{k,2}(\mathbb{R}^2, \mathbb{C}) \subset C^{k-2}(\mathbb{R}^2, \mathbb{C})$. If $\varphi \in C^1(\mathbb{R}^2, \mathbb{C}) \cap H^1(\mathbb{R}^2, \mathbb{C})$, then $\varphi \in W^{1,\infty}(\mathbb{R}^2, \mathbb{C})$ follows since both φ and $\nabla\varphi$ have to decay at infinity. Thus, $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ implies $\varphi_t \in H^2(\mathbb{R}^2, \mathbb{C}) \cap W^{1,\infty}(\mathbb{R}^2, \mathbb{C})$, which suffices for our estimates. Since φ_t obeys a defocusing nonlinear Schrödinger equation, we expect the regularity of the solution φ_t to follow from the regularity of the initial datum φ_0. For a certain class of external potentials A_t this has been proven in [13]:

Lemma 4.7. *Let $\varphi_0 \in \Sigma^k(\mathbb{R}^2, \mathbb{C}) = \{f \in L^2(\mathbb{R}^2, \mathbb{C}) | \sum_{\alpha+\beta\leq k} \|x^\alpha\partial_x^\beta f\| < \infty\}$, for $k \geq 2$. Let, for $b > 0$, φ_t be the unique solution to*

$$i\partial_t\varphi_t = (-\Delta + A_t + b|\varphi_t|^2)\varphi_t.$$

Let $A. \in L^\infty_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x, \mathbb{C})$ real valued and smooth with respect to the space variable: for (almost) all $t \in \mathbb{R}$, the map $x \mapsto A_t(x)$ is C^∞. Moreover, A_t is at most quadratic in space, uniformly w.r.t. time t:

$$\forall\alpha \in \mathbb{N}^2, |\alpha| \geq 2, \quad \partial_x^\alpha A. \in L^\infty(\mathbb{R}_t \times \mathbb{R}^d_x, \mathbb{C}).$$

In addition, $t \mapsto \sup_{|x|\leq 1} |A_t(x)|$ belongs to $L^\infty(\mathbb{R}, \mathbb{C})$. Then

(a) $\varphi_t \in \Sigma^k(\mathbb{R}^2, \mathbb{C})$, *which implies* $\varphi_t \in H^k(\mathbb{R}^2, \mathbb{C})$.
(b) $\|\varphi_t\| = \|\varphi_0\|$.
(c) *Let* $\varphi_0 \in \Sigma^3(\mathbb{R}^2, \mathbb{C})$. *Assume in addition that* $A. \in C^1(\mathbb{R}, L^\infty(\mathbb{R}^2, \mathbb{R}))$. *Then, for any fixed* $t \geq 0$, $\mathcal{K}(\varphi_t, A_t) < \infty$ *follows.*

Proof. Part (a) is Corollary 1.4. in [13]. We like to remark that $\|\varphi_t\|_{H^k} \leq C$ holds, if $A_t = 0$, see Section 1.2. in [13]. The conditions on A_t are for example satisfied if $A_t \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ for all $t \in \mathbb{R}$, $A_t(x) = 0$, for all $|t| \geq T$. Part (b) can be verified directly, using the existence of global in time solutions. Part (c) follows from (a) and the embedding $H^3(\mathbb{R}^2, \mathbb{C}) \subset H^2(\mathbb{R}^2, \mathbb{C}) \cap W^{1,\infty}(\mathbb{R}^2, \mathbb{C})$. $\square$

5. Microscopic Structure in 2 Dimensions

5.1. The scattering state. In this section we analyze the microscopic structure which is induced by V_N. In particular, we explain why the dynamical properties of the system are determined by the low energy scattering regime.

Definition 5.1. Let $V_N \in \mathcal{V}_N$. For any $R \geq \mathrm{diam}(\mathrm{supp}(V_N))$, we define the zero energy scattering state $j_{N,R} \in C^1(\mathbb{R}^2, \mathbb{R})$ by

$$\begin{cases} \left(-\Delta_x + \frac{1}{2}e^{2N}V(e^N x)\right) j_{N,R}(x) = 0, \\ j_{N,R}(x) = 1 \text{ for } |x| = R. \end{cases} \tag{26}$$

Next, we want to recall some important properties of the scattering state $j_{N,R}$, see also Appendix C of [38].

Lemma 5.2. *Let* $V_N \in \mathcal{V}_N$. *Define* $I_R = \int_{\mathbb{R}^2} d^2x \, V_N(x) j_{N,R}(x)$. *For the scattering state defined previously the following relations hold:*

(a) *There exists a nonnegative number a, called scattering length of the potential V, such that*

$$I_R = \frac{4\pi}{\ln\left(\frac{e^N R}{a}\right)}$$

(in the case $a = 0$ *we have* $I_R = 0$*). The scattering length a does not depend on R and fulfills* $a \leq \mathrm{diam}(\mathrm{supp}(V))$. *Furthermore,* $I_R \geq 0$ *holds.*
(b) $j_{N,R}$ *is a nonnegative function which is spherically symmetric in* $|x|$. *For* $|x| \geq \mathrm{diam}(\mathrm{supp}(V_N))$, $j_{N,R}$ *is given by*

$$j_{N,R}(x) = 1 + \frac{1}{\ln\left(\frac{e^N R}{a}\right)} \ln\left(\frac{|x|}{R}\right).$$

Proof. (a)+(b) Rescaling $x \to e^N x = y$, we obtain, setting $\tilde{R} = e^N R$ and $s_{\tilde{R}}(y) = j_{0,e^N R}(y)$, the unscaled scattering equation

$$\begin{cases} \left(-\Delta_y + \frac{1}{2}V(y)\right) s_{\tilde{R}}(y) = 0, \\ s_{\tilde{R}}(y) = 1 \text{ for } |y| = \tilde{R}. \end{cases} \tag{27}$$

Since we assume V to be nonnegative, one can define the scattering state $s_{\tilde{R}}$ by a variational principle. Theorem C.1 in [38] then implies that $s_{\tilde{R}}$ is a nonnegative, spherically symmetric function in $|y|$. It is then easy to verify that for diam $(\text{supp}(V)) \leq |y|$ there exists a number $A \in \mathbb{R}$ such that

$$s_{\tilde{R}}(y) = 1 + \frac{A}{4\pi} \ln\left(\frac{|y|}{\tilde{R}}\right). \tag{28}$$

Next, we show that $A = \int_{\mathbb{R}^2} d^2 y V(y) s_{\tilde{R}}(y)$. This can be seen by noting that, for $r > \text{diam}(\text{supp}(V))$,

$$\int_{\mathbb{R}^2} d^2 y V(y) s_{\tilde{R}}(y) = 2 \int_{B_r(0)} d^2 y \Delta s_{\tilde{R}}(y) = 2 \int_{\partial B_r(0)} \nabla s_{\tilde{R}}(y) \cdot ds$$

$$= \frac{A}{2\pi} \int_{\partial B_r(0)} \nabla \ln(|y|) \cdot ds = \frac{A}{2\pi} \int_0^{2\pi} \frac{1}{r} r \, d\varphi$$

$$= A.$$

By Theorem C.1 in [38], there exists a number $a \geq 0$, not depending on $\tilde{R}$, such that for all $|y| \geq \text{diam}(\text{supp}(V))$

$$s_{\tilde{R}}(y) = \frac{\ln(|y|/a)}{\ln(\tilde{R}/a)}.$$

Comparing this with (28), we obtain

$$\int_{\mathbb{R}^2} V(y) s_{\tilde{R}}(y) dy^2 = \frac{4\pi}{\ln\left(\frac{\tilde{R}}{a}\right)}.$$

Since $s_{\tilde{R}}$ is nonnegative, it furthermore follows that $a \leq \text{diam}(\text{supp}(V))$. This directly implies $A \geq 0$. By scaling, we obtain

$$I_R = \int_{\mathbb{R}^2} V_N(y) j_{N,R}(y) dy^2 = \int_{\mathbb{R}^2} V(y) s_{\tilde{R}}(y) dy^2 = \frac{4\pi}{\ln\left(\frac{e^N R}{a}\right)}.$$

$\square$

Assuming that the energy per particle $\mathcal{E}_{V_N}(\Psi)$ is of order one, the wave function Ψ will have a microscopic structure near the interactions V_N, given by $j_{N,R}$. The interaction among two particles is then determined by $\frac{4\pi}{N + \ln\left(\frac{R}{a}\right)} \approx \frac{4\pi}{N}$. Keeping in mind that each particle interacts with all other $N - 1$ particles, we obtain the effective Gross–Pitaevskii equation, for $\varphi_t \in H^2(\mathbb{R}^2, \mathbb{C})$

$$i \partial_t \varphi_t(x) = (-\Delta + A_t + 4\pi |\varphi_t(x)|^2)\varphi_t(x).$$

Thus, choosing $V_N(x) = e^{2N} V(e^N x)$ leads in our setting to an effective one-particle equation which is determined by the low energy scattering behavior of the particles. We remark that, for any $s > 0$, the potential $e^{2Ns} V(e^{Ns} x)$ yields to the coupling $4\pi/s$.

5.2. *Properties of the scattering state.* Note that the potential V_N is strongly peaked within an exponentially small region. In order to control the short scale structure of Ψ_t, we define a potential M_μ with softer scaling behaviour in such a way that the potential $V_N - M_\mu$ has scattering length zero. This allows us to "replace" V_N by M_μ, which has better scaling behavior and is easier to control. In particular, $\|M_\mu\| \leq CN^{-1+\mu}$ can be controlled for μ sufficiently small.

Definition 5.3. Let $V_N \in \mathcal{V}_N$. For any $\mu > 0$ and any $R_\mu \geq N^{-\mu}$ we define the potential M_μ via

$$M_\mu(x) = \begin{cases} 4\pi N^{-1+2\mu} & \text{if } N^{-\mu} < |x| \leq R_\mu, \\ 0 & \text{else .} \end{cases} \tag{29}$$

Furthermore, we define the zero energy scattering state $f_\mu \in C^1(\mathbb{R}^2, \mathbb{R})$ of the potential $\frac{1}{2}(V_N - M_\mu)$, that is

$$\begin{cases} \left(-\Delta_x + \frac{1}{2}\left(V_N(x) - M_\mu(x)\right)\right) f_\mu(x) = 0, \\ f_\mu(x) = 1 \text{ for } |x| = R_\mu. \end{cases} \tag{30}$$

Note that M_μ and f_μ depend on R_μ.

Remark 5.4. In the following, we choose R_μ to be the smallest value such that the scattering length of the potential $(V_N - M_\mu)$ is zero which is equivalent to the condition $\int_{\mathbb{R}} d^2x(V_N(x) - M_\mu(x))f_\mu(x) = 0$. The existence of such $R_\mu < \infty$ will be proven in Lemma 5.5.

Note, that choosing R_μ to be the minimal value such that $(V_N - M_\mu)$ has scattering length zero excludes the possibility for bound states for the potential. This will be shown in Lemma 7.10 (a). Heuristically speaking, the absence of bound states can be seen in the following way: The attractive part of the potential, i.e. $-M_\mu$, is chosen to be as small as possible, i.e. just to compensates the repulsive part. Then, there is not enough attractiveness left to form a bound state.

Lemma 5.5. *For the scattering state f_μ, defined by* (30), *the following relations hold*:

(a) *There exists a minimal value $R_\mu < \infty$ such that $\int_{\mathbb{R}^2} d^2x(V_N(x) - M_\mu(x))f_\mu(x) = 0$.*

For the rest of the paper we assume that R_μ is the minimum we get in (a).

(b) *There exists $K_\mu \in \mathbb{R}$, $K_\mu > 0$ such that $K_\mu f_\mu(x) = j_{N, R_\mu}(x) \; \forall |x| \leq N^{-\mu}$.*
(c) *For N sufficiently large the supports of V_N and M_μ do not overlap.*
(d) *f_μ is a positive, monotone nondecreasing function in $|x|$.*
(e)

$$f_\mu(x) = 1 \quad \text{for } |x| \geq R_\mu. \tag{31}$$

(f)

$$1 \geq K_\mu \geq 1 + \frac{1}{N + \ln\left(\frac{R_\mu}{a}\right)} \ln\left(\frac{N^{-\mu}}{R_\mu}\right). \tag{32}$$

(g) $R_\mu \leq CN^{-\mu}$.

For any fixed $0 < \mu$, N sufficiently large such that V_N and M_μ do not overlap, we obtain

(h)

$$|N\|V_N f_\mu\|_1 - 4\pi| = |N\|M_\mu f_\mu\|_1 - 4\pi| \leq C \frac{\ln(N)}{N}.$$

(i) *Define*

$$g_\mu(x) = 1 - f_\mu(x).$$

Then,

$$\|g_\mu\|_1 \leq CN^{-1-2\mu}\ln(N), \quad \|g_\mu\| \leq CN^{-1-\mu}\ln(N), \quad \|g_\mu\|_\infty \leq 1.$$

(j)

$$|N\|M_\mu\|_1 - 4\pi| \leq C \frac{\ln(N)}{N}.$$

(k)

$$M_\mu \in \mathcal{W}_\mu, \, M_\mu f_\mu \in \mathcal{W}_\mu.$$

Proof. (a) In the following, we will sometimes denote, with a slight abuse of notation, $f_\mu(x) = f_\mu(r)$ and $j_{N,R}(x) = j_{N,R}(r)$ for $r = |x|$ (for this, recall that f_μ and $j_{N,R}$ are radially symmetric). We further denote by $f'_\mu(r)$ the derivative of f_μ with respect to r.

We first show by contradiction that there exists a $x_0 \in \mathbb{R}^2$, $|x_0| \leq N^{-\mu}$, such that $f_\mu(x_0) \neq 0$. For this, assume that $f_\mu(x) = 0$ for all $|x| \leq N^{-\mu}$. Since f_μ is continuous, there exists a maximal value $r_0 \geq N^{-\mu}$ such that the scattering equation (30) is equivalent to

$$\begin{cases} \left(-\Delta_x - \frac{1}{2}M_\mu(x)\right) f_\mu(x) = 0, \\ f_\mu(x) = 1 \text{ for } |x| = R_\mu, \\ f_\mu(x) = 0 \text{ for } |x| \leq r_0. \end{cases} \tag{33}$$

Using (30) and Gauss'-theorem, we further obtain

$$f'_\mu(r) = \frac{1}{4\pi r} \int_{B_r(0)} d^2x (V_N(x) - M_\mu(x)) f_\mu(x). \tag{34}$$

(33) and (34) then imply for $r > r_0$

$$\begin{aligned} |f'_\mu(r)| &= \frac{1}{4\pi r} \left| \int_{B_r(0)} d^2x M_\mu(x) f_\mu(x) \right| = \frac{2\pi N^{-1+2\mu}}{r} \left| \int_{r_0}^r dr' r' f_\mu(r') \right| \\ &\leq \frac{2\pi N^{-1+2\mu}}{r} \left| \int_{r_0}^r dr' r' (r' - r_0) \sup_{r_0 \leq s \leq r} |f'_\mu(s)| \right|. \end{aligned}$$

Taking the supreme over the interval $[r_0, r]$, the inequality above then implies that there exists a constant $C(r, r_0) \neq 0$, $\lim_{r \to r_0} C(r, r_0) = 0$ such that

$$\sup_{r_0 \leq s \leq r} |f'_\mu(s)| \leq C(r, r_0)N^{-1+2\mu} \sup_{r_0 \leq s \leq r} |f'_\mu(s)|.$$

Thus, for r close enough to r_0, the inequality above can only hold if $f'_\mu(s) = 0$ for $s \in [r_0, r]$, yielding a contradiction to the choice of r_0.

Consequently, there exists a $x_0 \in \mathbb{R}^2$, $|x_0| \leq N^{-\mu}$, such that $f_\mu(x_0) \neq 0$. We can thus define

$$h(x) = f_\mu(x) \frac{j_{N,R}(x_0)}{f_\mu(x_0)}$$

on the compact set $\overline{B_{x_0}(0)}$. One easily sees that $h(x) = j_{N,R}(x)$ on $\partial \overline{B_{x_0}(0)}$ and satisfies the zero energy scattering equation (26) for $x \in \overline{B_{N^{-\mu}}(0)}$. Note that the scattering equations (26) and (30) have a unique solution on any compact set. It then follows that $h(x) = j_{N,R}(x) \ \forall x \in \overline{B_{N^{-\mu}}(0)}$. Since $j_{N,R}(N^{-\mu}) \neq 0$, we then obtain $f_\mu(N^{-\mu}) \neq 0$. Applying Theorem C.1 in [38] once more, it then follows that either f_μ or $-f_\mu$ is a nonnegative, monotone nondecrasing function in $|x|$ for all $|x| \leq N^{-\mu}$.

Recall that M_μ and hence $f_\mu(x)$ depend on $R_\mu \in [N^{-\mu}, \infty[$. For conceptual clarity, we denote $M_\mu^{(R_\mu)}(x) = M_\mu(x)$ and $f_\mu^{(R_\mu)}(x) = f_\mu(x)$ for the rest of the proof of part (a). For μ fixed, consider the function

$$s : [N^{-\mu}, \infty[\to \mathbb{R}$$
$$R_\mu \mapsto \int_{B_{R_\mu}(0)} d^2x (V_N(x) - M_\mu^{(R_\mu)}(x)) f_\mu^{(R_\mu)}(x).$$

We show by contradiction that the function s has at least one zero. Assume $s \neq 0$ were to hold. We can assume w.l.o.g. $s > 0$. It then follows from Gauss'-theorem that $f_\mu^{\prime (R_\mu)}(R_\mu) > 0$ for all $R_\mu \geq N^{-\mu}$. By uniqueness of the solution of the scattering equation (30), for $\tilde{R}_\mu < R_\mu$ there exists a constant $K_{\tilde{R}_\mu, R_\mu} \neq 0$, such that for all $|x| \leq \tilde{R}_\mu$ we have $f_\mu^{(\tilde{R}_\mu)}(x) = K_{\tilde{R}_\mu, R_\mu} f_\mu^{(R_\mu)}(x)$. Since $f_\mu^{(R_\mu)}$ and s are continuous, we can further conclude $K_{\tilde{R}_\mu, R_\mu} > 0$. From $s \neq 0$, it then follows that, for all $r \in [N^{-\mu}, \infty[$ and for all $R_\mu \in [N^{-\mu}, \infty[$, $f_\mu^{\prime (R_\mu)}(r) \neq 0$. Thus, for all $r \in [N^{-\mu}, \infty[$ and for all $R_\mu \in [N^{-\mu_1}, \infty[$, the function $f_\mu^{(R_\mu)}(r)$ doesn't change sign. From Lemma 5.2, the assumption $s(N^{-\mu}) > 0$ and $K_{\tilde{R}_\mu, R_\mu} > 0$, we obtain, for all $r \in [0, N^{-\mu}]$ and for all $R_\mu \in [N^{-\mu}, \infty[$, that $f_\mu^{(R_\mu)}(r) \geq 0$ holds. This, however, implies $\lim_{R_\mu \to \infty} s(R_\mu) = -\infty$ yielding to a contradiction. By continuity of s, there exists thus a minimal value $R_\mu \geq N^{-\mu}$ such that $s(R_\mu) = 0$.

Remark 5.6. As mentioned, we will from now on fix $R_\mu \in [N^{-\mu}, \infty[$ as the minimal value such that $s(R_\mu) = 0$. Furthermore, we may assume $a > 0$ and $R_\mu > N^{-\mu}$ in the following. For $a = 0$, we can choose $R_\mu = N^{-\mu}$, such that $f_\mu(x) = j_{N,R}(x)$. It is then easy to verify that the Lemma stated is valid.

(b) From (a), we can conclude that

$$K_\mu = \frac{j_{N, R_\mu}(N^{-\mu})}{f_\mu(N^{-\mu})}. \tag{35}$$

Next, we show that the constant K_μ is positive. Since $j_{N, R_\mu}(N^{-\mu})$ is positive, it follows from Eq. (35) that K_μ and $f_\mu(N^{-\mu})$ have equal sign. By (a), the sign of f_μ

is constant for $|x| \leq R_\mu$. Since j_{N,R_μ} and V_N are nonnegative functions, we obtain by Gauss-theorem and the scattering equation (30)

$$\text{sgn}\left(\frac{\partial f_\mu}{\partial r}|_{r=N^{-\mu}}\right) = \text{sgn}(K_\mu). \tag{36}$$

Recall that R_μ is the smallest value such that $\frac{\partial f_\mu}{\partial r}|_{r=R_\mu} = 0$. If it were now that K_μ is negative, we could conclude from (35) and (36) that $\frac{\partial f_\mu}{\partial r}|_{r=N^{-\mu}} < 0$ and $f_\mu(N^{-\mu}) < 0$. Since R_μ is by definition the smallest value where $\frac{\partial f_\mu}{\partial r} = 0$, we were able to conclude from the continuity of the derivative that $\frac{\partial f_\mu}{\partial r} < 0$ for all $r < R_\mu$ and hence $f(R_\mu) < 0$. However, this were in contradiction to the boundary condition of the zero energy scattering state [see (30)] and thus $K_\mu > 0$ follows.

(c) This directly follows from $e^{-N} < CN^{-\mu}$ for N sufficiently large.

(d) From the proof of property (b), we see that f_μ and its derivative is positive at $N^{-\mu}$. From (34), we obtain $f'_\mu(r) = 0$ for all $r > R_\mu$. Further (34) gives that R_μ is the smallest value such that $f'_\mu(R_\mu) = 0$. This and continuity imply that $f_\mu(r) > 0$ for all $r < R_\mu$. Since f_μ is continuous, positive at $N^{-\mu}$, and its derivative is a nonnegative function, it follows that f_μ is a positive, monotone nondecreasing function in $|x|$.

(e) By definition of R_μ, it follows that $\tilde{I} = \int_{\mathbb{R}^2} d^2x (V_N(x) - M_\mu(x)) f_\mu(x) = 0$. Therefore, for all $|x| \geq R_\mu$, f_μ solves $-\Delta f_\mu(x) = 0$, which has the solution

$$f_\mu(x) = 1 + \frac{\tilde{I}}{4\pi} \ln\left(\frac{|x|}{R_\mu}\right) = 1.$$

(f) Since f_μ is a positive monotone nondecreasing function in $|x|$, we obtain

$$1 \geq f_\mu(N^{-\mu}) = j_{N,R_\mu}(N^{-\mu})/K_\mu = \left(1 + \frac{1}{N + \ln\left(\frac{R_\mu}{a}\right)} \ln\left(\frac{N^{-\mu}}{R_\mu}\right)\right)/K_\mu.$$

We obtain the lower bound

$$K_\mu \geq 1 + \frac{1}{N + \ln\left(\frac{R_\mu}{a}\right)} \ln\left(\frac{N^{-\mu}}{R_\mu}\right).$$

For the upper bound we first prove that $f_\mu(x) \geq j_{N,R_\mu}(x)$ holds for all $|x| \leq R_\mu$. Using the scatting equations (26) and (30) we obtain

$$\Delta_x(f_\mu(x) - j_{N,R_\mu}(x)) = \frac{1}{2} V_N(x)(f_\mu(x) - j_{N,R_\mu}(x)) - \frac{1}{2} M_\mu(x) f_\mu(x)$$

as well as $f_\mu(R_\mu) - j_{N,R_\mu}(R_\mu) = 0$. Since $M_\mu(x) f_\mu(x) \geq 0$, we obtain that $\Delta_x(f_\mu(x) - j_{N,R_\mu}(x)) \leq 0$ for $N^{-\mu} \leq |x| \leq R_\mu$. That is, $f_\mu(x) - j_{N,R_\mu}(x)$ is superharmonic for $N^{-\mu} < |x| < R_\mu$. Using the minimum principle, we obtain, using that $f_\mu - j_{N,R_\mu}$ is spherically symmetric

$$\min_{N^{-\mu} \leq |x| \leq R_\mu} (f_\mu - j_{N,R_\mu}) = \min_{|x| \in \{N^{-\mu}, R_\mu\}} (f_\mu - j_{N,R_\mu}). \tag{37}$$

If it were now that $\min_{|x|\in\{N^{-\mu},R_\mu\}}(f_\mu - j_{N,R_\mu}) = f_\mu(N^{-\mu}) - j_{N,R_\mu}(N^{-\mu}) \le f_\mu(R_\mu) - j_{N,R_\mu}(R_\mu) = 0$, we could conclude that $f_\mu(x) - j_{N,R_\mu}(x) \le 0$ for all $N^{-\mu} \le |x| \le R_\mu$. Since $f_\mu(x) - j_{N,R_\mu}(x)$ then obeys

$$\begin{cases} -\Delta(f_\mu(x) - j_{N,R_\mu}(x)) + \frac{1}{2}V_N(x)(f_\mu(x) - j_{N,R_\mu}(x)) = 0 & \text{for } |x| \le N^{-\mu}, \\ f_\mu(x) - j_{N,R_\mu}(x) \le 0 & \text{for } |x| = N^{-\mu}, \end{cases}$$

we could then conclude that $f_\mu(x) - j_{N,R_\mu}(x) \le 0$ for all $|x| \le R_\mu$. From this, we obtain that $\Delta(f_\mu(x) - j_{N,R_\mu}(x)) \le 0$ for $|x| \le R_\mu$. That is, $f_\mu(x) - j_{N,R_\mu}(x)$ is superharmonic for all $|x| \le R_\mu$. Using the minimum principle once again, we then obtain

$$\min_{B_{R_\mu}(0)} (f_\mu - j_{N,R_\mu}) = f_\mu(R_\mu) - j_{N,R_\mu}(R_\mu) = 0$$

which contradicts $f_\mu(x) - j_{N,R_\mu}(x) \le 0$ for $|x| \le R_\mu$. Therefore, we can conclude in (37) that $\min_{N^{-\mu} \le |x| \le R_\mu}(f_\mu - j_{N,R_\mu}) = f_\mu(R_\mu) - j_{N,R_\mu}(R_\mu) = 0$ holds. Then, it follows that $f_\mu(x) - j_{N,R_\mu}(x) \ge 0$ for all $N^{-\mu} \le |x| \le R_\mu$. Using the zero energy scattering equation $-\Delta(f_\mu(x) - j_{N,R_\mu}(x)) + \frac{1}{2}V_N(x)(f_\mu(x) - j_{N,R_\mu}(x)) = 0$ for $|x| \le N^{-\mu}$, we can, together with $f_\mu(N^{-\mu}) - j_{N,R_\mu}(N^{-\mu}) \ge 0$, conclude that $f_\mu(x) - j_{N,R_\mu}(x) \ge 0$ for all $|x| \le R_\mu$.

As a consequence, we obtain the desired bound $K_\mu = \frac{j_{N,R_\mu}(N^{-\mu})}{f_\mu(N^{-\mu})} \le 1$.

(g) Since f_μ is a nonnegative, monotone nondecreasing function in $|x|$ with $f_\mu(x) = 1$ $\forall |x| \ge R_\mu$, it follows that

$$Cf_\mu(N^{-\mu}) = f_\mu(N^{-\mu}) \int_{\mathbb{R}^2} d^2x\, V_N(x) \ge \int_{\mathbb{R}^2} d^2x\, V_N(x) f_\mu(x)$$

$$= \int_{\mathbb{R}^2} d^2x\, M_\mu(x) f_\mu(x) \ge f_\mu(N^{-\mu}) \int_{\mathbb{R}^2} d^2x\, M_\mu(x).$$

Therefore, $\int_{\mathbb{R}^2} d^2x\, M_\mu(x) \le C$ holds, which implies that $R_\mu \le CN^{1/2-\mu}$. From

$$\frac{1}{K_\mu} \frac{4\pi}{N + \ln\left(\frac{R_\mu}{a}\right)} = \frac{1}{K_\mu} \int_{\mathbb{R}^2} d^2x\, V_N(x) j_{N,R_\mu}(x) = \int_{\mathbb{R}^2} d^2x\, V_N(x) f_\mu(x)$$

$$= \int_{\mathbb{R}^2} d^2x\, M_\mu(x) f_\mu(x) = 8\pi^2 N^{-1+2\mu} \int_{N^{-\mu}}^{R_\mu} dr\, r f_\mu(r)$$

we conclude that

$$\int_{N^{-\mu}}^{R_\mu} dr\, r f_\mu(r) = \frac{N^{1-2\mu}}{2\pi K_\mu \left(N + \ln\left(\frac{R_\mu}{a}\right)\right)}.$$

Since f_μ is a nonegative, monotone nondecreasing function in $|x|$,

$$\frac{1}{2}(R_\mu^2 - N^{-2\mu})\frac{j_{N,R_\mu}(N^{-\mu})}{K_\mu} = \frac{1}{2}(R_\mu^2 - N^{-2\mu})f_\mu(N^{-\mu}) \le \int_{N^{-\mu}}^{R_\mu} dr\, r f_\mu(r)$$

which implies

$$R_\mu^2 N^{2\mu} \leq \frac{N}{\pi \left(N + \ln \left(\frac{R_\mu}{a} \right) \right) j_{N,R_\mu}(N^{-\mu})} + 1.$$

Using $R_\mu \leq C N^{1/2-\mu}$, it then follows

$$j_{N,R_\mu}(N^{-\mu}) = 1 + \frac{1}{N + \ln \left(\frac{R_\mu}{a} \right)} \ln \left(\frac{N^{-\mu}}{R_\mu} \right) \geq 1 - \frac{C}{N},$$

which implies $R_\mu \leq C N^{-\mu}$.

(h) Using

$$\|M_\mu f_\mu\|_1 = \|V_N f_\mu\|_1 = K_\mu^{-1} \|V_N j_{N,R_\mu}\|_1 = K_\mu^{-1} \frac{4\pi}{N + \ln \left(\frac{R_\mu}{a} \right)},$$

we obtain

$$|N\|V_N f_\mu\|_1 - 4\pi| = |N\|M_\mu f_\mu\|_1 - 4\pi| = 4\pi \left| K_\mu^{-1} \frac{N}{N + \ln \left(\frac{R_\mu}{a} \right)} - 1 \right|$$

$$= \frac{4\pi}{K_\mu} \left| \frac{N - N K_\mu + K_\mu \ln \left(\frac{R_\mu}{a} \right)}{N + \ln \left(\frac{R_\mu}{a} \right)} \right| \leq C \frac{\ln(N)}{N}.$$

(i) Using for $|x| \leq R_\mu$ the inequalities $j_{N,R_\mu}(x) \geq 1 + \frac{1}{N+\ln \left(\frac{R_\mu}{a} \right)} \ln \left(\frac{|x|}{R_\mu} \right)$ as well as $1 \geq f_\mu(x) \geq j_{N,R_\mu}(x)$, it follows for $|x| \leq R_\mu$

$$0 \leq g_\mu(x) = 1 - f_\mu(x) \leq 1 - j_{N,R_\mu}(x) \leq -\frac{1}{N + \ln \left(\frac{R_\mu}{a} \right)} \ln \left(\frac{|x|}{R_\mu} \right)$$

$$\leq C N^{-1} |\ln (N|x|)|.$$

Since $g_\mu(x) = 0$ for $|x| > R_\mu$, we conclude with $R_\mu \leq C N^{-\mu}$ that

$$\|g_\mu\|_1 \leq \frac{C}{N} \int_0^{R_\mu} dr \, r |\ln (Nr)| \leq C N^{-1-2\mu} \ln N,$$

as well as

$$\|g_\mu\|^2 \leq \frac{C}{N^2} \int_0^{R_\mu} dr \, r \, (\ln (Nr))^2$$

$$= C N^{-4} \left[r^2 (2(\ln(r))^2 - 2 \ln(r) + 1) \right]_0^{N R_\mu}$$

$$\leq C N^{-2-2\mu} (\ln(N))^2.$$

$\|g_\mu\|_\infty = \|1 - f_\mu\|_\infty \leq 1$, since f_μ is a nonnegative, monotone nondecreasing function with $f_\mu(x) \leq 1$.

(j) Using (h) and (i), we obtain with $\|M_\mu\|_1 \leq CN^{-1}$

$$|N\|M_\mu\|_1 - 4\pi| \leq |N\|M_\mu f_\mu\|_1 - 4\pi| + N\|M_\mu g_\mu\|_1$$

$$\leq C\left(\frac{\ln(N)}{N} + \|\mathbb{1}_{|\cdot|\geq N^{-\mu}}g_\mu\|_\infty\right).$$

Since $g_\mu(x)$ is a nonnegative, monotone nonincreasing function, it follows with $K_\mu \leq 1$

$$\|\mathbb{1}_{|\cdot|\geq N^{-\mu}}g_\mu\|_\infty = g_\mu(N^{-\mu}) = 1 - f_\mu(N^{-\mu}) = 1 - \frac{j_{N,R_\mu}(N^{-\mu})}{K_\mu}$$

$$\leq 1 - \left(1 + \frac{1}{N + \ln\left(\frac{R_\mu}{a}\right)}\ln\left(\frac{N^{-\mu}}{R_\mu}\right)\right).$$

and (j) follows.

(k) $M_\mu \in \widetilde{\mathcal{W}}_\mu$ follows directly from $R_\mu \leq CN^{-\mu}$. From part (j) we then get $b_{M_\mu} = 4\pi$ and $M_\mu \in \mathcal{W}_\mu$. By means of part (d) we conclude $0 \leq M_\mu(x)f_\mu(x) \leq M_\mu(x)$ which together with part (h) implies $M_\mu f_\mu \in \widetilde{\mathcal{W}}_\mu, b_{M_\mu f_\mu} = 4\pi$ and $M_\mu f_\mu \in \mathcal{W}_\mu$. □

6. Proof of the Theorem

In this section, we present the proof of Theorem 2.4. We start with the NLS regime and then pursue with the exponential scaling. In both cases we follow the same strategy: After giving the precise definition of the functional we explain its connection to the notion of Bose–Einstein condensation in terms of reduced density matrices. Thereupon, we differentiate the functional with respect to its time variable, perform a Grönwall estimate and finally prove the respective part of the theorem.

6.1. Proof for the NLS scaling W_β with $\beta > 0$.

6.1.1. Definition of the functional The goal of this section is to define a functional $\alpha \colon L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ which is adapted to potentials with NLS scaling and which meets all the requirements stated in Sect. 3. In short, we demand the functional to converge to zero for properly chosen initial states and its time derivative to be controllable by means of a Grönwall estimate. Additionally, the functional should allow to prove both Bose–Einstein condensation and the convergence of the energy per particle of the many-body system to the effective energy functional.

While interactions in the mean-field scaling (W_β with $\beta = 0$) become weak for large particle numbers, potentials W_β with $\beta > 1/2$ are getting peaked as $N \to \infty$. This fact needs to be taken into account when defining a suitable counting functional. For small β and a large class of different choices of the weight $\widehat{m}^{\varphi_t}$ with φ_t being a solution of (3), it is possible to show that

$$\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle \leq \langle\!\langle \Psi_0, \widehat{m}^{\varphi_0}\Psi_0\rangle\!\rangle + \int_0^t ds$$

$$\left(\mathcal{K}(\varphi_s, A_s)\left(\langle\!\langle \Psi_s, \widehat{m}^{\varphi_s}\Psi_s\rangle\!\rangle + \mathcal{O}(1) + \langle\!\langle \Psi_s, \widehat{n}^{\varphi_s}\Psi_s\rangle\!\rangle + \left|\mathcal{E}_{W_\beta}(\Psi_s) - \mathcal{E}_{b_{W_\beta}}^{GP}(\varphi_s)\right|\right)\right).$$

This enables us to perform an integral type Grönwall estimate if we choose

$$\alpha(\Psi_t, \varphi_t) = \langle\!\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle\!\rangle + \left| \mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}_{b_{W_\beta}}^{GP}(\varphi_t) \right|.$$

Here, the smallness of the distance between the energies is used to control the kinetic energy per particle of the many-body system (Lemma 7.6). This prevents the wave function from being strongly localized in the support of the potential and in this way softens the effect of the interaction. Moreover, it allows us to bound the kinetic energy of the particles which are not in the condensate state φ_t by $\alpha(\Psi_t, \varphi_t)$, see Lemma 7.9.

For large β, the interaction is harder to control and several estimates break down, if one defines α as above. It is therefore necessary to redefine the functional $\alpha(\Psi_t, \varphi_t)$ and to carefully choose a new weight function m. Let us explain why this is necessary. To obtain an integral type Grönwall estimate, we will calculate the time derivative of $\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t} \Psi_t \rangle\!\rangle$. This time derivative will contain contributions of the form $\widehat{m} - \widehat{m}_1$ and $\widehat{m} - \widehat{m}_2$. To obtain sufficient error estimates for large β, it is necessary to choose a weight function m such that $\|\widehat{m} - \widehat{m}_i\|_{\mathrm{op}}$ with $i = 1, 2$ can be controlled sufficiently well (one can infer from the proof below that $\|\widehat{n} - \widehat{n}_i\|_{\mathrm{op}} = \mathcal{O}(N^{-1/2})$ with $i = 1, 2$ is not decaying sufficiently in N, see part (b) of Lemma 7.7). For the Grönwall estimate, we require in addition $\|\widehat{m} - \widehat{n}\|_{\mathrm{op}} \to 0$, as $N \to \infty$.

In total, this suggests the following form of the functional

Definition 6.1. For $0 < \xi < \frac{1}{3}$ define

$$m(k) = \begin{cases} \sqrt{k/N}, & \text{for } k \geq N^{1-2\xi}; \\ 1/2(N^{-1+\xi} k + N^{-\xi}), & \text{else.} \end{cases}$$

and

$$\alpha^<(\Psi, \varphi) = \langle\!\langle \Psi, \widehat{m}^\varphi \Psi \rangle\!\rangle + \left| \mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{b_{W_\beta}}^{GP}(\varphi) \right|.$$

Remark 6.2. It should be noted, that $\alpha^<$ depends on the parameter ξ which will be chosen later. For better readability, we disregard the ξ dependence in the notation.

The counting measure can be related to the trace norm distance of the one-particle reduced density matrix.

Lemma 6.3. *Let $0 < \xi < 1/3$, $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$, $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ and $\alpha^<(\Psi, \varphi)$ be defined as in Definition 6.1. Then,*

$$Tr \left| \gamma_\Psi^{(1)} - |\varphi\rangle\langle\varphi| \right| \leq \sqrt{8\alpha^<(\Psi, \varphi)}, \tag{38}$$

$$\alpha^<(\Psi, \varphi) \leq \sqrt{Tr \left| \gamma_\Psi^{(1)} - |\varphi\rangle\langle\varphi| \right|} + \left| \mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{b_{W_\beta}}^{GP}(\varphi) \right| + \frac{1}{2} N^{-\xi}. \tag{39}$$

Proof. We would like to mention, that this Lemma has been proven in [6, Lemma 3.3]. For sake of completeness, we briefly recall the argument. From [30, Lemma 2.3] and [50, eq. (6)] one concludes

$$\langle\!\langle \Psi, (\widehat{n}^\varphi)^2 \Psi \rangle\!\rangle \leq Tr \left| \gamma_\Psi^{(1)} - |\varphi\rangle\langle\varphi| \right| \leq \sqrt{8 \langle\!\langle \Psi, (\widehat{n}^\varphi)^2 \Psi \rangle\!\rangle}. \tag{40}$$

If one then uses that $n(k)^2 \leq n(k) \leq m(k)$ and $m(k) \leq n(k) + \frac{1}{2} N^{-\xi}$ imply the relations

$$\langle\!\langle \Psi, (\widehat{n}^\varphi)^2 \Psi \rangle\!\rangle \leq \langle\!\langle \Psi, \widehat{m}^\varphi \Psi \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \Psi, \widehat{m}^\varphi \Psi \rangle\!\rangle \leq \sqrt{\langle\!\langle \Psi, (\widehat{n}^\varphi)^2 \Psi \rangle\!\rangle} + \frac{1}{2} N^{-\xi},$$

the Lemma follows. $\square$

6.1.2. Preliminaries for the Grönwall estimate Subsequently, we will perform a Grönwall estimate for $\alpha^<$ and prove part (a) of Theorem 2.4. For this, we define

Definition 6.4. Let $0 < \xi < 1/3$ and $W_\beta \in \mathcal{W}_\beta$. Define

$$Z_\beta^\varphi(x_j, x_k) = W_\beta(x_j - x_k) - \frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_j) - \frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_k). \quad (41)$$

Note, for $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, we have $N\|W_\beta\|_1 = \|W\|_1$. With

$$m^a(k) = m(k) - m(k+1), \quad m^b(k) = m(k) - m(k+2)$$

and

$$\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2),$$

we define for $l \in \{a, b, c\}$ the functionals $\gamma_l^<: L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ by

$$\gamma_a^<(\Psi, \varphi) = \langle\!\langle \Psi, \dot{A}_t(x_1)\Psi \rangle\!\rangle - \langle \varphi, \dot{A}_t\varphi \rangle \quad (42)$$

$$\gamma_b^<(\Psi, \varphi) = N(N-1)\Im\left(\langle\!\langle \Psi, Z_\beta^\varphi(x_1, x_2)\widehat{r}\Psi \rangle\!\rangle \right) \quad (43)$$

$$= -2N(N-1)\Im\left(\langle\!\langle \Psi, p_1 q_2 \widehat{m}_{-1}^a Z_\beta^\varphi(x_1, x_2) p_1 p_2 \Psi \rangle\!\rangle \right)$$

$$- N(N-1)\Im\left(\langle\!\langle \Psi, q_1 q_2 \widehat{m}_{-2}^b W_\beta(x_1 - x_2) p_1 p_2 \Psi \rangle\!\rangle \right)$$

$$- 2N(N-1)\Im\left(\langle\!\langle \Psi, q_1 q_2 \widehat{m}_{-1}^a Z_\beta^\varphi(x_1, x_2) p_1 q_2 \Psi \rangle\!\rangle \right) \quad (44)$$

$$\gamma_c^<(\Psi, \varphi) = i N(N\|W_\beta\|_1 - b_{W_\beta})\langle\!\langle \Psi, (q_1|\varphi(x_1)|^2\widehat{m}^a p_1 - p_1\widehat{m}^a|\varphi(x_1)|^2 q_1)\Psi \rangle\!\rangle. \quad (45)$$

The value of the functional $\alpha^<(\Psi_t, \varphi_t)$ at time t is then bounded by

Lemma 6.5. *Let $W_\beta \in \mathcal{W}_\beta$. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{W_\beta}\Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h_{b_{W_\beta}}^{GP}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$, $\|\varphi_0\| = 1$. Let $\alpha^<(\Psi_t, \varphi_t)$ be defined as in Definition 6.1. Then*

$$\alpha^<(\Psi_t, \varphi_t) \le \alpha^<(\Psi_0, \varphi_0) + \int_0^t ds \left(\left|\gamma_a^<(\Psi_s, \varphi_s)\right| + \left|\gamma_b^<(\Psi_s, \varphi_s)\right| + \left|\gamma_c^<(\Psi_s, \varphi_s)\right| \right). \quad (46)$$

Proof. For the proof of the Lemma we restore the upper index φ_t in order to pay respect to the time dependence of $\widehat{m}^{\varphi_t}$. The time derivative of φ_t is given by (3), i.e. $i\partial_t\varphi_t(x_j) = h_{b_{W_\beta}, j}^{GP}\varphi_t(x_j)$. Here, $h_{b_{W_\beta}, j}^{GP}$ denotes the operator $h_{b_{W_\beta}}^{GP}$ acting on the jth coordinate x_j.

We then obtain

$$\frac{d}{dt}\langle\!\langle\Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle = i\langle\!\langle H_{W_\beta}\Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle - i\langle\!\langle\Psi_t, \widehat{m}^{\varphi_t}H_{W_\beta}\Psi_t\rangle\!\rangle$$

$$- i\langle\!\langle\Psi_t, [\sum_{j=1}^{N} h^{GP}_{bw_\beta,j}, \widehat{m}^{\varphi_t}]\Psi_t\rangle\!\rangle$$

$$= i\langle\!\langle\Psi_t, [H_{W_\beta} - \sum_{j=1}^{N} h^{GP}_{bw_\beta,j}, \widehat{m}^{\varphi_t}]\Psi_t\rangle\!\rangle$$

$$= i\langle\!\langle\Psi_t, \left[\left(\frac{1}{2}N(N-1)W_\beta(x_1-x_2) - Nb_{W_\beta}|\varphi_t(x_1)|^2\right), \widehat{m}^{\varphi_t}\right]\Psi_t\rangle\!\rangle$$

$$= iN(N\|W_\beta\|_1 - b_{W_\beta})\langle\!\langle\Psi_t, [|\varphi_t(x_1)|^2, \widehat{m}^{\varphi_t}]\Psi_t\rangle\!\rangle$$

$$+ i\frac{N(N-1)}{2}\langle\!\langle\Psi_t, [Z^{\varphi_t}_\beta(x_1,x_2), \widehat{m}^{\varphi_t}]\Psi_t\rangle\!\rangle,$$

where we used the symmetry of Ψ_t. Using Lemma 4.2 (d), it follows that (dropping the explicit dependence on φ_t from now on)

$$\frac{d}{dt}\langle\!\langle\Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle = iN(N\|W_\beta\|_1 - b_{W_\beta})\langle\!\langle\Psi_t, (q_1|\varphi_t(x_1)|^2\widehat{m}^a p_1$$

$$- p_1\widehat{m}^a|\varphi_t(x_1)|^2 q_1)\Psi_t\rangle\!\rangle$$

$$+ i\frac{N(N-1)}{2}\langle\!\langle\Psi_t, [Z^{\varphi_t}_\beta(x_1,x_2), p_1 p_2(\widehat{m} - \widehat{m}_2)]\Psi_t\rangle\!\rangle$$

$$+ i\frac{N(N-1)}{2}\langle\!\langle\Psi_t, [Z^{\varphi_t}_\beta(x_1,x_2), (p_1 q_2 + q_1 p_2)(\widehat{m} - \widehat{m}_1)]\Psi_t\rangle\!\rangle.$$

Since $Z^{\varphi_t}_\beta$ and $p_1 p_2(\widehat{m} - \widehat{m}_2)$ as well as $p_1 q_2(\widehat{m} - \widehat{m}_1)$ are selfadjoint, we obtain

$$\frac{d}{dt}\langle\!\langle\Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle = \gamma_c^<(\Psi_t, \varphi_t) - N(N-1)$$

$$\times \Im\left(\langle\!\langle\Psi_t, (p_1 p_2 + p_1 q_2 + q_1 p_2 + q_1 q_2)Z^{\varphi_t}_\beta(x_1,x_2)(\widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2))\Psi_t\rangle\!\rangle\right).$$

Note that in view of Lemma 4.2 (c) $\widehat{r}Q_j Z^{\varphi_t}_\beta(x_1,x_2)Q_j = Q_j Z^{\varphi_t}_\beta(x_1,x_2)Q_j\widehat{r}$ for any $j \in \{0,1,2\}$ and any weight r. Therefore,

$$\Im\left(\langle\!\langle\Psi_t, p_1 p_2 Z^{\varphi_t}_\beta(x_1,x_2)\widehat{m}^b p_1 p_2\Psi_t\rangle\!\rangle\right) = 0$$

$$\Im\left(\langle\!\langle\Psi_t, (p_1 q_2 + q_1 p_2)Z^{\varphi_t}_\beta(x_1,x_2)\widehat{m}^a(p_1 q_2 + q_1 p_2)\Psi_t\rangle\!\rangle\right) = 0.$$

Using Symmetry and Lemma 4.2 (c), we obtain the first line (43). Furthermore,

$$\frac{d}{dt}\langle\!\langle\Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle = \gamma_c^<(\Psi_t, \varphi_t) - 2N(N-1)\Im\left(\langle\!\langle\Psi_t, \widehat{m}^b_{-1}p_1 q_2 Z^{\varphi_t}_\beta(x_1,x_2)p_1 p_2\Psi_t\rangle\!\rangle\right)$$

$$- N(N-1)\Im\left(\langle\!\langle\Psi_t, \widehat{m}^b_{-2}q_1 q_2 Z^{\varphi_t}_\beta(x_1,x_2)p_1 p_2\Psi_t\rangle\!\rangle\right)$$

$$- 2N(N-1)\Im\left(\langle\!\langle\Psi_t, p_1 p_2 Z^{\varphi_t}_\beta(x_1,x_2)\widehat{m}^a p_1 q_2\Psi_t\rangle\!\rangle\right)$$

$$- 2N(N-1)\Im\left(\langle\!\langle\Psi_t, \widehat{m}^a_{-1}q_1 q_2 Z^{\varphi_t}_\beta(x_1,x_2)p_1 q_2\Psi_t\rangle\!\rangle\right).$$

Since $p_1 p_2 |\varphi_t^2|(x_1) q_1 q_2 = p_1 p_2 q_2 |\varphi_t^2|(x_1) q_1 = 0 = p_1 p_2 |\varphi_t^2|(x_2) q_1 q_2$, we can replace $Z_\beta^{\varphi_t}(x_1, x_2)$ in the second line by $W_\beta(x_1 - x_2)$.

The third line equals $2N(N-1)\Im\left(\langle\!\langle \Psi_t, \widehat{m}^a p_1 q_2 Z_\beta^{\varphi_t}(x_1, x_2) p_1 p_2 \Psi_t \rangle\!\rangle\right)$. Since

$$m(k-1) - m(k+1) - (m(k) - m(k+1)) = m(k-1) - m(k)$$

it follows that $\widehat{m}^b_{-1} - \widehat{m}^a = \widehat{m}^a_{-1} - (m(0) - m(1)) P_0$ and we get

$$\frac{d}{dt}\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t} \Psi_t \rangle\!\rangle = \gamma_c^<(\Psi_t, \varphi_t) - 2N(N-1)\Im\left(\langle\!\langle \Psi_t, p_1 q_2 \widehat{m}^a_{-1} Z_\beta^{\varphi_t}(x_1, x_2) p_1 p_2 \Psi_t \rangle\!\rangle\right)$$

$$- N(N-1)\Im\left(\langle\!\langle \Psi_t, q_1 q_2 \widehat{m}^b_{-2} W_\beta(x_1 - x_2) p_1 p_2 \Psi_t \rangle\!\rangle\right)$$

$$- 2N(N-1)\Im\left(\langle\!\langle \Psi_t, q_1 q_2 \widehat{m}^a_{-1} Z_\beta^{\varphi_t}(x_1, x_2) p_1 q_2 \Psi_t \rangle\!\rangle\right)$$

$$= \gamma_b^<(\Psi_t, \varphi_t) + \gamma_c^<(\Psi_t, \varphi_t).$$

For the second summand of $\alpha^<(\Psi_t, \varphi_t)$ we have

$$\frac{d}{dt}\left(\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{bw_\beta}(\varphi_t)\right) = \langle\!\langle \Psi_t, \dot{A}_t(x_1)\Psi_t \rangle\!\rangle - \langle \varphi_t, \dot{A}_t \varphi_t \rangle$$

$$- i\left\langle \varphi_t, \left[h^{GP}_{bw_\beta}, \left(h^{GP}_{bw_\beta} - \frac{bw_\beta}{2}|\varphi_t|^2 \right) \right] \varphi_t \right\rangle$$

$$- \left\langle \varphi_t, \frac{bw_\beta}{2}\left(\frac{d}{dt}|\varphi_t|^2 \right) \varphi_t \right\rangle$$

$$= \langle\!\langle \Psi_t, \dot{A}_t(x_1)\Psi_t \rangle\!\rangle - \langle \varphi_t, \dot{A}_t \varphi_t \rangle$$

$$+ i\left\langle \varphi_t, \left[h^{GP}_{bw_\beta}, \frac{bw_\beta}{2}|\varphi_t|^2 \right] \varphi_t \right\rangle$$

$$- i\left\langle \varphi_t, \left[h^{GP}_{bw_\beta}, \frac{bw_\beta}{2}|\varphi_t|^2 \right] \varphi_t \right\rangle$$

$$= \gamma_a^<(\Psi_t, \varphi_t). \tag{47}$$

By explicit estimates, one can show that the functions $\gamma_j^<(\Psi_\cdot, \varphi_\cdot) : \mathbb{R} \to \mathbb{R}, t \mapsto \gamma_j^<(\Psi_t, \varphi_t)$ with $j \in \{a, b, c\}$ are continuous if $A_\cdot \in C^1(\mathbb{R}, L^\infty(\mathbb{R}^2, \mathbb{R}))$. The Lemma then follows using that $|f(x)| \leq |f(0)| + \int_0^x dy |f'(y)|$ holds for any $f \in C^1(\mathbb{R}, \mathbb{R})$. $\quad\square$

6.1.3. The Grönwall estimate In order to establish a Grönwall estimate for $\alpha^<$, we have to find a suitable bound for the right hand side of (46).

Lemma 6.6. Let $W_\beta \in \mathcal{W}_\beta$. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H_{W_\beta} \Psi_t$ with initial datum $\Psi_0 \in L^2_s(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C}), \|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t \varphi_t = h^{GP}_{bw_\beta} \varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C}), \|\varphi_0\| = 1$. Let $\mathcal{E}_{W_\beta}(\Psi_0) \leq C$.

(a) Let $\beta < 1/12$. Moreover, let $\alpha^<(\Psi_t, \varphi_t)$, $\gamma_a^<(\Psi_t, \varphi_t)$ and $\gamma_b^<(\Psi_t, \varphi_t)$ be defined as in Definitions 6.1 and 6.4 with $\xi = 1/6$. Then

$$|\gamma_a^<(\Psi_t, \varphi_t) + \gamma_b^<(\Psi_t, \varphi_t) + \gamma_c^<(\Psi_t, \varphi_t)| \leq \mathcal{K}(\varphi_t, A_t)\left(\alpha^<(\Psi_t, \varphi_t) + N^{-2\beta}\ln(N)\right). \tag{48}$$

(b) *Let $\beta \geq 1/12$. Moreover, let $\alpha^<(\Psi_t, \varphi_t)$, $\gamma_a^<(\Psi_t, \varphi_t)$ and $\gamma_b^<(\Psi_t, \varphi_t)$ be defined as in Definitions* 6.1 *and* 6.4 *with $\xi = 1/10$. Then*

$$|\gamma_a^<(\Psi_t, \varphi_t) + \gamma_b^<(\Psi_t, \varphi_t) + \gamma_c^<(\Psi_t, \varphi_t)| \leq \mathcal{K}(\varphi_t, A_t)\left(\alpha^<(\Psi_t, \varphi_t) + N^{-1/10}\right). \tag{49}$$

The proof of Lemma 6.6 is given in Sect. 7.3.

At this point, we only consider the most relevant term $\gamma_b^<(\Psi_t, \varphi_t)$ and explain on a heuristic level why it is small. The principle argument follows the ideas and estimates of [49]. The first line in (44) is the most important one. This expression is only small if the correct coupling parameter $b_{W_\beta} \approx N\|W_\beta\|_1$ is used in the mean-field equation (3). Then,

$$Np_1^{\varphi_t} W_\beta(x_1 - x_2) p_1^{\varphi_t} = Np_1^{\varphi_t} W_\beta * |\varphi|^2(x_2) p_1^{\varphi_t} \to p_1^{\varphi_t} |\varphi|^2(x_2)\|W\|_1 p_1^{\varphi_t}$$

converges against the mean-field potential, and hence the first expression of (44) is small.

In order to estimate the second and third line of (44), one tries to bound $N^2 \langle\!\langle \Psi_t, q_1^{\varphi_t} q_2^{\varphi_t} \widehat{m}_{-2}^b W_\beta(x_1 - x_2) p_1^{\varphi_t} p_2^{\varphi_t} \Psi_t \rangle\!\rangle$ and $N^2 \langle\!\langle \Psi_t, q_1^{\varphi_t} q_2^{\varphi_t} \widehat{m}_{-1}^a Z_\beta^\varphi(x_1 - x_2) p_1^{\varphi_t} q_2^{\varphi_t} \Psi_t \rangle\!\rangle$ in terms of $\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle\!\rangle + \mathcal{O}(N^{-\eta})$ for some $\eta > 0$. By means of

$$|\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle\!\rangle - \langle\!\langle \Psi_t, \widehat{m}^{\varphi_t} \Psi_t \rangle\!\rangle| \leq \|\widehat{n}^{\varphi_t} - \widehat{m}^{\varphi_t}\|_{\mathrm{op}} = N^{-\xi}$$

this can then be bounded by $\alpha^<(\Psi_t, \varphi_t) + \mathcal{O}(N^{-\eta})$ for some $\eta > 0$.

With the help of Lemma 6.5, Lemma 6.6 and Grönwall's Lemma, we obtain

Lemma 6.7. *Let $W_\beta \in \mathcal{W}_\beta$. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H_{W_\beta}\Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t \varphi_t = h_{b_{W_\beta}}^{GP} \varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$, $\|\varphi_0\| = 1$.*

(a) *Let $\beta < 1/12$ and $\alpha^<(\Psi_t, \varphi_t)$ be defined as in Definition* 6.1 *with $\xi = 1/6$. Then,*

$$\alpha^<(\Psi_t, \varphi_t) \leq e^{\int_0^t ds\, \mathcal{K}(\varphi_s, A_s)}\left(\alpha^<(\Psi_0, \varphi_0) + N^{-2\beta}\ln(N)\right). \tag{50}$$

(b) *Let $\beta \geq 1/12$ and $\alpha^<(\Psi_t, \varphi_t)$ be defined as in Definition* 6.1 *with $\xi = 1/10$. Then,*

$$\alpha^<(\Psi_t, \varphi_t) \leq e^{\int_0^t ds\, \mathcal{K}(\varphi_s, A_s)}\left(\alpha^<(\Psi_0, \varphi_0) + N^{-1/10}\right). \tag{51}$$

Proof. From Lemmas 6.5 and 6.6, we have

$$\alpha^<(\Psi_t, \varphi_t) \leq \alpha^<(\Psi_0, \varphi_0) + \int_0^t ds\, \mathcal{K}(\varphi_s, A_s)\left(\alpha^<(\Psi_s, \varphi_s) + N^{-2\beta}\ln(N)\right)$$

in the case of $\beta < 1/12$. Thus if we apply Grönwall's Lemma, we get

$$\alpha^<(\Psi_s, \varphi_s) \leq \alpha^<(\Psi_0, \varphi_0) + \int_0^t ds\, \mathcal{K}(\varphi_s, A_s)N^{-2\beta}\ln(N)$$

$$+ \int_0^t ds\, \mathcal{K}(\varphi_s, A_s)e^{\int_s^t d\tau\, \mathcal{K}(\varphi_\tau, A_\tau)}\left(\alpha^<(\Psi_0, \varphi_0)\right.$$

$$\left. + \int_0^s du\, \mathcal{K}(\varphi_u, A_u)N^{-2\beta}\ln(N)\right).$$

With the help of the relation $|x| \leq e^{|x|}$ this can be further simplified and one obtains (50). Part (b) of the Lemma is shown in complete analogy. $\square$

Proof of Theorem 2.4: Part (a). Note that under the assumptions $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ and $A. \in C^1(\mathbb{R}, L^\infty(\mathbb{R}^2, \mathbb{R}))$ there exists a constant $C_t < \infty$, depending on t, φ_0 and A_t, such that $\int_0^t ds \mathcal{K}(\varphi_s, A_s) \leq C_t$, see Sect. 4. Let $\beta < 1/12$ and $\xi = 1/6$. We now combine Lemmas 6.3 and 6.7 to estimate

$$\mathrm{Tr}\left|\gamma_{\Psi_t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \leq C\sqrt{\alpha^<(\Psi_t, \varphi_t)} \leq e^{C_t}\sqrt{\alpha^<(\Psi_0, \varphi_0) + N^{-2\beta}\ln(N)}$$

$$\leq e^{C_t}\left(\sqrt[4]{\mathrm{Tr}\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|} + \sqrt{\left|\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}_{bw_\beta}^{GP}(\varphi_0)\right|}\right.$$

$$\left. + N^{-\beta}\sqrt{\ln(N)}\right).$$

Here, we have used $N^{-1/6} \leq N^{-2\beta}\ln(N)$ and $\sqrt{|a| + |b|} \leq \sqrt{|a|} + \sqrt{|b|}$ to obtain the last line. In a similar way, one shows

$$\left|\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}_{bw_\beta}^{GP}(\varphi_t)\right| \leq \alpha^<(\Psi_t, \varphi_t)$$

$$\leq e^{C_t}\left(\sqrt{\mathrm{Tr}\left|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|\right|} + \left|\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}_{bw_\beta}^{GP}(\varphi_0)\right|\right.$$

$$\left. + N^{-2\beta}\ln(N)\right).$$

In total, this shows part (a) of Theorem 2.4 for $\beta < 1/12$. The estimates for $\beta \geq 1/12$ are shown in exactly the same manner. $\quad\square$

6.2. Proof for the exponential scaling V_N.

6.2.1. Definition of the functional

In case of the exponential scaling, the interaction is so strong such that the many-body wave function develops a non-negligible short scale correlation structure which prevents the particles from being localized too close to each other. These correlations determine the statical and dynamical properties of the condensate in a crucial manner and need to be taken into account explicitly. It is therefore reasonable to expect that the counting measure needs to be modified, too.

In order to motivate how the correlation structure will appear in the definition of the functional we think for the moment of the most simple counting measure, namely $\langle\!\langle\Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle = 1 - \langle\!\langle\Psi_t, p_1^{\varphi_t}\Psi_t\rangle\!\rangle$. This functional counts the relative number of particles which are not in the state φ_t and consequently measures if the many-body state is approximately given by the product state $\varphi_t^{\otimes N}$, in the sense of reduced density matrices. However, in the face of the exponential scaling, one should picture the many-body state not as the product of one-particle states but rather as a wave function of Jastrow-type, i.e.

$$\Psi_t(x_1, \ldots, x_N) \approx \prod_{1 \leq i < j \leq N} j_{N,R}(x_i - x_j)\prod_{k=1}^N \varphi_t(x_k)$$

$$= \prod_{l=2}^N j_{N,R}(x_1 - x_l)\varphi_t(x_1)\left(\prod_{2 \leq i < j \leq N} j_{N,R}(x_i - x_j)\prod_{k=1}^N \varphi_t(x_k)\right)$$

with $j_{N,R}$ being the zero energy scattering state as defined in (26).

In the following, we will consider the correlation structure to be induced by f_μ (see Definition 5.3) rather than by $j_{N,R}$. This replacement does not change the heuristic discussion above, since $f_\mu(x) \approx j_{N,R_\mu}(x) \ \forall |x| \le N^{-\mu}$ for N large (see Lemma 5.5), but will allow us to smoothen the singular interaction, as we will explain in the following.

Instead of projecting onto the state φ_t, the previous discussion suggests to replace $p_1^{\varphi_t}$ by $|\prod_{k=2}^N f_\mu(x_1 - x_k)\varphi_t(x_1)\rangle\langle\prod_{l=2}^N f_\mu(x_1 - x_l)\varphi_t(x_1)|$. The counting measure would then be given by

$$1 - \left\langle\!\!\!\left\langle \Psi_t, |\prod_{k=2}^N f_\mu(x_1 - x_k)\varphi_t(x_1)\rangle\langle\prod_{l=2}^N f_\mu(x_1 - x_l)\varphi_t(x_1)|\Psi_t\right\rangle\!\!\!\right\rangle$$

$$= 1 - \left\langle\!\!\!\left\langle \Psi_t, \prod_{k=2}^N f_\mu(x_1 - x_k)p_1^{\varphi_t}\prod_{l=2}^N f_\mu(x_1 - x_l)\Psi_t\right\rangle\!\!\!\right\rangle.$$

This expression can be further simplified, if we use $g_\mu = 1 - f_\mu$ and only keep the terms which are at most linear in g_μ

$$1 - \left\langle\!\!\!\left\langle \Psi_t, \left(1 - \sum_{k=2}^N g_\mu(x_1 - x_k)\right)p_1^{\varphi_t}\left(1 - \sum_{l=2}^N g_\mu(x_1 - x_l)\right)\Psi_t\right\rangle\!\!\!\right\rangle$$

$$\approx 1 - \langle\!\langle \Psi_t, p_1^{\varphi_t}\Psi_t\rangle\!\rangle + 2(N-1)\Re\langle\!\langle \Psi_t, g_\mu(x_1 - x_2)p_1^\varphi\Psi_t\rangle\!\rangle$$

$$= \langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle + 2(N-1)\Re\langle\!\langle \Psi_t, g_\mu(x_1 - x_2)p_1^\varphi\Psi_t\rangle\!\rangle. \tag{52}$$

With the help of the symmetry of the many-body wave function and the identity $q_1^{\varphi_t} = 1 - p_1^{\varphi_t}$, we compute

$$\frac{d}{dt}\langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle = 2\Im\left(\langle\!\langle \Psi_t, \left((N-1)V_N(x_1 - x_2) - 4\pi|\varphi_t(x_1)|^2\right)p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right).$$

Defining $h_{4\pi}^{GP}(x_1) = (-\Delta_1 + A_t(x_1)) + 4\pi|\varphi_t(x_1)|^2$, we further compute

$$\frac{d}{dt}2(N-1)\Re\left(\langle\!\langle \Psi_t, g_\mu(x_1 - x_2)p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right)$$

$$= -2(N-1)\Im\left(\langle\!\langle \Psi_t, [H_{V_N}, g_\mu(x_1 - x_2)]p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right)$$

$$- 2(N-1)\Im\left(\langle\!\langle \Psi_t, g_\mu(x_1 - x_2)\left[(H_{V_N} - h_{4\pi}^{GP}(x_1)), p_1^{\varphi_t}\right]\Psi_t\rangle\!\rangle\right).$$

Using (30) and neglecting the mixed derivatives we get that $[H_{V_N}, g_\mu(x_1 - x_2)] \approx (V_N - M_\mu)(x_1 - x_2)f_\mu(x_1 - x_2)$. Further one can show that the leading order of $g_\mu(x_1 - x_2)\left[(H_{V_N} - h_{4\pi}^{GP}(x_1)), p_1^{\varphi_t}\right]$ is given by $g_\mu(x_1 - x_2)V_N(x_1 - x_2)p_1^{\varphi_t}$. This is due to the smallness of the support of g_μ and V_N which significantly overlap only for this term.

Hence the leading order of $\frac{d}{dt}$ (52) is given by

$$2\Im\left(\langle\!\langle \left((N-1)M_\mu(x_1 - x_2)f_\mu(x_1 - x_2) - 4\pi|\varphi_t(x_1)|^2\right)p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right). \tag{53}$$

Summarizing we can say that due to this adjustment and by means of the scattering equation (30), the interaction V_N got replaced by the less singular potential $M_\mu f_\mu$ in in

the first line of the equation above. It is this less singular potential that can be controlled using the results from the previous chapter. $M_\mu f_\mu$ has the properties of the W_β considered above [see Lemma 5.5 (k)], making (53) controllable. This explains why we chose to use f_μ in the definition of the modified counting functional instead of $j_{N,R}$. In return we obtain additional error terms, which, however, can be estimated sufficiently well, see Lemma 6.13.

Making use of Lemma 4.2 (c) and (d) this idea can also be used for weight functions different from $\frac{k}{N}$. Note that due to symmetry the correction term in (52) can be written as

$$2(N-1)\Re\langle\!\langle\Psi_t, g_\mu(x_1 - x_2)p_1^\varphi \Psi_t\rangle\!\rangle$$
$$= -N(N-1)\Re\langle\!\langle\Psi_t, g_\mu(x_1 - x_2)\left(-N^{-1}\right)\left(p_1^\varphi q_2^\varphi + q_1^\varphi p_2^\varphi\right)\Psi_t\rangle\!\rangle$$
$$- N(N-1)\Re\langle\!\langle\Psi_t, g_\mu(x_1 - x_2)\left(-2N^{-1}\right)p_1^\varphi p_2^\varphi \Psi_t\rangle\!\rangle.$$

Moreover, N^{-1} can be viewed as the discrete time derivative of the weight $\frac{k}{N}$, in other words

$$-N^{-1} = \frac{k}{N} - \frac{k+1}{N} = n^2(k) - n^2(k+1) \quad \text{and}$$
$$-2N^{-1} = \frac{k}{N} - \frac{k+2}{N} = n^2(k) - n^2(k+2).$$

We will use this insight to modify the functional $\alpha^<(\Psi_t, \varphi_t)$ from Definition 6.1. We first compute the time derivative of $\langle\!\langle\Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle$ and then add an additional term to the counting measure in a way such that the interaction V_N gets replaced by the potential $M_\mu f_\mu$.

Pursuing this approach results in the following definition.

Definition 6.8. Let $0 < \xi < \frac{1}{3}$, $\mu > 0$ and $m(k)$ be defined as in Definition 6.1. Moreover, let

$$m^a(k) = m(k) - m(k+1), \tag{54}$$
$$m^b(k) = m(k) - m(k+2) \quad \text{and} \tag{55}$$
$$\widehat{r} = \widehat{m^b}p_1 p_2 + \widehat{m^a}(p_1 q_2 + q_1 p_2). \tag{56}$$

Then, $\alpha: L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ is defined by

$$\alpha(\Psi, \varphi) = \langle\!\langle\Psi, \widehat{m}\Psi\rangle\!\rangle + \left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\right| - N(N-1)\Re\left(\langle\!\langle\Psi, g_\mu(x_1 - x_2)\widehat{r}\Psi\rangle\!\rangle\right). \tag{57}$$

Remark 6.9. It should be noted that $\widehat{r}$ depends on ξ and the functional α depends on ξ and μ. Both parameters are later chosen in a way such that we can establish an integral type Grönwall estimate.

If one recalls Definition 6.3, one sees that α is obtained from $\alpha^<$ by adding an additional correction term. It is important to note that (see proof of Lemma 6.10)

$$N(N-1)|\Re\left(\langle\!\langle\Psi, g_\mu(x_1 - x_2)\widehat{r}\Psi\rangle\!\rangle\right)| \leq C\|\varphi\|_\infty N^{-\mu+\xi}\ln(N). \tag{58}$$

For μ chosen large enough this allows us to show that the convergence of α to zero can be related to the notion of complete Bose–Einstein condensation in terms of reduced density matrices.

Lemma 6.10. *Let* $0 < \xi < 1/3$, $\mu > 0$, $\Psi \in L^2_s(\mathbb{R}^{2N}, \mathbb{C})$, $\varphi \in L^2(\mathbb{R}^2, \mathbb{C}) \cap L^\infty(\mathbb{R}^2, \mathbb{C})$ *and* $\alpha(\Psi, \varphi)$ *be defined as in Definition* 6.1. *Then, there exists a constant* $C \in (0, \infty)$ *such that*

$$Tr \left| \gamma_\Psi^{(1)} - |\varphi\rangle\langle\varphi| \right| \leq \sqrt{8\alpha(\Psi, \varphi)} + C\|\varphi\|_\infty N^{-1/2(\mu-\xi)} \sqrt{\ln(N)}, \tag{59}$$

$$\alpha(\Psi, \varphi) \leq \sqrt{Tr \left| \gamma_\Psi^{(1)} - |\varphi\rangle\langle\varphi| \right|} + \left| \mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \right|$$

$$+ \frac{1}{2}N^{-\xi} + C\|\varphi\|_\infty N^{-\mu+\xi} \ln(N). \tag{60}$$

Proof. Using $\|\widehat{m}^a\|_{op} + \|\widehat{m}^b\|_{op} \leq CN^{-1+\xi}$, see (76), together with Eq. (16) and Lemma 5.5 (i), we obtain

$$\|g_\mu(x_1 - x_2)\widehat{r}\|_{op} \leq \|g_\mu(x_1 - x_2)p_1(\widehat{m}^b p_2 + \widehat{m}^a q_2)\|_{op} + \|g_\mu(x_1 - x_2)p_2 q_1 \widehat{m}^a\|_{op}$$

$$\leq \|\varphi\|_\infty \|g_\mu\|(\|\widehat{m}^a\|_{op} + \|\widehat{m}^b\|_{op})$$

$$\leq \|\varphi\|_\infty N^{\xi-2-\mu} \ln(N).$$

Therefore, we bound $N(N-1)|\Re\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)| \leq \|\varphi\|_\infty N^{-\mu+\xi} \ln(N)$. By means of Lemma 6.3 the Lemma follows. □

6.2.2. Preliminaries for the Grönwall estimate

Definition 6.11. Let $0 < \xi < 1/3$, $\mu > 0$ and $\widehat{r}$ be defined as in Definition 6.8. Then, $\gamma \colon L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$ is defined by

$$\gamma(\Psi, \varphi) = |\gamma_a(\Psi, \varphi)| + |\gamma_b(\Psi, \varphi)| + |\gamma_c(\Psi, \varphi)| + |\gamma_d(\Psi, \varphi)| + |\gamma_e(\Psi, \varphi)| + |\gamma_f(\Psi, \varphi)|, \tag{61}$$

where the different summands are:

(a) The change in the energy-difference

$$\gamma_a(\Psi, \varphi) = \langle\!\langle\Psi, \dot{A}_t(x_1)\Psi\rangle\!\rangle - \langle\varphi, \dot{A}_t\varphi\rangle.$$

(b) The new interaction term

$$\gamma_b(\Psi, \varphi) = -N(N-1)\Im\left(\langle\!\langle\Psi, \widetilde{Z}_\mu^\varphi(x_1, x_2)\widehat{r}\,\Psi\rangle\!\rangle\right)$$

$$- N(N-1)\Im\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\widehat{r}\,\mathcal{Z}^\varphi(x_1, x_2)\Psi\rangle\!\rangle\right),$$

where, using M_μ from Definition 5.3,

$$\widetilde{Z}_\mu^\varphi(x_1, x_2) = \left(M_\mu(x_1 - x_2) - 4\pi\frac{|\varphi|^2(x_1) + |\varphi|^2(x_2)}{N-1}\right) f_\mu(x_1 - x_2)$$

$$\mathcal{Z}^\varphi(x_1, x_2) = V_N(x_1 - x_2) - \frac{4\pi}{N-1}|\varphi|^2(x_1) - \frac{4\pi}{N-1}|\varphi|^2(x_2). \tag{62}$$

(c) The mixed derivative term

$$\gamma_c(\Psi, \varphi) = -4N(N-1)\langle\!\langle\Psi, (\nabla_1 g_\mu(x_1 - x_2))\nabla_1\widehat{r}\Psi\rangle\!\rangle.$$

(d) Three particle interactions

$$\gamma_d(\Psi, \varphi) = 2N(N-1)(N-2)\Im\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\left[V_N(x_1-x_3),\widehat{r}\right]\Psi\rangle\!\rangle\right)$$
$$- N(N-1)(N-2)\Im\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\left[4\pi|\varphi|^2(x_3),\widehat{r}\right]\Psi\rangle\!\rangle\right).$$

(e) Interaction terms of the correction

$$\gamma_e(\Psi, \varphi) = \frac{1}{2}N(N-1)(N-2)(N-3)\Im\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi\rangle\!\rangle\right).$$

(f) Correction terms of the mean field

$$\gamma_f(\Psi, \varphi) = -2N(N-2)\Im\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\left[4\pi|\varphi|^2(x_1),\widehat{r}\right]\Psi\rangle\!\rangle\right).$$

The value of $\alpha(\Psi_t, \varphi_t)$ at time t is then bounded by

Lemma 6.12. *Let $V_N \in \mathcal{V}_N$ and let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h_{4\pi}^{GP}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$, $\|\varphi_0\| = 1$. Let $\alpha(\Psi_t, \varphi_t)$ and $\gamma(\Psi_t, \varphi_t)$ be defined as in (57) and (61). Then*

$$\alpha(\Psi_t, \varphi_t) \leq \alpha(\Psi_0, \varphi_0) + \int_0^t ds\,\gamma(\Psi_s, \varphi_s).$$

Proof. We first calculate

$$\frac{d}{dt}\left(\langle\!\langle\Psi, \widehat{m}\Psi\rangle\!\rangle - N(N-1)\Re\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)$$
$$= -N(N-1)\Im\left(\langle\!\langle\Psi_t, \mathcal{Z}^{\varphi_t}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)$$
$$- N(N-1)\Re\left(i\langle\!\langle\Psi_t, g_\mu(x_1-x_2)\left[H_{V_N} - \sum_{i=1}^N h_{4\pi,i}^{GP},\widehat{r}\right]\Psi_t\rangle\!\rangle\right)$$
$$- N(N-1)\Re\left(i\langle\!\langle\Psi_t, \left[H_{V_N}, g_\mu(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right).$$

Using symmetry and $\Re(iz) = -\Im(z)$, we obtain

$$\frac{d}{dt}\left(\langle\!\langle\Psi, \widehat{m}\Psi\rangle\!\rangle - N(N-1)\Re\left(\langle\!\langle\Psi, g_\mu(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)$$
$$= -N(N-1)\Im\left(\langle\!\langle\Psi_t, \mathcal{Z}^{\varphi_t}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)$$
$$+ N(N-1)\Im\left(\langle\!\langle\Psi_t, g_\mu(x_1-x_2)\left[\mathcal{Z}^{\varphi_t}(x_1,x_2),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)$$
$$+ 2N(N-1)(N-2)\Im\left(\langle\!\langle\Psi_t, g_\mu(x_1-x_2)\left[V_N(x_1-x_3),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)$$
$$- N(N-1)(N-2)\Im\left(\langle\!\langle\Psi_t, g_\mu(x_1-x_2)\left[4\pi|\varphi_t|^2(x_3),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)$$
$$+ \frac{1}{2}N(N-1)(N-2)(N-3)\Im\left(\langle\!\langle\Psi_t, g_\mu(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)$$
$$+ N(N-1)\Im\left(\langle\!\langle\Psi_t, \left[H_{V_N}, g_\mu(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right).$$
$$- 2N(N-2)\Im\left(\langle\!\langle\Psi_t, g_\mu(x_1-x_2)\left[4\pi|\varphi_t|^2(x_1),\widehat{r}\right]\Psi_t\rangle\!\rangle\right).$$

The third and fourth lines equal γ_d (recall that Ψ is symmetric), the fifth line equals γ_e and the seventh line equals γ_f. Using that $(1 - g_\mu(x_1 - x_2))\mathcal{Z}^\varphi(x_1, x_2) = \widetilde{Z}^\varphi_\mu(x_1, x_2) + (V_N(x_1 - x_2) - M_\mu(x_1 - x_2))f_\mu(x_1 - x_2)$ we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\!\langle\Psi, \widehat{m}\Psi\rangle\!\rangle - N(N-1)\Re\left(\langle\!\langle\Psi, g_\mu(x_1 - x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)
$$
$$
\leq \gamma_d(\Psi_t, \varphi_t) + \gamma_e(\Psi_t, \varphi_t) + \gamma_f(\Psi_t, \varphi_t)
$$
$$
- N(N-1)\Im\left(\langle\!\langle\Psi_t, \widetilde{Z}^{\varphi_t}_\mu(x_1, x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)
$$
$$
- N(N-1)\Im\left(\langle\!\langle\Psi_t, (V_N(x_1 - x_2) - M_\mu(x_1 - x_2))f_\mu(x_1 - x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)
$$
$$
- N(N-1)\Im\left(\langle\!\langle\Psi_t, g_\mu(x_1 - x_2)\widehat{r}\mathcal{Z}^{\varphi_t}(x_1, x_2)\Psi_t\rangle\!\rangle\right)
$$
$$
+ N(N-1)\Im\left(\langle\!\langle\Psi_t, \left[H_{V_N}, g_\mu(x_1 - x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right). \tag{63}
$$

The first, second and the fourth line give $\gamma_b + \gamma_d + \gamma_e + \gamma_f$. Using Definition (5.3) the commutator in the fifth line equals

$$
[H_{V_N}, g_\mu(x_1 - x_2)] = -[H_{V_N}, f_\mu(x_1 - x_2)]
$$
$$
= [\Delta_1 + \Delta_2, f_\mu(x_1 - x_2)]
$$
$$
= (\Delta_1 + \Delta_2)f_\mu(x_1 - x_2)
$$
$$
+ (2\nabla_1 f_\mu(x_1 - x_2))\nabla_1 + (2\nabla_2 f_\mu(x_1 - x_2))\nabla_2
$$
$$
= (V_N(x_1 - x_2) - M_\mu(x_1 - x_2))f_\mu(x_1 - x_2)
$$
$$
- (2\nabla_1 g_\mu(x_1 - x_2))\nabla_1 - (2\nabla_2 g_\mu(x_1 - x_2))\nabla_2.
$$

Using symmetry the third and fifth line in (63) give

$$
-4N(N-1)\langle\!\langle\Psi_t, (\nabla_1 g_\mu(x_1 - x_2))\nabla_1\widehat{r}\Psi_t\rangle\!\rangle = \gamma_c(\Psi_t, \varphi_t).
$$

By means of

$$
\frac{d}{dt}\left(\mathcal{E}_{M_\mu}(\Psi_t) - \mathcal{E}^{GP}_{N\|M_\mu\|_1}(\varphi_t)\right) = \gamma_a(\Psi_t, \varphi_t)
$$

and the fundamental theorem of calculus the result follows. □

6.2.3. The Grönwall estimate Again, we will bound the time derivative of $\alpha(\Psi_t, \varphi_t)$ such that we can employ a Grönwall estimate.

Lemma 6.13. *Let $V_N \in \mathcal{V}_N$. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum $\Psi_0 \in L^2_s(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ and $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{4\pi}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ and $\|\varphi_0\| = 1$. Let $\mathcal{E}_{V_N}(\Psi_0) \leq C$. Let $\alpha(\Psi_t, \varphi_t)$, $\gamma_i(\Psi_t, \varphi_t), i \in \{a, b, c, d, e, f\}$ be defined as in Definitions 6.8 and 6.11 with $\xi = 1/10$ and $\mu = 10$. Then,*

$$
\sum_{i\in\{a,b,c,d,e,f\}} |\gamma_i(\Psi_t, \varphi_t)| \leq \mathcal{K}(\varphi_t, A_t)\left(\alpha(\Psi_t, \varphi_t) + N^{-1/10}\right). \tag{64}
$$

The proof of the Lemma can be found in Sect. 7.4. By means of Lemma 5.5 (h) and (i), the terms γ_a and γ_b can be estimated in the same way as $\gamma^<_a$ and $\gamma^<_b$. The estimates for γ_c, γ_d, γ_e and γ_f are based on the smallness of the L^p-norms of g_μ, see Lemma 5.5 (i).

Thus, combining Lemmas 6.12 and 6.13, we obtain the following estimate for $\alpha(\Psi_t, \varphi_t)$ by means of Grönwall's Lemma

Lemma 6.14. *Let $V_N \in \mathcal{V}_N$. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H_{V_N} \Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ and $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t \varphi_t = h_{4\pi}^{GP} \varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$. Let $\mathcal{E}_{V_N}(\Psi_0) \leq C$. Let $\alpha(\Psi_t, \varphi_t)$ be defined as in Definition 6.8 with $\xi = 1/10$ and $\mu = 10$. Then,*

$$\alpha(\Psi_t, \varphi_t) \leq e^{\int_0^t ds \, \mathcal{K}(\varphi_s, A_s)} \big(\alpha(\Psi_0, \varphi_0) + N^{-1/10} \big). \tag{65}$$

Proof. This is proven in the same way as Lemma 6.7. $\quad\square$

Proof of Theorem 2.4: Part (b). Again, we note that under the assumptions $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ and $A. \in C^1(\mathbb{R}, L^\infty(\mathbb{R}^2, \mathbb{R}))$ there exists a constant $C_t < \infty$, depending on t, φ_0 and A_t, such that $\int_0^t ds \mathcal{K}(\varphi_s, A_s) \leq C_t$, see Sect. 4.

Let $\xi = 1/10$ and $\mu = 10$. If we then combine Lemmas 6.10 and 6.14 to estimate

$$\text{Tr} \left| \gamma_{\Psi_t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| \right| \leq C\sqrt{\alpha(\Psi_t, \varphi_t)} + C\|\varphi_t\|_\infty N^{-1} \leq e^{C_t} \big(\sqrt{\alpha(\Psi_0, \varphi_0)} + N^{-1/20} \big)$$

$$\leq e^{C_t} \left(\sqrt[4]{\text{Tr} \left| \gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0| \right|} + \sqrt{\left| \mathcal{E}_{V_N}(\Psi_0) - \mathcal{E}_{4\pi}^{GP}(\varphi_0) \right|} + N^{-1/20} \right).$$

Moreover, one obtains

$$\left| \mathcal{E}_{V_N}(\Psi_t) - \mathcal{E}_{4\pi}^{GP}(\varphi_t) \right| \leq \alpha(\Psi_t, \varphi_t) + C\|\varphi_t\|_\infty N^{-1}$$

$$\leq e^{C_t} \left(\sqrt{\text{Tr} \left| \gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0| \right|} + \left| \mathcal{E}_{V_N}(\Psi_0) - \mathcal{E}_{4\pi}^{GP}(\varphi_0) \right| + N^{-1/10} \right).$$

Finally, this shows part (b) of Theorem 2.4. $\quad\square$

7. Rigorous Estimates

7.1. Smearing out the potential W_β. To control the potential W_β for β large, we use a technique which allows us to replace the potential W_β by some potential $U_{\beta_1,\beta} \in \mathcal{W}_{\beta_1}, \beta_1 < \beta$ with $\|W_\beta\|_1 = \|U_{\beta_1,\beta}\|_1$. For this, define $h_{\beta_1,\beta}$ by $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. The function $h_{\beta_1,\beta}$ can be thought as an electrostatic potential which is caused by the charge $W_\beta - U_{\beta_1,\beta}$. It is then possible to rewrite

$$\langle\!\langle \chi, W_\beta(x_1 - x_2)\Omega \rangle\!\rangle = \langle\!\langle \chi, U_{\beta_1,\beta}(x_1 - x_2)\Omega \rangle\!\rangle$$
$$- \langle\!\langle \nabla_1\chi, (\nabla_1 h_{\beta_1,\beta})(x_1 - x_2)\Omega \rangle\!\rangle - \langle\!\langle \chi, (\nabla_1 h_{\beta_1,\beta})(x_1 - x_2)\nabla_1\Omega \rangle\!\rangle,$$

for $\chi, \omega \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$. We will verify that the L^p-norms of $h_{\beta_1,\beta}$ and $\nabla h_{\beta_1,\beta}$ are better to control than the respective L^p-norm of W_β. With additional control of $\nabla_1\Omega$ and $\nabla_1\chi$, it is therefore possible to obtain a sufficient bound for $\langle\!\langle \chi, W_\beta(x_1 - x_2)\Omega \rangle\!\rangle$ for large β.

Definition 7.1. For any $0 \leq \beta_1 < \beta$ and any $W_\beta \in \mathcal{W}_\beta$ we define

$$U_{\beta_1,\beta}(x) = \begin{cases} \frac{4}{\pi}\|W_\beta\|_1 N^{2\beta_1} & \text{for } |x| < 1/2N^{-\beta_1}, \\ 0 & \text{else.} \end{cases}$$

and

$$h_{\beta_1,\beta}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - y|(W_\beta(y) - U_{\beta_1,\beta}(y))d^2y. \tag{66}$$

Lemma 7.2. *Let* $0 \leq \beta_1 < \beta$, $W_\beta \in \mathcal{W}_\beta$ *and* $N \in \mathbb{N}$ *large enough such that* $supp(W_\beta) \subseteq supp(U_{\beta_1, \beta})$. *Then,*

(a)

$$U_{\beta_1, \beta} \in \mathcal{W}_{\beta_1},$$
$$\Delta h_{\beta_1, \beta} = W_\beta - U_{\beta_1, \beta}.$$

(b) *Pointwise estimates*

$$h_{\beta_1, \beta}(x) = 0 \, for \, |x| \geq 1/2 \, N^{-\beta_1}, \quad |h_{\beta_1, \beta}(x)| \leq CN^{-1} \ln(N), \quad (67)$$

$$|\nabla h_{\beta_1, \beta}(x)| \leq CN^{-1} \left(|x|^2 + N^{-2\beta} \right)^{-\frac{1}{2}}. \quad (68)$$

(c) *Norm estimates*

$$\|h_{\beta_1, \beta}\|_\infty \leq CN^{-1} \ln(N),$$

$$\|h_{\beta_1, \beta}\|_\lambda \leq CN^{-1 - \frac{2}{\lambda}\beta_1} \ln(N) \, for \, 1 \leq \lambda \leq \infty,$$

$$\|\nabla h_{\beta_1, \beta}\|_\lambda \leq CN^{-1 + \beta - \frac{2}{\lambda}\beta_1} \quad for \, 1 \leq \lambda \leq \infty.$$

Furthermore, for $\lambda = 2$*, we obtain the improved bounds*

$$\|h_{0, \beta}\| \leq CN^{-1}, \quad (69)$$

$$\|\nabla h_{\beta_1, \beta}\| \leq CN^{-1} (\ln(N))^{1/2}. \quad (70)$$

Proof. (a) $U_{\beta_1, \beta} \in \widetilde{\mathcal{W}}_{\beta_1}$ follows directly from the definition of $U_{\beta_1, \beta}$. Since $W_\beta \in \mathcal{W}_\beta$ one has $\left| N \| U_{\beta_1, \beta} \|_1 - b_{W_\beta} \right| \leq CN^{-1} \ln(N)$ and consequently $U_{\beta_1, \beta} \in \mathcal{W}_{\beta_1}$. Furthermore, $h_{\beta_1, \beta}$ is a solution of Poisson's equation because $-\frac{1}{2\pi} \ln |x - y|$ is the radially symmetric Green's function of the Laplacian in two dimensions [36, Theorem 6.21].

(b) The first statement is a well known result from standard electrodynamics. It follows from Newton's theorem [36, Theorem 9.7] and $\|U_{\beta_1, \beta}\|_1 = \|W_\beta\|_1$. Heuristically speaking, W_β can be understood as a charge density and $-U_{\beta_1, \beta}$ as a smeared out charge density of opposite sign such that the "total charge" is zero. Moreover if we use that $W_\beta(x) = U_{\beta_1, \beta}(x) = 0$ for all $|x| \geq 1/2N^{-\beta_1}$, we obtain the pointwise estimate

$$|h_{\beta_1, \beta}(x)| \leq \frac{1}{2\pi} \int_{B_{1/2N^{-\beta_1}(0)}} d^2 y \, \left| \ln |x - y| W_\beta(y) \right|$$

$$+ \frac{1}{2\pi} \int_{B_{1/2N^{-\beta_1}(0)}} d^2 y \, \left| \ln |x - y| U_{\beta_1, \beta}(y) \right|.$$

Subsequently, we estimate each term separately. Therefore, it is useful to recall that there exists an $R \in (0, \infty)$ such that $W_\beta(x) = 0$ for all $|x| \geq RN^{-\beta}$. This allows us to bound the first summand by

$$\int_{B_{1/2N^{-\beta_1}(0)}} d^2 y | \ln |x - y| | W_\beta(y) \leq \int_{B_{RN^{-\beta}(0)}} d^2 y | \ln |x - y| | W_\beta(y).$$

For $2RN^{-\beta} < |x| < 1/2N^{-\beta_1}$ one has $|x - y| \leq N^{-\beta_1} \leq 1$ in the integral above. This implies $|\ln|x - y|| = -\ln|x - y|$ and leads to

$$\int_{B_{1/2N^{-\beta_1}(0)}} d^2y |\ln|x - y||W_\beta(y) \leq -\|W_\beta\|_1 \ln(|x| - RN^{-\beta}) \leq -\|W_\beta\|_1 \ln(RN^{-\beta})$$

$$\leq C\|W_\beta\|_1 \ln N^\beta \leq CN^{-1} \ln(N)$$

for all $2RN^{-\beta} < |x| < 1/2N^{-\beta_1}$.

Let next $|x| \leq 2RN^{-\beta}$. We again have $|x - y| \leq 1$ for all $y \in B_{1/2N^{-\beta_1}(0)}$ and obtain

$$\int_{B_{1/2N^{-\beta_1}(0)}} |\ln|x - y||W_\beta(y)d^2y \leq C\|W_\beta\|_\infty \int_{B_{RN^{-\beta}(0)}} -\ln|x - y|d^2y$$

$$\leq CN^{-1+2\beta} \int_{B_{RN^{-\beta}(x)}} -\ln|y|d^2y$$

$$\leq CN^{-1+2\beta} \int_{B_{4RN^{-\beta}(0)}} -\ln|y|d^2y$$

$$= CN^{-1+2\beta} \left[-|y|^2(2\ln|y| - 1) \right]_0^{4RN^{-\beta}}$$

$$\leq CN^{-1} \ln(N^\beta)$$

for all $|x| \leq 2RN^{-\beta}$. If we repeat the same estimate for $|x| \leq 1/2N^{-\beta_1}$ and $U_{\beta_1,\beta}$ with $\|U_{\beta_1,\beta}\|_\infty \leq CN^{-1+2\beta_1}$ we get

$$\int_{B_{1/2N^{-\beta_1}(0)}} |\ln|x - y||U_{\beta_1,\beta}(y)d^2y \leq C\|U_{\beta_1,\beta}\|_\infty \int_{B_{1/2N^{-\beta_1}(0)}} -\ln|x - y|d^2y$$

$$\leq CN^{-1} \ln(N^{\beta_1}),$$

which proves the first statement.

For the gradient, we estimate the two terms on the r.h.s. of

$$|\nabla h_{\beta_1,\beta}(x)| \leq \frac{1}{2\pi} \int \frac{1}{|x - y|} W_\beta(y)d^2y + \frac{1}{2\pi} \int \frac{1}{|x - y|} U_{\beta_1,\beta}(y)d^2y$$

separately. Let first $2RN^{-\beta} \leq |x|$. Similarly as in the previous argument, one finds

$$\int \frac{1}{|x - y|} W_\beta(y)d^2y \leq \int_{B_{RN^{-\beta}(0)}} \frac{1}{|x - y|} W_\beta(y)d^2y \leq \frac{\|W_\beta\|_1}{|x| - RN^{-\beta}}$$

for $RN^{-\beta} \leq |x|$, which implies that

$$\int \frac{1}{|x - y|} W_\beta(y)d^2y \leq \frac{C\|W_\beta\|_1}{(|x|^2 + N^{-2\beta})^{\frac{1}{2}}} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta})^{\frac{1}{2}}}$$

for all $2RN^{-\beta} \leq |x|$. For $|x| \leq 2RN^{-\beta}$, we make use of

$$N^\beta \leq \frac{C}{(|x|^2 + N^{-2\beta})^{1/2}}$$

and estimate

$$\int \frac{1}{|x-y|} W_\beta(y) d^2y \le \|W_\beta\|_\infty \int_{B_{RN^{-\beta}(0)}} \frac{1}{|x-y|} d^2y$$

$$\le CN^{2\beta-1} \int_0^{RN^{-\beta}} d|y| = CN^{-1+\beta} \le \frac{CN^{-1}}{\left(|x|^2 + N^{-2\beta}\right)^{1/2}}.$$

Equivalently, we obtain

$$\int \frac{1}{|x-y|} U_{\beta_1,\beta}(y) d^2y \le \|U_{\beta_1,\beta}\|_\infty \int_{B_{N^{-\beta_1}(0)}} \frac{1}{|x-y|} d^2y$$

$$= CN^{-1+\beta_1} \le \frac{CN^{-1}}{\left(|x|^2 + N^{-2\beta_1}\right)^{1/2}} \le \frac{CN^{-1}}{\left(|x|^2 + N^{-2\beta}\right)^{\frac{1}{2}}},$$

for $|x| \le N^{-\beta_1}$. Since $\nabla h_{\beta_1,\beta}(x) = 0$ for $|x| \ge N^{-\beta_1}$, the second statement of (b) follows.

(c) The first part of (c) follows from (b) and the fact that the support of $h_{\beta_1,\beta}$ and $\nabla h_{\beta_1,\beta}$ has radius $\le CN^{-\beta_1}$. The bounds on the L^2-norm can be improved by

$$\|\nabla h_{\beta_1,\beta}\|_2^2 \le C \int_0^{CN^{-\beta_1}} dr\, r |\nabla h_{\beta_1,\beta}(r)|^2 \le \frac{C}{N^2} \int_0^{CN^{-\beta_1}} dr \frac{r}{r^2 + N^{-2\beta}}$$

$$= \frac{C}{N^2} \ln\left(\frac{N^{-2\beta_1} + N^{-2\beta}}{N^{-2\beta}}\right) \le \frac{C}{N^2} \ln(N).$$

By means of [36, Theorem 9.7] we obtain

$$|h_{0,\beta}(x)| \le \frac{1}{2\pi} |\ln(x)| \int \left(U_{0,\beta}(y) + W_\beta(y)\right) d^2y \le CN^{-1} |\ln(x)|$$

and

$$\|h_{0,\beta}\|_2^2 \le CN^{-2} \int_0^1 dr\, r \ln^2(r) \le CN^{-2},$$

where we have used that $h_{0,\beta}(x) = 0$ for all $|x| \ge 1$. □

7.2. *Estimates on the cutoff.* In order to smear out singular potentials as explained in the previous section and to obtain sufficient bounds, it seems at first necessary to show that $\|\nabla_1 q_1 \Psi_t\|$ decays in N. However, this term will in fact not be small for the dynamic generated by V_N. There, we rather expect that $\|\nabla_1 q_1 \Psi_t\| = \mathcal{O}(1)$ holds. It has been shown in [18,37] that the interaction energy is purely kinetic in the Gross–Pitaevskii regime, which implies that a relevant part of the kinetic energy is concentrated around the scattering centers. We must thus cutoff the part which is used to form the microscopic structure. For this, we define the set $\overline{\mathcal{A}}_j^{(d)}$ which includes all configurations where the distance between particle x_i and particle x_j, $j \ne i$ is smaller than N^{-d}. It is then possible to prove that the kinetic energy concentrated on the complement of $\overline{\mathcal{A}}_j^{(d)}$, i.e. $\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|$, is small, see Lemma 7.9.

Definition 7.3. For any $j, k = 1, \ldots, N$ and $d > 0$ let

$$a_{j,k}^{(d)} = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{2N} : |x_j - x_k| < N^{-d}\} \subseteq \mathbb{R}^{2N}$$

$$\overline{\mathcal{A}}_j^{(d)} = \bigcup_{k \neq j} a_{j,k}^{(d)} \qquad \mathcal{A}_j^{(d)} = \mathbb{R}^{2N} \backslash \overline{\mathcal{A}}_j^{(d)} \qquad \overline{\mathcal{B}}_j^{(d)} = \bigcup_{k \neq l \neq j} a_{k,l}^{(d)} \qquad \mathcal{B}_j^{(d)} = \mathbb{R}^{2N} \backslash \overline{\mathcal{B}}_j^{(d)}.$$

$$(71)$$

Lemma 7.4. (a) *For all $j \neq k$ with $1 \leq j, k \leq N$,*

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{op} \leq C \|\varphi\|_\infty N^{1/2-d},$$

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} \nabla_j p_j\|_{op} \leq C \|\nabla\varphi\|_\infty N^{1/2-d},$$

$$\|\mathbb{1}_{a_{j,k}^{(d)}} p_j\|_{op} \leq C \|\varphi\|_\infty N^{-d}.$$

(b) *Let $\Psi \in H^1(\mathbb{R}^{2N}, \mathbb{C})$. For any $1 < p < \infty$, there exists a positive constant C_p, such that*

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \Psi\|^2 \leq C_p N^{(1-2d)\frac{p-1}{p}} \|\nabla_1 \Psi\|^{2\frac{p-1}{p}} \|\Psi\|^{\frac{2}{p}},$$

(c) *Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi\|_{H^1} \leq C$. For any $\epsilon > 0$, there exists a positive constant C_ϵ such that*

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j^{(d)}} \Psi\| \leq C_\epsilon N^{1-d+\epsilon}.$$

(d) *For any $k \neq j$*

$$\|[\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}}, p_k]\|_{op} = \|[\mathbb{1}_{a_{j,k}^{(d)}}, p_k]\|_{op} = \|[\mathbb{1}_{\mathcal{A}_j^{(d)}}, p_k]\|_{op} \leq C \|\varphi\|_\infty N^{-d}.$$

Proof. (a) First note that the volume of the sets $a_{j,k}^{(d)}$ introduced in Definition 7.3 are $|a_{j,k}^{(d)}| = \pi N^{-2d}$.

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{op} = \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_1\|_{op} = \|p_1 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_1\|_{op}^{\frac{1}{2}} \leq \left(\|\varphi\|_\infty^2 \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{1,\infty}\right)^{1/2}$$

where we defined

$$\|f\|_{p,\infty} = \sup_{x_2,\ldots,x_N \in \mathbb{R}^2} \left(\int dx_1 |f(x_1, \ldots, x_N)|^p\right)^{\frac{1}{p}}.$$

Using $\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \leq \sum_{k=2}^N \mathbb{1}_{a_{1,k}^{(d)}}$ as well as $\left(\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\right)^p = \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}$, we obtain

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{p,\infty} \leq \sup_{x_2,\ldots,x_N \in \mathbb{R}^2} \left(\int dx_1 \sum_{k=2}^N \mathbb{1}_{a_{1,k}^{(d)}}\right)^{\frac{1}{p}} \leq \left(N|a_{1,k}|\right)^{\frac{1}{p}} \leq C N^{(1-2d)\frac{1}{p}}.$$

This implies

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{\mathrm{op}} \leq C\|\varphi\|_\infty N^{\frac{1}{2}-d}.$$

The second statement of (a) can be proven similarly. Analogously, we obtain

$$\|\mathbb{1}_{a_{j,k}^{(d)}} p_j\|_{\mathrm{op}} \leq \|\varphi\|_\infty |a_{j,k}^{(d)}|^{1/2} \leq C\|\varphi\|_\infty N^{-d}.$$

(b) Without loss of generality, we can set $j = 1$. Recall the two-dimensional Sobolev inequality, for $\varrho \in H^1(\mathbb{R}^2, \mathbb{C})$ and for any $2 < m < \infty$, there exists a positive constant C_m, such that $\|\varrho\|_m \leq C_m \|\nabla \varrho\|^{\frac{m-2}{m}} \|\varrho\|^{\frac{2}{m}}$ holds. Using Hölder and Sobolev for the x_1-integration, we get, for $p > 1$

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \Psi\|^2 = \langle\!\langle \Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \Psi \rangle\!\rangle$$

$$= \int d^2x_2 \ldots d^2x_N \int d^2x_1 |\Psi(x_1, \ldots, x_N)|^2 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}(x_1, \ldots, x_N)$$

$$\leq \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{\frac{p}{p-1},\infty} \int d^2x_2 \ldots d^2x_N \left(\int d^2x_1 |\Psi(x_1, \ldots, x_N)|^{2p}\right)^{1/p}$$

$$\leq C_p N^{(1-2d)\frac{p-1}{p}} \int d^2x_2 \ldots d^2x_N \left(\int d^2x_1 |\nabla_1 \Psi(x_1, \ldots, x_N)|^2\right)^{\frac{p-1}{p}}$$

$$\times \left(\int d^2\tilde{x}_1 |\Psi(\tilde{x}_1, \ldots, x_N)|^2\right)^{\frac{1}{p}},$$

where C_p denotes a positive constant, depending on p.
Using Hölder for the $x_2, \ldots x_N$-integration with the conjugate pair $r = \frac{p}{p-1}$ and $s = p$, we then obtain

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \Psi\|^2 \leq C_p N^{(1-2d)\frac{p-1}{p}} \|\nabla_1 \Psi\|^{2\frac{p-1}{p}} \|\Psi\|^{\frac{2}{p}}.$$

(c) We use that $\overline{\mathcal{B}}_j^{(d)} \subset \bigcup_{k=1}^N \overline{\mathcal{A}}_k^{(d)}$. Hence one can find pairwise disjoint sets $\mathcal{C}_k \subset \overline{\mathcal{A}}_k^{(d)}$, $k = 1, \ldots, N$ such that $\overline{\mathcal{B}}_j^{(d)} \subset \bigcup_{k=1}^N \mathcal{C}_k$. Since the sets $\mathcal{C}_k$ are pairwise disjoint, the $\mathbb{1}_{\mathcal{C}_k} \Psi$ are pairwise orthogonal and we get

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j^{(d)}} \Psi\|^2 = \sum_{k=1}^N \|\mathbb{1}_{\mathcal{C}_k} \Psi\|^2 \leq \sum_{k=1}^N \|\mathbb{1}_{\overline{\mathcal{A}}_k^{(d)}} \Psi\|^2.$$

(d)

$$\|[\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}, p_2]\|_{\mathrm{op}} \leq \|[\mathbb{1}_{a_{1,2}}, p_2]\|_{\mathrm{op}} \leq \|\mathbb{1}_{a_{1,2}} p_2\|_{\mathrm{op}} + \|p_2 \mathbb{1}_{a_{1,2}}\|_{\mathrm{op}}$$

$$\leq 2\|\varphi\|_\infty |a_{1,2}|^{\frac{1}{2}} \leq C\|\varphi\|_\infty N^{-d}.$$

$\square$

7.3. Proof of Lemma 6.6. The goal of this section is to prove Lemma 6.6. To this end, we bound each of the functionals $\gamma_a^<$, $\gamma_b^<$ and $\gamma_c^<$ separately and then collect the estimates. In view of the conditions required in Lemma 6.6, the following is assumed in the rest of this section:

Let $\beta > 0$, $W_\beta \in \mathcal{W}_\beta$, $\varphi \in H^3(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi\| = 1$ and $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi\| = 1$ such that $\mathcal{E}_{W_\beta}(\Psi) \leq C$.

Control of $\gamma_a^<$

Lemma 7.5. *For any function $B \in L^\infty(\mathbb{R}^2, \mathbb{R})$, any $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi\| = 1$ and any $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi\| = 1$ we have*

$$|\langle\!\langle \Psi, B(x_1)\Psi \rangle\!\rangle - \langle \varphi, B\varphi \rangle| \leq C\|B\|_\infty(\langle\!\langle \Psi, \widehat{n}^\varphi \Psi \rangle\!\rangle + N^{-\frac{1}{2}}).$$

Proof. Using $1 = p_1 + q_1$,

$$\langle\!\langle \Psi, B(x_1)\Psi \rangle\!\rangle - \langle \varphi, B\varphi \rangle$$
$$= \langle\!\langle \Psi, p_1 B(x_1) p_1 \Psi \rangle\!\rangle + 2\Re\langle\!\langle \Psi, q_1 B(x_1) p_1 \Psi \rangle\!\rangle + \langle\!\langle \Psi, q_1 B(x_1) q_1 \Psi \rangle\!\rangle - \langle \varphi, B\varphi \rangle$$
$$\leq \langle \varphi, B\varphi \rangle(\|p_1\Psi\|^2 - 1) + 2\Re\langle\!\langle \Psi, \widehat{n}^{-1/2} q_1 B(x_1) p_1 \widehat{n}_1^{1/2} \Psi \rangle\!\rangle + \langle\!\langle \Psi, q_1 B(x_1) q_1 \Psi \rangle\!\rangle,$$

where we used Lemma 4.2 (c). Since $\|p_1\Psi\|^2 - 1 = -\|q_1\Psi\|^2$ it follows that

$$|\langle\!\langle \Psi, B(x_1)\Psi \rangle\!\rangle - \langle \varphi, B\varphi \rangle| \leq C\|B\|_\infty \left(\langle\!\langle \Psi, \widehat{n}^2 \Psi \rangle\!\rangle + \langle\!\langle \Psi, \widehat{n}_1 \Psi \rangle\!\rangle + \langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle \right)$$

$$\leq C\|B\|_\infty(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-\frac{1}{2}}). \tag{72}$$

$\square$

Using Lemma 7.5, $\|\widehat{n} - \widehat{m}\|_{op} \leq CN^{-\xi}$ and setting $B = \dot{A}_t$, we get

$$|\gamma_a^<(\Psi, \varphi)| \leq C\|\dot{A}_t\|_\infty(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-\frac{1}{2}}) \leq C\|\dot{A}_t\|_\infty(\alpha^<(\Psi, \varphi) + N^{-\xi}). \tag{73}$$

Control of $\gamma_b^<$ To control $\gamma_b^<$ we will first prove that $\|\nabla_1 \Psi_t\|$ is uniformly bounded in N, if initially the energy per particle $\mathcal{E}_U(\Psi_0)$ is of order one.

Lemma 7.6. *Let $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. For any $U \in L^2(\mathbb{R}^2, \mathbb{R})$, $U(x) \geq 0$, let Ψ_t the unique solution to $i\partial_t \Psi_t = H_U \Psi_t$ with initial datum Ψ_0. Let $\mathcal{E}_U(\Psi_0) \leq C$. Then*

$$\|\nabla_1 \Psi_t\| \leq \mathcal{K}(\varphi_t, A_t).$$

Proof. Using $\frac{d}{dt}\mathcal{E}_U(\Psi_t) \leq \|\dot{A}_t\|_\infty$, we obtain $\mathcal{E}_U(\Psi_t) \leq \mathcal{K}(\varphi_t, A_t)$. This yields

$$\|\nabla_1 \Psi_t\|^2 \leq \mathcal{K}(\varphi_t, A_t) - \frac{N-1}{2}\|\sqrt{U}(x_1 - x_2)\Psi_t\|^2 + \|A_t\|_\infty \leq \mathcal{K}(\varphi_t, A_t).$$

$\square$

Next, we control $\widehat{m}^a$ and $\widehat{m}^b$ which were defined in Definition 6.4. The difference $m(k) - m(k+1)$ and $m(k) - m(k+2)$ is of leading order given by the derivative of the function $m(k)$—k understood as real variable—with respect to k. The k-derivative of $m(k)$ equals

$$m(k)' = \begin{cases} 1/(2\sqrt{kN}), & \text{for } k \geq N^{1-2\xi}; \\ 1/2(N^{-1+\xi}), & \text{else.} \end{cases} \tag{74}$$

It is then easy to show that, for any $j \in \mathbb{Z}$, there exists a $C_j < \infty$ such that

$$\widehat{m}_j^x \leq C_j N^{-1}\widehat{n}^{-1} \text{ for } x \in \{a, b\} \tag{75}$$

$$\|\widehat{m}_j^x\|_{\text{op}} \leq C_j N^{-1+\xi} \text{ for } x \in \{a, b\} \tag{76}$$

$$\|\widehat{n}\widehat{m}_j^x\|_{\text{op}} \leq C_j N^{-1} \text{ for } x \in \{a, b\} \tag{77}$$

$$\|\widehat{r}\|_{\text{op}} \leq \|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}} \leq CN^{-1+\xi}. \tag{78}$$

Now, we prove some general bounds, which will allow us to estimate the different terms of $\gamma_b^<$ in (44). In order to facilitate the notation, let $\widehat{w} \in \{N\widehat{m}_{-1}^a, N\widehat{m}_{-2}^b\}$. Then $w(k) < Cn(k)^{-1}$ and $\|\widehat{w}_1\|_{\text{op}} \leq C\|\widehat{w}\|_{\text{op}} \leq CN^{\xi}$ follows.

Lemma 7.7. *Let* $\beta > 0$ *and* $W_\beta \in \mathcal{W}_\beta$. *Let* $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ *with* $\|\Psi\| = 1$ *and let* $\|\nabla_1\Psi\| \leq \mathcal{K}(\varphi, A_t)$. *Let* $w(k) < n(k)^{-1}$ *and* $\|\widehat{w}_1\|_{op} \leq C\|\widehat{w}\|_{op} \leq CN^{\xi}$ *for some* $0 < \xi < 1/3$. *Then,*

(a)

$$N\left|\langle\!\langle\Psi, p_1 p_2 Z_\beta^\varphi(x_1, x_2)q_1 p_2\widehat{w}\Psi\rangle\!\rangle\right| \leq \mathcal{K}(\varphi, A_t)\left(N^{-1} + N^{-2\beta}\ln(N)\right).$$

(b)

$$N\left|\langle\!\langle\Psi, p_1 p_2 W_\beta(x_1 - x_2)\widehat{w}q_1 q_2\Psi\rangle\!\rangle\right| \leq \mathcal{K}(\varphi, A_t)\Big(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle$$

$$+ \inf_{\min\{\beta, 1/2\} > \beta_1 > 0} \inf_{\eta > 0}\left(N^{\eta - 2\beta_1}\ln(N)^2 + \|\widehat{w}\|_{op}N^{-1+2\beta_1} + \|\widehat{w}\|_{op}^2 N^{-\eta}\right)\Big).$$

In addition, we have the slightly improved bound

$$N|\langle\!\langle\Psi, p_1 p_2 W_\beta(x_1 - x_2)q_1 q_2\widehat{w}\Psi\rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + \|\widehat{w}\|_{op}N^{-1+2\beta}\right) \tag{79}$$

for all $\beta < 1/2$.

(c)

$$N|\langle\!\langle\Psi, p_1 q_2 Z_\beta^\varphi(x_1, x_2)\widehat{w}q_1 q_2\Psi\rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\Big(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N)$$

$$+ \inf\left\{\left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\right|, \left|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{bw_\beta}^{GP}(\varphi)\right| + N^{-2\beta}\ln(N)\right\}\Big).$$

Proof. Since the left hand sides of all these statements are bounded, it follows that all these estimates hold uniformly in N being in any finite subset of $\mathbb{N}$. Hence it suffices to prove the validity of (a), (b) and (c) for sufficiently large $N \in \mathbb{N}$.

(a) In view of Lemma 4.4, we obtain

$$N \left| \langle\!\langle \Psi, p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \widehat{w} \Psi \rangle\!\rangle \right| \leq N \| p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \|_{\mathrm{op}} \| \widehat{n} \widehat{w} \Psi \|$$

$$\leq C N \| p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \|_{\mathrm{op}}.$$

$\| p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \|_{\mathrm{op}}$ can be estimated using $p_1 q_1 = 0$ and (19):

$$N \left\| p_1 p_2 \left(W_\beta(x_1 - x_2) - \frac{N \|W_\beta\|_1}{N - 1} |\varphi(x_1)|^2 - \frac{N \|W_\beta\|_1}{N - 1} |\varphi(x_2)|^2 \right) q_1 p_2 \right\|_{\mathrm{op}}$$

$$\leq \| p_1 p_2 (N W_\beta(x_1 - x_2) - N \|W_\beta\|_1 |\varphi(x_1)|^2) p_2 \|_{\mathrm{op}} + C \|\varphi\|_\infty^2 N^{-1}$$

$$\leq \|\varphi\|_\infty \| N(W_\beta * |\varphi|^2) - \|N W_\beta\|_1 |\varphi|^2 \| + C \|\varphi\|_\infty^2 N^{-1}.$$

Let h be given by

$$h(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} d^2 y \ln |x - y| N W_\beta(y) + \frac{1}{2\pi} \|N W_\beta\|_1 \ln |x|,$$

which implies

$$\Delta h(x) = N W_\beta(x) - \|N W_\beta\|_1 \delta(x).$$

As above (see Lemma 7.2), we obtain $h(x) = 0$ for $x \notin B_{RN^{-\beta}}(0)$, where $RN^{-\beta}$ is the radius of the support of W_β. Thus,

$$\|h\|_1 \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2 x \int_{\mathbb{R}^2} d^2 y |\ln |x - y|| \mathbb{1}_{B_{RN^{-\beta}}(0)}(x) N W_\beta(y)$$

$$+ \frac{1}{2\pi} N \|W_\beta\|_1 \int_{\mathbb{R}^2} d^2 x |\ln |x|| \mathbb{1}_{B_{RN^{-\beta}}(0)}(x) \leq C N^{-2\beta} \ln(N). \qquad (80)$$

Integration by parts and Young's inequality give that

$$\| N(W_\beta * |\varphi|^2) - \|N W_\beta\|_1 |\varphi|^2 \| = \| (\Delta h) * |\varphi|^2 \|$$

$$\leq \|h\|_1 \|\Delta |\varphi|^2 \|_2 \leq \mathcal{K}(\varphi, A_t) N^{-2\beta} \ln(N).$$

Thus, we obtain the bound

$$N \left| \langle\!\langle \Psi, p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \widehat{w} \Psi \rangle\!\rangle \right| \leq \mathcal{K}(\varphi, A_t) \left(N^{-1} + N^{-2\beta} \ln(N) \right), \qquad (81)$$

which then proves (a).

(b) We will first consider $\beta < 1/2$.

Using Lemmas 4.2 (c) and 4.6 with $O_{1,2} = q_2 W_\beta(x_1 - x_2) p_2$, $\Omega = N^{-1/2} (\widehat{w})^{1/2} q_1 \Psi$

and $\chi = N^{1/2} p_1 (\widehat{w}_2)^{1/2} \Psi$ we get

$$
\begin{aligned}
&|\langle\!\langle \Psi, \, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle| \\
&= |\langle\!\langle \Psi, \, (\widehat{w})^{1/2} q_1 q_2 W_\beta(x_1 - x_2) p_1 p_2 (\widehat{w}_2)^{1/2} \Psi \rangle\!\rangle| \\
&\leq N^{-1} \left\| (\widehat{w})^{1/2} q_1 \Psi \right\|^2 + N \left| \langle\!\langle q_2 (\widehat{w}_2)^{1/2} \Psi, \, p_1 \sqrt{W_\beta}(x_1 - x_2) p_3 \sqrt{W_\beta}(x_1 - x_3) \right. \\
&\quad \left. \sqrt{W_\beta}(x_1 - x_2) p_2 \sqrt{W_\beta}(x_1 - x_3) p_1 q_3 (\widehat{w}_2)^{1/2} \Psi \rangle\!\rangle \right| \\
&\quad + N(N-1)^{-1} \| q_2 W_\beta(x_1 - x_2) p_2 p_1 (\widehat{w}_2)^{1/2} \Psi \|^2 \\
&\leq N^{-1} \left\| (\widehat{w})^{1/2} q_1 \Psi \right\|^2 + N \| \sqrt{W_\beta}(x_1 - x_2) p_1 \|_{\mathrm{op}}^4 \, \| q_2 (\widehat{w}_2)^{1/2} \Psi \|^2 \\
&\quad + 2N(N-1)^{-1} \| p_1 q_2 (\widehat{w}_1)^{1/2} W_\beta(x_1 - x_2) p_2 p_1 \Psi \|^2 \\
&\quad + 2N(N-1)^{-1} \| q_1 q_2 (\widehat{w})^{1/2} W_\beta(x_1 - x_2) p_2 p_1 \Psi \|^2 .
\end{aligned}
$$

With Lemma 4.2 (e) we get the bound

$$
\begin{aligned}
&\leq N^{-1} \| (\widehat{w})^{1/2} \widehat{n} \Psi \|^2 + N \| \varphi \|_\infty^4 \| W_\beta \|_1^2 \, \| \widehat{n} (\widehat{w}_2)^{1/2} \Psi \|^2 \\
&\quad + 2N(N-1)^{-1} \| W_\beta \|^2 \| \varphi \|_\infty^2 \left(\| \widehat{w}_1 \|_{\mathrm{op}} + \| \widehat{w} \|_{\mathrm{op}} \right) .
\end{aligned}
$$

Note, that $\| W_\beta \|_1 \leq C N^{-1}$, $\| W_\beta \|^2 \leq C N^{-2+2\beta}$. Furthermore, using $\widehat{w}_2 \leq (\widehat{n})^{-1}$, we have under the conditions on $\widehat{w}$

$$
\| \widehat{n} (\widehat{w}_2)^{1/2} \Psi \| \leq \| \widehat{n} (\widehat{n})^{-1/2} \Psi \| = \| (\widehat{n})^{1/2} \Psi \| = \sqrt{\langle \Psi, \widehat{n} \Psi \rangle} .
$$

In total, we obtain

$$
N |\langle\!\langle \Psi, \, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle + \| \widehat{w} \|_{\mathrm{op}} N^{-1+2\beta} \right)
$$

and we get (b) for the case $\beta < 1/2$.

(b) We prove part (b) for general $\beta > 0$. We use $U_{\beta_1, \beta}$ from Definition 7.1 for some $0 < \beta_1 < \min\{\beta, 1/2\}$. We then obtain

$$
\begin{aligned}
N \langle\!\langle \Psi, \, p_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \\
= N \langle\!\langle \Psi, \, p_1 p_2 U_{\beta_1, \beta}(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \quad &(82) \\
+ N \langle\!\langle \Psi, \, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1, \beta}(x_1 - x_2) \right) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle . \quad &(83)
\end{aligned}
$$

Term (82) has been controlled above. So we are left to control (83).
Let $\Delta h_{\beta_1, \beta} = W_\beta - U_{\beta_1, \beta}$. Integrating by parts and using that $\nabla_1 h_{\beta_1, \beta}(x_1 - x_2) = -\nabla_2 h_{\beta_1, \beta}(x_1 - x_2)$ gives

$$
\begin{aligned}
N \left| \langle\!\langle \Psi, \, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1, \beta}(x_1 - x_2) \right) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \\
\leq N \left| \langle\!\langle \nabla_1 p_1 \Psi, \, p_2 \nabla_2 h_{\beta_1, \beta}(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \quad &(84) \\
+ N \left| \langle\!\langle \Psi, \, p_1 p_2 \nabla_2 h_{\beta_1, \beta}(x_1 - x_2) \nabla_1 \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| . \quad &(85)
\end{aligned}
$$

Let $t_1 \in \{p_1, \nabla_1 p_1\}$ and let $\Gamma \in \{\widehat{w}q_1\Psi, \nabla_1\widehat{w}q_1\Psi\}$.
For both (84) and (85), we use Lemma 4.6 with $O_{1,2} = N^{1+\eta/2}q_2\nabla_2 h_{\beta_1,\beta}(x_1 - x_2)p_2$, $\chi = t_1\Psi$ and $\Omega = N^{-\eta/2}\Gamma$. This yields

$$(84) + (85) \leq 2 \sup_{t_1 \in \{p_1,\nabla_1 p_1\}, \Gamma \in \{\widehat{w}q_1\Psi, \nabla_1\widehat{w}q_1\Psi\}} \left(N^{-\eta}\|\Gamma\|^2 \right. \tag{86}$$

$$+ \frac{N^{2+\eta}}{N-1}\|q_2\nabla_2 h_{\beta_1,\beta}(x_1 - x_2)t_1 p_2\Psi\|^2 \tag{87}$$

$$+ N^{2+\eta}\left| \langle\!\langle \Psi, t_1 p_2 q_3 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2)\nabla_3 h_{\beta_1,\beta}(x_1 - x_3)t_1 q_2 p_3 \Psi \rangle\!\rangle \right| \Big). \tag{88}$$

The first term can be bounded using Corollary 4.5 by

$$N^{-\eta}\|\nabla_1\widehat{w}q_1\Psi\|^2 \leq 4N^{-\eta}\|\widehat{w}\|_{\mathrm{op}}^2\|\nabla_1 q_1\Psi\|^2$$
$$N^{-\eta}\|\widehat{w}q_1\Psi\|^2 \leq CN^{-\eta}.$$

Thus $(86) \leq \mathcal{K}(\varphi, A_t)N^{-\eta}\|\widehat{w}\|_{\mathrm{op}}^2$ using that $\|\nabla_1 q_1\Psi\| \leq \mathcal{K}(\varphi, A_t)$. By $\|t_1\Psi\|^2 \leq \mathcal{K}(\varphi, A_t)$, we obtain

$$(87) \leq \mathcal{K}(\varphi, A_t)\frac{N^{2+\eta}}{N-1}\|\nabla_2 h_{\beta_1,\beta}(x_1 - x_2)p_2\|_{\mathrm{op}}^2 \leq \mathcal{K}(\varphi, A_t)\frac{N^{2+\eta}}{N-1}\|\varphi\|_{\infty}^2\|\nabla h_{\beta_1,\beta}\|^2$$
$$\leq \mathcal{K}(\varphi, A_t)N^{\eta-1}\ln(N),$$

where we used Lemma 7.2 in the last step.
Next, we estimate

$$(88) \leq N^{2+\eta}\|p_2\nabla_2 h_{\beta_1,\beta}(x_1 - x_2)t_1 q_2\Psi\|^2$$
$$\leq 2N^{2+\eta}\|p_2 h_{\beta_1,\beta}(x_1 - x_2)t_1\nabla_2 q_2\Psi\|^2$$
$$\quad + 2N^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1 - x_2)t_1 q_2\Psi\|^2$$
$$\leq 2N^{2+\eta}\|p_2 h_{\beta_1,\beta}(x_1 - x_2)\|_{\mathrm{op}}^2\|t_1\nabla_2 q_2\Psi\|^2$$
$$\quad + 2N^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1 - x_2)\|_{\mathrm{op}}^2\|t_1 q_2\Psi\|^2$$
$$\leq \mathcal{K}(\varphi, A_t)N^{2+\eta}\|h_{\beta_1,\beta}\|^2$$
$$\leq \mathcal{K}(\varphi, A_t)N^{\eta-2\beta_1}\ln(N)^2.$$

Thus, for all $\eta \in \mathbb{R}$

$$N\langle\!\langle \Psi, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1,\beta}(x_1 - x_2) \right)\widehat{w}q_1 q_2\Psi \rangle\!\rangle$$
$$\leq \mathcal{K}(\varphi, A_t)\left(\|\widehat{w}\|_{\mathrm{op}}^2 N^{-\eta} + N^{\eta-1}\ln(N) + N^{\eta-2\beta_1}\ln(N)^2 \right).$$

Combining the estimates and using $N^{\eta-1}\ln(N) < N^{\eta-2\beta_1}\ln(N)^2$, we obtain

$$N\langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2)\widehat{w}q_1 q_2\Psi \rangle\!\rangle \leq \mathcal{K}(\varphi, A_t)\Big(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle$$

$$+ \inf_{\min\{\beta, 1/2\} > \beta_1 > 0} \inf_{\eta > 0} \left(N^{\eta-2\beta_1}\ln(N)^2 + \|\widehat{w}\|_{\mathrm{op}}N^{-1+2\beta_1} + \|\widehat{w}\|_{\mathrm{op}}^2 N^{-\eta} \right) \Big).$$

(c) We note that $q_1 p_2 |\varphi|^2(x_1) q_1 q_2 = 0$ and estimate

$$N \left| \langle\!\langle \Psi, q_1 p_2 \frac{N \|W_\beta\|_1}{N-1} |\varphi|^2(x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \leq C \|\varphi\|_\infty^2 \|\widehat{w}\,\widehat{n}\|_{\mathrm{op}} \|q_1 \Psi\|^2$$
$$\leq \mathcal{K}(\varphi, A_t) \langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle.$$

Hence, it is left to estimate $N \left| \langle\!\langle \Psi, q_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right|$. Let $U_{0,\beta}$ be given as in Definition 7.1. Moreover, let $\mathcal{A}_1^{(d)}$ and $\overline{\mathcal{A}}_1^{(d)}$ be defined as in Definition 7.3 with $d \geq \max\{7, 3+\beta\}$. We use Lemma 4.2 (c) and integrating by parts to get

$$N \left| \langle\!\langle \Psi, q_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right|$$
$$\leq N \left| \langle\!\langle \Psi, q_1 p_2 U_{0,\beta}(x_1 - x_2) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle \right|$$
$$+ N \left| \langle\!\langle \Psi, q_1 p_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle \right|$$
$$\leq \|U_{0,\beta}\|_\infty N \|q_1 \Psi\| \, \|\widehat{w} q_1 q_2 \Psi\|$$
$$+ N \left| \langle\!\langle \nabla_1 q_1 p_2 \Psi, (\nabla_1 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right|$$
$$+ N \left| \langle\!\langle \Psi, \widehat{w}_1 q_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 q_1 q_2 \Psi \rangle\!\rangle \right|$$
$$\leq N \|U_{0,\beta}\|_\infty \|q_1 \Psi\| \, \|\widehat{w} q_1 q_2 \Psi\| \tag{89}$$
$$+ N \left| \langle\!\langle \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi, p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \tag{90}$$
$$+ N \left| \langle\!\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle \right| \tag{91}$$
$$+ N \left| \langle\!\langle \Psi, \widehat{w}_1 q_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_2 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi \rangle\!\rangle \right| \tag{92}$$
$$+ N \left| \langle\!\langle \Psi, \widehat{w}_1 q_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_2 \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi \rangle\!\rangle \right|. \tag{93}$$

In the following, we will estimate each term separately.

Estimate of (89):
Lemma 4.4 and Definition 7.1 yields the bound

$$(89) \leq C \langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle.$$

Estimate of (90):
For (90) we use that $\nabla_2 h_{0,\beta}(x_1 - x_2) = -\nabla_1 h_{0,\beta}(x_1 - x_2)$, Cauchy Schwarz and $ab \leq a^2 + b^2$ and get

$$(90) \leq \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + N^2 \|p_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi\|^2. \tag{94}$$

$\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2$ can be bounded using Lemma 7.9.
Integration by parts and Lemma 4.2 (c) as well as $(a+b)^2 \leq 2a^2 + 2b^2$ gives for the

second summand

$$N^2 \|p_1(\nabla_1 h_{0,\beta}(x_1 - x_2))q_1 q_2 \widehat{w}\Psi\|^2$$

$$\leq 2N^2 \|p_1 h_{0,\beta}(x_1 - x_2)\nabla_1 q_1 q_2 \widehat{w}\Psi\|^2$$

$$+ 2N^2 \||\varphi(x_1)\rangle\langle\nabla_1\varphi(x_1)|h_{0,\beta}(x_1 - x_2)q_1 q_2 \widehat{w}\Psi\|^2$$

$$\leq CN^2 \|p_1 h_{0,\beta}(x_1 - x_2)q_2(p_1\widehat{w}_1 + q_1\widehat{w})\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1 \Psi\|^2 \tag{95}$$

$$+ CN^2 \|p_1 h_{0,\beta}(x_1 - x_2)q_2 p_1 \widehat{w}_1 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1 \Psi\|^2 \tag{96}$$

$$+ CN^2 \|p_1 h_{0,\beta}(x_1 - x_2)q_2 q_1 \widehat{w}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1 \Psi\|^2 \tag{97}$$

$$+ 2N^2 \||\varphi(x_1)\rangle\langle\nabla_1\varphi(x_1)|h_{0,\beta}(x_1 - x_2)q_1 q_2 \widehat{w}\Psi\|^2. \tag{98}$$

For (95) we use Lemma 4.4, Lemma 4.2 (e) with Lemma 7.2 (c) and then Lemma 7.9.

$$(95) \leq CN^2 \|p_1 h_{0,\beta}(x_1 - x_2)\|_{\mathrm{op}}^2 \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1 \Psi\|^2$$

$$\leq \mathcal{K}(\varphi, A_t)\Big(\big\langle\Psi, \widehat{n}^\varphi\Psi\big\rangle + N^{-1/6}\ln(N)$$

$$+ \inf\Big\{\big|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\big|, \big|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{bw_\beta}^{GP}(\varphi)\big| + N^{-2\beta}\ln(N)\Big\}\Big).$$

Let $s_1 \in \{p_1, q_1\}$ and let $\widehat{d} \in \{\widehat{w}, \widehat{w}_1\}$. Note that $\|\widehat{d}\|_{\mathrm{op}} = \|\widehat{w}\|_{\mathrm{op}}$. Then, (96) and (97) can be estimated with help of Lemma 7.4, part (b)

$$(96), (97) \leq CN^2 \|\nabla_1 q_1 \Psi\|\|\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\widehat{d}s_1 q_2 h_{0,\beta}(x_1 - x_2)p_1 h_{0,\beta}(x_1 - x_2)q_2 s_1 \widehat{d}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1 \Psi\|$$

$$\leq C_p N^{2 + \frac{1-2d}{2}\frac{p-1}{p}}\|\nabla_1 q_1 \Psi\|\|\|\nabla_1 \widehat{d}s_1 q_2 h_{0,\beta}(x_1 - x_2)p_1 h_{0,\beta}(x_1 - x_2)q_2 s_1 \widehat{d}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1 \Psi\|^{\frac{p-1}{p}}$$

$$\times \|\widehat{d}s_1 q_2 h_{0,\beta}(x_1 - x_2)p_1 h_{0,\beta}(x_1 - x_2)q_2 s_1 \widehat{d}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1 \Psi\|^{\frac{1}{p}}$$

$$\leq C_p N^{2 + \frac{1-2d}{2}\frac{p-1}{p}}\|\nabla_1 q_1 \Psi\|\|\|\widehat{w}\|_{\mathrm{op}}\|p_1 h_{0,\beta}(x_1 - x_2)\|_{\mathrm{op}}\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1 \Psi\|$$

$$\times \|\nabla_1 \widehat{d}s_1 q_2 h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}}^{\frac{p-1}{p}}\|\widehat{d}s_1 q_2 h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}}^{\frac{1}{p}}$$

$$\leq C_p \mathcal{K}(\varphi, A_t) N^{1 + \frac{1-2d}{2}\frac{p-1}{p}}\|\widehat{w}\|_{\mathrm{op}}^2 \|\nabla_1 s_1 h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}}^{\frac{p-1}{p}}\|h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}}^{\frac{1}{p}}$$

$$\leq C_p \mathcal{K}(\varphi, A_t) N^{1 + \frac{1-2d}{2}\frac{p-1}{p}}\|\widehat{w}\|_{\mathrm{op}}^2 \big(\|\nabla\varphi\|\|h_{0,\beta}\| + \|\nabla_1 h_{0,\beta}\|\big)^{\frac{p-1}{p}}\|h_{0,\beta}\|^{\frac{1}{p}}$$

$$\leq C_p \mathcal{K}(\varphi, A_t)\|\widehat{w}\|_{\mathrm{op}}^2 (1 + \ln(N))^{\frac{p-1}{2p}} N^{\frac{1-2d}{2}\frac{p-1}{p}}.$$

Here, we used, for $s_1 \in \{p_1, 1 - p_1\}$,

$$\|\nabla_1 s_1 h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}} \leq \|\nabla_1 p_1 h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}} + \|\nabla_1 h_{0,\beta}(x_1 - x_2)p_1\|_{\mathrm{op}}$$

$$\leq \|\varphi\|_\infty\big(\|\nabla\varphi\|\|h_{0,\beta}\| + \|\nabla h_{0,\beta}\|\big)$$

and then applied Lemma 4.2 (e). With $\|\widehat{w}\|_{\mathrm{op}} \leq N^{1/3}$, we obtain

$$(96) + (97) \leq C_p \mathcal{K}(\varphi, A_t)\ln(N)^{\frac{p-1}{2p}} N^{\frac{2}{3}} N^{\frac{1-2d}{2}\frac{p-1}{p}}$$

$$\leq C_p \mathcal{K}(\varphi, A_t)\ln(N)^{\frac{1}{2}} N^{2 + \frac{d}{p} - d - \frac{1}{2p}}.$$

For $p = 2$ and $d \geq \max\{7, 3 + \beta\}$, we obtain

$$(96) + (97) \leq C_2 \mathcal{K}(\varphi, A_t) N^{-1} \leq \mathcal{K}(\varphi, A_t) N^{-1}.$$

Line (98) can be bounded by

$$\begin{aligned}
(98) &\leq CN^2 \|h_{0,\beta}(x_1 - x_2)\nabla_1 p_1\|_{\text{op}}^2 \|q_1 q_2 \widehat{w} \Psi\|^2 \\
&\leq CN^2 \|h_{0,\beta}\|^2 \|\nabla \varphi\|_\infty^2 \|\widehat{n}\,\widehat{w}\|_{\text{op}}^2 \|q_1 \Psi\|^2 \\
&\leq C\|\nabla \varphi\|_\infty^2 \langle\!\langle \Psi \widehat{n} \Psi \rangle\!\rangle.
\end{aligned}$$

Estimate of (91) and (92):
For (91) and (92) we use Cauchy–Schwarz and then Sobolev inequality as in Lemma 7.4 implies that for any $p > 1$, there exists a constant C_p such that

$$\begin{aligned}
(91) + (92) &\leq N \|\nabla_1 q_1 \Psi\| \left\| \mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}} p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi \right\| \\
&\quad + N \|\nabla_1 q_1 \Psi\| \left\| \mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}} q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi \right\| \\
&\leq C_p N \|\nabla_1 q_1 \Psi\| \, N^{\frac{1-2d}{2}\frac{p-1}{p}} \|\nabla_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\|^{\frac{p-1}{p}} \\
&\quad \times \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\|^{1/p} \\
&\quad + C_p N \|\nabla_1 q_1 \Psi\| \, N^{\frac{1-2d}{2}\frac{p-1}{p}} \|\nabla_1 q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\|^{\frac{p-1}{p}} \\
&\quad \times \|q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\|^{1/p}.
\end{aligned}$$

Using Lemma 4.2, Lemma 4.4, Corollary 4.5 and Lemma 7.2, we obtain

$$\begin{aligned}
&\|\nabla_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\| \\
&\quad \leq \|p_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\| + \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 q_1 q_2 \widehat{w} \Psi\| \\
&\quad \leq C \left(\|p_2 (W_\beta - U_{0,\beta})(x_1 - x_2)\|_{\text{op}} + \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2))\|_{\text{op}} \right) \\
&\quad \leq C\|\varphi\|_\infty \left(N^{-1+\beta} + N^{-1}(\ln(N))^{1/2} \right),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\|\nabla_1 q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\| \\
&\quad \leq \|q_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\| + \|q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 q_1 p_2 \widehat{w}_1 \Psi\| \\
&\quad \leq C \left(\|p_2 (W_\beta - U_{0,\beta})(x_1 - x_2)\|_{\text{op}} + \|\widehat{w}_1\|_{\text{op}} \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2))\|_{\text{op}} \right) \\
&\quad \leq C\|\varphi\|_\infty \left(N^{-1+\beta} + \|\widehat{w}\|_{\text{op}} N^{-1}(\ln(N))^{1/2} \right).
\end{aligned}$$

Moreover, we estimate

$$\|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\| \leq C\|\varphi\|_\infty \|\nabla_1 h_{0,\beta}\|_2 \leq C\|\varphi\|_\infty N^{-1}(\ln(N))^{1/2}$$

$$\|q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\| \leq C\|\varphi\|_\infty \|\nabla_1 h_{0,\beta}\|_2 \leq C\|\varphi\|_\infty N^{-1}(\ln(N))^{1/2}.$$

Thus if we choose $p = 2$ and recall that $\xi < 1/3$ and $d \geq \max\{7, 3 + \beta\}$, we obtain

$$\begin{aligned}
(91) + (92) &\leq C_2 \|\varphi\|_\infty N^{1 + \frac{1-2d}{4}} \left(N^{-1+\beta} + \|\widehat{w}\|_{\text{op}} N^{-1}(\ln(N))^{1/2} \right)^{\frac{1}{2}} \left(N^{-1}(\ln(N))^{1/2} \right)^{1/2} \\
&\leq C_2 \|\varphi\|_\infty \left(N^{\frac{1}{2}+\beta-d}(\ln(N))^{1/2} + N^{\frac{1}{3}+\frac{1}{2}-d}\ln(N) \right)^{\frac{1}{2}} \leq C\|\varphi\|_\infty N^{-1}.
\end{aligned}$$

Estimate of (93):
For (93) we use Lemma 4.6 with $\Omega = \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi$, $O_{1,2} = N q_2(\nabla_2 h_{0,\beta}(x_1 - x_2))p_2$
and $\chi = \widehat{w}_1 q_1 \Psi$.

$$(93) \leq \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 \tag{99}$$

$$+ 2N \|q_2(\nabla_2 h_{0,\beta}(x_1 - x_2))\widehat{w}_1 q_1 p_2 \Psi\|^2 \tag{100}$$

$$+ N^2 \big|\langle\!\langle \Psi, q_1 q_3 \widehat{w}_1 (\nabla_2 h_{0,\beta}(x_1 - x_2))p_2 p_3 (\nabla_3 h_{0,\beta}(x_1 - x_3))\widehat{w}_1 q_1 q_2 \Psi \rangle\!\rangle\big|. \tag{101}$$

Line (100) is bounded by

$$(100) \leq CN \|(\nabla_2 h_{0,\beta}(x_1 - x_2))p_2\|_{\mathrm{op}}^2 \|\widehat{w}_1 \widehat{n}\|_{\mathrm{op}}^2$$
$$\leq C\|\varphi\|_\infty^2 N \|\nabla_2 h_{0,\beta}(x_1 - x_2)\|^2 \leq C\|\varphi\|_\infty^2 N^{-1} \ln(N).$$

(99)+(101) is bounded by

$$\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + N^2 \|p_2(\nabla_2 h_{0,\beta}(x_1 - x_2))\widehat{w}_1 q_1 q_2 \Psi\|^2.$$

Both terms can be controlled analogously to (94).
Complete estimate:
In total, we obtain

$$N |\langle\!\langle \Psi, p_1 q_2 Z_\beta^\varphi(x_1, x_2)\widehat{w}q_1 q_2 \Psi \rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\Big(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-1/6} \ln(N)$$

$$+ \inf \left\{ \left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\right|, \left|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{bW_\beta}^{GP}(\varphi)\right| + N^{-2\beta} \ln(N) \right\}\Big).$$

$\square$

To estimate $\gamma_b^<$ we recall that $\widehat{w} \in \{N\widehat{m}_{-1}^a, N\widehat{m}_{-2}^b\}$ with $w(k) < n(k)^{-1}$ and $\|\widehat{w}_1\|_{\mathrm{op}} \leq$
$C\|\widehat{w}\|_{\mathrm{op}} \leq CN^\xi$. Lemma 7.7, $\|\widehat{n} - \widehat{m}\|_{op} \leq CN^{-\xi}$ and $\xi < 1/3$ imply

$$\gamma_b^<(\Psi, \varphi) \leq \mathcal{K}(\varphi, A_t)\Big(\alpha^<(\Psi, \varphi) + N^{-1/3}$$

$$+ \inf_{\min\{\beta, 1/2\} > \beta_1 > 0} \inf_{\eta > 0} \left(N^{\eta - 2\beta_1} \ln(N)^2 + N^{-1+\xi+2\beta_1} + N^{2\xi - \eta}\right)\Big).$$

In addition, we have the improved bound

$$\gamma_b^<(\Psi, \varphi) \leq \mathcal{K}(\varphi, A_t)\left(\alpha^<(\Psi, \varphi) + N^{-\xi} + N^{-1+\xi+2\beta} + (N^{-1/6} + N^{-2\beta})\ln(N)\right)$$

for al $\beta < 1/2$.
Control of $\gamma_c^<$ With Definition 2.1 and (76) we estimate

$$|\gamma_c^<(\Psi, \varphi)| \leq N\big|N\|W_\beta\|_1 - b_{W_\beta}\big| \big|\langle\!\langle \Psi, (q_1|\varphi(x_1)|^2\widehat{m}^a p_1 - p_1\widehat{m}^a |\varphi(x_1)|^2 q_1)\Psi \rangle\!\rangle\big|$$
$$\leq N\big|N\|W_\beta\|_1 - b_{W_\beta}\big| \|\varphi\|_\infty^2 \|\widehat{m}^a\|_{\mathrm{op}}\|q_1\Psi\|$$
$$\leq \mathcal{K}(\varphi, A_t)N^{-1+\xi} \ln(N).$$

Collecting all the estimates for $\gamma_a^<$, $\gamma_b^<$ and $\gamma_c^<$ then proves Lemma 6.6.

Proof of Lemma 6.6. Let the assumptions of Lemma 6.6 be satisfied. By the previous we have

$$\sum_{k\in\{a,b,c\}} |\gamma_k^<(\Psi_t,\varphi_t)| \leq \mathcal{K}(\varphi_t,A_t)\Big(\alpha^<(\Psi_t,\varphi_t) + N^{-\xi} + \Big(N^{-1/6} + N^{-2\beta}\Big)\ln(N)$$

$$+ \inf_{\min\{\beta,1/2\}>\beta_1>0}\inf_{\eta>0}\Big(N^{\eta-2\beta_1}\ln(N)^2 + N^{-1+\xi+2\beta_1} + N^{2\xi-\eta}\Big)\Big)$$

and the slightly stronger estimate

$$\sum_{k\in\{a,b,c\}} |\gamma_k^<(\Psi_t,\varphi_t)| \leq \mathcal{K}(\varphi_t,A_t)\Big(\alpha^<(\Psi_t,\varphi_t) + N^{-\xi} + N^{-1+\xi+2\beta}$$

$$+ \Big(N^{-1/6} + N^{-2\beta}\Big)\ln(N)\Big)$$

if $\beta < 1/2$. Inequality (49) follows for $1/3 \leq \beta$ from the first bound (with $\beta_1 = 3/10$ and $\eta = 3/10$) and for $1/12 \leq \beta < 1/3$ from the second relation. Moreover, if we choose $\beta < 1/12$ and $\xi = 1/6$ we obtain (48). □

7.4. Proof of Lemma 6.13.

Next, we prove Lemma 6.13. We will proceed in a similar way as in the previous section and consecutively estimate the functionals γ_i with $i \in \{a,b,c,d,e,f\}$. In the rest of this section we assume that $V_N \in \mathcal{V}_N$, $\varphi \in H^3(\mathbb{R}^2,\mathbb{C})$ with $\|\varphi\| = 1$ and that $\Psi \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap H^2(\mathbb{R}^{2N},\mathbb{C})$ with $\|\Psi\| = 1$ such that $\mathcal{E}_{V_N}(\Psi) \leq C$.

For the most involved scaling which is induced by V_N, we need to control $\|p_1 V_N(x_1 - x_2)\Psi\|$.

Lemma 7.8. *Let $V_N \in \mathcal{V}_N$, $\Psi \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap H^1(\mathbb{R}^{2N},\mathbb{C})$, $\varphi \in H^3(\mathbb{R}^2,\mathbb{C})$ with $\|\varphi\| = 1$ and $\mathcal{E}_{V_N}(\Psi) \leq C$. Then*

$$\|p_1 V_N(x_1 - x_2)\Psi\| \leq \mathcal{K}(\varphi,A_t)N^{-\frac{1}{2}}. \tag{102}$$

Proof. We estimate

$$\|p_1 V_N(x_1 - x_2)\Psi\| = \|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_2)V_N(x_1 - x_2)\Psi\|$$
$$\leq \|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_2)\|_{\mathrm{op}}\|V_N(x_1 - x_2)\Psi\|.$$

With Lemma 4.2 (e) we get

$$\|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_2)\|_{\mathrm{op}}^2 \leq \|\varphi\|_\infty^2 \|\mathbb{1}_{\mathrm{supp}(V_N)}\|_1 \leq C\|\varphi\|_\infty^2 e^{-2N}.$$

Using

$$C \geq \mathcal{E}_{V_N}(\Psi) = \|\nabla\Psi\|^2 + \frac{(N-1)}{2}\|\sqrt{V_N}(x_1 - x_2)\Psi\|^2 + \langle\!\langle \Psi, A_t(x_1)\Psi \rangle\!\rangle$$

as well as

$$\|V_N(x_1 - x_2)\Psi\|^2 = \|\sqrt{V_N}(x_1 - x_2)\sqrt{V_N}(x_1 - x_2)\Psi\|^2$$
$$\leq \|\sqrt{V_N}\|_\infty^2 \|\sqrt{V_N}(x_1 - x_2)\Psi\|^2$$
$$\leq Ce^{2N}\frac{\mathcal{E}_{V_N}(\Psi) + \|A_t\|_\infty}{N} \leq C(1 + \|A_t\|_\infty)\frac{e^{2N}}{N},$$

we obtain

$$\| p_1 V_N(x_1 - x_2)\Psi \| \leq \mathcal{K}(\varphi, A_t) N^{-\frac{1}{2}}.$$

$\square$

Control of γ_a

In total analogy to (73) we get

$$|\gamma_a(\Psi, \varphi)| \leq C\|\dot{A}_t\|_\infty (\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-\frac{1}{2}}) \leq C\|\dot{A}_t\|_\infty (\langle\!\langle \Psi, \widehat{m}\Psi \rangle\!\rangle + N^{-\xi}).$$

With Definition 6.8 and (58) we have

$$|\gamma_a(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t)(\alpha(\Psi, \varphi) + N^{-\xi} + N^{-\mu+\xi} \ln(N)).$$

Control of γ_b

Recall that

$$\gamma_b(\Psi, \varphi) = -N(N-1)\Im\left(\langle\!\langle \Psi, \widetilde{Z}_\mu^\varphi(x_1, x_2)\widehat{r}\,\Psi \rangle\!\rangle\right)$$
$$- N(N-1)\Im\left(\langle\!\langle \Psi, g_\mu(x_1 - x_2)\widehat{r}\,\mathcal{Z}^\varphi(x_1, x_2)\Psi \rangle\!\rangle\right).$$

Estimate (102) yields to the bound $\| p_1 \mathcal{Z}^\varphi(x_1, x_2)\Psi \| \leq \mathcal{K}(\varphi, A_t)N^{-1/2}$. Thus, if we use Lemma 5.5 and $||\widehat{m}^a||_{\mathrm{op}} + ||\widehat{m}^b||_{\mathrm{op}} \leq CN^{-1+\xi}$ [see (76)] the second line is controlled by

$$N^2\left(\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}}\right)\|g_\mu(x_1 - x_2)p_1\|_{\mathrm{op}}\|p_1 \mathcal{Z}^\varphi(x_1, x_2)\Psi\|$$
$$\leq \mathcal{K}(\varphi, A_t)N^{1/2+\xi}\|g_\mu\| \leq \mathcal{K}(\varphi, A_t)N^{\xi-1/2-\mu}\ln(N).$$

The first line of γ_b can be bounded with (62) and $f_\mu = 1 - g_\mu$ by

$$N(N-1)|\Im\left(\langle\!\langle \Psi, \widetilde{Z}_\mu^\varphi(x_1, x_2)\widehat{r}\,\Psi \rangle\!\rangle\right)|$$
$$\leq N^2|\Im\left(\langle\!\langle \Psi, \left(M_\mu(x_1 - x_2)f_\mu(x_1 - x_2)\right.\right.$$
$$\left.\left. - \frac{N}{N-1}\|M_\mu f_\mu\|_1 \left(|\varphi(x_1)|^2 + |\varphi(x_2)|^2\right)\right)\widehat{r}\,\Psi \rangle\!\rangle\right)| \tag{103}$$
$$+ \frac{N^2}{N-1}|\langle\!\langle \Psi, \left(\|N M_\mu f_\mu\|_1 - 4\pi\right)\left(|\varphi(x_1)|^2 + |\varphi(x_2)|^2\right)\widehat{r}\,\Psi \rangle\!\rangle| \tag{104}$$
$$+ \frac{N^2}{N-1}|\langle\!\langle \Psi, 4\pi\left(|\varphi(x_1)|^2 + |\varphi(x_2)|^2\right)g_\mu(x_1 - x_2)\widehat{r}\,\Psi \rangle\!\rangle|. \tag{105}$$

Since $M_\mu f_\mu \in \mathcal{W}_\mu$, (103) is of the same form as $\gamma_b^<(\Psi, \varphi)$. By means of Lemma 7.7, $\|\widehat{n} - \widehat{m}\|_{op} \leq CN^{-\xi}$ and (58), we obtain

$$|(103)| \leq \mathcal{K}(\varphi, A_t)\left(\alpha(\Psi, \varphi) + N^{-\xi} + \left(N^{-1/6} + N^{-\mu+\xi}\right)\ln(N)\right.$$
$$\left. + \inf_{\min\{\beta, 1/2\} > \beta_1 > 0} \inf_{\eta > 0}\left(N^{\eta-2\beta_1}\ln(N)^2 + N^{-1+2\beta_1+\xi} + N^{2\xi-\eta}\right)\right).$$

Using Lemma 5.5 (h), the second term is controlled by

$$(104) \leq C\|\varphi\|_\infty^2 N\left|N\|M_\mu f_\mu\|_1 - 4\pi\right|\|\widehat{r}\|_{\mathrm{op}} \leq C\|\varphi\|_\infty^2 N^{-1+\xi}\ln(N).$$

The last term is controlled by

$$(105) \leq CN\|\varphi\|_\infty^2 \|g_\mu(x_1 - x_2)p_1\|_{op}\big(\|\widehat{m}^a\|_{op} + \|\widehat{m}^b\|_{op}\big) \leq C\|\varphi\|_\infty^3 N^{-1-\mu+\xi} \ln(N)$$

which implies the bound

$$|\gamma_b(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t)\Big(\alpha(\Psi, \varphi) + N^{-\xi} + \Big(N^{-1+\xi} + N^{-1/6} + N^{-\mu+\xi}\Big)\ln(N)$$

$$+ \inf_{\min\{\beta,1/2\}>\beta_1>0} \inf_{\eta>0} \Big(N^{\eta-2\beta_1}\ln(N)^2 + N^{-1+2\beta_1+\xi} + N^{2\xi-\eta}\Big)\Big).$$

Control of γ_c

Recall that

$$\gamma_c(\Psi, \varphi) = -4N(N-1)\langle\!\langle \Psi, (\nabla_1 g_\mu(x_1 - x_2))\nabla_1\widehat{r}\Psi\rangle\!\rangle.$$

Using $\widehat{r} = (p_2 + q_2)\widehat{r} = p_2\widehat{r} + p_1 q_2\widehat{m}^a$ and $\nabla_1 g_\mu(x_1 - x_2) = -\nabla_2 g_\mu(x_1 - x_2)$, integration by parts yields to

$$|\gamma_c(\Psi, \varphi)| \leq 4N^2 |\langle\!\langle \Psi, g_\mu(x_1 - x_2)\nabla_1\nabla_2(p_2\widehat{r} + p_1 q_2\widehat{m}^a)\Psi\rangle\!\rangle| \tag{106}$$

$$+ 4N^2 |\langle\!\langle \nabla_2\Psi, g_\mu(x_1 - x_2)\nabla_1 p_2\widehat{r}\Psi\rangle\!\rangle| \tag{107}$$

$$+ 4N^2 |\langle\!\langle \nabla_2\Psi, g_\mu(x_1 - x_2)\nabla_1 p_1 q_2\widehat{m}^a\Psi\rangle\!\rangle|. \tag{108}$$

We begin with

$$(106) \leq CN^2\|g_\mu\|\|\nabla\varphi\|_\infty\big(\|\nabla_1\widehat{r}\Psi\| + \|\nabla_2 q_2\widehat{m}^a\Psi\|\big)$$

$$\leq CN^{1-\mu}\ln(N)\|\nabla\varphi\|_\infty\big(\|\nabla_1\widehat{r}\Psi\| + \|\nabla_2 q_2\widehat{m}^a\Psi\|\big).$$

Let $s_1, t_1 \in \{p_1, q_1\}$. Inserting the identity $1 = p_1 + q_1$, we obtain for $a \in \{-1, 0, 1\}$,

$$\|\nabla_1\widehat{r}\Psi\| \leq C\sup_{s_1,t_1,a}\|\widehat{r}_a s_1\nabla_1 t_1\Psi\| \leq C\sup_{t_1,a}\|\widehat{r}_a\|_{op}\|\nabla_1 t_1\Psi\| \leq CN^{-1+\xi}.$$

In analogy $\|\nabla_2 q_2\widehat{m}^a\Psi\| \leq C\|\widehat{m}^a\|_{op} \leq CN^{-1+\xi}$. This yields the bound

$$(106) \leq \mathcal{K}(\varphi, A_t)N^{-\mu+\xi}\ln(N).$$

Furthermore, (107) is bounded by

$$(107) \leq 4N^2\|\nabla_2\Psi\|\,\|g_\mu\|\,\|\varphi\|_\infty\|\nabla_1\widehat{r}\Psi\| \leq C\|\varphi\|_\infty N^{\xi-\mu}\ln(N). \tag{109}$$

Similarly, we obtain

$$(108) \leq 4N^2\|\nabla_2\Psi\|\,\|g_\mu\|\,\|\nabla\varphi\|_\infty\|q_2\widehat{m}^a\Psi\| \leq C\|\nabla\varphi\|_\infty N^{\xi-\mu}\ln(N).$$

It follows that $|\gamma_c(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t)N^{\xi-\mu}\ln(N)$.

Control of γ_d

To control γ_d and γ_e we will use the notation

$$\begin{aligned} m^c(k) &= m^a(k) - m^a(k+1) & m^d(k) &= m^a(k) - m^a(k+2) \\ m^e(k) &= m^b(k) - m^b(k+1) & m^f(k) &= m^b(k) - m^b(k+2). \end{aligned} \tag{110}$$

Since the second k-derivative of m is given by (see (74) for the first derivative)

$$m(k)'' = \begin{cases} -1/(4\sqrt{k^3 N}), & \text{for } k \geq N^{1-2\xi}; \\ 0, & \text{else.} \end{cases}$$

it is easy to verify that

$$\|\widehat{m}_j^x\|_{\text{op}} \leq CN^{-2+3\xi} \text{ for } x \in \{c, d, e, f\}. \tag{111}$$

Recall that

$$\gamma_d(\Psi, \varphi) = 2N(N-1)(N-2)\Im\left(\langle\!\langle \Psi, g_\mu(x_1 - x_2)\left[V_N(x_1 - x_3), \widehat{r}\right]\Psi\rangle\!\rangle\right)$$
$$- N(N-1)(N-2)\Im\left(\langle\!\langle \Psi, g_\mu(x_1 - x_2)\left[4\pi|\varphi|^2(x_3), \widehat{r}\right]\Psi\rangle\!\rangle\right).$$

Since $p_j + q_j = 1$, we can rewrite $\widehat{r}$ as

$$\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2) = (\widehat{m}^b - 2\widehat{m}^a)p_1 p_2 + \widehat{m}^a(p_1 + p_2).$$

Thus,

$$|\gamma_d(\Psi, \varphi)| \leq CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)\left[V_N(x_1 - x_3), (\widehat{m}^b - 2\widehat{m}^a)p_1 p_2\right.\right.$$
$$\left.\left. + \widehat{m}^a(p_1 + p_2)\right]\Psi\rangle\!\rangle\right|$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)\left[4\pi|\varphi|^2(x_3), \widehat{r}\right]\Psi\rangle\!\rangle\right|$$
$$\leq CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)p_2\left[V_N(x_1 - x_3), \widehat{m}^a\right]\Psi\rangle\!\rangle\right| \tag{112}$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)V_N(x_1 - x_3)(\widehat{m}^b - 2\widehat{m}^a)p_1 p_2\Psi\rangle\!\rangle\right| \tag{113}$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)(\widehat{m}^b - 2\widehat{m}^a)p_1 p_2 V_N(x_1 - x_3)\Psi\rangle\!\rangle\right| \tag{114}$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)\widehat{m}^a p_1 V_N(x_1 - x_3)\Psi\rangle\!\rangle\right| \tag{115}$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)V_N(x_1 - x_3)\widehat{m}^a p_1\Psi\rangle\!\rangle\right| \tag{116}$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)\left[4\pi|\varphi|^2(x_3), \widehat{r}\right]\Psi\rangle\!\rangle\right|. \tag{117}$$

Using Lemma 4.2 (d), we obtain the following estimate:

$$(112) = CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)p_2\left[V_N(x_1 - x_3), p_1 p_3\widehat{m}^d + p_1 q_3\widehat{m}^c + q_1 p_3\widehat{m}^c\right]\Psi\rangle\!\rangle\right|$$
$$\leq CN^3\left|\langle\!\langle \Psi, V_N(x_1 - x_3)g_\mu(x_1 - x_2)p_2\mathbb{1}_{\text{supp}(V_N)}(x_1 - x_3)\right.$$
$$\left.\left(p_1 p_3\widehat{m}^d + p_1 q_3\widehat{m}^c + q_1 p_3\widehat{m}^c\right)\Psi\rangle\!\rangle\right|$$
$$+ CN^3 \left|\langle\!\langle \Psi, g_\mu(x_1 - x_2)p_2\left(p_1 p_3\widehat{m}^d + p_1 q_3\widehat{m}^c + q_1 p_3\widehat{m}^c\right)V_N(x_1 - x_3)\Psi\rangle\!\rangle\right|.$$

Both lines are bounded by

$$CN^3\|V_N(x_1 - x_3)\Psi\| \, \|g_\mu(x_1 - x_2)p_2\|_{\text{op}}$$
$$\times \left(2\|\mathbb{1}_{\text{supp}(V_N)}(x_1 - x_3)p_1\|_{\text{op}} + \|\mathbb{1}_{\text{supp}(V_N)}(x_1 - x_3)p_3\|_{\text{op}}\right)\left(\|\widehat{m}^d\|_{\text{op}} + \|\widehat{m}^c\|_{\text{op}}\right).$$

In view of Lemmas 4.2 (e) and 5.5 (i), $\|g_\mu(x_1-x_2)p_2\|_{op} \leq \|\varphi\|_\infty\|g_\mu\| \leq C\|\varphi\|_\infty N^{-1-\mu}$ $\ln(N)$. Using (111), together with $\|\mathbb{1}_{\text{supp}(V_N)}(x_1-x_3)p_1\|_{op}\|V_N(x_1-x_3)\Psi\| \leq \mathcal{K}(\varphi, A_t)$ $N^{-1/2}$, we obtain, using $\xi < 1/2$,

$$(112) \leq \mathcal{K}(\varphi, A_t)N^{-1/2+3\xi-\mu}\ln(N) \leq \mathcal{K}(\varphi, A_t)N^{1/2+\xi-\mu}\ln(N).$$

We continue with

$$(113) + (114) + (115) \leq CN^3\|V_N(x_1-x_3)\Psi\|\|g_\mu(x_1-x_2)p_2\|_{op}$$
$$\times \|\mathbb{1}_{\text{supp}(V_N)}(x_1-x_3)p_1\|_{op}\|(\widehat{m}^b - 2\widehat{m}^a)\|_{op}$$
$$+ CN^3\|g_\mu(x_1-x_2)p_2\|_{op}\|\widehat{m}^b - 2\widehat{m}^a\|_{op}\|p_1V_N(x_1-x_3)\Psi\|$$
$$+ CN^3\|g_\mu(x_1-x_2)p_1\|_{op}\|\widehat{m}^a\|_{op}\|p_1V_N(x_1-x_3)\Psi\|$$
$$\leq \mathcal{K}(\varphi, A_t)N^{1/2+\xi-\mu}\ln(N).$$

Next, we estimate (116). The support of the function $g_\mu(x_1-x_2)V_N(x_1-x_3)$ is such that $|x_1-x_2| \leq CN^{-\mu}$, as well as $|x_1-x_3| \leq Ce^{-N}$. Therefore, $g_\mu(x_1-x_2)V_N(x_1-x_3) \neq 0$ implies $|x_2-x_3| \leq CN^{-\mu}$. We estimate

$$(116) = CN^3\left|\langle\!\langle\Psi, g_\mu(x_1-x_2)V_N(x_1-x_3)p_1\mathbb{1}_{B_{CN^{-\mu}}(0)}(x_2-x_3)\widehat{m}^a\Psi\rangle\!\rangle\right|$$
$$\leq CN^3\|p_1V_N(x_1-x_3)g_\mu(x_1-x_2)\Psi\|\|\mathbb{1}_{B_{CN^{-\mu}}(0)}(x_2-x_3)\widehat{m}^a\Psi\|$$
$$\leq CN^3\|p_1\mathbb{1}_{\text{supp}(V_N)}(x_1-x_3)\|_{op}\|g_\mu(x_1-x_2)V_N(x_1-x_3)\Psi\|$$
$$\times \|\mathbb{1}_{B_{CN^{-\mu}}(0)}(x_2-x_3)\widehat{m}^a\Psi\|$$
$$\leq C_pN^{5/2}\|g_\mu\|_\infty\|\mathbb{1}_{B_{CN^{-\mu}}(0)}\|_{\frac{p}{p-1}}^{\frac{1}{2}}\|\nabla_1\widehat{m}^a\Psi\|^{\frac{p-1}{p}}\|\widehat{m}^a\Psi\|^{\frac{1}{p}}$$
$$\leq CN^{5/2}\|g_\mu\|_\infty N^{-\mu/2}\|\nabla_1\widehat{m}^a\Psi\|^{1/2}\|\widehat{m}^a\Psi\|^{1/2}$$
$$\leq CN^{3/2+\xi-\mu/2}.$$

In the fourth line, we applied Sobolev inequality as in the proof of Lemma 7.4, then setting $p = 2$. Furthermore, we used $\|\nabla_1\widehat{m}^a\Psi\|^{1/2}\|\widehat{m}^a\Psi\|^{1/2} \leq CN^{-1+\xi}$, as well as $\|g_\mu\|_\infty \leq C$, see Lemma 5.5.

Using Lemma 4.2 (d), (117) can be bounded by

$$CN^3\left|\langle\!\langle\Psi, g_\mu(x_1-x_2)\left[4\pi|\varphi|^2(x_3), p_1p_2(\widehat{r}-\widehat{r}_2) + (p_1q_2 + q_1p_2)(\widehat{r}-\widehat{r}_1)\right]\Psi\rangle\!\rangle\right|$$
$$\leq CN^3\|\varphi\|_\infty^2\left(\|\widehat{r}-\widehat{r}_2\|_{op} + \|\widehat{r}-\widehat{r}_1\|_{op}\right)\|g_\mu(x_1-x_2)p_2\|_{op}.$$

Note that $\|\widehat{r}-\widehat{r}_2\|_{op} + \|\widehat{r}-\widehat{r}_1\|_{op} \leq \sum_{j\in\{c,d,e,f\}}\|\widehat{m}^j\|_{op} \leq CN^{-2+3\xi}$ holds. With $\|g_\mu(x_1-x_2)p_2\|_{op} \leq CN^{-1-\mu}\ln(N)$, it then follows that

$$|(117)| \leq C\|\varphi\|_\infty^2 N^{3\xi-\mu}\ln(N).$$

In total, we obtain

$$|\gamma_d(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t)\left(N^{3/2+\xi-\mu/2} + N^{1/2+3\xi-\mu}\ln(N)\right).$$

Control of γ_e

Recall that

$$\gamma_e(\Psi, \varphi) = -\frac{1}{2} N(N-1)(N-2)(N-3) \Im \left(\langle\!\langle \Psi, g_\mu(x_1 - x_2) \left[V_N(x_3 - x_4), \widehat{r} \right] \Psi \rangle\!\rangle \right).$$

Using symmetry, Lemma 4.2 (d) and notation (110), γ_e is bounded by

$$\begin{aligned}
\gamma_e(\Psi, \varphi) &\leq N^4 \left| \langle\!\langle \Psi, g_\mu(x_1 - x_2) \left[V_N(x_3 - x_4), \widehat{m}^c \, p_1 p_2 p_3 p_4 + 2\widehat{m}^d \, p_1 p_2 p_3 q_4 \right. \right. \\
&\quad \left. \left. + 2\widehat{m}^e \, p_1 q_2 p_3 p_4 + 4\widehat{m}^f \, p_1 q_2 p_3 q_4 \right] \Psi \rangle\!\rangle \right| \\
&\leq 4N^4 \| V_N(x_3 - x_4) \Psi \| \| \mathbb{1}_{\text{supp}(V_N)}(x_3 - x_4) p_3 \|_{\text{op}} \| g_\mu(x_1 - x_2) p_1 \|_{\text{op}} \\
&\quad \times (\| \widehat{m}^c \|_{\text{op}} + \| \widehat{m}^d \|_{\text{op}} + \| \widehat{m}^e \|_{\text{op}} + \| \widehat{m}^f \|_{\text{op}}).
\end{aligned}$$

We get with (111), Lemma 5.5 and Lemma 4.2 that

$$|\gamma_e(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t) N^{1/2 + 3\xi - \mu} \ln(N).$$

Control of γ_f

Recall that

$$\gamma_f(\Psi, \varphi) = 2N(N-1) \frac{N-2}{N-1} \Im \left(\langle\!\langle \Psi, g_\mu(x_1 - x_2) \left[4\pi |\varphi|^2(x_1), \widehat{r} \right] \Psi \rangle\!\rangle \right).$$

We obtain the estimate

$$|\gamma_f(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t) N^2 \| g_\mu \| \| \widehat{r} \|_{\text{op}} \leq \mathcal{K}(\varphi, A_t) N^{\xi - \mu} \ln(N).$$

Proof of Lemma 6.13. Let the assumptions of Lemma 6.13 be satisfied. With the previous estimates and $\xi < 1/3$ we get

$$\begin{aligned}
\sum_{k \in \{a,b,c,d,e,f\}} |\gamma_k(\Psi_t, \varphi_t)| &\leq \mathcal{K}(\varphi_t, A_t) \Big(\alpha(\Psi_t, \varphi_t) + N^{-\xi} + \left(N^{2-\mu/2} + N^{-1/6} \right) \ln(N) \\
&\quad + \inf_{\min\{\beta, 1/2\} > \beta_1 > 0} \inf_{\eta > 0} \left(N^{\eta - 2\beta_1} \ln(N)^2 + N^{-1 + 2\beta_1 + \xi} + N^{2\xi - \eta} \right) \Big).
\end{aligned}$$

Choosing $\xi = 1/10$, $\mu = 10$, $\eta = 3/10$ and $\beta_1 = 3/10$, we obtain (64). $\qquad \square$

7.5. Energy estimates. In this section we show that $\| \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi \|^2$ can be controlled sufficiently well in terms of the counting functionals $\alpha^<$ and α. If Ψ_t is evolving according to W_β, one could actually show that $\| \nabla_1 q_1 \Psi_t \|^2$ is small already without cutoff. While such a proof would be less involved, we chose a unified presentation which both covers the Gross–Pitaevskii scaling and the NLS scaling.

Lemma 7.9. Let $W_\beta \in \mathcal{W}_\beta$, $V_N \in \mathcal{V}_N$ and $A_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$, $\| \Psi \| = 1$ with $\| \nabla_1 \Psi \| \leq C$. Let $\varphi \in H^3(\mathbb{R}^2, \mathbb{C})$, $\| \varphi \| = 1$. For $d \geq 3$, define the sets $\mathcal{A}_1^{(d)}, \overline{\mathcal{B}}_1^{(d)}$ as in Definition 7.3. Then, for N large enough and $d \geq 3$,

$$\begin{aligned}
\| \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi \|^2 &\leq \mathcal{K}(\varphi, A_t) \Big(\langle\!\langle \Psi, \widehat{n}^\varphi \Psi \rangle\!\rangle + N^{-1/6} \ln(N) \\
&\quad + \inf \left\{ \left| \mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \right|, \left| \mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{b_{W_\beta}}^{GP}(\varphi) \right| + N^{-2\beta} \ln(N) \right\} \Big).
\end{aligned}$$

Proof. We start with expanding $\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)$. This yields

$$
\begin{aligned}
\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi) &= \|\nabla_1 \Psi\|^2 + \frac{N-1}{2}\|\sqrt{W_\beta}(x_1 - x_2)\Psi\|^2 \\
&\quad - \|\nabla\varphi\|^2 - \frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2 + \langle\!\langle \Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle \varphi, A_t\varphi\rangle \\
&= \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 + M(\Psi,\varphi) + Q_\beta(\Psi,\varphi),
\end{aligned}
$$

where we have defined

$$
M(\Psi,\varphi) = 2\Re\left(\langle\!\langle \nabla_1 q_1\Psi, \mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\rangle\!\rangle\right) \tag{118}
$$

$$
+ \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2 \tag{119}
$$

$$
+ \langle\!\langle \Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle \varphi, A_t\varphi\rangle,
$$

$$
Q_\beta(\Psi,\varphi) = \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2
$$

$$
+ \frac{N-1}{2}\langle\!\langle \Psi, (1 - p_1 p_2)W_\beta(x_1 - x_2)(1 - p_1 p_2)\Psi\rangle\!\rangle
$$

$$
+ \frac{N-1}{2}\langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2)p_1 p_2\Psi\rangle\!\rangle - \frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2
$$

$$
+ (N-1)\Re\langle\!\langle \Psi, (1 - p_1 p_2)W_\beta(x_1 - x_2)p_1 p_2\Psi\rangle\!\rangle. \tag{120}
$$

Notice that the first two terms in $Q_\beta(\Psi,\varphi)$ are nonnegative. This yields to the bound

$$
S_\beta(\Psi,\varphi) = (N-1)|\langle\!\langle \Psi, (1 - p_1 p_2)W_\beta(x_1 - x_2)p_1 p_2\Psi\rangle\!\rangle| \tag{121}
$$

$$
+ \left|\frac{N-1}{2}\langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2)p_1 p_2\Psi\rangle\!\rangle - \frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2\right|
$$

$$
\geq -Q_\beta(\Psi,\varphi). \tag{122}
$$

We therefore obtain

$$
\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 \leq \left|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right| + |M(\Psi,\varphi)| + |S_\beta(\Psi,\varphi)|. \tag{123}
$$

Thus if we use that Definition 2.1 implies the estimate

$$
\left|\mathcal{E}^{GP}_{b_{W_\beta}}(\varphi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right| \leq \frac{1}{2}\left|b_{W_\beta} - N\|W_\beta\|_1\right|\,\|\varphi^2\|^2 \leq \mathcal{K}(\varphi, A_t)N^{-1}\ln(N),
$$

we get the bound:

$$
\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 \leq \left|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{b_{W_\beta}}(\varphi)\right| + \mathcal{K}(\varphi, A_t)N^{-1}\ln(N)
$$

$$
+ |M(\Psi,\varphi)| + |S_\beta(\Psi,\varphi)|.
$$

Next, we split up the energy difference $\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi)$,

$$
\begin{aligned}
\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi) &= \|\nabla_1\Psi\|^2 + \frac{N-1}{2}\|\sqrt{V_N}(x_1 - x_2)\Psi\|^2 - \|\nabla\varphi\|^2 \\
&\quad - 2\pi\|\varphi^2\|^2 + \langle\!\langle \Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle \varphi, A_t\varphi\rangle.
\end{aligned}
$$

In order to better estimate the terms corresponding to the two-particle interactions, we introduce, for $\nu > d$, the potential $M_\nu(x)$, defined in Definition 5.3. Note, that $\nu > d$ assures that that part of the interaction M_ν which lies within the set $\mathcal{A}_1^{(d)}$ will be negligible. We continue with

$$\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) = \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2$$
$$+ \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2$$
$$+ \frac{1}{2} \langle\!\langle \Psi, \sum_{j\neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} (V_N - M_\nu)(x_1 - x_j)\Psi \rangle\!\rangle$$
$$+ \frac{1}{2} \langle\!\langle \Psi, \sum_{j\neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} M_\nu(x_1 - x_j)\Psi \rangle\!\rangle - \|\nabla\varphi\|^2 - 2\pi \|\varphi^2\|^2$$
$$+ \langle\!\langle \Psi, A_t(x_1)\Psi \rangle\!\rangle - \langle \varphi, A_t\varphi \rangle.$$

After reordering, the identity $q_1 = 1 - p_1$, together with the symmetry of $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$ gives

$$\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) = \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2$$
$$+ \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2$$
$$+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2)M_\nu(x_1 - x_2)(1 - p_1 p_2)\mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle$$
$$+ \|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j\neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} (V_N - M_\nu)(x_1 - x_j)\Psi \rangle\!\rangle$$
$$+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\nu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2$$
$$+ 2\Re \left(\langle\!\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 p_1 \Psi \rangle\!\rangle \right)$$
$$+ (N-1)\Re \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2)M_\nu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle$$
$$+ \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 p_1 \Psi\|^2 - \|\nabla\varphi\|^2$$
$$+ \langle\!\langle \Psi, A_t(x_1)\Psi \rangle\!\rangle - \langle \varphi, A_t\varphi \rangle$$
$$= \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + M(\Psi, \varphi) + \tilde{Q}_\nu(\Psi, \varphi).$$

with

$$\tilde{Q}_\nu(\Psi, \varphi) = \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2$$
$$+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2)M_\nu(x_1 - x_2)(1 - p_1 p_2)\mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle$$
$$+ \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2$$
$$+ \|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j\neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} (V_N - M_\nu)(x_1 - x_j)\Psi \rangle\!\rangle$$

$$+ (N-1)\Re\langle\!\langle\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}}(1 - p_1 p_2)M_\nu(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle$$

$$+ \frac{N-1}{2}\langle\!\langle\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\nu(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle - 2\pi\|\varphi^2\|^2. \quad (124)$$

The first three terms in $\tilde{Q}_\nu(\Psi, \varphi)$ are nonnegative. For $\nu > d$ and N large enough, Lemma 7.10 implies that (124) is also nonnegative. Thus, for $\nu > d$, we obtain the bound

$$\tilde{S}_\nu(\Psi, \varphi) = (N-1)\left|\langle\!\langle\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}}(1 - p_1 p_2)M_\nu(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right| \quad (125)$$

$$+ \left|\frac{N-1}{2}\langle\!\langle\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\nu(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle - 2\pi\|\varphi^2\|^2\right|$$

$$\geq -\tilde{Q}_\nu(\Psi, \varphi). \quad (126)$$

In total, we obtain

$$\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1 \Psi\|^2 \leq |M(\Psi, \varphi)| + \tilde{S}_\nu(\Psi, \varphi) + \left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\right|. \quad (127)$$

It is therefore left to estimate $M(\Psi, \varphi)$, $S_\beta(\Psi, \varphi)$ and $\tilde{S}_\nu(\Psi, \varphi)$.

Estimate of $S_\beta(\Psi, \varphi)$ and $\tilde{S}_\nu(\Psi, \varphi)$.

The contributions (121) and (125) are estimated in Lemma 7.11.

$$(121), (125) \leq \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \hat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N)).$$

We are thus left to estimate (122) and (126). We begin with the estimate for (126). As in (80), we can write

$$\langle\!\langle\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\nu(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle = \langle\varphi, M_\nu * |\varphi|^2 \varphi\rangle\langle\!\langle\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle.$$

With Lemma 7.3 (c) with $\epsilon = 1/2$, we get $\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\| \leq CN^{3/2-d}$. Together with $\|p_1 p_2 \Psi\|^2 = 1 + 2\|p_1 q_2 \Psi\|^2 + \|q_1 q_2 \Psi\|^2$, we therefore obtain

$$(126) \leq 3\|q_1 \Psi\|^2 + C\left(N^{3/2-d} + N^{3-2d}\right) + \frac{1}{2}|N\langle\varphi, M_\nu * |\varphi|^2 \varphi\rangle - N\|M_\nu\|_1\|\varphi^2\|^2|$$

$$+ \frac{1}{2}|4\pi - N\|M_\nu\|_1|\|\varphi^2\|^2 + \frac{1}{2}\langle\varphi, M_\nu * |\varphi|^2 \varphi\rangle.$$

Note that, using Young's inequality and (80)

$$|\langle\varphi, NM_\nu * |\varphi|^2 \varphi\rangle - N\|M_\nu\|_1\|\varphi^2\|^2|$$

$$= \left|\int_{\mathbb{R}^2} d^2 x |\varphi(x)|^2 \left(N(M_\nu * |\varphi|^2)(x) - N\|M_\nu\|_1|\varphi(x)|^2\right)\right|$$

$$\leq \|\varphi\|_\infty^2\|N(M_\nu * |\varphi|^2) - \|NM_\nu\|_1|\varphi|^2\|_1$$

$$\leq C\|\varphi\|_\infty^2\|\Delta|\varphi|^2\|_1 N^{-2\nu}\ln(N)$$

$$\leq \mathcal{K}(\varphi, A_t)N^{-2\nu}\ln(N).$$

Since $|N\|M_\nu\|_1 - 4\pi| \leq C\frac{\ln(N)}{N}$ (see Lemma 5.5) and $\langle\varphi, M_\nu * |\varphi|^2\varphi\rangle \leq \|\varphi\|_\infty^4\|M_\nu\|_1 \leq C\|\varphi\|_\infty^4 N^{-1}$, it follows that

$$|(126)| \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}^\varphi\Psi\rangle\!\rangle + N^{3/2-d} + N^{3-2d+2} + N^{-2\nu}\ln(N) + N^{-1}\ln(N)\right)$$

$$\leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}^\varphi\Psi\rangle\!\rangle + N^{-1}\ln(N)\right), \tag{128}$$

where $\nu > d \geq 3$ was used in the last inequality.

Using the same estimates, we obtain

$$(122) \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}^\varphi\Psi\rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1}\ln(N)\right).$$

In total, we obtain, for any $\nu > d \geq 1$, the bound

$$S_\beta(\Psi, \varphi) \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1/6}\ln(N)\right)$$

$$\tilde{S}_\nu(\Psi, \varphi) \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N)\right).$$

Estimate of $M(\Psi, \varphi)$. We need to estimate (118), (119) and (120). We start with

$$|(118)| \leq 2|\langle\!\langle\nabla_1 q_1\Psi, \mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1 p_1\Psi\rangle\!\rangle| + 2|\langle\!\langle\nabla_1 q_1\Psi, \nabla_1 p_1\Psi\rangle\!\rangle|$$

$$\leq 2\|\nabla_1 q_1\Psi\|\ \|\mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1 p_1\|_{op} + 2|\langle\!\langle\widehat{n}^{-1/2}q_1\Psi, \Delta_1 p_1\widehat{n}_1^{1/2}\Psi\rangle\!\rangle|.$$

By Lemma 7.4, we obtain $\|\mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1 p_1\|_{op} \leq C\|\nabla\varphi\|_\infty N^{1/2-d}$.

Furthermore, we use $\|\nabla_1 q_1\Psi\| \leq \|\nabla_1\Psi\| + \|\nabla_1 p_1\Psi\| \leq \mathcal{K}(\varphi, A_t)$ (see also Lemma 7.6) and $|\langle\!\langle\widehat{n}^{-1/2}q_1\Psi, \Delta_1 p_1\widehat{n}_1^{1/2}\Psi\rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\|\widehat{n}_1^{1/2}\Psi\|\|\widehat{n}^{1/2}\Psi\| \leq \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1})$. Hence, for $d \geq 3$,

$$|(118)| \leq \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{1-d} + N^{-1}) \leq \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1}).$$

With $\|\nabla_1 p_1\Psi\|^2 = \|\nabla\varphi\|^2\|p_1\Psi\|^2$ line (119) is estimated by

$$(119) = \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2$$

$$\leq |\|\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2| + \|\mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1 p_1\Psi\|^2$$

$$\leq C\left(\|\nabla\varphi\|^2\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle + \|\nabla\varphi\|_\infty^2 N^{1-2d}\right)$$

$$\leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1}\right).$$

For line (120), we use Lemma 7.5 to obtain

$$(120) \leq C\|A_t\|_\infty\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/2}\right).$$

In total, we obtain

$$M(\Psi, \varphi) \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/2}\right).$$

$$\square$$

Lemma 7.10. (a) *Let $V_N \in \mathcal{V}_N$ and let R_ν and M_ν be defined as in Definition 5.3. Then, for any $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C}) \cap \mathcal{D}(\nabla_1)$*

$$\|\mathbb{1}_{|x_1 - x_2| \leq R_\nu} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, (V_N - M_\nu)(x_1 - x_2)\Psi \rangle\!\rangle \geq 0. \tag{129}$$

(b) *Let $V_N \in \mathcal{V}_N$ and let M_ν be defined as in Definition 5.3. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$. Then, for sufficiently large N and for $\nu > d$,*

$$\|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j \neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} (V_N - M_\nu)(x_1 - x_j)\Psi \rangle\!\rangle \geq 0. \tag{130}$$

Proof. (a) We first show nonnegativity of the one-particle operator $H^{Z_n} : H^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ given by

$$H^{Z_n} = -\Delta + \frac{1}{2} \sum_{z_k \in Z_n} (V_N(\cdot - z_k) - M_\nu(\cdot - z_k))$$

for any $n \in \mathbb{N}$ and any n-elemental subset $Z_n \subset \mathbb{R}^2$ which is such that the supports of the potentials $M_\nu(\cdot - z_k)$ are pairwise disjoint for any two $z_k \in Z_n$.
Since $f_\nu(\cdot - z_k)$ is the the zero energy scattering state of the potential $1/2 V_N(\cdot - z_k) - 1/2 M_\nu(\cdot - z_k)$, it follows that

$$F_\nu^{Z_n} = \prod_{z_k \in Z_n} f_\nu(\cdot - z_k).$$

fulfills $H^{Z_n} F_\nu^{Z_n} = 0$ for any such Z_n. By construction f_ν is a nonnegative function, so is $F_\nu^{Z_n}$. Since $\frac{1}{2} \sum_{z_k \in Z_n} (V_N(\cdot - z_k) - M_\nu(\cdot - z_k)) \in L^\infty(\mathbb{R}^2, \mathbb{C})$, this potential is a infinitesimal perturbation of $-\Delta$, thus $\sigma_{\text{ess}}(H^{Z_n}) = [0, \infty)$. Assume now that H^{Z_n} is not nonnegative. Then, there exists a ground state $\Psi_G \in H^2(\mathbb{R}^2, \mathbb{C})$ of H^{Z_n} of negative energy $E < 0$. The phase of the ground state can be chosen such that the ground state is real and positive a.e. (see e.g. [52], Theorem 10.12.). Since $f_\nu(\cdot - z_k)$ is positive outside $\text{supp}(V_N)$, the following inequality is valid[6]

$$\langle F_\nu^{Z_n}, H^{Z_n} \Psi_G \rangle = \langle F_\nu^{Z_n}, E \Psi_G \rangle < 0. \tag{131}$$

On the other hand we have since $F_\nu^{Z_n}$ is the zero energy scattering state

$$\langle F_\nu^{Z_n}, H^{Z_n} \Psi_G \rangle = \langle H^{Z_n} F_\nu^{Z_n}, \Psi_G \rangle = 0.$$

This contradicts (131) and the nonnegativity of H^{Z_n} follows.
Now, assume that there exists a $\psi \in H^2(\mathbb{R}^2, \mathbb{C})$ such that the quadratic form

$$Q(\psi) = \|\mathbb{1}_{|\cdot| \leq R_\nu} \nabla \psi\|^2 + \frac{1}{2} \langle \psi, (V_N(\cdot) - M_\nu(\cdot))\psi \rangle < 0.$$

Since V_N and M_ν are spherically symmetric we can assume that ψ is spherically symmetric. Substituting $\psi \to a\psi$, $a \in \mathbb{R}$, we can furthermore assume that, for all

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[6] Note that a one particle ground state of negative energy decays exponentially, that is $\Psi_G(x) \leq C_1 e^{-C_2|x|}$, $C_1, C_2 > 0$. Hence, (131) is well defined, although $F_\nu^{Z_n} \notin L^2(\mathbb{R}^2, \mathbb{C})$.

$|x| = R_\nu$, $\psi(x) = 1 - \epsilon$ for $\epsilon > 0$.

Define $\tilde{\psi}$ such that $\tilde{\psi}(x) = \psi(x)$ for $|x| \leq R_\nu$ and $\tilde{\psi}(x) = 1$ for $|x| > R_\nu + \epsilon$ and $\epsilon > 0$. Furthermore, $\tilde{\psi}$ can be constructed such that $\|\mathbb{1}_{|\cdot| \geq R_\nu} \nabla \tilde{\psi}\|^2 \leq C(\epsilon + \epsilon^2)$.

Then $Q(\tilde{\psi}) = Q(\psi) < 0$ holds, because the operator associated with the quadratic form is supported inside the ball $B_0(R_\nu)$.

Using $\tilde{\psi}$, we can construct a set of points Z_n and a $\chi \in H^2(\mathbb{R}^2, \mathbb{C})$ such that $\langle \chi, H^{Z_n} \chi \rangle < 0$, contradicting to nonnegativity of H^{Z_n}.

For $R > 1$ let

$$\xi_R(x) = \begin{cases} R^2/x^2, & \text{for } |x| > R; \\ 1, & \text{else.} \end{cases}$$

Let now Z_n be a subset $Z_n \subset \mathbb{R}^2$ with $|Z_n| = n$ which is such that the supports of the potentials $M_\nu(\cdot - z_k)$ lie within the Ball around zero with radius R and are pairwise disjoint for any two $z_k \in Z_n$. Since we are in two dimensions we can choose a n which is of order R^2.

Let now $\chi_R(x) = \xi_R(x) \prod_{z_k \in Z_n} \tilde{\psi}(x - z_k)$. By construction, there exists a $D = \mathcal{O}(1)$ such that $\chi_R(x) = \tilde{\psi}(x - z_k)$ for $|x - z_k| \leq D$. From this, we obtain

$$\langle \chi_R, H^{Z_n} \chi_R \rangle = \|\nabla \chi_R\|^2 + n \frac{1}{2} \langle \psi, (V_N(\cdot) - M_\nu(\cdot))\psi \rangle$$

$$= n Q(\psi) + \sum_{z_k \in Z_n} \|\mathbb{1}_{|\cdot - z_k| \geq R_\nu} \nabla \chi_R\|^2$$

$$\leq n Q(\psi) + Cn(\epsilon + \epsilon^2) + \|\nabla \xi_R\|^2$$

$$= n Q(\psi) + Cn(\epsilon + \epsilon^2) + C.$$

Choosing R and hence n large enough and ϵ small, we can find a Z_n such that $\langle \chi_R, H^{Z_n} \chi_R \rangle$ is negative, contradicting nonnegativity of H^{Z_n}.

Now, we can prove that

$$\|\mathbb{1}_{|x_1 - x_2| \leq R_\nu} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, (V_N - M_\nu)(x_1 - x_2)\Psi \rangle\!\rangle \geq 0. \qquad (132)$$

holds for any $\Psi \in H^2(\mathbb{R}^{2N}, \mathbb{C})$. Using the coordinate transformation $\tilde{x}_1 = x_1 - x_2$, $\tilde{x}_i = x_i \; \forall i \geq 2$, we have $\nabla_{x_1} = \nabla_{\tilde{x}_1}$. Thus (132) is equivalent to $\tilde{Q}(\Psi) :=$ $\|\mathbb{1}_{|x_1| \leq R_\nu} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, (V_N - M_\nu)(x_1)\Psi \rangle\!\rangle \geq 0 \; \forall \Psi \in H^2(\mathbb{R}^{2N}, \mathbb{C})$.

If it were now that $\tilde{Q}(\Psi)$ is not nonnegative, then there exists a $\Gamma \in H^2(\mathbb{R}^{2N}, \mathbb{C})$ such that $\tilde{Q}(\Gamma) < 0$. By the Schmidt decomposition theorem, there exist two orthonormal bases $\{\Phi_k\}_{k \in \mathbb{N}} \subset H^2(\mathbb{R}^{2N-2}, \mathbb{C})$, $\{\varphi_l\}_{l \in \mathbb{N}} \subset H^2(\mathbb{R}^2, \mathbb{C})$ and nonnegative numbers $\{\lambda_k\}_{k \in \mathbb{N}}$ such that

$$\Gamma = \sum_{k \in \mathbb{N}} \lambda_k \varphi_k \otimes \Phi_k.$$

By this

$$\tilde{Q}(\Gamma) = \sum_{k \in \mathbb{N}} |\lambda_k|^2 Q(\varphi_k) \geq 0,$$

which in turn yields to a contradictions. Therefore, $Q(\Psi) \geq 0$ for all $\Psi \in H^2(\mathbb{R}^2, \mathbb{C})$. By a standard density argument, we can conclude that $Q(\Psi) \geq 0 \; \forall \Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C}) \cap \mathcal{D}(\nabla_1)$.

(b) Define $c_k = \{(x_1, \ldots, x_N) \in \mathbb{R}^{2N} | |x_1 - x_k| \leq R_v\}$ and $\mathcal{C}_1 = \cup_{k=2}^N c_k$. For $(x_1, \ldots, x_N)$ $\in \mathcal{B}_1^{(d)}$ it holds that $|x_i - x_j| \geq N^{-d}$ for $2 \leq i, j \leq N$. Let $v > d$. Assume that $N^{-d} > 2R_v$, which hold for N sufficiently large, since $R_v \leq CN^{-v}$. Then, it follows that, for $i \neq j$, $\left(c_i \cap \mathcal{B}_1^{(d)}\right) \cap \left(c_j \cap \mathcal{B}_1^{(d)}\right) = \emptyset$. Under the same conditions, we also have $\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \geq \mathbb{1}_{\mathcal{C}_1}$. Therefore

$$\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \mathbb{1}_{\mathcal{B}_1^{(d)}} \geq \mathbb{1}_{\mathcal{C}_1} \mathbb{1}_{\mathcal{B}_1^{(d)}} = \mathbb{1}_{\mathcal{C}_1 \cap \mathcal{B}_1^{(d)}} = \mathbb{1}_{\cup_{k=2}^N \left(c_k \cap \mathcal{B}_1^{(d)}\right)} = \sum_{k=2}^N \mathbb{1}_{c_k \cap \mathcal{B}_1^{(d)}} = \mathbb{1}_{\mathcal{B}_1^{(d)}} \sum_{k=2}^N \mathbb{1}_{c_k}.$$

Note that $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ depends only on $x_2, \ldots, x_N$. By this

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \mathbb{1}_{\mathcal{B}_1^{(d)}} \nabla_1 \Psi\|^2 \geq \sum_{k=2}^N \|\mathbb{1}_{c_k} \nabla_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\|^2 = (N-1)\|\mathbb{1}_{|x_1 - x_2| \leq R_\beta} \nabla_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\|^2.$$

This yields

$$(130) \geq (N-1)\left(\|\mathbb{1}_{|x_1 - x_2| \leq R_v} \nabla_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\|^2 + \frac{1}{2}\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, (V_N - M_v)(x_1 - x_2)\mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\rangle\!\rangle\right)$$

$$\geq 0.$$

where the last inequality follows from (a), using $\mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C}) \cap \mathcal{D}(\nabla_1)$.
$\square$

Lemma 7.11. *Let* $W_\beta \in \mathcal{W}_\beta$. *Let* $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$ *and* $\|\nabla_1 \Psi\|$ *be bounded uniformly in* N. *Let* d *in Definition* 7.3 *of* $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ *sufficiently large. Let* $\Gamma \in$ $\{\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\}$. *Then, for all* $\beta > 0$,

(a)

$$N\left|\langle\!\langle \Gamma, q_1 p_2 W_\beta(x_1 - x_2)p_1 p_2 \Gamma\rangle\!\rangle\right| \leq C\|\varphi\|_\infty^2 \left(\langle\!\langle \Psi, \hat{n}\Psi\rangle\!\rangle + N^{-1}\right),$$

$$N\left|\langle\!\langle \Gamma, p_1 q_2 W_\beta(x_1 - x_2)p_1 p_2 \Gamma\rangle\!\rangle\right| \leq C\|\varphi\|_\infty^2 \left(\langle\!\langle \Psi, \hat{n}\Psi\rangle\!\rangle + N^{-1}\right).$$

(b)

$$N|\langle\!\langle \Gamma, p_1 p_2 W_\beta(x_1 - x_2)q_1 q_2 \Gamma\rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6} \ln(N)\right).$$

(c)

$$N|\langle\!\langle \Gamma, (1 - p_1 p_2)W_\beta(x_1 - x_2)p_1 p_2 \Gamma\rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6} \ln(N)\right).$$

Proof. (a) We will only consider the first inequality of (a). The second inequality of (a) can be proven analogously. Let first $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi$. Then,

$$N\left|\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, q_1 p_2 W_\beta(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\rangle\!\rangle\right|$$

$$\leq N\left|\langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi, q_1 p_2 W_\beta(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\rangle\!\rangle\right| \tag{133}$$

$$+ N\left|\langle\!\langle \Psi, q_1 p_2 W_\beta(x_1 - x_2)p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\rangle\!\rangle\right|. \tag{134}$$

Using Lemma 7.4 (c) with $\epsilon = 1$, together with $\| p_2 W_\beta(x_1 - x_2) p_2 \|_{\mathrm{op}} \leq \| \varphi \|_\infty^2 \| W_\beta \|_1$, the first line can be bounded by

$$(133) \leq \mathcal{K}(\varphi, A_t) N \| \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \| \| W_\beta \|_1 \leq \mathcal{K}(\varphi, A_t) N^{2-d}. \tag{135}$$

The second term is bounded by

$$(134) = N \left| \langle\!\langle \sqrt{W_\beta(x_1 - x_2)} q_1 p_2(\hat{n})^{-\frac{1}{2}} \Psi, \sqrt{W_\beta(x_1 - x_2)} p_1 p_2 \hat{n}_1^{\frac{1}{2}} \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi \rangle\!\rangle \right|$$

$$\leq C N \| \sqrt{W_\beta(x_1 - x_2)} p_2 \|_{\mathrm{op}}^2 \left(\| q_1(\hat{n})^{-\frac{1}{2}} \Psi \|^2 + \| \hat{n}_1^{\frac{1}{2}} \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi \|^2 \right)$$

$$\leq C N \| \sqrt{W_\beta(x_1 - x_2)} p_2 \|_{\mathrm{op}}^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + \| \hat{n}_1^{\frac{1}{2}} \Psi \|^2 + \| \hat{n}_1^{\frac{1}{2}} \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \|^2 \right)$$

$$\leq C N \| W_\beta \|_1 \| \varphi \|_\infty^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + \| \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \|^2 \right)$$

$$\leq C \| \varphi \|_\infty^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + N^{4-2d} \right) \leq C \| \varphi \|_\infty^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + N^{-1} \right).$$

This yields (a) in the case $\Gamma = \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi$. The inequality (a) can be proven analogously for $\Gamma = \Psi$.

(b) Let $\Gamma = \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi$. We first consider (b) for potentials with $\beta < 1/4$. We have to estimate

$$N | \langle\!\langle \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi \rangle\!\rangle |$$

$$\leq N | \langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \Psi \rangle\!\rangle |$$
$$+ N | \langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \Psi \rangle\!\rangle |$$
$$+ N | \langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \rangle\!\rangle |$$
$$+ N | \langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \rangle\!\rangle |$$

$$\leq N | \langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \Psi \rangle\!\rangle | \tag{136}$$
$$+ C N \| \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \| \| W_\beta \|_\infty. \tag{137}$$

The last term is bounded using Lemma 7.4 (c) with $\epsilon = 1$

$$(137) \leq C N N^{2-d} N^{-1+2\beta} \leq N^{-1/2},$$

where the last inequality holds choosing $d \geq 3$.

Using Lemmas 4.2 (c) and 4.6 with $O_{1,2} = q_2 W_\beta(x_1 - x_2) p_2$, $\Omega = N^{-1/2} q_1 \Psi$ and $\chi = N^{1/2} p_1 \Psi$ we get

$$(136) \leq \| q_1 \Psi \|^2 + N^2 \big| \langle\!\langle q_2 \Psi, p_1 \sqrt{W_\beta(x_1 - x_2)} p_3 \sqrt{W_\beta(x_1 - x_3)}$$
$$\sqrt{W_\beta(x_1 - x_2)} p_2 \sqrt{W_\beta(x_1 - x_3)} p_1 q_3 \Psi \rangle\!\rangle \big|$$
$$+ N^2 (N-1)^{-1} \| q_2 W_\beta(x_1 - x_2) p_2 p_1 \Psi \|^2$$
$$\leq \| q_1 \Psi \|^2 + N^2 \| \sqrt{W_\beta(x_1 - x_2)} p_1 \|_{\mathrm{op}}^4 \| q_2 \Psi \|^2$$
$$+ C N \| W_\beta(x_1 - x_2) p_2 \|_{\mathrm{op}}^2.$$

With Lemma 4.2 (e) we get the bound

$$(136) \leq \|q_1\Psi\|^2 + N^2\|\varphi\|_\infty^4\|W_\beta\|_1^2\|q_1\Psi\|^2 + CN\|W_\beta\|^2\|\varphi\|_\infty^2.$$

Note, that $\|W_\beta\|_1 \leq CN^{-1}$, $\|W_\beta\|^2 \leq CN^{-2+2\beta}$. Hence

$$(136) \leq C\left(\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle + \mathcal{K}(\varphi, A_t)N^{-1+2\beta}\right).$$

Note that, for $\beta < 1/4$, $N^{-1+2\beta} \leq N^{-1/6}\ln(N)$. Using the same bounds for $\Gamma = \Psi$, we obtain (b) for the case $\beta < 1/4$.

b) for $1/4 \leq \beta$:

We use $U_{\beta_1,\beta}$ from Definition 7.1 for some $0 < \beta_1 < 1/4$.

By the estimate above, it is left to control

$$N\left|\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right|.$$

Let $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. Integrating by parts and using that $\nabla_1 h_{\beta_1,\beta}(x_1-x_2) = -\nabla_2 h_{\beta_1,\beta}(x_1-x_2)$ gives

$$N\left|\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right|$$

$$= N\left|\langle\!\langle\nabla_1 p_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right| \tag{138}$$

$$+ N\left|\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)\nabla_1 q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right|. \tag{139}$$

Let $(a_1, b_1) = (q_1, \nabla p_1)$ or $(a_1, b_1) = (\nabla q_1, p_1)$. Then, both terms can be estimated as follows:

We use Lemma 4.6 with $\Omega = N^{-\eta/2}a_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$, $O_{1,2} = N^{1+\eta/2}q_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)p_2$ and $\chi = b_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. We choose $\eta < 2\beta_1$.

$$N\left|\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, a_1p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right|$$

$$\leq N^{-\eta}\|a_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 \tag{140}$$

$$+ \frac{N^{2+\eta}}{N-1}\|q_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1p_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 \tag{141}$$

$$+ N^{2+\eta}\left|\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, b_1p_2q_3\nabla_2 h_{\beta_1,\beta}(x_1-x_2)\nabla_3 h_{\beta_1,\beta}(x_1-x_3)b_1q_2p_3\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right|^{1/2}. \tag{142}$$

We obtain (note that $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ does not depend on x_1)

$$(140) \leq N^{-\eta}\|a_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 = N^{-\eta}\|\mathbb{1}_{\mathcal{B}_1^{(d)}}a_1\Psi\|^2 \leq \mathcal{K}(\varphi, A_t)N^{-\eta}$$

since both $\|\nabla q_1\Psi\|$ and $\|q_1\Psi\|$ are bounded uniformly in N. Since q_2 is a projector it follows that

$$(141) \leq \frac{N^{2+\eta}}{N-1}\|\nabla_2 h_{\beta_1,\beta}(x_1-x_2)p_2\|_{op}^2\|b_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2$$

$$\leq C\frac{N^{2+\eta}}{N-1}\|\varphi\|_\infty^2\|\nabla h_{\beta_1,\beta}\|^2\|b_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2$$

$$\leq \mathcal{K}(\varphi, A_t)N^{\eta-1}\ln(N)\|\varphi\|_\infty^2,$$

where we used Lemma 7.2 in the last step.

Next, we estimate

$$(142) \leq N^{2+\eta} \| p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \mathbb{1}_{\overline{B}_1^{(d)}} \Psi \|^2$$

$$\leq 2N^{2+\eta} \| p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \mathbb{1}_{\overline{B}_1^{(d)}} \Psi \|^2 \tag{143}$$

$$+ 2N^{2+\eta} \| p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \Psi \|^2. \tag{144}$$

The first term can be estimated as

$$(143) \leq C N^{2+\eta} \| \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 \|_{\mathrm{op}}^2 \| \mathbb{1}_{\overline{B}_1^{(d)}} \Psi \|^2$$

$$\leq C N^{2+\eta} \| \nabla_2 h_{\beta_1,\beta} \|^2 (\| \varphi \|_\infty^2 + \| \nabla \varphi \|_\infty^2) \| \mathbb{1}_{\overline{B}_1^{(d)}} \Psi \|^2$$

$$\leq \mathcal{K}(\varphi, A_t) N^{2+\eta} N^{-2} \ln(N) N^{4-2d}$$

$$\leq \mathcal{K}(\varphi, A_t) N^{-2+\eta} \ln(N),$$

for any $d \geq 3$. In the last line we used Lemma 7.4 (c) with $\epsilon = 1$. The last term can be estimated as

$$(144) \leq 2N^{2+\eta} \| p_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 \nabla_2 q_2 \Psi \|^2$$

$$+ 2N^{2+\eta} \| |\varphi(x_2)\rangle \langle \nabla\varphi(x_2)| h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \Psi \|^2$$

$$\leq C N^{2+\eta} \| p_2 h_{\beta_1,\beta}(x_1 - x_2) \|_{\mathrm{op}}^2 \| b_1 \nabla_2 q_2 \Psi \|^2$$

$$+ C N^{2+\eta} \| |\varphi(x_2)\rangle \langle \nabla\varphi(x_2)| h_{\beta_1,\beta}(x_1 - x_2) \|_{\mathrm{op}}^2 \| b_1 q_2 \Psi \|^2$$

$$\leq C N^{2+\eta} \left(\| \nabla\varphi \|_\infty^2 + \| \varphi \|_\infty^2 \right) \| h_{\beta_1,\beta} \|^2 (1 + \| \nabla\varphi \|^2)$$

$$\leq \mathcal{K}(\varphi, A_t) N^{\eta - 2\beta_1} \ln(N)^2.$$

Combining both estimates we obtain, for any $\beta > 1/4$,

$$N \left| \langle\!\langle \mathbb{1}_{\overline{B}_1^{(d)}} \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \mathbb{1}_{\mathcal{B}^{(d)}_1} \Psi \rangle\!\rangle \right|$$

$$\leq \inf_{\eta > 0} \inf_{0 < \beta_1 < 1/4} \left(\mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-1+2\beta_1} + N^{-\eta} \right. \right.$$

$$\left. \left. + N^{\eta-1} \ln(N) + N^{\eta-2\beta_1} \ln(N) \right) \right)$$

$$\leq \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-1/6} \ln(N) \right).$$

where the last inequality comes from choosing $\eta = 1/3$ and $\beta_1 = 1/4$. For $\Gamma = \Psi$, (b) can be estimated the same way, yielding the same bound.

(c) This follows from (a) and (b), using that $1 - p_1 p_2 = q_1 q_2 + p_1 q_2 + q_1 p_2$. □

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