

EXISTENCE OF POSITIVE SOLUTIONS TO STOCHASTIC THIN-FILM EQUATIONS*

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Abstract. We construct martingale solutions to stochastic thin-film equations by introducing a (spatial) semidiscretization and establishing convergence. The discrete scheme allows for variants of the energy and entropy estimates in the continuous setting as long as the discrete energy does not exceed certain threshold values depending on the spatial grid size h . Using a stopping time argument to prolongate high-energy paths constant in time, arbitrary moments of coupled energy/entropy functionals can be controlled. Having established Hölder regularity of approximate solutions, the convergence proof is then based on compactness arguments—in particular on Jakubowski’s generalization of Skorokhod’s theorem—weak convergence methods, and recent tools on martingale convergence.

Key words. stochastic thin-film equation, thin-film flow, thermal noise, nonnegativity-preserving scheme, stochastic partial differential equation

AMS subject classifications. 60H15, 76D08, 73D03, 35G20, 35K25, 35K65, 35Q35

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1. Introduction. For fluids consisting of just a small number of molecules, the validity range of models from fluid mechanics may be enhanced by incorporating thermal fluctuations into the model.

In the case of a thin liquid film with a thickness corresponding to just around 10^1 – 10^3 molecule layers (cf. Figure 1), classical models of continuum mechanics do not always give a precise description of thin-film evolution. While morphologies of film dewetting can be captured by thin-film models (see [2]), discrepancies arise with respect to time-scales of dewetting. To put it concisely, certain effects, which accelerate film rupture at its very onset, seem not to be included in these models. Based on physical considerations, thermal fluctuations may have a strong influence during the first stage of rupture of very thin films—cf. [47]. This fact motivated Grün, Mecke, and Rauscher [37] to formally derive a stochastic thin-film equation which reads

$$(1.1) \quad du = -\operatorname{div}(m(u)\nabla(\Delta u - \mathcal{W}'(u)))dt + \operatorname{div}(\sqrt{m(u)}dS)$$

and which is considered on rectangular spatial domains subjected to periodic boundary conditions. Parallel in time, Davidovitch, Moro, and Stone [17] presented a different derivation under the perspective of investigating the influence of fluctuations on droplet spreading. In both cases, a no-slip boundary condition has been assumed at the liquid-solid interface, leading to a degeneracy $m(u) = u^3$. Numerical simulations in [37] indicate that thermal noise is indeed able to overcome the aforementioned discrepancies with respect to time-scales of dewetting.

In (1.1), u corresponds to the height of the thin liquid film, and $m(u)$ is the thickness-dependent *mobility function* which depends on the flow condition at the liquid-solid interface. The effective interface potential $\mathcal{W}(u)$ reflects the effect of

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(attractive and repulsive) interaction forces between liquid and substrate molecules, and dS denotes vector-valued white noise.

In the present work, we study the existence of nonnegative solutions in the one-dimensional case. We focus on a mobility $m(u) = u^2$, which—with a grain of salt—corresponds to a Navier slip condition at the liquid-solid interface (cf. [49]). For the potential \mathcal{W} , a prototypical example compatible with our assumption (H2) below is given by $\mathcal{W}(u) = u^{-8} - u^{-2} + 1$. This potential stands for disjoining and conjoining van der Waals interactions between fluid and solid based on 6-12-Lennard–Jones pair potentials.

In Remark 3.3, we will show that space-time white noise is not compatible with finiteness of the physical energies encountered in thin-film flow—hence rendering the mathematical analysis of the stochastic thin-film equation infeasible due to the strong nonlinearities in the equation.

Therefore, we will consider Q -Wiener processes. Denoting the eigenfunctions of the Laplacian on the spatial domain $\mathcal{O} = (0, L)$ subjected to periodic boundary conditions by g_ℓ , $\ell \in \mathbb{N}$, introducing a sequence of independent Brownian motions $(\beta_\ell)_{\ell \in \mathbb{N}}$ as well as a sequence $(\lambda_\ell)_{\ell \in \mathbb{N}}$ of nonnegative real numbers that converges to zero sufficiently fast, we assume that the noise is of the form

$$(1.2) \quad dW = \sum_{\ell=1}^{\infty} \lambda_\ell g_\ell d\beta_\ell.$$

It is well-known (see, e.g., [51]) that the increments of $W = \sum_{\ell=1}^{\infty} \lambda_\ell g_\ell \beta_\ell$ come along with Gaussian laws

$$\mathbb{P} \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q) \quad \forall 0 \leq s \leq t \leq T,$$

where the self-adjointed operator Q is defined by

$$(1.3) \quad Qg_\ell = \lambda_\ell^2 g_\ell \quad \forall \ell \in \mathbb{N}.$$

Examples of such covariance operators Q —and hence, examples of such noise—are provided by Hilbert–Schmidt operators of the form

$$(1.4) \quad (Qf)(x) := \int_0^L q(y-x)f(y)dy$$

with nonnegative, symmetric, L -periodic, and sufficiently smooth kernels q : Following the results of Blömker [10], such operators share the system of orthonormal eigenfunctions with the Laplacian.

Recall that the decay of the eigenvalues is related to the smoothness of q . Choosing q compactly supported, we infer from Theorem 3.3 in [10] that the correlation of the (physical) noise $\xi(t, x) := \partial_t W$ is given by

$$\mathbb{E}(\xi(t, x)\xi(s, y)) = \delta(t-s)q(x-y),$$

where $\delta(\cdot)$ denotes the Dirac δ -distribution. In this spirit, the size of the support of q gives twice the correlation length of the noise. Assuming a finite interaction length, Q -Wiener processes seem to be a reasonable ansatz from a physical point of view, too.

Altogether, in our setting the stochastic thin-film equation reads

$$(1.5) \quad du = -\left(m(u)(u_{xx} - \mathcal{W}'(u))_x\right)_x dt + \sum_{\ell=1}^{\infty} (\lambda_\ell \sqrt{m(u)} g_\ell)_x d\beta_\ell$$

on $\mathcal{O} \times [0, \infty)$ subjected to periodic boundary conditions.

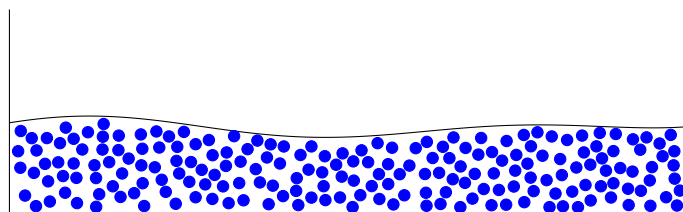


FIG. 1. A thin liquid film consisting only of a limited number of fluid molecules.

The main result of the present contribution is the existence of a weak martingale solution to the stochastic thin-film equation (1.5); see Definition 3.1 and Theorem 3.2 below. For properly chosen initial data, the solution turns out to be positive on $\mathcal{O} \times [0, T]$ almost surely. The assumptions on the probabilistic setting, initial data, and the effective interface potential \mathcal{W} will be made precise in section 2; see (H1)–(H4).

For the deterministic thin-film equation

$$(1.6) \quad u_t = -\operatorname{div}(m(u)\nabla\Delta u)$$

and its variants, in recent decades an extensive mathematical theory has been developed. Despite the lack of comparison principles, it allows for globally nonnegative solutions. The first result on existence and nonnegativity of weak solutions in the case of one spatial dimension is due to Bernis and Friedman [6]. More refined results—addressing in particular positivity properties and the regularity of solutions at $\partial\{u(\cdot, t) > 0\}$ —were obtained in [3] and [7]. Without further conditions, weak solutions to the thin-film equation are in general nonunique, at least for initial data with compact support. Well-posedness might hold true if another condition at the free boundary $\partial\{u(\cdot, t) > 0\}$ is imposed in addition to the natural condition $u = 0$. Physical considerations suggest prescribing the contact angle (or equivalently, $|\nabla u|$) at the free boundary. In the case of complete wetting $|\nabla u| = 0$, this boundary condition may be enforced implicitly by additional regularity constraints on the solution, so-called entropy estimates [3, 7]. In the case of partial wetting $|\nabla u| = \alpha > 0$, enforcing the contact angle condition for weak solutions is significantly more complicated; see [9, 48, 50].

Equation (1.6) has been studied in multiple space dimensions, too. Results on existence and nonnegativity of entropy solutions can be found in [15] and [36]—see also [20] and [32] for basic results on fourth-order degenerate parabolic equations.

For local-in-time results on existence and uniqueness, we refer to [25, 27, 28, 29, 30, 43, 46].

There is a rich qualitative theory of solutions to the thin-film equation: Finite speed of propagation of solutions and upper bounds on the propagation of free boundaries have been established in space dimension $d = 1$ in [4, 5] and in [41]. Results in multiple space dimensions were obtained in [8, 34, 33]. The large-time behavior of solutions to the Cauchy problem has been analyzed in the case $m(u) = u$ in [14]. Waiting time phenomena were studied in [16, 26]. Rigorous lower bounds on the propagation of the free boundary and sufficient conditions for instantaneous forward motion of the free boundary—based on the discovery of certain monotonicity formulas for the thin-film equation—have been deduced in [22, 23, 21].

Quite recently, much progress has been made in the analysis of nonlinear parabolic stochastic partial differential equations. Debussche, Hofmanová, and Vovelle study quasi-linear degenerate second-order parabolic stochastic partial differential equations

in [18], Hofmanová, Röger, and von Renesse prove the existence of weak solutions for stochastic mean curvature flow of two-dimensional graphs [39], and Breit, Feireisl, and Hofmanová identify incompressible limits of solutions to stochastic compressible Navier–Stokes equations [11]. These papers take advantage of novel approaches to construct martingale solutions introduced in [13] and in [40].

The general strategy for the proof of our existence result is to combine these probabilistic techniques with methods used in the numerical analysis of thin-film equations (cf. [38, 35, 52]). The latter are based on discrete versions of some integral estimates for the thin-film equation—the energy estimate and the entropy estimate.

It is worth mentioning that we only discretize in space, which allows us to control the degeneracies and singularities in the equation from the very beginning using Ito’s lemma. For fully discrete approaches to stochastic partial differential equations, we mention the papers [12] and [1] on stochastic Navier–Stokes equations and stochastic Landau–Lifshitz–Gilbert equations, respectively. For a first result on fully discrete convergent schemes for stochastic versions of degenerate parabolic equations, we refer to [31], which is about the discretization of the stochastic porous-medium equation with linear multiplicative noise inside a source term.

Let us specify the fundamental integral quantities to be used in the paper. The energy $E[u]$ of a thin liquid film is given by the sum of the surface energies associated with the interfaces between liquid and ambient fluid/vacuum and between liquid and solid. In lubrication approximation, the former is given by $\int_{\mathcal{O}} \sqrt{1 + |u_x|^2} dx \approx \int_{\mathcal{O}} 1 + \frac{1}{2}|u_x|^2 dx$, the latter by $\int_{\mathcal{O}} \mathcal{W}(u) dx$. One may subtract the constant $\int_{\mathcal{O}} 1 dx$, resulting in the expression

$$(1.7) \quad E[u] := \int_{\mathcal{O}} \frac{1}{2}|u_x|^2 + \mathcal{W}(u) dx.$$

A second integral expression useful for analysis and numerics of thin-film equations is the so-called mathematical entropy

$$(1.8) \quad S[u] := \int_{\mathcal{O}} G(u) dx$$

with its density given by

$$G(u) := \int_1^u \int_1^s \frac{1}{m(r)} dr ds.$$

Especially in our setting, we get

$$S[u] = \int_{\mathcal{O}} -\log u + u - 1 dx.$$

A formal application of Ito’s lemma yields for solutions to the stochastic thin-film equation (1.5)

$$\begin{aligned} & \mathbb{E} \int_{\mathcal{O}} \frac{1}{2}|u_x|^2 + \mathcal{W}(u) dx \Big|_0^T \\ &= -\mathbb{E} \int_0^T \int_{\mathcal{O}} m(u) |(u_{xx} - \mathcal{W}'(u))_x|^2 dx dt \\ & \quad + \frac{1}{2} \sum_{\ell \in \mathbb{N}} \lambda_{\ell}^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} |(\sqrt{m(u)} g_{\ell})_{xx}|^2 dx dt \\ & \quad + \frac{1}{2} \sum_{\ell \in \mathbb{N}} \lambda_{\ell}^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} \mathcal{W}''(u) |(\sqrt{m(u)} g_{\ell})_x|^2 dx dt \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \int_{\mathcal{O}} G(u) dx \Big|_0^T &= -\mathbb{E} \int_0^T \int_{\mathcal{O}} (u_{xx} - \mathcal{W}'(u)) u_{xx} dx dt \\ &\quad + \frac{1}{2} \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} u^{-2} |(\sqrt{m(u)} g_\ell)_x|^2 dx dt \\ &= -\mathbb{E} \int_0^T \int_{\mathcal{O}} |u_{xx}|^2 + \mathcal{W}''(u) |u_x|^2 dx dt \\ &\quad + \frac{1}{2} \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} u^{-2} |(\sqrt{m(u)} g_\ell)_x|^2 dx dt. \end{aligned}$$

Recalling the relation $m(u) = u^2$, we therefore obtain for the weighted sum $E[u] + \kappa S[u]$ for κ large enough

$$\begin{aligned} \mathbb{E} E[u] + \kappa \mathbb{E} S[u] \Big|_0^T &\leq -\mathbb{E} \int_0^T \int_{\mathcal{O}} \frac{\kappa}{2} |u_{xx}|^2 + \kappa \mathcal{W}''(u) |u_x|^2 + m(u) |(u_{xx} - \mathcal{W}'(u))_x|^2 dx dt \\ &\quad + \frac{1}{2} \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} \mathcal{W}''(u) |(\sqrt{m(u)} g_\ell)_x|^2 dx dt \\ &\quad + \frac{1}{2} \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} |(\sqrt{m(u)} g_\ell)_{xx}|^2 dx dt \\ &\quad + \frac{\kappa}{2} \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \mathbb{E} \int_0^T \int_{\mathcal{O}} u^{-2} |(\sqrt{m(u)} g_\ell)_x|^2 dx dt. \end{aligned}$$

Further estimates and an application of the Gronwall lemma then yield a suitable energy-entropy estimate which in particular gives a uniform bound of the form

$$\begin{aligned} (1.9) \quad &\mathbb{E} \left\{ \sup_{t \in [0, T]} E[u] + \sup_{t \in [0, T]} S[u] \right\} \\ &+ \mathbb{E} \int_0^T \int_{\mathcal{O}} |u_{xx}|^2 dx dt + \mathbb{E} \int_0^T \int_{\mathcal{O}} u^2 |(u_{xx} - \mathcal{W}'(u))_x|^2 dx dt \\ &+ \mathbb{E} \int_0^T \int_{\mathcal{O}} u^{-p-2} u_x^2 dx dt \\ &\leq C(p, T, u_0) < \infty. \end{aligned}$$

We emphasize that this estimate is a special case of a much more general result on formal integral estimates for stochastic thin-film equations obtained by Dirr and Grün (see [19]).

Let us give the outline of the paper. In section 2, we formulate a semidiscrete scheme for the stochastic thin-film equation. Aiming at discrete counterparts of the formal integral estimate (1.9), the mobility $m(u) = u^2$ is discretized following ideas from numerical analysis. In section 3, our result on existence and positivity, Theorem 3.2, is stated. Sections 4 and 5 are devoted to its proof.

In Lemma 4.1, we show that a bound on the discretization-adapted energy $E_h[v]$ (which is a slight modification of $E[v]$; see (4.1)) for a function $v \in X_h$ entails a strictly positive lower bound on v , uniformly with respect to h . Based on this observation, solutions u^h, p^h to our semidiscrete scheme (2.1) are constructed in Lemma 4.2 by reducing the problem to a classical existence result for SDEs. In

Proposition 4.4, we derive uniform estimates on the combined energy-entropy functional $R(t) := E_h[u^h(t)] + \kappa S_h[u^h(t)]$ as well as on its dissipation, the most important terms of which are $\int_0^{T_{max}} \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 dx dt$ and $\int_0^{T_{max}} \int_{\mathcal{O}} |\Delta_h u^h|^2 dx dt$ where p^h is an appropriate discretization of the pressure $-u_{xx} + \mathcal{W}'(u)$. The derivation of these bounds makes use of Ito's formula and a subtle discretization of the degenerate mobility $m(u) = u^2$ which is inspired by numerical analysis (cf. [38], [35]). Based on the Burkholder-Davis-Gundy inequality, we also obtain estimates on stochastic moments of the pathwise supremum $\sup_{t \in [0, T_{max}]} R(t)$ of the combined energy-entropy functional. Starting from these uniform energy-entropy estimates, in Lemma 4.11 we obtain uniform estimates on the solution u^h in an appropriately chosen Hölder space. The passage to the limit $h \rightarrow 0$ is based on tightness results for height, pressure, and flux of the thin film in appropriate function spaces. Jakubowski's generalization of Skorokhod's theorem to nonmetric spaces [42] and recent strategies on martingale convergence (cf. [11, 13, 18, 40]) turn out to be crucial tools to succeed. Convergence in the deterministic terms follows by classical arguments of pde-theory.

Notation. Throughout the paper, we use standard notation for Sobolev spaces and from stochastic analysis. The spatial domain \mathcal{O} is given by the interval $(0, L)$, and we abbreviate $I := [0, T]$. The notation $a \wedge b$ stands for the minimum of a and b , and $L_2(X, Y)$ denotes the set of Hilbert-Schmidt operators from X to Y . For values of $\gamma_s, \gamma_t \in (0, 1)$, $C^{\gamma_s, \gamma_t}(\bar{\mathcal{O}} \times [0, T])$ denotes the space of continuous functions on $\bar{\mathcal{O}} \times [0, T]$ which are Hölder-continuous with exponent γ_s (respectively, γ_t) with respect to space (respectively, time). In particular, the exponent γ will exclusively be used for Hölder properties related to the martingale solution for the stochastic thin film equation. For a stopping time T , we write χ_T to denote the (ω -dependent) characteristic function of the time interval $[0, T]$. The abbreviation $(v)_{\mathcal{O}}$ is used for the mean value of a function v over a domain \mathcal{O} .

Further notation related to the discretization will be introduced in section 2.

2. Preliminaries on the discretization. In this section, we will introduce a semi-discrete scheme which will serve to obtain spatially discrete approximate solutions to the stochastic thin-film equation. Existence of those approximate solutions will be established in section 4 applying a stopping time argument to solutions of an appropriate system of ordinary stochastic differential equations (SDEs).

- Given an integer fraction h of a real number $L > 0$, by X_h we denote the space of periodic linear finite elements, i.e., the space of periodic continuous functions on $[0, L]$ that are linear on each of the intervals $[0, h], [h, 2h], \dots, [L-h, L]$.
- By e_i , we denote the function in X_h that equals 1 at $x = ih$ and that vanishes for all other $x = kh, k \neq i$.
- Let $C_{per}([0, L])$ be the space of periodic continuous functions on $[0, L]$. By $\mathcal{I}_h : C_{per}([0, L]) \rightarrow X_h$, we denote the nodal interpolation operator uniquely defined by $(\mathcal{I}_h \psi)(ih) := \psi(ih)$ for all $i \in \{1, \dots, L_h\}$, where $L_h := Lh^{-1}$ is the dimension of X_h .
- On the Hilbert space X_h , we introduce the scalar product

$$(\phi^h, \psi^h)_h := \sum_{i=1}^{L_h} h \phi^h(ih) \psi^h(ih)$$

and the corresponding norm

$$\|\psi^h\|_h := \left(\sum_{i=1}^{L_h} h |\phi^h(ih)|^2 \right)^{1/2}.$$

Note that the norm $\|\cdot\|_h$ is equivalent to the $L^2(\mathcal{O})$ -norm on X_h , uniformly in h . With a slight misuse of notation, we will frequently abbreviate $(\mathcal{I}_h\phi, \psi_h)_h$ for functions $\phi \in C_{per}([0, L])$ and $\psi_h \in X_h$ by $(\phi, \psi_h)_h$.

- By ∂_x^{+h} and ∂_x^{-h} , we denote the forward and backward difference quotients, respectively, i.e., $\partial_x^{+h}f(x) := h^{-1}(f(x+h) - f(x))$ (with f extended outside of $[0, L]$ by periodicity).
- The discrete Laplacian $\Delta_h v^h$ of a function $v^h \in X_h$ is defined by the variational formulation

$$(\Delta_h v^h, \psi^h)_h := - \int_{\mathcal{O}} v_x^h \cdot \psi_x^h dx \quad \forall \psi^h \in X_h.$$

We note the identity $\Delta_h v^h = \partial_x^{+h}(\partial_x^{-h}u)$.

- Sometimes, we abbreviate $v_i := v(ih)$ for functions $v \in C^0(\mathcal{O})$ and $i = 1, \dots, L_h$.

Now we are in position to formulate the general assumptions on the data.

- (H1) The mobility is given by $m(u) = u^2$.
- (H2) The effective interface potential $\mathcal{W}(u)$ has continuous second-order derivatives on \mathbb{R}^+ and satisfies for some $p > 2$ and $u > 0$ the following estimates with appropriate positive constants:

$$c_1 u^{-p-2} - c_2 \leq \mathcal{W}''(u) \leq C_1 u^{-p-2},$$

$$\mathcal{W}(u) \geq C u^{-p}.$$

For nonpositive u , we define $\mathcal{W}(u) := +\infty$.

- (H3) Let Λ be a probability measure on $H_{per}^1(\mathcal{O})$ equipped with the Borel σ -algebra which is supported on the subset of strictly positive functions such that there is a positive constant C with the property that

$$\text{esssup}_{v \in \text{supp } \Lambda} \left\{ E[\mathcal{I}_h v] + \left(\int_{\mathcal{O}} v dx \right) + \left(\int_{\mathcal{O}} v dx \right)^{-1} \right\} \leq C$$

for any $h > 0$ with $E[\cdot]$ as defined in (1.7).

- (H4) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration such that
 - W is a Q -Wiener process on Ω adapted to $(\mathcal{F}_t)_{t \geq 0}$ which admits a decomposition of the form $W = \sum_{\ell=1}^{\infty} \lambda_{\ell} g_{\ell} \beta_{\ell}$ for a sequence of independent standard Brownian motions β_{ℓ} and nonnegative real numbers $(\lambda_{\ell})_{\ell \in \mathbb{N}}$,
 - the noise W is colored in the sense that $\sum_{\ell=1}^{\infty} \ell^4 \lambda_{\ell}^2 < \infty$,
 - there exists a \mathcal{F}_0 -measurable random variable u_0 such that $\Lambda = \mathbb{P} \circ u_0^{-1}$.

We refer to Remark 3.3 for a discussion of why we do not expect existence results under the assumption of space-time white noise. Finally, with respect to (H3) it is worth mentioning that we do not know of any stochastic thin-film-type equation which would allow for (at least formal) integral estimates compatible with just nonnegative initial data.

Let us define our scheme for approximation. On a stochastic basis satisfying (H4), given a positive time T_{max} and introducing $E_{max,h} := \frac{1}{2}h^{-(p-2)/(p+2)}$, we consider solutions

$$\begin{aligned} u^h &\in L^2(\Omega; C([0, T_{max}]; X_h)), \\ p^h &\in L^2(\Omega; L^\infty((0, T_{max}); X_h)) \end{aligned}$$

to the system of SDEs

$$\begin{aligned} (2.1a) \quad (u^h(T), \phi^h)_h &= (u_0, \phi^h)_h \\ &\quad - \int_0^{T \wedge T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h \phi_x^h \, dx \, dt \\ &\quad - \sum_{\ell=1}^{N_h} \int_0^{T \wedge T_h} \int_{\mathcal{O}} \lambda_\ell u^h g_\ell \phi_x^h \, dx \, d\beta_\ell \quad \forall \phi^h \in X_h, \end{aligned}$$

$$(2.1b) \quad (p^h, \phi^h)_h = \chi_{T_h} \int_{\mathcal{O}} u_x^h \phi_x^h \, dx + \chi_{T_h} (\mathcal{W}'(u^h), \phi^h)_h \quad \forall \phi^h \in X_h,$$

where $\chi_{T_h} = \chi_{T_h}(t)$ is an abbreviation for the characteristic function of the time interval $[0, T_h]$ and where T_h is the stopping time defined by $T_h := T_{max} \wedge \inf\{t \geq 0 : E_h[u^h(t)] \geq E_{max,h}\}$. Here, $N_h \in \mathbb{N}$ is a cutoff for the noise for the purpose of discretization, subject only to the condition $N_h \rightarrow \infty$ for $h \rightarrow 0$. Furthermore, $M_h(u^h)$ is a suitable modification of the pointwise mobility $m(u^h)$; see below.

Discrete initial data are computed by the formula $u_0^h(\omega) := \mathcal{I}_h u_0(\omega)$. We note the following result, which can be established in a standard way.

LEMMA 2.1. *Let (u^h, p^h) be a solution of (2.1). Then,*

$$(2.2) \quad (u^h(t, \omega))_{\mathcal{O}} = \frac{1}{L} (u^h(t, \omega), 1)_h = \frac{1}{L} (u_0^h(\omega), 1)_h = (u_0^h(\omega))_{\mathcal{O}} \quad \forall t \geq 0,$$

$$(2.3) \quad |(u_0^h)_h(\omega) - (u_0(\omega))_{\mathcal{O}}| \leq Ch \left(\int_{\mathcal{O}} |(u_0)_x(\omega)|^2 \right)^{1/2},$$

where $(v)_{\mathcal{O}}$ denotes the mean value of a function v over \mathcal{O} .

The discrete mobility $M_h(u^h)$ is defined as follows. Choose $\sigma := \frac{1}{2}h^{2/(p+2)}$ and consider the shifted mobility $m_\sigma(u) := m(\max(\sigma, u))$. Then, for an element $v^h \in X_h$, the discrete mobility $M_h(v^h)$ is given as the elementwise constant function defined by

$$(2.4) \quad M_h(v^h)|_{(ih, (i+1)h]} := \begin{cases} m_\sigma(v_i^h) & \text{if } v_i^h = v_{i+1}^h, \\ \left(\int_{v_i^h}^{v_{i+1}^h} \frac{1}{m_\sigma(s)} \, ds \right)^{-1} & \text{if } v_i^h \neq v_{i+1}^h. \end{cases}$$

Related to the discrete mobility $M_h(\cdot)$, we introduce the nonnegative discrete entropy density

$$(2.5) \quad G_h(s) := \int_1^s \int_1^\mu \frac{1}{m_\sigma(\tau)} \, d\tau \, d\mu$$

and similarly $g_h(s) := G_h'(s)$. For further reference, we note the identity

$$(2.6) \quad \int_{\mathcal{O}} M_h(u^h) p_x^h \partial_x (\mathcal{I}_h g_h(u^h)) \, dx = \int_{\mathcal{O}} p_x^h u_x^h \, dx,$$

which is commonly referred to as *entropy consistency of the discrete mobility*; cf. [38] and [52]. Moreover, observe that

$$(2.7) \quad M_h(u^h)|_{(ih, (i+1)h]} = u_i^h \cdot u_{i+1}^h$$

if $\min(u_i^h, u_{i+1}^h) \geq \sigma = \frac{1}{2}h^{\frac{2}{p+2}}$.

3. Main results.

DEFINITION 3.1. Let Λ be a probability measure on $H^1_{per}(\mathcal{O})$. We will call a triple $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), \tilde{u}, \tilde{W})$ a weak martingale solution to the stochastic thin-film equation (1.5) with initial data Λ on the time interval $[0, T]$ provided

- (i) $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ is a stochastic basis with a complete, right-continuous filtration,
- (ii) \tilde{W} satisfies (H4) with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$,
- (iii) $\tilde{u} \in L^2(\tilde{\Omega}; L^2([0, T]; H^3_{per}(\mathcal{O}))) \cap L^2(\tilde{\Omega}; C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T]))$ with $\gamma \in (0, 1/2)$ is positive $\tilde{\mathbb{P}}$ -almost surely,
- (iv) there exists an $\tilde{\mathcal{F}}_0$ -measurable $H^1_{per}(\mathcal{O})$ -valued random variable \tilde{u}_0 such that $\Lambda = \tilde{\mathbb{P}} \circ \tilde{u}_0^{-1}$, and the equation

$$(3.1) \quad \int_{\mathcal{O}} \tilde{u}(t) \phi \, dx = \int_{\mathcal{O}} \tilde{u}_0 \phi \, dx + \int_0^t \int_{\mathcal{O}} m(\tilde{u})(\tilde{u}_{xx} - \mathcal{W}'(\tilde{u}))_x \phi_x \, ds - \sum_{\ell=1}^{\infty} \lambda_{\ell} \int_0^t \int_{\mathcal{O}} \sqrt{m(\tilde{u})} g_{\ell} \phi_x \, dx \, d\beta_{\ell}$$

holds $\tilde{\mathbb{P}}$ -almost surely for all $t \in [0, T]$ and all $\phi \in H^1_{per}(\mathcal{O})$.

We are going to establish the existence of a weak martingale solution via approximation by solutions to the semidiscrete scheme (2.1).

THEOREM 3.2. Let the assumptions (H1)–(H4) be satisfied and let $T_{max} > 0$ be given. Assume u^h, p^h (where $h \rightarrow 0$) to be a sequence of solutions to the Faedo-Galerkin scheme (2.1) for the stochastic thin-film equation (1.5) with $E_{max,h} = \frac{1}{2} h^{-(p-2)/(p+2)}$. Let $0 < \gamma < 1/2$ be given.

Then there exist a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ as well as processes \tilde{u}^h, \tilde{p}^h , and $\tilde{u} \in L^2(\tilde{\Omega}; L^2([0, T]; H^1_{per}(\mathcal{O})))$ such that the following holds: The processes \tilde{u}^h, \tilde{p}^h have the same law as the processes u^h, p^h and for a subsequence we $\tilde{\mathbb{P}}$ -almost surely have the convergence $\tilde{u}^h \rightarrow \tilde{u}$ strongly in $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])$ and $\sqrt{M_h(\tilde{u}^h)} \tilde{p}^h_x \rightharpoonup -\tilde{u}((\tilde{u}_{xx} - \mathcal{W}'(\tilde{u}))_x)$ weakly in $L^2(\mathcal{O} \times [0, T_{max}])$. Furthermore, \tilde{u} is a weak martingale solution to the stochastic thin-film equation in the sense of Definition 3.1 satisfying the additional bound

$$\mathbb{E} \left[\sup_{t \in [0, T_{max}]} E[\tilde{u}]^{\bar{p}} \right] + \mathbb{E} \left[\int_0^T \int_{\mathcal{O}} m(\tilde{u}) |(\tilde{u}_{xx} - \mathcal{W}'(\tilde{u}))_x|^2 \, dx \, dt \right] \leq C(u_0, \bar{p}, T_{max})$$

for any $\bar{p} \geq 1$. In particular, \tilde{u} is positive $\tilde{\mathbb{P}}$ -almost surely.

Remark 3.3. Note that the regularization of the noise (see hypothesis (H4)) is actually necessary: Due to the strong nonlinearity in the leading-order term in the thin-film equation (1.1) with $m(u) = u^n$, even for $\mathcal{W} = 0$ any weak formulation of the thin-film equation involves either terms like the Dirichlet energy density $|\nabla u|^2$ or even higher derivatives. Therefore, without any control of the Dirichlet energy of solutions, establishing a weak formulation of the thin-film equation seems to be impossible.

However, in the case of space-time white noise dS , even for linearized model equations, the finiteness of the Dirichlet energy fails: Considering the one-dimensional model equation

$$(3.2) \quad du = -\Delta^2 u dt + \nabla \cdot dS$$

with white noise dS on $(0, 2\pi)$ subjected to periodic boundary conditions, we may reformulate it as

$$du = -u_{xxxx}dt + \sum_{\ell=1}^{\infty} \left(\frac{1}{\sqrt{\pi}} \sin(\ell x) \right)_x d\beta_{2\ell} + \sum_{\ell=0}^{\infty} \left(\frac{1}{\sqrt{\pi}} \cos(\ell x) \right)_x d\beta_{2\ell+1}$$

for a sequence of independent standard Brownian motions $(\beta_\ell)_{\ell \in \mathbb{N}}$. For the “excitation” of the mode $\sin(\ell x)/\sqrt{\pi}$, we infer

$$\begin{aligned} d \int_{(0,2\pi)} \frac{1}{\sqrt{\pi}} \sin(\ell x) u(x, t) dx \\ = -\ell^4 \int_{(0,2\pi)} \frac{1}{\sqrt{\pi}} \sin(\ell x) u(x, t) dx dt - \ell \int_{(0,2\pi)} \frac{1}{\pi} \sin^2(\ell x) dx d\beta^{2\ell+1}. \end{aligned}$$

Solving this SDE and computing the variance shows that at time $T := 1$, the probability distribution of the excitation of the mode $\sin(\ell x)/\sqrt{\pi}$ is a Gaussian with vanishing expectation and standard deviation

$$\sigma \left(\int_{(0,2\pi)} \frac{1}{\sqrt{\pi}} \sin(\ell x) u(x, t) dx \right) \sim \frac{1}{\ell}.$$

As the excitations of the different modes are stochastically independent, even at the level of the linear fourth-order model equation (3.2) forced by the spatial derivative of space-time white noise, one may not expect finiteness of the Dirichlet energy.

4. A priori estimates.

4.1. Discretization of the energy and entropy functionals. To establish a discrete counterpart of the combined energy-entropy estimate, we introduce discretization-adapted variants of the energy $E[u]$ and of the entropy $S[u]$. We set

$$(4.1) \quad E_h[v] := \int_{\mathcal{O}} \frac{1}{2} |v_x|^2 + \mathcal{I}_h[\mathcal{W}(v)] dx$$

and

$$(4.2) \quad S_h[v] := \int_{\mathcal{O}} \mathcal{I}_h[G_h(v)] dx.$$

A bound on the energy $E_h[v]$ entails a lower bound on the thickness of the thin film as well as on its oscillation.

LEMMA 4.1. *Let $u^h \in X_h$ be strictly positive. We then have the estimate*

$$(4.3) \quad \sup_{x \in \mathcal{O}} (u^h)^{-1} \leq C \left(\int_{\mathcal{O}} u^h dx \right)^{-1} + C E_h[u^h]^{2/(p-2)}.$$

If in addition the bounds

$$(4.4) \quad E_h[u^h] \leq h^{-\frac{p-2}{p+2}}$$

and

$$(4.5) \quad \int_{\mathcal{O}} u^h dx \geq h^{\frac{2}{2+p}}$$

hold, we have the estimates

$$(4.6) \quad \min_{x \in \mathcal{O}} u^h \geq ch^{\frac{2}{2+p}}$$

and

$$(4.7) \quad \left| \frac{u^h(ih)}{u^h((i+1)h)} \right| \leq C$$

for all $i = 1, \dots, L_h$.

Proof. To establish (4.3), we first estimate

$$(4.8) \quad \begin{aligned} \int_{\mathcal{O}} |(u^h)^{-p/2+1}|_x dx &= C(p) \int_{\mathcal{O}} (u^h)^{-p/2} |u_x^h| dx \leq C(p) \left(\int_{\mathcal{O}} (u^h)^{-p} dx \right)^{1/2} \left(\int_{\mathcal{O}} |u_x^h|^2 dx \right)^{1/2} \\ &\leq 2C(p) \left(h \sum_{i=1}^{L_h} (u^h(ih))^{-p} \right)^{1/2} E_h[u^h]^{1/2} \leq CE_h[u^h]. \end{aligned}$$

Note that we use hypothesis (H2) in the last step, while the second inequality is built upon

$$\int_{\mathcal{O}} (u^h)^{-p} dx \leq 2h \sum_{i=1}^{L_h} (u^h(ih))^{-p},$$

which in turn follows from the estimate

$$\begin{aligned} \int_{[ih, (i+1)h]} (u^h)^{-p} dx &= \int_{[ih, (i+1)h]} \left(u^h(ih) \frac{(i+1)h-x}{h} + u^h((i+1)h) \frac{x-ih}{h} \right)^{-p} dx \\ &\leq h \cdot (\min\{u^h(ih), u^h((i+1)h)\})^{-p} \\ &\leq h \cdot ((u^h(ih))^{-p} + (u^h((i+1)h))^{-p}). \end{aligned}$$

Using (4.8), we infer

$$\begin{aligned} \sup_{x \in \mathcal{O}} (u^h)^{-1} &= \left(\sup_{x \in \mathcal{O}} (u^h)^{-(p-2)/2} \right)^{2/(p-2)} \\ &\leq \left(\inf_{x \in \mathcal{O}} (u^h)^{-(p-2)/2} + \int_{\mathcal{O}} |(u^h)^{-(p-2)/2}|_x dx \right)^{2/(p-2)} \\ &\leq C \inf_{x \in \mathcal{O}} (u^h)^{-1} + CE_h[u^h]^{2/(p-2)} \\ &\leq C \left(\int_{\mathcal{O}} u^h dx \right)^{-1} + CE_h[u^h]^{2/(p-2)}. \end{aligned}$$

Having established (4.3), (4.6) is an easy consequence of the assumption (4.4). To see (4.7), we use the embedding

$$\sup_{x, y \in \mathcal{O}} \frac{|u^h(x) - u^h(y)|}{|x - y|^{1/2}} \leq C \|u_x^h\|_{L^2(\mathcal{O})} \leq C(E_h[u^h])^{1/2}$$

to compute

$$\left| \frac{u^h(ih)}{u^h((i+1)h)} - 1 \right| = \frac{|u^h(ih) - u^h((i+1)h)|}{u^h((i+1)h)} \leq \frac{Ch^{1/2} \sqrt{E_h[u^h]}}{ch^{\frac{2}{2+p}}} \leq C. \quad \square$$

4.2. Existence of solutions for the semidiscrete scheme. Let us now show that our Faedo–Galerkin scheme (2.1) admits a solution.

LEMMA 4.2. *Let T_{max} be a positive real number and $E_{max,h} = \frac{1}{2}h^{-(p-2)/(p+2)}$. Then there exist a stopping time T_h and stochastic processes $u^h \in L^2(\Omega; C([0, T_{max}]; X_h))$, $p^h \in L^2(\Omega; L^\infty((0, T_{max}); X_h))$ with the following properties:*

- *Almost surely, we have $T_h = T_{max} \wedge \inf\{t \in [0, \infty) : E_h[u^h(\cdot, t)] \geq E_{max,h}\}$.*
- *Almost surely, the process p^h solves (2.1b) for $t \leq T_{max}$ and is contained in $C([0, T_h]; X_h)$.*
- *Almost surely, the process u^h solves (2.1a) for $t \leq T_h$ and is constant for $t \in [T_h, T_{max}]$ (and thus solves (2.1a) for $t \leq T_{max}$).*

Proof. The proof proceeds by reducing the assertion to a standard existence result for SDEs. For given $\hat{u}^h(t) \in X_h$, consider the associated $\hat{p}^h(t)$ defined through

$$(4.9) \quad (\hat{p}^h, \phi^h)_h = \int_{\mathcal{O}} \hat{u}_x^h \phi_x^h dx + (\mathcal{W}'(\hat{u}^h), \phi^h)_h \quad \forall \phi^h \in X_h.$$

First notice that $\hat{p}^h(t)$ (as an element of the finite-dimensional vector space X_h) is uniquely determined by $\hat{u}^h(t)$. Moreover, it is a Lipschitz continuous function of $\hat{u}^h(t) \in X_h$ as long as we have $E_h[\hat{u}^h(t)] \leq 3E_{max,h}$ (as the latter condition implies a positive lower bound for $\hat{u}^h(t)$ and therefore differentiability of \mathcal{W}' at $\hat{u}^h(x, t)$). Denote by $\mathcal{P}[\hat{u}^h(t)]$ the function $\hat{p}^h(t)$ associated to $\hat{u}^h(t)$ via (4.9).

Let $\theta : \mathbb{R} \rightarrow [0, 1]$ be a cutoff with $\theta(s) \equiv 1$ for $s \leq E_{max,h}$ and $\theta(s) \equiv 0$ for $s \geq 2E_{max,h}$. Note that $E_h[\hat{u}^h(t)]$ is a Lipschitz continuous function of $\hat{u}^h(t) \in X_h$ as long as we have $E_h[\hat{u}^h(t)] \leq 2E_{max,h}$.

We then solve the SDE

$$(4.10) \quad \begin{aligned} du^h(t, ih) = & -\frac{1}{h}\theta(E_h[u^h(t)]) \int_{\mathcal{O}} M_h(u^h(t, x)) \mathcal{P}[u^h(t)]_x (e_i^h)_x dx dt \\ & -\frac{1}{h}\theta(E_h[u^h(t)]) \sum_{\ell=1}^{N_h} \lambda_\ell \int_{\mathcal{O}} u^h(t, x) g_\ell(e_i^h)_x dx d\beta_\ell \end{aligned}$$

with initial data $u^h(0, ih) = \mathcal{I}_h u_0(ih)$, $i = 1, \dots, L_h$. This is possible as for $E_h[u^h(t)] \leq 2E_{max,h}$ (which implies a positive lower bound on $u^h(t)$) the coefficients of the SDE depend in a Lipschitz-continuous way on $u^h(t)$ and for $E_h[u^h(t)] \geq 2E_{max,h}$ the coefficients are zero.

Now, define T_h as $T_h := T_{max} \wedge \inf\{t \in [0, T_{max}] : E_h[u^h(\cdot, t)] \geq E_{max,h}\}$ and modify $u^h(t)$ to be constant for $t \in [T_h, T_{max}]$. Finally, we define $p^h(t)$ by (2.1b). We then see that for this choice, the assertions of the lemma are satisfied. \square

4.3. The energy and entropy estimates. We now demonstrate that our spatial semidiscretization preserves the combined energy-entropy estimate as long as the energy remains below a critical threshold. As before, we choose $E_{max,h} = \frac{1}{2}h^{-(p-2)/(p+2)}$ to be the threshold energy. In particular, it becomes infinite in the limit $h \rightarrow 0$.

Writing $u^h(x, t) = \sum_{i=1}^{L_h} a_i(t) e_i(x)$, we first note that (2.1a) may be rewritten as

$$(4.11) \quad da_i = \frac{1}{h} L_i(s) ds + \sum_{\ell=1}^{N_h} Z_i(\lambda_\ell g_\ell) d\beta_\ell,$$

where we have introduced

$$(4.12) \quad L_i(s) := -\chi_{T_h}(s) \int_{\mathcal{O}} M_h(u^h(s)) p_x^h(s) (e_i^h)_x dx$$

and $Z_i : L^2(\mathcal{O}) \rightarrow \mathbb{R}$ defined by

$$(4.13) \quad Z_i(w) := \chi_{T_h} \frac{1}{h} \sum_{\ell=1}^{\infty} \int_{\mathcal{O}} ((g_{\ell}, w)_{L^2(\mathcal{O})} u^h g_{\ell})_x e_i^h dx.$$

For given positive parameters α and κ , we consider the integral quantity

$$(4.14) \quad R(\alpha, \kappa, h, s) := \alpha + E_h[u^h(s)] + \kappa S_h[u^h(s)].$$

For ease of presentation, we will often abbreviate $R(s) := R(\alpha, \kappa, h, s)$.

The following result of calculus is immediate.

LEMMA 4.3. *Let $\bar{p} \geq 1$ be given. Let $\mathcal{R}(s) := R(\alpha, \kappa, h, s)^{\bar{p}}$. The first and second variations of $\mathcal{R}(s)$ are given by*

$$(4.15) \quad D\mathcal{R}(s) = \bar{p}R(s)^{\bar{p}-1}(DE_h(s) + \kappa DS_h(s))$$

and

$$(4.16) \quad \begin{aligned} D^2\mathcal{R}(s) &= \bar{p}R(s)^{\bar{p}-1}(D^2E_h(s) + \kappa D^2S_h(s)) \\ &+ \bar{p}(\bar{p}-1)R(s)^{\bar{p}-2}(DE_h(s) + \kappa DS_h(s)) \otimes (DE_h(s) + \kappa DS_h(s)), \end{aligned}$$

respectively, with

$$(4.17) \quad DE_h(h)\phi^h = \int_{\mathcal{O}} u_x^h \phi_x^h dx + (\mathcal{W}'(u^h), \phi^h)_h,$$

$$(4.18) \quad (D^2E_h(s)\phi^h)\psi^h = \int_{\mathcal{O}} \phi_x^h \psi_x^h dx + (\mathcal{W}''(u^h)\phi^h, \psi^h)_h,$$

$$(4.19) \quad DS_h(s)\phi^h = (g_h(u^h), \phi^h)_h,$$

where g_h is the derivative of G_h , and

$$(4.20) \quad (D^2S_h(s)\phi^h, \psi^h) = \left(\frac{1}{m_{\sigma}(u^h)} \phi^h, \psi^h \right)_h.$$

Using Ito's formula, we derive the following integral estimates.

PROPOSITION 4.4. *Let $\bar{p} \geq 1$ be arbitrary and let u^h, p^h be a solution to (2.1a) and (2.1b) for a parameter $0 < h < 1$. Then, for sufficiently large α and κ depending only on $(\lambda_{\ell})_{\ell}$ and on \bar{p} , there exist positive constants C_1 and C_2 independent of h*

and independent of initial data such that for all $t \in [0, T_{max}]$ the following inequality holds:

$$\begin{aligned}
 & \mathbb{E}[R(t \wedge T_h)^{\bar{p}}] \\
 & + C_1 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} M_h(u^h(s)) |p_x^h(s)|^2 dx ds \right] \\
 & + C_1 \kappa \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u^h\|_h^2 ds \right] \\
 & + C_1 \kappa \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{u_{i-1}^h}^{u_i^h} |\tau|^{-p-2} d\tau \int_{(i-1)h}^{ih} |u_x^h(s)|^2 dx ds \right] \\
 & \leq \mathbb{E}[R(0)^{\bar{p}}] + \bar{\gamma}(\kappa, \alpha, \lambda, \bar{p}) \left(\mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \right] + t \mathbb{E} \left[(u_0)_{\mathcal{O}}^{2\bar{p}} \right] \right) \\
 (4.21) \quad & \leq \mathbb{E}[R(0)^{\bar{p}} + \bar{\gamma} t (u_0)_{\mathcal{O}}^{2\bar{p}}] \exp(\bar{\gamma} t)
 \end{aligned}$$

with

$$\bar{\gamma} := C_2 \left(\frac{\kappa^2 \bar{p}(\bar{p} - 1)}{\alpha} \sum_{\ell=1}^{\infty} \lambda_{\ell}^2 + (\bar{p} + \bar{p}\kappa + \kappa^2 \bar{p}(\bar{p} - 1)) \sum_{\ell=1}^{\infty} \ell^2 \lambda_{\ell}^2 + \bar{p} \sum_{\ell=1}^{\infty} \ell^4 \lambda_{\ell}^4 \right).$$

Moreover, for $T_{max} > 0$ arbitrary but fixed and sufficiently large α and κ , there exists a positive constant \bar{C} depending only on \bar{p} , $(\lambda_{\ell})_{\ell \in \mathbb{N}}$, T_{max} , and initial data such that

$$(4.22) \quad \mathbb{E} \left[\sup_{t \in [0, T_{max}]} R(t \wedge T_h)^{\bar{p}} \right] \leq \bar{C}.$$

Proof. Using the notation

$$(4.23) \quad \varphi(h, s) := \frac{1}{h} \sum_{i=1}^{L_h} L_i(s) e_i$$

and

$$(4.24) \quad \Phi(h, s)(w) := \sum_{i=1}^{L_h} Z_i(w) e_i,$$

we may rewrite (2.1) as

$$(4.25) \quad du^h = \varphi(h, s) ds + \Phi(h, s)(dW_Q^h)$$

with

$$(4.26) \quad W_Q^h := \sum_{\ell=1}^{N_h} \lambda_{\ell} g_{\ell} \beta_{\ell}.$$

By Ito's formula, we deduce

$$\begin{aligned}
 R(t \wedge T_h)^{\bar{p}} &= R(0)^{\bar{p}} + \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1}(DE_h + \kappa DS_h)\varphi(h, s) ds \\
 &\quad + \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1}(DE_h + \kappa DS_h)\Phi(h, s) dW_Q^h \\
 &\quad + \frac{1}{2} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} \bar{p}R(s)^{\bar{p}-1}(D^2E_h + \kappa D^2S_h)(\Phi(h, s)(\lambda_\ell g_\ell), \Phi(h, s)(\lambda_\ell g_\ell)) ds \\
 &\quad + \frac{1}{2} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} \bar{p}(\bar{p} - 1)R(s)^{\bar{p}-2}(DE_h + \kappa DS_h) \otimes (DE_h + \kappa DS_h) \\
 &\quad \quad \quad (\Phi(h, s)(\lambda_\ell g_\ell), \Phi(h, s)(\lambda_\ell g_\ell)) ds \\
 &= R(0)^{\bar{p}} + \bar{p}\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \frac{1}{h} \left(g_h(u^h), \sum_{i=1}^{L_h} L_i(s)e_i \right)_h ds \\
 &\quad + \bar{p}\kappa \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(g_h(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)e_i \right)_h d\beta_\ell \\
 &\quad + \bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \frac{1}{h} \left(\mathcal{W}'(u^h), \sum_{i=1}^{L_h} L_i(s)e_i \right)_h ds \\
 &\quad + \bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(\mathcal{W}'(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)e_i \right)_h d\beta_\ell \\
 &\quad + \bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \frac{1}{h} \int_{\mathcal{O}} u_x^h \sum_{i=1}^{L_h} L_i(s)(e_i)_x dx ds \\
 &\quad + \bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} u_x^h \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)(e_i)_x dx d\beta_\ell \\
 &\quad + \frac{\bar{p}\kappa}{2} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(\frac{1}{m_\sigma(u^h)}, \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)e_i \right)^2 \right)_h ds \\
 &\quad + \frac{\bar{p}}{2} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(\mathcal{W}''(u^h), \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)e_i \right)^2 \right)_h ds \\
 &\quad + \frac{\bar{p}}{2} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \left| \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)(e_i)_x \right|^2 dx ds \\
 &\quad + \frac{\bar{p}(\bar{p} - 1)}{2} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \left[\left(\kappa g_h(u^h) + \mathcal{W}'(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)e_i \right)_h \right. \\
 &\quad \quad \quad \left. + \int_{\mathcal{O}} u_x^h \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)(e_i)_x \right) dx \right]^2 ds \\
 (4.27) \quad &=: R^{\bar{p}}(0) + I_1 + \dots + I_{10}.
 \end{aligned}$$

In what follows, we will frequently take advantage of the identities

$$(4.28) \quad \left(w, \sum_{i=1}^{L_h} Z_i(g_\ell) e_i \right)_h = \chi_{T_h} \int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{I}_h[w] dx \quad \forall w \in C_{per}^0(\mathcal{O})$$

and

$$(4.29) \quad \frac{1}{h} \left(w, \sum_{i=1}^{L_h} L_i(s) e_i \right)_h = -\chi_{T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h \mathcal{I}_h[w]_x dx,$$

which can easily be deduced from (4.12), (4.13) together with the fact that the $h^{-1/2} e_i$ form an orthonormal basis with respect to the scalar product $(\cdot, \cdot)_h$.

By (4.29), we infer

$$\frac{1}{h} \left(g_h(u^h), \sum_{i=1}^{L_h} L_i(s) e_i \right)_h = - \int_{\mathcal{O}} M_h(u^h) p_x^h \mathcal{I}_h[g_h(u^h)]_x dx.$$

Noting the identity $-\Delta_h u^h = p^h - \mathcal{I}_h[\mathcal{W}'(u^h)]$ and using entropy consistency of M_h (cf. (2.6)) and (2.1b) gives for $s \leq T_h$

$$\begin{aligned} \frac{1}{h} \left(g_h(u^h), \sum_{i=1}^{L_h} L_i(s) e_i \right)_h &= - \int_{\mathcal{O}} u_x^h p_x^h dx \\ &= - \|p^h - \mathcal{I}_h[\mathcal{W}'(u^h)]\|_h^2 + \|\mathcal{I}_h[\mathcal{W}'(u^h)]\|_h^2 - (\mathcal{I}_h[\mathcal{W}'(u^h)], p^h)_h \\ &= - \|\Delta_h u^h\|_h^2 - (u_x^h, \mathcal{I}_h[\mathcal{W}'(u^h)]_x)_h \\ &= - \|\Delta_h u^h\|_h^2 - \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} \mathcal{W}''(s) ds \int_{ih}^{(i+1)h} |u_x^h|^2 dx, \end{aligned}$$

where in the last step we have exploited that u_x^h is constant on the interval $(ih, (i+1)h)$. Hence, using (H2) and $\int_{\mathcal{O}} |u_x^h(\cdot, s)|^2 dx \leq R(s)$ we infer

$$\begin{aligned} (4.30) \quad I_1 &= -\bar{p}\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u^h\|_h^2 ds \\ &\quad - \bar{p}\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} \mathcal{W}''(s) ds \int_{ih}^{(i+1)h} |u_x^h|^2 dx ds \\ &\leq -\bar{p}\kappa \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \|\Delta_h u^h\|_h^2 ds \\ &\quad - \bar{p}\kappa c_1 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} |s|^{-p-2} ds \int_{ih}^{(i+1)h} |u_x^h|^2 dx ds \\ &\quad + \bar{p}\kappa c_2 \int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \\ &=: -I_{1a} + I_{1b}. \end{aligned}$$

Obviously, $\mathbb{E}[I_{1a}]$ and $\mathbb{E}[I_{1b}]$ have a good sign and may be used for absorption purposes, while $\mathbb{E}[I_{1c}]$ becomes a Gronwall term. In the same spirit, we have for $s \leq T_h$

$$\begin{aligned} &\frac{1}{h} \left(\mathcal{W}'(u^h), \sum_{i=1}^{L_h} L_i(s) e_i \right)_h + \frac{1}{h} \int_{\mathcal{O}} u_x^h \sum_{i=1}^{L_h} L_i(s) (e_i)_x dx \\ &\stackrel{(2.1b)}{=} \frac{1}{h} \left(p^h, \sum_{i=1}^{L_h} L_i e_i \right)_h \stackrel{(4.29)}{=} - \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 dx. \end{aligned}$$

Hence,

$$(4.31) \quad I_3 + I_5 = -\bar{p} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 dx ds.$$

To verify (4.21), let us take the expectation in the terms on the right-hand side of (4.27). Observe that $\mathbb{E}[I_2] = \mathbb{E}[I_4] = \mathbb{E}[I_6] = 0$ due to the martingale property of I_2, I_4, I_6 which follows using Corollary 21.76 in [45] and the definition of the stopping time T_h (see Lemma 4.2) combined with Lemma 4.1.

Ad I_7 : By Lemma 4.5, we find

$$\begin{aligned} & \frac{\bar{p}\kappa}{2} \sum_{\ell=1}^{N_h} \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(\frac{1}{m_\sigma(u^h)}, \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h \right)^2 ds \right] \\ & \leq \bar{p}\kappa \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \right. \\ & \quad \times \sum_{i=1}^{L_h} \frac{1}{m_\sigma(u^h)_i} \left\{ \int_{(i-1)h}^{ih} \left| \frac{u^h(x+h) - u^h(x)}{h} \right|^2 |g_\ell(x)|^2 dx \right. \\ & \quad \left. \left. + \int_{(i-1)h}^{ih} |u^h(x+h)|^2 \left| \frac{g_\ell(x+h) - g_\ell(x)}{h} \right|^2 dx \right\} ds \right] =: (*)_1. \end{aligned}$$

By convexity, we have

$$\begin{aligned} & \int_{(i-1)h}^{ih} \left| \frac{u^h(x+h) - u^h(x)}{h} \right|^2 dx \\ & \leq \frac{h}{2} \left(\left| \frac{u^h(ih) - u^h((i-1)h)}{h} \right|^2 + \left| \frac{u^h((i+1)h) - u^h(ih)}{h} \right|^2 \right). \end{aligned}$$

Using in addition the estimate $\left| \frac{g_\ell(x+h) - g_\ell(x)}{h} \right|^2 \leq C\ell^2$, we obtain

$$\begin{aligned} (*)_1 & \leq \bar{p}\kappa C \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} h \sum_{i=1}^{L_h} \frac{1}{2} \left(\frac{1}{m_\sigma(u^h)_i} + \frac{1}{m_\sigma(u^h)_{i-1}} \right) \left| \frac{u_i^h - u_{i-1}^h}{h} \right|^2 ds \right] \\ & \quad + \bar{p}\kappa C \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} + R(s)^{\bar{p}-1} (u_0)_{\mathcal{O}}^2 ds \right] =: (*)_2. \end{aligned}$$

For $p > 2$, for any $\delta > 0$, there exists $C_\delta > 0$ such that $\frac{1}{m_\sigma(s)} \leq \delta s^{-p} + C_\delta$ for all $s > 0$. Together with (4.7) and $\int_{\mathcal{O}} |u_x^h(\cdot, s)|^2 dx \leq R(s)$, we infer

$$\begin{aligned} (*)_2 & \leq \delta \bar{p}\kappa \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{(u^h)_i}^{(u^h)_{i+1}} |\tau|^{-p-2} d\tau \int_{ih}^{(i+1)h} |u_x^h|^2 dx ds \right] \\ (4.32) \quad & + C_\delta \bar{p}\kappa \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} + R(s)^{\bar{p}-1} (u_0)_{\mathcal{O}}^2 ds \right] =: I_{7a} + I_{7b}. \end{aligned}$$

Note that for δ sufficiently small, I_{7a} can be absorbed in $\mathbb{E}[I_{1a}]$, while I_{7b} will become a Gronwall term due to Lemma 2.1 and (H3).

Ad I_8 : In the same spirit, using in particular inequality (4.7), we have

$$\begin{aligned}
 & \frac{\bar{p}}{2} \sum_{\ell=1}^{N_h} \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(\mathcal{W}''(u^h), \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h \right)^2 ds \right] \\
 & \leq C(\bar{p}) \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} |\tau|^{-p-2} d\tau \int_{ih}^{(i+1)h} |u_x^h|^2 dx ds \right] \\
 (4.33) \quad & + C(\bar{p}) \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \right] =: I_{8a} + I_{8b}.
 \end{aligned}$$

For sufficiently large κ , I_{8a} can be absorbed by $\mathbb{E}[I_{1a}]$, while I_{8b} will become a Gronwall term.

Ad I_9 : Using periodicity and the special form of the stiffness matrix, we obtain

$$\begin{aligned}
 & \frac{\bar{p}}{2} \sum_{\ell=1}^{N_h} \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) (e_i)_x \right)^2 dx ds \right] \\
 & = \frac{\bar{p}}{2h} \sum_{\ell=1}^{N_h} \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} (2Z_i^2 - Z_i Z_{i-1} - Z_i Z_{i+1}) (\lambda_\ell g_\ell) ds \right] \\
 & = \frac{\bar{p}}{2h} \mathbb{E} \left[\sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} (Z_{i+1} - Z_i)^2 (\lambda_\ell g_\ell) ds \right] \\
 & = \frac{\bar{p}}{2h} \mathbb{E} \left[\sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \left(\int_{\mathcal{O}} (u^h \lambda_\ell g_\ell)_x \frac{e_{i+1} - e_i}{h} dx \right)^2 ds \right] \\
 & = \frac{\bar{p}}{2h} \mathbb{E} \left[\sum_{\ell=1}^{N_h} \lambda_\ell^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \left(\int_{\mathcal{O}} \partial_h^- ((u^h g_\ell)_x) e_{i+1} dx \right)^2 ds \right] \\
 & \leq \frac{\bar{p}}{2h} \mathbb{E} \left[\sum_{\ell=1}^{\infty} \lambda_\ell^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} 2h \int_{(i-1)h}^{ih} |\partial_h^- ((u^h g_\ell)_x)|^2 dx ds \right] \\
 & \leq 2\bar{p} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} |\partial_h^- (u_x^h g_\ell)|^2 dx ds \right] \\
 & \quad + 4\bar{p} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} |(g_\ell)_x \partial_h^- u^h|^2 dx ds \right] \\
 & \quad + 4\bar{p} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} |u^h|^2 |\partial_h^+ ((g_\ell)_x)|^2 dx ds \right] \\
 & \leq C\bar{p} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} |\Delta_h u^h|^2 dx ds \right] \\
 & \quad + C\bar{p} \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} |u_x^h|^2 dx ds \right] \\
 (4.34) \quad & + C\bar{p} \sum_{\ell=1}^{\infty} \ell^4 \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} \left\{ R(s)^{\bar{p}} + R(s)^{\bar{p}-1} (u_0)_{\mathcal{O}}^2 \right\} ds \right] =: I_{9a} + I_{9b} + I_{9c}.
 \end{aligned}$$

Note that Lemma 2.1 has been applied in the last step. The term I_{9a} can be absorbed in $\mathbb{E}[I_{1a}]$ provided κ is sufficiently large. The remaining two terms are Gronwall terms.

Ad I_{10} : By (4.28), we have

$$(4.35) \quad \left(g_h(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h = \lambda_\ell \int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{I}_h[g_h(u^h)] dx.$$

Combining (4.28) and (2.1b), we identify

$$(4.36) \quad \begin{aligned} & \left(\mathcal{W}'(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h + \int_{\mathcal{O}} u_x^h \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)(e_i)_x dx \\ &= -\lambda_\ell \left(p^h, \sum_{i=1}^{L_h} Z_i(g_\ell) e_i \right)_h = -\lambda_\ell \int_{\mathcal{O}} g_\ell u^h p_x^h dx. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \frac{\bar{p}(\bar{p}-1)}{2} \mathbb{E} \left[\sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \left[\left(\kappa g_h(u^h) + \mathcal{W}'(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h \right. \right. \\ & \qquad \qquad \qquad \left. \left. + \int_{\mathcal{O}} u_x^h \left(\sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell)(e_i)_x \right) dx \right]^2 ds \right] \\ & \leq \bar{p}(\bar{p}-1) \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\kappa^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \left(\int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{I}_h(g_h(u^h)) dx \right)^2 ds \right. \\ & \qquad \qquad \left. + \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \left(\int_{\mathcal{O}} g_\ell u^h p_x^h dx \right)^2 ds \right] \\ & \leq \bar{p}(\bar{p}-1) C(\mathcal{O}) \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\kappa^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \int_{\mathcal{O}} |u_x^h|^2 |\mathcal{I}_h[g_h(u^h)]|^2 |g_\ell|^2 dx ds \right. \\ & \qquad \qquad \left. + \kappa^2 \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \int_{\mathcal{O}} |u^h|^2 |\mathcal{I}_h[g_h(u^h)]|^2 |(g_\ell)_x|^2 dx ds \right. \\ & \qquad \qquad \left. + \int_0^{t \wedge T_h} R(s)^{\bar{p}-2} \int_{\mathcal{O}} |g_\ell|^2 M_h(u^h) |p_x^h|^2 dx ds \right] =: (*)_3. \end{aligned}$$

By Lemmas 4.6 and 2.1 as well as $R(s) \geq \alpha$, we have

$$(4.37) \quad \begin{aligned} (*)_3 & \leq \frac{\kappa^2 \bar{p}(\bar{p}-1)}{\alpha} C(\mathcal{O}) \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \left\{ \mathbb{E} \left[\int_0^{t \wedge T_h} \left\{ R(s)^{\bar{p}} + R(s)^{\bar{p}-1} (u_0)_{\mathcal{O}}^2 \right\} ds \right] \right. \\ & \qquad \left. + \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} |\tau|^{-p-2} d\tau \int_{i_h}^{(i+1)h} |u_x^h|^2 dx ds \right] \right\} \\ & \quad + \frac{\bar{p}(\bar{p}-1)}{\alpha} C(\mathcal{O}) \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 dx ds \right] \\ & =: I_{10a} + I_{10b} + I_{10c}. \end{aligned}$$

For $\alpha > 1$ sufficiently large, I_{10b} is readily absorbed in $\mathbb{E}[I_{1a}]$, and I_{10c} is absorbed in $\mathbb{E}[I_3 + I_5]$. The remaining term is a Gronwall term.

Taking the expectation in (4.27), moving all the terms with a negative sign to the left-hand side, choosing α and κ sufficiently large for the sake of absorption, and estimating all the remaining terms as suggested by the estimates (4.32), (4.33), (4.34), and (4.37), we infer the first part of (4.21), i.e., with the right-hand side given by

$$\mathbb{E}[R(0)^{\bar{p}}] + \bar{\gamma}(\kappa, \alpha, \lambda, \bar{p}) \left(\mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \right] + t \mathbb{E} \left[(u_0)_{\mathcal{O}}^{2\bar{p}} \right] \right).$$

Combining the estimate

$$\int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \leq \int_0^t R(s \wedge T_h)^{\bar{p}} ds$$

with Gronwall’s lemma and the first part of (4.21), we get

$$\mathbb{E}[R(t \wedge T_h)^{\bar{p}}] \leq \mathbb{E}[R(0)^{\bar{p}} + t\bar{\gamma} (u_0)_{\mathcal{O}}^{2\bar{p}}] \exp(\bar{\gamma}t)$$

for all $t \in [0, T_{max}]$. Hence, the left-hand side of (4.21) is bounded by

$$\begin{aligned} & \mathbb{E}[R(0)^{\bar{p}}] + \bar{\gamma}(\kappa, \alpha, \lambda, \bar{p}) \left(\mathbb{E} \left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} ds \right] + t\bar{\gamma} \mathbb{E} \left[(u_0)_{\mathcal{O}}^{2\bar{p}} \right] \right) \\ & \leq \mathbb{E} \left[R(0)^{\bar{p}} + t\bar{\gamma} (u_0)_{\mathcal{O}}^{2\bar{p}} \right] \exp(\bar{\gamma}t), \end{aligned}$$

which gives the second part of (4.21), too.

To establish (4.22), by (4.27) we may estimate

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, T_{max} \wedge T_h]} R(s)^{\bar{p}} \right] \\ & \leq \mathbb{E} \left[R(0)^{\bar{p}} \right] + \mathbb{E} \left[\sup_{s \in [0, T_{max} \wedge T_h]} (I_1 + I_3 + I_5 + I_7 + I_8 + I_9 + I_{10}) \right] \\ & \quad + \mathbb{E} \left[\sup_{s \in [0, T_{max} \wedge T_h]} (I_2 + I_4 + I_6) \right]. \end{aligned}$$

To establish (4.22), we only have to estimate the expected values of the suprema with respect to time of the absolute values of the stochastic integrals, i.e., the terms I_2 , I_4 , and I_6 , as we have already established the desired estimates on the remaining terms above. We note that $\mathbb{E}[\sup_{s \in [0, T_{max} \wedge T_h]} R(s)^{\bar{p}}]$ is finite due to the cut-off mechanism applied.

We begin with I_4 and I_6 . Using (4.36), we get

$$\begin{aligned} (4.38) \quad I_4 + I_6 &= \bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(\mathcal{W}'(u^h), \sum_{i=1}^{N_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h d\beta_\ell \\ & \quad + \bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} u_x^h \sum_{i=1}^{N_h} Z_i(\lambda_\ell g_\ell) (e_i)_x dx d\beta_\ell \\ &= -\bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \lambda_\ell \int_{\mathcal{O}} u^h p_x^h g_\ell dx d\beta_\ell. \end{aligned}$$

Consider the Hilbert–Schmidt operator $\mathcal{T}_1(s) : Q^{1/2}L^2(\mathcal{O}) \rightarrow \mathbb{R}$ for $s \in [0, T_{max}]$ defined by

$$\mathcal{T}_1(s)(w) := \chi_{T_h}(s)R(s)^{\bar{p}-1} \int_{\mathcal{O}} u^h p_x^h \mathcal{P}_{N_h}[w] dx,$$

where $\mathcal{P}_{N_h} : L^2(\mathcal{O}) \rightarrow \text{span}\{g_1, \dots, g_{N_h}\}$ is the orthogonal L^2 -projection. Using (4.7) gives

$$\begin{aligned} \left(\sum_{\ell=1}^{\infty} |\mathcal{T}_1(s)(Q^{1/2}g_{\ell})|^2 \right)^{1/2} &= \left(\sum_{\ell=1}^{N_h} |\mathcal{T}_1(s)(Q^{1/2}g_{\ell})|^2 \right)^{1/2} \\ &= \chi_{T_h}(s)R(s)^{\bar{p}-1} \left(\sum_{\ell=1}^{N_h} \lambda_{\ell}^2 \left| \int_{\mathcal{O}} (u^h p_x^h)(s)g_{\ell} dx \right|^2 \right)^{1/2} \\ (4.39) \quad &\leq C\chi_{T_h}(s)R(s)^{\bar{p}-1} \left(\sum_{\ell=1}^{N_h} \lambda_{\ell}^2 \int_{\mathcal{O}} M_h(u^h)|p_x^h|^2 dx \right)^{1/2}. \end{aligned}$$

By the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned} &\bar{p}\mathbb{E} \left[\sup_{t \in [0, T_{max}]} \left| \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \sum_{\ell=1}^{N_h} \lambda_{\ell} \int_{\mathcal{O}} u^h p_x^h g_{\ell} dx d\beta_{\ell} \right| \right] \\ &\leq \bar{p}C_{BDG}\mathbb{E} \left[\left(\int_0^{T_{max} \wedge T_h} R(s)^{2\bar{p}-2} \sum_{\ell=1}^{N_h} \lambda_{\ell}^2 \int_{\mathcal{O}} M_h(u^h)|p_x^h|^2 dx ds \right)^{1/2} \right] \\ &\leq \bar{p}C_{BDG}\mathbb{E} \left[\sup_{s \in [0, T_{max} \wedge T_h]} R(s)^{\bar{p}/2} \right. \\ &\quad \left. \times \left(\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-2} \sum_{\ell=1}^{N_h} \lambda_{\ell}^2 \int_{\mathcal{O}} M_h(u^h)|p_x^h|^2 dx ds \right)^{1/2} \right] \\ &\leq \frac{1}{4}\mathbb{E} \left[\sup_{s \in [0, T_{max} \wedge T_h]} R(s)^{\bar{p}} \right] \\ &\quad + C_{BDG}^2 \bar{p}^2 \sum_{\ell=1}^{\infty} \lambda_{\ell}^2 \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-2} \int_{\mathcal{O}} M_h(u^h)|p_x^h|^2 dx ds \right] \\ (4.40) \quad &=: I_{4a} + I_{4b}. \end{aligned}$$

Note that I_{4a} can be absorbed by $\mathbb{E}[\sup_{s \in [0, T_{max} \wedge T_h]} R(s)^{\bar{p}}]$. For I_{4b} , we estimate

$$I_{4b} \leq \frac{C_{BDG}^2}{\alpha} \bar{p}^2 \sum_{\ell=1}^{\infty} \lambda_{\ell}^2 \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} M_h(u^h)|p_x^h|^2 dx ds \right].$$

This term can be controlled by the bound on the right-hand side of (4.21) and therefore by a constant depending on T_{max} , $\sum_{\ell \in \mathbb{N}} \ell^4 \lambda_{\ell}^2$, \bar{p} , α , κ , and initial data.

Finally, let us discuss I_2 . Using (4.35), we get

$$\begin{aligned} I_2 &= \kappa \bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(g_h(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h d\beta_\ell \\ &= \kappa \bar{p} \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \lambda_\ell \int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{I}_h[g_h(u^h)] dx d\beta_\ell. \end{aligned}$$

Consider $\mathcal{T}_2(s) : Q^{1/2}L^2(\mathcal{O}) \rightarrow \mathbb{R}$ defined for $s \in [0, T_{max}]$ by

$$(4.41) \quad \mathcal{T}_2(s)(w) := \chi_{T_h}(s) R(s)^{\bar{p}-1} \int_{\mathcal{O}} (u^h \mathcal{P}_{N_h}[w])_x \mathcal{I}_h[g_h(u^h)] dx.$$

Similarly as in (4.39), the Hilbert–Schmidt norm of \mathcal{T}_2 is estimated by

$$\|\mathcal{T}_2(s)\|_{L_2(Q^{1/2}L^2(\mathcal{O});\mathbb{R})} \leq \chi_{T_h}(s) C(\mathcal{O}) R(s)^{\bar{p}-1} \left(\sum_{\ell=1}^{\infty} \lambda_\ell^2 \int_{\mathcal{O}} |(u^h g_\ell)_x \mathcal{I}_h[g_h(u^h)]|^2 dx \right)^{1/2}.$$

By Burkholder–Davis–Gundy, we get as above

$$\begin{aligned} &\kappa \bar{p} \mathbb{E} \left[\sup_{0 \leq t \leq T_{max} \wedge T_h} \left| \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \left(g_h(u^h), \sum_{i=1}^{L_h} Z_i(\lambda_\ell g_\ell) e_i \right)_h d\beta_\ell \right| \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\sup_{0 \leq s \leq T_{max} \wedge T_h} R(s)^{\bar{p}} \right] \\ &\quad + 2\kappa^2 \bar{p}^2 C \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-2} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \int_{\mathcal{O}} |u_x^h|^2 |\mathcal{I}_h[g_h(u^h)]|^2 dx ds \right] \\ &\quad + 2\kappa^2 \bar{p}^2 C \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-2} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \int_{\mathcal{O}} |u^h \mathcal{I}_h[g_h(u^h)]|^2 |(g_\ell)_x|^2 dx ds \right] \\ (4.42) \quad &\leq I_{2a} + I_{2b} + I_{2c}. \end{aligned}$$

The term I_{2a} can be absorbed similarly as I_{4a} . Combining Poincaré’s inequality with Lemma 4.6 and taking expectations, we find

$$\begin{aligned} I_{2b} + I_{2c} &\leq C \kappa^2 \bar{p}^2 \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-2} \sum_{\ell=1}^{\infty} \lambda_\ell^2 \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} |\tau|^{-p-2} d\tau \int_{ih}^{(i+1)h} |u_x^h|^2 dx ds \right] \\ &\quad + C \kappa^2 \bar{p}^2 \sum_{\ell=1}^{\infty} \lambda_\ell^2 \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-1} ds \right] \\ &\quad + C \kappa^2 \bar{p}^2 \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} |u^h|^2 dx ds \right] \\ &\leq \frac{C(\kappa, \bar{p}, \lambda)}{\alpha} \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}-1} \sum_{i=1}^{L_h} \int_{u_i^h}^{u_{i+1}^h} |\tau|^{-p-2} d\tau \int_{ih}^{(i+1)h} |u_x^h|^2 dx ds \right] \\ &\quad + \frac{C(\kappa, \bar{p}, \lambda)}{\alpha} \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}} ds \right] \\ (4.43) \quad &\quad + C \kappa^2 \bar{p}^2 \sum_{\ell=1}^{\infty} \ell^2 \lambda_\ell^2 \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s)^{\bar{p}} ds + T_{max} (u_0)_{\mathcal{O}}^{2\bar{p}} \right]. \end{aligned}$$

Note that the first term on the right-hand side can be controlled by the right-hand side of (4.21) multiplied by a constant factor depending on $\kappa, \bar{p}, \lambda,$ and α . Similarly, the remaining terms can be bounded using (H3) and (4.21). Summing up and noting that α and κ have to be chosen depending on $(\lambda_\ell)_{\ell \in \mathbb{N}}$ and \bar{p} , (4.22) is established. \square

In the proof of the combined energy-entropy estimate, we have used the following auxiliary lemmas.

LEMMA 4.5. *We have the estimate*

$$\begin{aligned} & \left(\frac{1}{m_\sigma(u^h)}, \left(\sum_{i=1}^{L_h} Z_i(g_\ell) e_i \right) \right)_h \\ & \leq 2 \sum_{i=1}^{L_h} \frac{1}{(m_\sigma(u^h)_i)} \left(\int_{(i-1)h}^{ih} \left| \frac{u^h(x+h) - u^h(x)}{h} \right|^2 g_\ell(x)^2 dx \right. \\ & \quad \left. + \int_{(i-1)h}^{ih} (u^h(x+h))^2 \left| \frac{g_\ell(x+h) - g_\ell(x)}{h} \right|^2 dx \right) \end{aligned}$$

for arbitrary $\ell \in \mathbb{N}$ and positive $u^h \in X_h$.

Proof. By the identity $Z_i(g_\ell) = \frac{1}{h} \int_{\mathcal{O}} (u^h g_\ell)_x e_i dx$, we have

$$\begin{aligned} & \left(\frac{1}{m_\sigma(u^h)}, \left(\sum_{i=1}^{L_h} Z_i(g_\ell) e_i \right) \right)_h = h \sum_{i=1}^{L_h} \frac{1}{(m_\sigma(u^h)_i)} \frac{1}{h^2} \left(\int_{\mathcal{O}} (u^h g_\ell)_x e_i dx \right)^2 \\ & = h \sum_{i=1}^{L_h} \frac{1}{m_\sigma((u^h)_i)} \frac{1}{h^2} \left(-\frac{1}{h} \int_{(i-1)h}^{ih} u^h g_\ell dx + \frac{1}{h} \int_{ih}^{(i+1)h} u^h g_\ell dx \right)^2 \\ & = h \sum_{i=1}^{L_h} \frac{1}{m_\sigma((u^h)_i)} \frac{1}{h^2} \left(\int_{(i-1)h}^{ih} \frac{u^h(x+h) - u^h(x)}{h} g_\ell(x) dx \right. \\ & \quad \left. + \int_{(i-1)h}^{ih} u^h(x+h) \frac{g_\ell(x+h) - g_\ell(x)}{h} dx \right)^2, \end{aligned}$$

which implies the assertion of the lemma. \square

LEMMA 4.6. *Assuming u^h to be strictly positive and to satisfy the estimate $E_h[u_h] \leq \frac{1}{2} h^{-(p-2)/(p+2)}$, there exists a positive constant C independent of h and u^h such that the estimate*

$$\begin{aligned} & \int_{\mathcal{O}} |u^h|^2 |\mathcal{I}_h[g_h(u^h)]|^2 |(g_\ell)_x|^2 dx + \int_{\mathcal{O}} |u_x^h|^2 |\mathcal{I}_h[g_h(u^h)]|^2 |g_\ell|^2 dx \\ & \leq C \left(\ell^2 \int_{\mathcal{O}} |u^h|^2 dx + \sum_{i=1}^{L_h} \int_{(u^h)_{i-1}}^{(u^h)_i} |\tau|^{-p-2} d\tau \int_{(i-1)h}^{ih} |u_x^h|^2 dx + \int_{\mathcal{O}} |u_x^h|^2 dx \right) \end{aligned}$$

holds for each $\ell \in \mathbb{N}$.

Proof. By the boundedness of $E_h[u^h]$ and (4.6), we have

$$g_h(u^h)(x) = \int_1^{u^h(x)} \frac{1}{s^2} ds = 1 - \frac{1}{u^h(x)}.$$

Hence,

$$\begin{aligned} u^h \mathcal{I}_h[g_h(u^h)]|_{((i-1)h, ih)} &= \frac{(x - (i-1)h)^2}{h^2} (u_i^h - 1) + \frac{(ih - x)^2}{h^2} (u_{i-1}^h - 1) \\ &\quad + \frac{(x - (i-1)h) \cdot (ih - x)}{h^2} \left(u_i^h + u_{i-1}^h - \frac{u_{i-1}^h}{u_i^h} - \frac{u_i^h}{u_{i-1}^h} \right). \end{aligned}$$

By (4.7), we get

$$|u^h \mathcal{I}_h[g_h(u^h)]| \leq C(1 + u_i^h + u_{i-1}^h).$$

Hence,

$$(4.44) \quad \int_{\mathcal{O}} |u^h \mathcal{I}_h[g_h(u^h)]|^2 |(g_\ell)_x|^2 dx \leq C \ell^2 \int_{\mathcal{O}} |u_h|^2 dx.$$

Furthermore, one has

$$|g_h(s)|^2 \leq C(s^{-p-2} + 1)$$

for any $s > 0$. Therefore, the estimate

$$\int_{\mathcal{O}} |u_x^h|^2 |\mathcal{I}_h[g_h(u^h)]|^2 |g_\ell|^2 dx \leq C \int_{\mathcal{O}} |u_x^h|^2 (\mathcal{I}_h[|u_h|^{-p-2}] + 1) dx$$

holds, which in connection with (4.44) and (4.7) yields the assertion of the lemma. \square

4.4. Uniform Hölder continuity. Let us prove that appropriate Hölder norms (with respect to space and time) of solutions to our semidiscrete scheme are square-integrable with respect to the probability measure. We begin with an auxiliary result on the stochastic integral.

LEMMA 4.7. *Let $h \in (0, 1]$, $T_{max} > 0$, $\bar{p} > 1$, $\alpha \in (0, \frac{1}{2})$. Assume u^h, p^h to be a solution to (2.1a) and (2.1b) with initial data satisfying (H3) and (H4).*

If $2\alpha\bar{p} > 1$ holds, the stochastic integral

$$(4.45) \quad I_h(t) := \sum_{i=1}^{L_h} \sum_{\ell=1}^{N_h} \frac{1}{h} \int_0^{t \wedge T_h} \int_{\mathcal{O}} (u^h \lambda_\ell g_\ell)_x e_i dx d\beta_\ell(s) e_i$$

is contained in $L^{2\bar{p}}(\Omega; C^\beta([0, T_{max}]; L^2(\mathcal{O})))$ with $\beta := \alpha - \frac{1}{2\bar{p}}$ and there exists a constant C_1 independent of $h > 0$ such that

$$(4.46) \quad \|I_h\|_{L^{2\bar{p}}(\Omega; C^\beta([0, T_{max}]; L^2(\mathcal{O})))} \leq C_1$$

holds.

Remark 4.8. Choosing α and \bar{p} sufficiently large, we infer the estimate

$$(4.47) \quad \|I_h\|_{L^2(\Omega; C^{1/4}([0, T_{max}]; L^2(\mathcal{O})))} \leq C_1$$

with an h -independent positive constant C_1 .

The proof of Lemma 4.7 makes use of the following lemma from [24].

LEMMA 4.9 (Lemma 2.1 in [24]). *Let $p \geq 2$ and $\alpha < \frac{1}{2}$; let H be a Hilbert space. Then, for any progressively measurable process $f \in L^p(\Omega \times [0, T]; L_2(L^2(\mathcal{O}); H))$, we have*

$$I(f) := \int_0^t f dW \in L^p(\Omega; W^{\alpha,p}([0, T]; H))$$

with an estimate of the form

$$\mathbb{E}[\|I(f)\|_{W^{\alpha,p}([0,T];H)}^p] \leq C(p, \alpha)\mathbb{E}\left[\int_0^T \|f(t)\|_{L_2(L^2(\mathcal{O});H)}^p dt\right].$$

Proof of Lemma 4.7. In light of Lemma 4.9, it is sufficient to show that

$$\tilde{Z}(s)(w) = \chi_{T_h}(s) \frac{1}{h} \sum_{i=1}^{L_h} \int_{\mathcal{O}} \left(u^h \sum_{\ell=1}^{N_h} \lambda_{\ell}(g_{\ell}, w)_{L^2(\mathcal{O})} g_{\ell} \right)_x e_i dx e_i$$

is progressively measurable and contained in

$$L^{2\bar{p}}(\Omega \times [0, T_{max}]; L_2(L^2(\mathcal{O}); L^2(\mathcal{O})))$$

with a uniform bound in h . Indeed, this result would imply $I_h(\cdot)$ to be contained in $L^{2\bar{p}}(\Omega; W^{\alpha,2\bar{p}}([0, T_{max}]; L^2(\mathcal{O})))$, again with a uniform bound in h . Hence, (4.46) would follow by the continuous embedding

$$W^{\alpha,2\bar{p}}([0, T_{max}]; L^2(\mathcal{O})) \hookrightarrow C^{\alpha-\frac{1}{2\bar{p}}}([0, T_{max}]; L^2(\mathcal{O}))$$

for α and \bar{p} satisfying the assumptions of the lemma.

Note that we have $\|g_{\ell}\|_{L^{\infty}(\mathcal{O})} \leq C$, as the g_{ℓ} are basically sine functions (since \mathcal{O} is a real interval of finite length). Computing the Hilbert–Schmidt norm of \tilde{Z} , using in particular the equivalence of the norm $\|\cdot\|_h$ on X_h with the L^2 -norm (with constants independent of h), we get

$$\begin{aligned} \|\tilde{Z}(s)\|_{L_2(L^2(\mathcal{O});L^2(\mathcal{O}))}^2 &= \sum_{\ell=1}^{\infty} \|\tilde{Z}(s)(g_{\ell})\|_{L^2(\mathcal{O})}^2 \\ &\leq C \sum_{\ell=1}^{\infty} \|\tilde{Z}(s)(g_{\ell})\|_h^2 = \frac{C}{h^2} \sum_{\ell=1}^{\infty} \chi_{T_h}(s) h \sum_{i=1}^{L_h} \lambda_{\ell}^2 \left| \int_{\mathcal{O}} (u^h g_{\ell})_x e_i dx \right|^2 \\ &\leq \frac{C}{h} \chi_{T_h}(s) \sum_{\ell=1}^{\infty} \lambda_{\ell}^2 \sum_{i=1}^{L_h} \int_{(i-1)h}^{ih} |(u^h g_{\ell})_x|^2 dx \int_{\mathcal{O}} e_i^2 dx \\ &\leq C \chi_{T_h}(s) \sum_{\ell=1}^{\infty} \lambda_{\ell}^2 \int_{\mathcal{O}} |(u^h g_{\ell})_x|^2 dx \\ &\leq C \chi_{T_h}(s) \left(\sum_{\ell=1}^{\infty} \ell^2 \lambda_{\ell}^2 \int_{\mathcal{O}} |u^h|^2 dx + \sum_{\ell=1}^{\infty} \lambda_{\ell}^2 \int_{\mathcal{O}} |u_x^h|^2 dx \right). \end{aligned}$$

By Proposition 4.4, Lemma 2.1, (H3), and $\sum_{\ell=1}^{\infty} \ell^2 \lambda_{\ell}^2 < \infty$, we infer

$$\tilde{Z}(\cdot, \cdot) \in L^{2\bar{p}}(\Omega \times [0, T_{max}]; L_2(L^2(\mathcal{O}); L^2(\mathcal{O})))$$

with a uniform bound in h . Finally, we note that \tilde{Z} is progressively measurable as it is a continuous composition of terms having this property. This gives the assertion of the lemma. \square

LEMMA 4.10. *Under the assumptions of Lemma 4.7 solutions u^h of (2.1a) are contained and uniformly bounded in the space*

$$L^2(\Omega; C^{1/4}([0, T_{max}]; L^2(\mathcal{O}))).$$

In particular, a positive constant C_2 independent of $h > 0$ exists such that

$$(4.48) \quad \mathbb{E} \left[\sup_{t_1, t_2 \in [0, T_{max}]} \frac{\|u^h(t_1) - u^h(t_2)\|_h^2}{|t_1 - t_2|^{2/4}} \right] \leq C_2.$$

Proof. Starting from the weak formulation

$$\begin{aligned} & (u^h(t_2) - u^h(t_1), \phi_h)_h + \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h \phi_x^h \, dx \, ds \\ & = (I_h(t_2) - I_h(t_1), \phi_h)_h \quad \forall \phi_h \in X_h \end{aligned}$$

for any $t_1, t_2 \in [0, T_{max}]$ with $t_1 \leq t_2$ and \mathbb{P} -a. e. $\omega \in \Omega$ (note that this follows from (2.1a) and the definition (4.45)), we obtain \mathbb{P} -almost surely

$$\begin{aligned} & (u^h(t_2) - u^h(t_1), \phi_h)_h + \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h \phi_x^h \, dx \, ds \\ & \leq \sup_{\|\psi_h\| \leq 1} |(I_h(t_2) - I_h(t_1), \psi_h)_h| \end{aligned}$$

for all $\phi_h \in X_h$ with $\|\phi_h\|_{L^2(\mathcal{O})} \leq 1$. Choosing $\phi_h := \frac{u^h(t_2) - u^h(t_1)}{\|u^h(t_2) - u^h(t_1)\|_{L^2(\mathcal{O})}}$, we have

$$\begin{aligned} & \frac{\|u^h(t_2) - u^h(t_1)\|_h^2}{\|u^h(t_2) - u^h(t_1)\|_{L^2(\mathcal{O})}} \\ & \leq \frac{1}{\|u^h(t_2) - u^h(t_1)\|_{L^2(\mathcal{O})}} \left| \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h (u^h(t_2) - u^h(t_1))_x \, dx \, ds \right| \\ & \quad + \|I_h(t_2) - I_h(t_1)\|_{L^2(\mathcal{O})}. \end{aligned}$$

Multiplying by $\|u^h(t_2) - u^h(t_1)\|_{L^2(\mathcal{O})}$, using the h -independent equivalence of $\|\cdot\|_h$ and $\|\cdot\|_{L^2(\mathcal{O})}$ on X_h , and applying Young's inequality, we see that there exists a constant C independent of h such that

$$(4.49) \quad \|u^h(t_2) - u^h(t_1)\|_h^2 \leq C \left| \int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h (u^h(t_2) - u^h(t_1))_x \, dx \, ds \right| + \|I_h(t_2) - I_h(t_1)\|_{L^2(\mathcal{O})}^2$$

is satisfied.

From Remark 4.8, we infer the existence of a function $\mathcal{C} \in L^2(\Omega)$ such that

$$(4.50) \quad \|I_h(t_2)(\omega) - I_h(t_1)(\omega)\|_{L^2(\mathcal{O})} \leq \mathcal{C}(\omega) |t_2 - t_1|^{1/4}$$

holds for all $t_1, t_2 \in [0, T_{max}]$ with $t_1 \leq t_2$, \mathbb{P} -almost surely.

Inequality (4.49) and Poincaré's inequality entail \mathbb{P} -almost surely

$$\begin{aligned} & \|u^h(t_2, \omega) - u^h(t_1, \omega)\|_h^2 \\ & \leq C \left(1 + T_{max} \left(\int_{\mathcal{O}} u_0 \right)^2 + \sup_t \|u_x^h(\omega)\|_{L^2(\mathcal{O})}^2 \right) \left(\int_{t_1 \wedge T_h}^{t_2 \wedge T_h} \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 \right)^{1/2} \sqrt{t_2 - t_1} \\ & \quad + C \mathcal{C}^2(\omega) |t_2 - t_1|^{1/2}. \end{aligned}$$

Dividing by $|t_2 - t_1|^{1/2}$, taking the supremum with respect to t_1 and t_2 , and taking expectations, we get

$$\begin{aligned} & \mathbb{E} \left[\sup_{t_1, t_2 \in [0, T_{max}]} \frac{\|u^h(t_2 \wedge T_h) - u^h(t_1 \wedge T_h)\|_h^2}{|t_2 - t_1|^{2/4}} \right] \\ & \leq C \mathbb{E} \left[\sup_{s \in [0, T_{max} \wedge T_h]} R^2(s) + T_{max}^2 \left(\int_{\mathcal{O}} u_0 \, dx \right)^4 \right] \\ & \quad + C \mathbb{E} \left[\int_0^{T_{max} \wedge T_h} R(s) \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 \, dx \, ds \right] + C \mathbb{E}[\mathcal{C}^2(\omega)]. \end{aligned}$$

By Proposition 4.4, the result follows, as the spatial Hölder property is a consequence of the standard embedding $H^1(\mathcal{O}) \subset C^{1/2}(\mathcal{O})$. \square

LEMMA 4.11. *Under the assumptions of Lemma 4.10, solutions u^h to (2.1a) are space-time Hölder-continuous almost surely. In particular, there is a positive constant C_3 independent of $h > 0$ for which we have*

$$(4.51) \quad \mathbb{E}[\|u^h\|_{C^{1/2, 1/8}(\mathcal{O} \times [0, T_{max}])}^2] \leq C_3.$$

Proof. We apply a standard interpolation argument—combining (4.48) with the embedding of $H_{per}^1(\mathcal{O})$ into $C^{1/2}(\mathcal{O})$. Using the notation of Lemma 4.10, taking $\delta > 0$ sufficiently small, we estimate for $t_1, t_2 \in [0, T_{max}]$ and $x \in \mathcal{O}$

$$\begin{aligned} & |u^h(x, t_1) - u^h(x, t_2)| \\ & = \left| \int_x^{x+\delta} u^h(x, t_1) - u^h(y, t_1) \, dy + \int_x^{x+\delta} u^h(y, t_1) - u^h(y, t_2) \, dy \right. \\ & \quad \left. + \int_x^{x+\delta} u^h(y, t_2) - u^h(x, t_2) \, dy \right| =: |I + II + III|. \end{aligned}$$

Note that $|I + III| \leq \delta^{1/2} \sup_t \|u_x^h\|_{L^2(\mathcal{O})}$ and

$$|II| \leq \delta^{-1/2} \left(\int_{\mathcal{O}} |u^h(x, t_1) - u^h(x, t_2)|^2 \, dx \right)^{1/2} \leq \mathcal{C}(\omega) \frac{|t_1 - t_2|^{1/4}}{\delta^{1/2}}$$

with $\mathcal{C} \in L^2(\Omega)$. Choosing $\delta := |t_1 - t_2|^{1/4}$, we get

$$|I + II + III| \leq \tilde{\mathcal{C}}(\omega) \left(|t_1 - t_2|^{1/8} + \frac{|t_1 - t_2|^{1/4}}{|t_1 - t_2|^{1/8}} \right) \leq \tilde{\mathcal{C}}(\omega) |t_1 - t_2|^{1/8}$$

with $\tilde{\mathcal{C}} \in L^2(\Omega)$, where we have used the property $\sup_t R(t) \in L^2(\Omega)$. This entails the result. \square

4.5. Estimates on the pressure. Finally, in our passage to the limit we need the following uniform estimate on the pressures p^h .

LEMMA 4.12. *For any $q \in [1, 2)$ there exists some $C > 0$ such that*

$$\mathbb{E} \left[\left(\int_0^{T_{max}} \int_{\mathcal{O}} |p_x^h|^2 + |p^h|^2 \, dx \, dt \right)^{q/2} \right] \leq C$$

holds for all $h \in (0, 1]$.

Proof. The proof only makes use of the energy-entropy estimates (4.21) and (4.22) as well as the lower bound on u^h in terms of $E[u^h]$ provided by (4.3). Fixing $q > 1$, we have by Hölder's inequality

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^{T_{max}} \int_{\mathcal{O}} |p_x^h|^2 dx dt \right)^{q/2} \right] \\ & \leq \mathbb{E} \left[\left(\int_0^{T_{max}} \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 dx dt \sup_{x \in \mathcal{O}, t \in [0, T_{max}]} \frac{1}{M_h(u^h(x, t))} \right)^{q/2} \right] \\ & \leq \left(\mathbb{E} \left[\int_0^{T_{max}} \int_{\mathcal{O}} M_h(u^h) |p_x^h|^2 dx dt \right] \right)^{q/2} \left(\mathbb{E} \left[\sup_{t \in [0, T_{max}]} \sup_{x \in \mathcal{O}} \frac{1}{|u^h(x, t)|^{2q/(2-q)}} \right] \right)^{(2-q)/2}. \end{aligned}$$

The first factor in this estimate is bounded by an h -independent constant due to (4.21) (applied for $\bar{p} = 1$). The second factor is also bounded by an h -independent constant, as may be seen by applying (4.3) and then (4.22) for appropriate \bar{p} .

By Poincaré's inequality, it only remains to establish a bound of the form

$$\mathbb{E} \left[\left(\int_0^{T_{max}} \left| \int_{\mathcal{O}} p^h dx \right|^2 dt \right)^{q/2} \right] \leq C.$$

Using the results of Lemma 2.1, this may be easily accomplished by testing the weak formulation (2.1b) of p^h by a constant as test function, resulting in the bound

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^{T_{max}} (p^h, 1)_h^2 dt \right)^{q/2} \right] & \leq \mathbb{E} \left[\left(\int_0^{T_h \wedge T_{max}} (\mathcal{I}_h \mathcal{W}'(u^h), 1)_h^2 dt \right)^{q/2} \right] \\ & \leq C + C \mathbb{E} \left[\sup_{t \in [0, T_{max}]} \sup_{x \in \mathcal{O}} \frac{1}{|u^h(x, t)|^{(p+1)q}} \right], \end{aligned}$$

where in the last step we have used hypothesis (H2) for \mathcal{W} . The second term on the right-hand side may be estimated by (4.3) and (4.22) for appropriate \bar{p} . \square

5. Convergence of the scheme.

5.1. Compactness. We intend to apply the Jakubowski–Skorokhod theorem [42] to identify a stochastic basis such that a subsequence of the solutions to the semidiscrete scheme (2.1a), (2.1b) almost surely converges in topologies which are appropriate for a passage to the limit in the nonlinearities of (1.5).

In the subsections to follow, we shall show that this limit is indeed a weak martingale solution to the stochastic thin-film equation (1.1) in the sense of Definition 3.1.

THEOREM 5.1 (Jakubowski [42]). *Let (\mathcal{X}, τ) be a topological space and assume that there exists a countable family $\{f_i : \mathcal{X} \rightarrow [-1, 1]\}_{i \in \mathcal{I}}$ of τ -continuous functions which separate points of \mathcal{X} .*

Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{X} -valued random variables. Suppose for each $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \mathcal{X}$ such that

$$(5.1) \quad \mathbb{P}\{X_n \in K_\epsilon\} > 1 - \epsilon \quad \forall n \in \mathbb{N}.$$

Then, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a sequence $(X_{n_k})_{k \in \mathbb{N}}$, and a sequence $(Y_k)_{k \in \mathbb{N}}$ of \mathcal{X} -valued random variables on $\tilde{\Omega}$ with the following properties: The law of X_{n_k} on \mathcal{X} coincides with the law of Y_k for all $k \in \mathbb{N}$. Furthermore, there exists a random variable $Y_\infty : \tilde{\Omega} \rightarrow \mathcal{X}$ such that for almost every $\omega \in \tilde{\Omega}$ the convergence $Y_k(\omega) \rightarrow Y_\infty(\omega)$ holds in the topology of \mathcal{X} .

In our setting, we consider for $\gamma \in (0, 1/2)$ the path spaces

$$\begin{aligned} \mathcal{X}_u &:= C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}]), \\ \mathcal{X}_p &:= (L^2([0, T_{max}]; H^1_{per}(\mathcal{O})))_{weak}, \\ \mathcal{X}_J &:= L^2_{weak}(\mathcal{O} \times [0, T_{max}]) \end{aligned}$$

associated with the solutions to our semidiscrete scheme u^h, p^h , and the corresponding pseudofluxes

$$(5.2) \quad J^h := \chi_{T_h} \sqrt{M_h(u^h)} p_x^h,$$

respectively. Denoting the laws of u^h, p^h , and J^h by μ_{u^h}, μ_{p^h} , and μ_{J^h} , respectively, we obtain the following result on tightness.

LEMMA 5.2. *Let $T_{max} > 0$ be arbitrary but fixed. Let (u^h, p^h, J^h) be a sequence of discrete solutions as constructed in Lemma 4.2. Then, the families of laws $(\mu_{u^h})_h, (\mu_{p^h})_h$, and $(\mu_{J^h})_h$ (for $h \in (0, 1]$) are tight.*

Proof. We begin with $(\mu_{u^h})_h$. From Lemma 4.11 we infer uniform boundedness of $(u_h)_h$ in $L^2(\Omega; C^{1/2, 1/8}(\mathcal{O} \times [0, T_{max}]))$. As $C^{1/2, 1/8}(\mathcal{O} \times [0, T_{max}])$ is compactly embedded in $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])$ with $\gamma \in (0, 1/2)$, the ball \bar{B}_R in $C^{1/2, 1/8}(\mathcal{O} \times [0, T_{max}])$ is a compact subset of $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])$. Furthermore, we have for any $R > 0$

$$\begin{aligned} &\mu_{u^h} \left(C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}]) \setminus \bar{B}_R \right) \\ &= \mathbb{P}[\|u^h\|_{C^{1/2, 1/8}} > R] \leq \frac{\mathbb{E}[\|u^h\|_{C^{1/2, 1/8}}^2]}{R^2}. \end{aligned}$$

Thus, choosing R large enough (independent of h), we have found a compact set $K \subset \mathcal{X}_u$ with $\mu_{u^h}(K) \geq 1 - \epsilon$ for all h .

Concerning $(\mu_{J^h})_h$, we argue as follows: Denote by \bar{B}_R the subset $\{f \in L^2(\mathcal{O} \times [0, T_{max}]) : \|f\|_{L^2} \leq R\}$. Due to the bound

$$(5.3) \quad \mathbb{E} \left[\int_0^{T_{max}} \int_{\mathcal{O}} |J^h|^2 dx dt \right] \leq C,$$

we have

$$\mathbb{P}[J^h \notin \bar{B}_R] \leq \frac{C}{R^2}.$$

As closed balls in L^2 are compact in the weak topology, the conclusion follows by choosing R large enough depending on ϵ .

To see the tightness of $(\mu_{p^h})_h$, one may argue similarly: Closed balls in the space $L^2([0, T_{max}]; H^1_{per}(\mathcal{O}))$ are compact in the weak topology. It is therefore sufficient to show that the probability that p^h is not contained in a ball \bar{B}_R in $L^2([0, T_{max}]; H^1_{per}(\mathcal{O}))$

converges to zero as $R \rightarrow \infty$, the convergence being uniformly with respect to h . By Lemma 4.12 (applied for example for $q = 1$), we have

$$\mathbb{P}[\|p^h\|_{L^2([0, T_{max}]; H_{per}^1(\mathcal{O}))} > R] \leq \frac{\mathbb{E}[\|p^h\|_{L^2([0, T_{max}]; H_{per}^1(\mathcal{O}))}]^2}{R} \leq \frac{C}{R},$$

the constant C being independent of h . \square

As a fourth path space, we introduce

$$(5.4) \quad \mathcal{X}_W := C([0, T]; L^2(\mathcal{O})).$$

Let μ_W be the law of W (i.e., the law of $\sum_{\ell \in \mathbb{N}} \lambda_\ell g_\ell \beta_\ell$). As $C([0, T]; L^2(\mathcal{O}))$ is a Polish space (completely metrizable and separable), μ_W is a regular measure and therefore a Radon measure (see [45, Theorem 13.6]). Being a Radon measure means in particular regularity from the interior, i.e.,

$$\mu(C([0, T]; L^2(\mathcal{O}))) = \sup\{\mu(K) : K \subset C([0, T]; L^2(\mathcal{O})) \text{ compact}\}.$$

Similarly, initial data are treated by the space $\mathcal{X}_{u_0} := H_{per}^1(\mathcal{O})$. We conclude together with Lemma 5.2.

LEMMA 5.3. *On the path space $\mathcal{X} := \mathcal{X}_u \times \mathcal{X}_p \times \mathcal{X}_J \times \mathcal{X}_W \times \mathcal{X}_{u_0}$, the joint laws μ_h , $\mu_h(A \times B \times C \times D \times E) := \mathbb{P}[\{u^h \in A\} \cap \{p^h \in B\} \cap \{J^h \in C\} \cap \{W \in D\} \cap \{u_0 \in E\}]$, for $h \in (0, 1]$ are tight.*

Following the strategy sketched at the beginning of this section, we apply the generalization of the Skorokhod theorem due to Jakubowski (i.e., Theorem 5.1) to obtain the following.

PROPOSITION 5.4. *Let $\gamma \in (0, 1/2)$ be given and assume u^h, p^h, T_h to be a sequence of solutions to our semidiscrete scheme (2.1) in the sense of Lemma 4.2, defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with respect to the Wiener process W . Then there exist a subsequence (not relabeled), a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, sequences of random variables*

$$\begin{aligned} \tilde{u}^h &: \tilde{\Omega} \rightarrow C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}]), \\ \tilde{p}^h &: \tilde{\Omega} \rightarrow L^2([0, T_{max}]; H_{per}^1(\mathcal{O})), \\ \tilde{J}^h &: \tilde{\Omega} \rightarrow L^2([0, T_{max}]; H_{per}^1(\mathcal{O})), \\ \tilde{u}_0^h &: \tilde{\Omega} \rightarrow H_{per}^1(\mathcal{O}), \end{aligned}$$

a sequence of $L^2(\mathcal{O})$ -valued processes \tilde{W}^h on $\tilde{\Omega}$, and random variables

$$\begin{aligned} \tilde{u} &\in L^2(\tilde{\Omega}; C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])), \\ \tilde{p} &\in L^{3/2}(\tilde{\Omega}; L^2([0, T_{max}]; H_{per}^1(\mathcal{O}))), \\ \tilde{J} &\in L^2(\tilde{\Omega}; L^2(\mathcal{O} \times [0, T_{max}])), \\ \tilde{u}_0 &\in L^2(\tilde{\Omega}; H_{per}^1(\mathcal{O})), \end{aligned}$$

as well as an $L^2(\mathcal{O})$ -valued process \tilde{W} on $\tilde{\Omega}$ such that the following holds:

- (i) The law of $(\tilde{u}^h, \tilde{p}^h, \tilde{J}^h, \tilde{W}^h, \tilde{u}_0^h)$ on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}]) \times L^2([0, T_{max}]; H_{per}^1(\mathcal{O})) \times L^2(\mathcal{O} \times [0, T_{max}]) \times C([0, T_{max}]; L^2(\mathcal{O})) \times H_{per}^1(\mathcal{O})$ under $\tilde{\mathbb{P}}$ coincides for any h with the law of (u^h, p^h, J^h, W, u_0) under \mathbb{P} .
- (ii) The sequence $(\tilde{u}^h, \tilde{p}^h, \tilde{J}^h, \tilde{W}^h, \tilde{u}_0^h)$ converges $\tilde{\mathbb{P}}$ -almost surely to $(\tilde{u}, \tilde{p}, \tilde{J}, \tilde{W}, \tilde{u}_0)$ in the topology of \mathcal{X} .

Furthermore, we introduce the random times

$$\tilde{T}_h := T_{max} \wedge \inf\{t \geq 0 : E_h[\tilde{u}^h(t)] \geq E_{max,h}\}.$$

Their behavior for $h \rightarrow 0$ is the content of the following lemma.

LEMMA 5.5. *Along a subsequence, the convergence $\lim_{h \rightarrow 0} \tilde{T}_h = T_{max}$ holds $\tilde{\mathbb{P}}$ -almost surely.*

Proof. By Markov's inequality and (4.22), we have for each $\tau \in (0, T_{max}]$ the estimate

$$\tilde{\mathbb{P}}(\{\tilde{T}_h < \tau\}) = \mathbb{P}(\{T_h < \tau\}) = \mathbb{P}\left(\left\{\omega \mid \sup_{t \in [0, \tau]} E_h(u^h(\cdot, t)) \geq \frac{1}{2}h^{-\frac{p-2}{p+2}}\right\}\right) \leq Ch^{\frac{p-2}{p+2}}.$$

Hence, $\tilde{T}_h \rightarrow T_{max}$ in probability for $h \rightarrow 0$. The assertion follows in a standard way. \square

The relationship of $J^h, p^h,$ and u^h is preserved for the $\tilde{J}^h, \tilde{p}^h,$ and \tilde{u}^h .

LEMMA 5.6. *Under the assumptions of Proposition 5.4, we identify \tilde{J}^h as*

$$(5.5) \quad \tilde{J}^h = \chi_{\tilde{T}_h} M_h(\tilde{u}^h)^{1/2} \tilde{p}_x^h.$$

Furthermore, \tilde{p}^h satisfies

$$(5.6) \quad (\tilde{p}^h, \phi)_h = \chi_{\tilde{T}_h} \int_{\mathcal{O}} \tilde{u}_x^h \phi_x dx + \chi_{\tilde{T}_h} (\mathcal{I}_h[\mathcal{W}'(\tilde{u}^h)], \phi)_h$$

for all $\phi \in X_h$ and $\tilde{\mathbb{P}}$ -almost all ω .

Proof. For arbitrary $\phi \in C^\infty(\mathcal{O} \times [0, T_{max}])$, by coincidence of laws (note that $T_h(\omega)$ and $\tilde{T}_h(\tilde{\omega})$ are measurable functions of $u^h(\omega) \in C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])$ and $\tilde{u}^h(\tilde{\omega}) \in C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])$, respectively) and Lemma 4.12 the expectation

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^{T_{max}} \int_{\mathcal{O}} \tilde{J}^h \phi dx dt - \int_0^{T_{max} \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h^{1/2}(\tilde{u}^h) \tilde{p}_x^h \phi dx dt \right| \right] \\ &= \mathbb{E} \left[\left| \int_0^{T_{max}} \int_{\mathcal{O}} J^h \phi dx dt - \int_0^{T_{max} \wedge T_h} \int_{\mathcal{O}} M_h^{1/2}(u^h) p_x^h \phi dx dt \right| \right] \end{aligned}$$

is well-defined and equal to zero (see (5.2)), which gives the claim regarding \tilde{J}^h .

Similarly, by (2.1b) we have for all $\phi \in X_h$ and all $0 \leq t_1 < t_2 \leq T_{max}$

$$\begin{aligned} & \mathbb{E} \left[\left| \int_{t_1}^{t_2} (\tilde{p}^h, \phi)_h - \chi_{\tilde{T}_h} \int_{\mathcal{O}} \tilde{u}_x^h \phi_x dx - \chi_{\tilde{T}_h} (\mathcal{I}_h[\mathcal{W}'(\tilde{u}^h)], \phi)_h dt \right| \right] \\ &= \mathbb{E} \left[\left| \int_{t_1}^{t_2} (p^h, \phi)_h - \chi_{T_h} \int_{\mathcal{O}} u_x^h \phi_x dx - \chi_{T_h} (\mathcal{I}_h[\mathcal{W}'(u^h)], \phi)_h dt \right| \right] = 0. \quad \square \end{aligned}$$

The next step is to verify that \tilde{W} and \tilde{W}^h are Q -Wiener processes adapted to suitably defined filtrations $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ and $(\tilde{\mathcal{F}}_t^h)_{t \geq 0}$.

We define $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ to be the $\tilde{\mathbb{P}}$ -augmented canonical filtration associated with $(\tilde{u}, \tilde{W}, \tilde{u}_0)$, i.e.,

$$(5.7) \quad \tilde{\mathcal{F}}_t := \sigma(\sigma(r_t \tilde{u}, r_t \tilde{W}) \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde{u}_0)).$$

Here, r_t is the restriction of a function defined on $[0, T_{max}]$ to the interval $[0, t]$, $t \in [0, T_{max}]$.

Note that we do not need an explicit dependence of the filtration on $r_t \tilde{J}$ and $r_t \tilde{p}$, as the fluxes \tilde{J}^h and the pressures \tilde{p}^h depend in a measurable way on \tilde{u}^h (cf. Lemma 5.6) and—later on—we will identify $\tilde{J} = \lim_{h \rightarrow 0} \tilde{J}^h = \lim_{h \rightarrow 0} \chi_{\tilde{T}_h} M_h^{1/2}(\tilde{u}^h) \tilde{p}_x^h = M^{1/2}(\tilde{u}) \tilde{p}_x$ and $\tilde{p} = \lim_{h \rightarrow 0} \tilde{p}^h = -\tilde{u}_{xx} + \mathcal{W}'(\tilde{u})$.

Analogously, we introduce the filtrations $(\tilde{\mathcal{F}}_t^h)_{t \geq 0}$ as the $\tilde{\mathbb{P}}$ -augmented canonical filtration associated with $(\tilde{u}^h, \tilde{W}^h, \tilde{u}_0^h)$

$$(5.8) \quad \tilde{\mathcal{F}}_t^h := \sigma(\sigma(r_t \tilde{u}^h, r_t \tilde{W}^h) \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde{u}_0^h)).$$

LEMMA 5.7. *The processes \tilde{W}^h and \tilde{W} are Q -Wiener processes adapted to the filtrations $(\tilde{\mathcal{F}}_t^h)_{t \geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t \geq 0}$, respectively. They can be written as*

$$(5.9) \quad \tilde{W}^h(t) = \sum_{\ell \in \mathbb{N}} \lambda_\ell \tilde{\beta}_\ell^h(t) g_\ell$$

and

$$(5.10) \quad \tilde{W}(t) = \sum_{\ell \in \mathbb{N}} \lambda_\ell \tilde{\beta}_\ell(t) g_\ell,$$

respectively. Here, $(\tilde{\beta}_\ell^h)_{\ell \in \mathbb{N}}$ and $(\tilde{\beta}_\ell)_{\ell \in \mathbb{N}}$ are families of independently and identically distributed Brownian motions with respect to $(\tilde{\mathcal{F}}_t^h)_{t \geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t \geq 0}$, respectively.

Proof. Recall that the \tilde{W}^h have the same law as W . Therefore, the \tilde{W}^h are Wiener processes having the same covariance operator as W . Hence, the decomposition in (5.9) holds true for the \tilde{W}^h . To prove that \tilde{W} is an $\tilde{\mathcal{F}}_t$ -martingale, we follow the ideas in [11, 13, 18, 40]. Consider for arbitrary but fixed $t_1 \in [0, T_{max}]$ continuous functions

$$(5.11) \quad \gamma : \mathcal{X}_u|_{[0, t_1]} \times \mathcal{X}_W|_{[0, t_1]} \rightarrow [0, 1].$$

Combining the martingale property of W with the identity of laws, we have for $t_2 \in [t_1, T_{max}]$

$$(5.12) \quad \begin{aligned} 0 &= \mathbb{E}[\gamma(u^h|_{[0, t_1]}, W|_{[0, t_1]})(W(t_2) - W(t_1))] \\ &= \mathbb{E}[\gamma(\tilde{u}^h|_{[0, t_1]}, \tilde{W}^h|_{[0, t_1]})(\tilde{W}^h(t_2) - \tilde{W}^h(t_1))]. \end{aligned}$$

Again by the identity of laws, we have

$$\sup_h \mathbb{E}[||\tilde{W}^h(t_2)||_{L^2(\mathcal{O})}^2] = \sup_h \mathbb{E}[||W(t_2)||_{L^2(\mathcal{O})}^2] < \infty.$$

Vitali’s convergence result and (5.12) entail

$$\mathbb{E}[\gamma(\tilde{u}|_{[0,t_1]}, \tilde{W}|_{[0,t_1]})(\tilde{W}(t_2) - \tilde{W}(t_1))] = 0,$$

which establishes the martingale property of \tilde{W} . Finally, the decomposition (5.10) with independent Brownian motions immediately follows—using Levy’s characterization of Brownian motion (cf. [45])—from the $C([0, T_{max}]; L^2(\mathcal{O}))$ -convergence $\tilde{\mathbb{P}}$ -almost surely of \tilde{W}^h combined with the fact that

$$\lambda_m^{-1} \lambda_\ell^{-1} (\tilde{W}^h(t), g_\ell)_{L^2(\mathcal{O})} (\tilde{W}^h(t), g_m)_{L^2(\mathcal{O})} - \delta_{\ell m} t$$

is a martingale. In fact, using higher moments of Brownian motion, the argumentation of (5.12) can be mimicked. \square

5.2. Convergence of the deterministic terms. In this subsection, we prove higher regularity of \tilde{u} as well as the identification of the pseudoflux \tilde{J}

$$(5.13) \quad \tilde{J} = \tilde{u} \tilde{p}_x$$

and the identification of the pressure \tilde{p}

$$(5.14) \quad \tilde{p} = -\tilde{u}_{xx} + \mathcal{W}'(\tilde{u}).$$

For ease of presentation, let us collect the convergence and boundedness results established so far:

$$(5.15) \quad \tilde{u}^h \rightarrow \tilde{u} \quad \text{in } C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}]) \text{ } \tilde{\mathbb{P}}\text{-almost surely,}$$

$$(5.16) \quad \tilde{p}^h \rightharpoonup \tilde{p} \quad \text{weakly in } L^2([0, T_{max}]; H^1_{per}(\mathcal{O})) \text{ } \tilde{\mathbb{P}}\text{-almost surely,}$$

$$(5.17) \quad \tilde{J}^h = \chi_{\tilde{T}_h} M_h(\tilde{u}^h)^{\frac{1}{2}} \tilde{p}_x^h \rightharpoonup \tilde{J} \quad \text{weakly in } L^2(\mathcal{O} \times [0, T_{max}]) \text{ } \tilde{\mathbb{P}}\text{-almost surely,}$$

$$(5.18) \quad \mathbb{E} \left\{ \sup_{t \in [0, T_{max}]} \left(\int_{\mathcal{O}} |\tilde{u}_x^h|^2(t) dx + \int_{\mathcal{O}} \mathcal{I}_h \mathcal{W}(\tilde{u}^h)(t) dx \right)^{\bar{p}} \right\} \leq C(\bar{p}, u_0) < \infty \text{ for every } \bar{p} \geq 1,$$

$$(5.19) \quad \mathbb{E} \left\{ \sup_{t \in [0, T_{max}]} \left(\int_{\mathcal{O}} \tilde{u}^h dx \right)^{\bar{p}} + \sup_{t \in [0, T_{max}]} \left(\int_{\mathcal{O}} \tilde{u}^h dx \right)^{-\bar{p}} \right\} \leq C(\bar{p}, u_0) < \infty \text{ for every } \bar{p} \geq 1,$$

$$(5.20) \quad \mathbb{E} \int_0^{T_h \wedge T_{max}} \|\Delta_h \tilde{u}^h\|_h^2 dt \leq C(u_0) < \infty,$$

$$(5.21) \quad \mathbb{E} \int_0^{T_{max}} \int_{\mathcal{O}} |\tilde{J}^h|^2 dx dt \leq C(u_0) < \infty,$$

$$(5.22) \quad \mathbb{E} \left[\left(\int_0^{T_{max}} \int_{\mathcal{O}} |\tilde{p}_x^h|^2 + |\tilde{p}^h|^2 dx dt \right)^{3/4} \right] \leq C(u_0) < \infty,$$

where $\Delta_h \tilde{u}^h$ satisfies the identity

$$(5.23) \quad \Delta_h \tilde{u}^h = \partial_x^{+h} (\partial_x^{-h} \tilde{u}^h) = -\tilde{p}^h + \mathcal{I}_h \mathcal{W}'(\tilde{u}^h)$$

for $t \in [0, T_h \wedge T_{max}]$; cf. (2.1b).

LEMMA 5.8. *We have almost surely*

$$\inf_{x \in \mathcal{O}, t \in [0, T_{max}]} \tilde{u}(x, t) > 0.$$

Proof. The estimates (5.18) and (5.19) imply by Fatou's lemma

$$\mathbb{E} \left[\liminf_{h \rightarrow 0} \sup_{t \in [0, T_{max}]} \left(E_h[\tilde{u}^h](t) + \left(\int_{\mathcal{O}} \tilde{u}^h dx \right)^{-1} \right) \right] \leq C < \infty$$

and therefore

$$\liminf_{h \rightarrow 0} \sup_{t \in [0, T_{max}]} \left(E_h[\tilde{u}^h](t) + \left(\int_{\mathcal{O}} \tilde{u}^h dx \right)^{-1} \right) < \infty$$

almost surely. Thus, by (4.3) we have almost surely

$$\limsup_{h \rightarrow 0} \min_{x \in \mathcal{O}, t \in [0, T_{max}]} \tilde{u}^h(x, t) > 0.$$

The almost sure uniform convergence (5.15) therefore entails strict positivity of the limit \tilde{u} , almost surely. \square

LEMMA 5.9. *For the limits \tilde{J} and \tilde{p} , we have the identification*

$$\tilde{J} = \tilde{u} \tilde{p}_x$$

and

$$\tilde{p} = -\tilde{u}_{xx} + \mathcal{W}'(\tilde{u})$$

pointwise a.e. almost surely. Furthermore, we have $\tilde{u}_{xxx} \in L^2(\mathcal{O} \times [0, T_{max}])$ almost surely.

Proof. For \tilde{J}^h , by (5.5) for any $\phi \in C^\infty(\mathcal{O} \times [0, T_{max}])$ the equality

$$\mathbb{E} \left[\left| \int_0^{T_{max}} \int_{\mathcal{O}} \tilde{J}^h \phi dx dt - \int_0^{T_{max} \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h^{1/2}(\tilde{u}^h) \tilde{p}_x^h \phi dx dt \right| \right] = 0$$

holds. In order to pass to the limit in this expression, we apply Fatou's lemma to the expectation: By our convergence properties, we have almost surely

$$\begin{aligned} \int_0^{T_{max}} \int_{\mathcal{O}} \tilde{J}^h \phi dx dt &\rightarrow \int_0^{T_{max}} \int_{\mathcal{O}} \tilde{J} \phi dx dt, \\ \int_0^{T_{max} \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h^{1/2}(\tilde{u}^h) \tilde{p}_x^h \phi dx dt &\rightarrow \int_0^{T_{max}} \int_{\mathcal{O}} M^{1/2}(\tilde{u}) \tilde{p}_x \phi dx dt \end{aligned}$$

for $h \rightarrow 0$; note that the restriction of the time integral to $T_{max} \wedge \tilde{T}_h$ is immaterial, as by (5.6) we have $\tilde{p}_x^h = 0$ for $t > \tilde{T}_h$. Therefore, Fatou's lemma yields

$$\mathbb{E} \left[\left| \int_0^{T_{max}} \int_{\mathcal{O}} \tilde{J} \phi dx dt - \int_0^{T_{max} \wedge \tilde{T}_h} \int_{\mathcal{O}} M^{1/2}(\tilde{u}) \tilde{p}_x \phi dx dt \right| \right] = 0$$

for any $\phi \in C^\infty(\mathcal{O} \times [0, T_{max}])$. This provides the identification of \tilde{J} .

Similarly, by (5.6) we have for all $\phi^h \in X_h$ and all $0 \leq t_1 < t_2 < T_{max}$

$$(5.24) \quad \mathbb{E} \left[\left| \int_{t_1}^{t_2} (\tilde{p}^h, \phi^h)_h - \chi_{\tilde{T}_h} \int_{\mathcal{O}} \tilde{u}_x^h \phi_x^h dx - \chi_{\tilde{T}_h} (\mathcal{I}_h[\mathcal{W}'(\tilde{u}^h)], \phi^h)_h dt \right| \right] = 0.$$

For a smooth test function $\phi \in C^\infty(\mathcal{O})$, consider the sequence of test functions $\phi^h := \mathcal{I}_h[\phi] \in X_h$. We then have the convergence $\phi^h \rightarrow \phi$ strongly in $L^\infty(\mathcal{O})$ and $\Delta_h \phi^h \rightarrow \Delta \phi$ strongly in $L^\infty(\mathcal{O})$ (the latter assertion is an easy consequence of the Taylor expansion). Furthermore, we have

$$\begin{aligned} & \left| \int_{t_1}^{t_2} (\tilde{p}^h, \phi^h)_h dt - \int_{t_1}^{t_2} \int_{\mathcal{O}} p^h \phi^h dx dt \right| \\ &= \left| \int_{t_1}^{t_2} h \sum_{i=1}^{L_h} \left(\tilde{p}^h(ih) \phi^h(ih) - \int_{((i-1)h, ih)} \tilde{p}^h \phi^h dx \right) \right| \\ &\leq Ch \int_{t_1}^{t_2} \|\tilde{p}_x^h\|_{L^1(\mathcal{O})} \|\phi^h\|_{L^\infty(\mathcal{O})} + \|\tilde{p}^h\|_{L^\infty(\mathcal{O})} \|\phi_x^h\|_{L^1(\mathcal{O})} dt. \end{aligned}$$

From Lemma 5.5, we infer $\chi_{\tilde{T}_h} \equiv 1$ on $[t_1, t_2]$ for h small enough (depending on $\tilde{\omega}$). Putting these considerations together, the convergences

$$\begin{aligned} & \int_{t_1}^{t_2} (\tilde{p}^h, \phi^h)_h dt \rightarrow \int_{t_1}^{t_2} \int_{\mathcal{O}} \tilde{p} \phi dx dt, \\ & \int_{t_1}^{t_2} \chi_{\tilde{T}_h} \int_{\mathcal{O}} \tilde{u}_x^h \phi_x^h dx dt = - \int_{t_1}^{t_2} \chi_{\tilde{T}_h} (\tilde{u}^h, \Delta_h \phi^h)_h dt \rightarrow - \int_{t_1}^{t_2} \int_{\mathcal{O}} \tilde{u} \phi_{xx} dx dt, \\ & \int_{t_1}^{t_2} \chi_{\tilde{T}_h} (\mathcal{I}_h[\mathcal{W}'(\tilde{u}^h)], \phi^h)_h dt \rightarrow \int_{t_1}^{t_2} \int_{\mathcal{O}} \mathcal{W}'(\tilde{u}) \phi dx dt \end{aligned}$$

hold almost surely (where for the last one we have used the $\tilde{\omega}$ -dependent lower bound for \tilde{u} from Lemma 5.8 and the uniform convergence almost surely).

Therefore, we may pass to the limit $h \rightarrow 0$ in (5.24) using Fatou's lemma for the expectation. This yields

$$\mathbb{E} \left[\left| \int_{t_1}^{t_2} \int_{\mathcal{O}} \tilde{p} \phi dx + \int_{\mathcal{O}} \tilde{u} \phi_{xx} dx - \int_{\mathcal{O}} \mathcal{W}'(\tilde{u}) \phi dx dt \right| \right] = 0$$

and thus provides the desired identification of \tilde{p} (first as a distribution, then due to $\tilde{p} \in L^2(\mathcal{O} \times [0, T_{max}])$ a.s. and $\mathcal{W}'(\tilde{u}) \in L^2(\mathcal{O} \times [0, T_{max}])$ a.s. also in the L^2 sense).

Using the lower bound from Lemma 5.8, the fact that $\tilde{J} \in L^2(\mathcal{O} \times [0, T_{max}])$ almost surely, the identity $\tilde{p}_x = \tilde{J}/M^{1/2}(\tilde{u})$, and the identity in the sense of distributions $\tilde{p}_x = -\tilde{u}_{xxx} + \mathcal{W}'(\tilde{u})$, we deduce that almost surely $\tilde{u}_{xxx} \in L^2(\mathcal{O} \times [0, T_{max}])$. \square

5.3. Convergence of the stochastic integral. Consider for $v \in H_{per}^2(\mathcal{O})$ arbitrary, but fixed, the operator $\mathcal{M}_{h,v} : \Omega \times [0, T_{max}] \rightarrow \mathbb{R}$ defined by

$$(5.25) \quad \begin{aligned} \mathcal{M}_{h,v}(t) &:= (u^h(t) - u_0^h, \mathcal{P}_h v)_h + \int_0^{t \wedge T_h} \int_{\mathcal{O}} M_h(u^h) p_x^h (\mathcal{P}_h v)_x dx ds \\ &= \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} \int_{\mathcal{O}} (u^h \lambda_{\ell} g_{\ell})_x \mathcal{P}_h v dx d\beta_{\ell}(s). \end{aligned}$$

Here, $\mathcal{P}_h : H_{per}^1(\mathcal{O}) \rightarrow X_h$ is a projection operator satisfying

$$(5.26) \quad \lim_{h \rightarrow 0} \|\mathcal{P}_h v - v\|_{H_{per}^1} = 0$$

for all $v \in H_{per}^1(\mathcal{O})$. Observe that by the optional stopping theorem, $\mathcal{M}_{h,v}$ is a real valued martingale; that is, denoting by r_s the restriction of a function on $[0, T_{max}]$ onto $[0, s]$, we have

$$(5.27) \quad \mathbb{E}([\mathcal{M}_{h,v}(t) - \mathcal{M}_{h,v}(s)]\Psi(r_s u^h, r_s W)) = 0$$

for all $0 \leq s \leq t \leq T_{max}$ and for all $[0, 1]$ -valued continuous functions Ψ defined on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, s]) \times C([0, s]; L^2(\mathcal{O}))$.

LEMMA 5.10. *For the quadratic variation of $\mathcal{M}_{h,v}$, we have*

$$(5.28) \quad \begin{aligned} \langle\langle \mathcal{M}_{h,v} \rangle\rangle_t &= \int_0^{t \wedge T_h} \sum_{\ell=1}^{N_h} \lambda_\ell^2 \left(\int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{P}_h v \, dx \right)^2 ds \\ &\leq C \|v\|_{H_{per}^1}^2 \int_0^{t \wedge T_h} \|u^h(s)\|_{L^2(\mathcal{O})}^2 ds. \end{aligned}$$

Proof. Consider $R(u^h, v) : \Omega \times [0, T_{max}] \times L^2(\mathcal{O}) \rightarrow \mathbb{R}$ defined by

$$(\omega, t, z) \mapsto \chi_{T_h}(t, \omega) \int_{\mathcal{O}} (u^h \mathcal{P}_{N_h} z)_x \mathcal{P}_h v \, dx(t, \omega),$$

where \mathcal{P}_{N_h} denotes the orthogonal L^2 -projection onto $\text{span}\{g_1, \dots, g_{N_h}\}$. For the Hilbert–Schmidt norm, we get using $\|g_\ell\|_{L^\infty} \leq C(L)$ and $\sum_{\ell=1}^\infty \lambda_\ell^2 < \infty$

$$\begin{aligned} \|R(u^h, v)(t, \omega)\|_{L_2(Q^{1/2}L^2(\mathcal{O}); \mathbb{R})}^2 &= \chi_{T_h}(t) \sum_{\ell=1}^{N_h} \lambda_\ell^2 \left(\int_{\mathcal{O}} (u^h g_\ell) \partial_x \mathcal{P}_h v \, dx \right)^2 \\ &\leq C \chi_{T_h}(t) \|u^h\|_{L^2(\mathcal{O})}^2(t, \omega) \|v\|_{H_{per}^1}^2. \end{aligned}$$

By Lemma 2.4.3 in [51], the result is obtained. □

Remark 5.11. From (4.21) and (5.28), we infer that $\mathcal{M}_{h,v}$ is a square-integrable martingale.

Similarly, $\mathcal{M}_{h,v}^2 - \int_0^{(\cdot) \wedge T_h} \sum_{\ell=1}^{N_h} \lambda_\ell^2 \left(\int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{P}_h v \, dx \right)^2 ds$ is a martingale. For the identification of the stochastic integral in the limit $h \rightarrow 0$, we will study the processes

$$(5.29) \quad \beta_\ell(t) = \int_{\mathcal{O}} \int_0^t \frac{1}{\lambda_\ell} g_\ell \, dW \, dx$$

and their cross variations with $\mathcal{M}_{h,v}$.

LEMMA 5.12. *For $\ell \in \mathbb{N}$ the cross variation $\langle\langle \mathcal{M}_{h,v}, \beta_\ell \rangle\rangle_t$ is given by the formula*

$$(5.30) \quad \langle\langle \mathcal{M}_{h,v}, \beta_\ell \rangle\rangle_t = \begin{cases} \lambda_\ell \int_0^{t \wedge T_h} \int_{\mathcal{O}} (u^h g_\ell)_x \mathcal{P}_h v \, dx \, ds, & \ell \leq N_h, \\ 0, & \ell > N_h. \end{cases}$$

Proof. Consider the mappings $S_\pm(u^h, v) : \Omega \times I \times Q^{1/2}L^2(\mathcal{O}) \rightarrow \mathbb{R}$ given by

$$z \mapsto \chi_{T_h} \left(\int_{\mathcal{O}} \partial_x (u \mathcal{P}_{N_h} z) \mathcal{P}_h v \, dx \pm \frac{1}{\lambda_\ell} \int_{\mathcal{O}} g_\ell z \, dx \right)$$

with \mathcal{P}_{N_h} as in the proof of Lemma 5.10. Obviously,

$$\begin{aligned} & \|S_{\pm}(u^h, v)(t, \omega)\|_{L_2(Q^{1/2}L^2(\mathcal{O});\mathbb{R})} \\ &= \chi_{T_h}(t) \sum_{k=1}^{\infty} \left(\lambda_k \int_{\mathcal{O}} \partial_x(u^h \mathcal{P}_{N_h} g_k) \mathcal{P}_h v \, dx \pm \int_{\mathcal{O}} g_{\ell} g_k \frac{\lambda_k}{\lambda_{\ell}} \, dx \right)^2 \\ &= \chi_{T_h}(t) \left(\sum_{k=1}^{N_h} \lambda_k \left(\int_{\mathcal{O}} \partial_x(u^h g_k) \mathcal{P}_h v \, dx \right)^2 \right. \\ &\quad \left. \pm 2 \sum_{k=1}^{\infty} \lambda_k \int_{\mathcal{O}} \partial_x(u^h \mathcal{P}_{N_h} g_k) \mathcal{P}_h v \, dx \int_{\mathcal{O}} g_{\ell} g_k \frac{\lambda_k}{\lambda_{\ell}} \, dx \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \left(\int_{\mathcal{O}} g_{\ell} g_k \frac{\lambda_k}{\lambda_{\ell}} \, dx \right)^2 \right) \\ &= \begin{cases} \chi_{T_h}(t) \left(\sum_{k=1}^{N_h} \left(\int_{\mathcal{O}} \partial_x(u^h \lambda_k g_k) \mathcal{P}_h v \, dx \right)^2 \pm 2 \lambda_{\ell} \int_{\mathcal{O}} \partial_x(u^h g_k) \mathcal{P}_h v \, dx + 1 \right), & \ell \leq N_h, \\ \chi_{T_h}(t) \left(\sum_{k=1}^{N_h} \left(\int_{\mathcal{O}} \partial_x(u^h \lambda_k g_k) \mathcal{P}_h v \, dx \right)^2 + 1 \right), & \ell > N_h. \end{cases} \end{aligned}$$

Using $\langle\langle \mathcal{M}_{h,v}, \beta_{\ell} \rangle\rangle_t = \frac{1}{4} (\langle\langle S_+(u^h, v) \rangle\rangle_t + \langle\langle S_-(u^h, v) \rangle\rangle_t)$, we get (5.30). \square

In particular, $\mathcal{M}_{h,v} \beta_{\ell} - \lambda_{\ell} \int_0^{(\cdot) \wedge T_h} \int_{\mathcal{O}} (u^h g_{\ell})_x \mathcal{P}_h v \, dx \, ds$ for $\ell \leq N_h$ and $\mathcal{M}_{h,v} \beta_{\ell}$ for $\ell > N_h$ are martingales, too.

By equality of laws, we deduce that

$$(5.31) \quad \widetilde{\mathcal{M}}_{h,v}(t) := (\tilde{u}^h(t) - \tilde{u}^h(0), \mathcal{P}_h v)_h + \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h(\tilde{u}^h) \tilde{p}_x^h(\mathcal{P}_h v)_x \, dx \, ds,$$

$$(5.32) \quad \widetilde{\mathcal{M}}_{h,v}^2(t) - \int_0^{t \wedge \tilde{T}_h} \sum_{\ell=1}^{N_h} \lambda_{\ell}^2 \left(\int_{\mathcal{O}} (\tilde{u}^h g_{\ell})_x \mathcal{P}_h v \, dx \right)^2 \, ds,$$

$$(5.33) \quad \widetilde{\mathcal{M}}_{h,v}(t) \tilde{\beta}_{\ell}^h(t) - \lambda_{\ell} \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} (\tilde{u}^h g_{\ell})_x \mathcal{P}_h v \, dx \, ds \quad \text{for } \ell \leq N_h$$

$$(5.34) \quad \widetilde{\mathcal{M}}_{h,v}(t) \tilde{\beta}_{\ell}^h(t) \quad \text{for } \ell > N_h$$

are $(\tilde{\mathcal{F}}_t^h)$ -martingales, where $\tilde{\beta}_{\ell}^h(t) := \int_{\mathcal{O}} \int_0^t \lambda_{\ell}^{-1} g_{\ell} d\tilde{W}^h \, dx$. In particular,

$$(5.35) \quad \langle\langle \widetilde{\mathcal{M}}_{h,v} \rangle\rangle_t = \int_0^{t \wedge \tilde{T}_h} \sum_{\ell=1}^{N_h} \lambda_{\ell}^2 \left(\int_{\mathcal{O}} \tilde{u}^h g_{\ell} (\mathcal{P}_h v)_x \, dx \right)^2 \, ds$$

and

$$(5.36) \quad \langle\langle \widetilde{\mathcal{M}}_{h,v}, \tilde{\beta}_{\ell}^h \rangle\rangle_t = \begin{cases} \lambda_{\ell} \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} (\tilde{u}^h g_{\ell})_x \mathcal{P}_h v \, dx \, ds & \text{if } \ell \leq N_h, \\ 0 & \text{if } \ell > N_h. \end{cases}$$

Exemplarily, for $\widetilde{\mathcal{M}}_{h,v}(t)$ we argue

$$\begin{aligned} 0 &= \mathbb{E} (\Psi(r_s u^h, r_s W) [\mathcal{M}_{h,v}(t) - \mathcal{M}_{h,v}(s)]) \\ &= \mathbb{E} (\Psi(r_s \tilde{u}^h, r_s \tilde{W}^h) [\widetilde{\mathcal{M}}_{h,v}(t) - \widetilde{\mathcal{M}}_{h,v}(s)]) \end{aligned}$$

for all $0 \leq s \leq t \leq T_{\max}$ and all $[0, 1]$ -valued continuous functions Ψ defined on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, s]) \times C([0, s]; L^2(\mathcal{O}))$.

Starting point for the passage to the limit $h \rightarrow 0$ are the identities

$$(5.37) \quad \mathbb{E}((\widetilde{\mathcal{M}}_{h,v}(t) - \widetilde{\mathcal{M}}_{h,v}(s))\Psi(r_s \tilde{u}^h, r_s \tilde{W}^h)) = 0,$$

$$(5.38) \quad \mathbb{E} \left(\left(\widetilde{\mathcal{M}}_{h,v}^2(t) - \widetilde{\mathcal{M}}_{h,v}^2(s) - \int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \sum_{\ell=1}^{N_h} \lambda_\ell^2 \left(\int_{\mathcal{O}} (\tilde{u}^h g_\ell)(\mathcal{P}_h v)_x dx \right)^2 d\tau \right) \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h) \right) = 0,$$

and

$$(5.39) \quad \mathbb{E} \left(\left(\widetilde{\mathcal{M}}_{h,v}(t) \tilde{\beta}_\ell^h(t) - \widetilde{\mathcal{M}}_{h,v}(s) \tilde{\beta}_\ell^h(s) - \int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \lambda_\ell \int_{\mathcal{O}} (\tilde{u}^h g_\ell)_x \mathcal{P}_h v dx d\tau \right) \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h) \right) = 0$$

for all $s \leq t \in [0, T_{\max}]$ and for all $[0, 1]$ -valued continuous functions Ψ defined on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, s]) \times C([0, s]; L^2(\mathcal{O}))$.

Let us pass to the limit in (5.37).

LEMMA 5.13. *For all $[0, 1]$ -valued continuous functions Ψ defined on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, s]) \times C([0, s]; L^2(\mathcal{O}))$, we have*

$$(5.40) \quad \mathbb{E} \left(\left(\int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(s))v dx + \int_s^t \int_{\mathcal{O}} m(\tilde{u}) \tilde{p}_x v_x dx d\tau \right) \Psi(r_s \tilde{u}, r_s \tilde{W}) \right) = 0$$

for all $0 \leq s \leq t < T_{\max}$.

Proof. Note that due to Lemma 5.5, we have $\chi_{\tilde{T}_h} = 1$ on $[s, t]$ for h sufficiently small depending on ω . By definition (5.31), we first discuss the term $(\tilde{u}^h(t) - \tilde{u}^h(s), \mathcal{P}_h v)_h$. By the strong convergence of \tilde{u}^h in $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{\max}])$ $\tilde{\mathbb{P}}$ -almost surely and of $\mathcal{P}_h v$ toward v in $L^2(\mathcal{O})$, we readily identify

$$(5.41) \quad \lim_{h \rightarrow 0} (\tilde{u}^h(t) - \tilde{u}^h(s), \mathcal{P}_h v)_h = \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(s))v dx.$$

In addition, $\Psi[r_s \tilde{u}^h, r_s \tilde{W}^h]$ converges $\tilde{\mathbb{P}}$ -almost surely to $\Psi[r_s \tilde{u}, r_s \tilde{W}]$ in \mathbb{R} by continuity of Ψ , the $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{\max}])$ -convergence of $\tilde{u}^h \rightarrow \tilde{u}$, and the $C([0, T_{\max}]; L^2(\mathcal{O}))$ -convergence of \tilde{W}^h .

To discuss the convergence behavior of

$$(5.42) \quad \int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h(\tilde{u}^h) \tilde{p}_x^h (\mathcal{P}_h v)_x dx d\tau \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h),$$

we proceed as follows. Due to (5.15), $M_h(\tilde{u}^h)$ converges to \tilde{u}^2 in $L^\infty(\mathcal{O} \times [0, T_{\max}])$ $\tilde{\mathbb{P}}$ -almost surely. By (5.18), Lemma 2.1, (H3), and Poincaré's inequality, we have \tilde{u}^h uniformly bounded in $L^{2\tilde{p}}(\tilde{\Omega}; L^\infty(\mathcal{O} \times [0, T_{\max}]))$. By Vitali's theorem,

$$(5.43) \quad M_h(\tilde{u}^h) \rightarrow \tilde{u}^2 \text{ in } L^{\tilde{p}}(\tilde{\Omega}; L^\infty(\mathcal{O} \times I)).$$

Next, we show that

$$(5.44) \quad \sqrt{M_h(\tilde{u}^h)}(\mathcal{P}_h v)_x \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h) \rightarrow \tilde{u} v_x \Psi(r_s \tilde{u}, r_s \tilde{W})$$

strongly in $L^2(\tilde{\Omega} \times \mathcal{O} \times [0, T_{max}])$. We note that in the discussion of L^2 -convergence, the Ψ -term is readily handled as it is uniformly bounded and converging $\tilde{\mathbb{P}}$ -almost surely. We have

$$\begin{aligned} & \mathbb{E} \left(\int_0^{T_{max}} \int_{\mathcal{O}} \left| \sqrt{M_h(\tilde{u}^h)}(\mathcal{P}_h v)_x - \tilde{u} v_x \right|^2 dx d\tau \right) \\ & \leq 2\mathbb{E} \left(\|\tilde{u}^h\|_{L^\infty(\mathcal{O} \times [0, T_{max}])}^2 \right) \|(\mathcal{P}_h v - v)_x\|_{L^2(\mathcal{O})}^2 \\ & \quad + 2\|(\mathcal{P}_h v)_x\|_{L^2(\mathcal{O})}^2 \mathbb{E} \left(\left\| \sqrt{M_h(\tilde{u}^h)} - \tilde{u} \right\|_{L^\infty(\mathcal{O} \times [0, T_{max}])}^2 \right), \end{aligned}$$

and the right-hand side converges to zero due to (5.15), (5.18), and (5.26). Hence, (5.44) is proven.

By the weak convergence of \tilde{J}^h toward \tilde{J} in $L^2(\mathcal{O} \times [0, T_{max}])$ $\tilde{\mathbb{P}}$ -almost surely, (see (5.17)), we infer

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h(\tilde{u}^h) \tilde{p}_x^h (\mathcal{P}_h v)_x dx d\tau \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h) \\ & = \lim_{h \rightarrow 0} \int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \sqrt{M_h(\tilde{u}^h)} \tilde{J}_h (\mathcal{P}_h v)_x dx d\tau \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h) \\ (5.45) \quad & = \int_s^t \int_{\mathcal{O}} \tilde{u} v_x \tilde{J} dx d\tau \Psi(r_s \tilde{u}, r_s \tilde{W}) \end{aligned}$$

$\tilde{\mathbb{P}}$ -almost surely. Setting

$$B_h(\tilde{u}^h, \tilde{p}^h, v, s, t) := \int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} \sqrt{M_h(\tilde{u}^h)} \tilde{J}_h (\mathcal{P}_h v)_x dx d\tau \Psi(r_s \tilde{u}^h, r_s \tilde{W}^h)$$

and using the estimate

$$\begin{aligned} & |B_h(\tilde{u}^h, \tilde{p}^h, v, s, t)| \\ & \leq \left(\int_{s \wedge \tilde{T}_h}^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} |\tilde{J}_h|^2 dx d\tau \right)^{1/2} \left(\sup_{\mathcal{O} \times [0, T_{max}]} M_h(\tilde{u}^h) \right)^{1/2} \left(\int_0^{T_{max}} \int_{\mathcal{O}} |(\mathcal{P}_h v)_x|^2 dx d\tau \right)^{1/2} \end{aligned}$$

as well as Lemma 2.1, (H3), and (4.22), we infer uniform integrability of a q -moment of $B_h(\tilde{u}^h, \tilde{p}^h, v, s, t)$ for a number $q > 1$. Vitali's theorem then entails

$$\lim_{h \rightarrow 0} \mathbb{E} (B_h(\tilde{u}^h, \tilde{p}^h, v, s, t)) = \mathbb{E} \left(\int_s^t \int_{\mathcal{O}} \tilde{u} v_x \tilde{J} dx d\tau \Psi(r_s \tilde{u}, r_s \tilde{W}) \right).$$

By Lemma 5.9, we get, in particular, $\tilde{J} = \tilde{u} \tilde{p}_x$. Together with (5.41), the lemma is proven. \square

LEMMA 5.14. For all $[0, 1]$ -valued continuous functions Ψ defined on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, s]) \times C([0, s]; L^2(\mathcal{O}))$, we have

$$(5.46) \quad \mathbb{E} \left(\left(\widetilde{\mathcal{M}}_v^2(t) - \widetilde{\mathcal{M}}_v^2(s) - \int_s^t \sum_{\ell=1}^{N_h} \lambda_\ell^2 \left(\int_{\mathcal{O}} \tilde{u} g_\ell v_x dx \right)^2 d\tau \right) \Psi(r_s \tilde{u}, r_s \tilde{W}) \right) = 0$$

for all $0 \leq s \leq t < T_{\max}$, where

$$\widetilde{\mathcal{M}}_v(t) := \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(0)) v dx + \int_0^t \int_{\mathcal{O}} \tilde{u}^2 \tilde{p}_x v_x dx d\tau.$$

Proof. Combining Lemma 5.5, (5.17), and the results of subsection 5.2, we have

$$(5.47) \quad \tilde{J}^h \rightharpoonup \tilde{u} \tilde{p}_x \text{ in } L^2(\mathcal{O} \times [0, T_{\max}]) \tilde{\mathbb{P}}\text{-almost surely}$$

for an appropriate subsequence. Mimicking the argument which entails (5.41) and using Lemma 5.5 as well as the convergence $\tilde{\mathbb{P}}$ -almost surely of \tilde{u}^h, \tilde{J}^h in \mathcal{X}_u and \mathcal{X}_J , respectively, we infer that

$$\widetilde{\mathcal{M}}_{h,v}^2(t) := \left((\tilde{u}^h(t) - \tilde{u}^h(0), \mathcal{P}_h v)_h + \int_0^{t \wedge \tilde{T}_h} \int_{\mathcal{O}} M_h(\tilde{u}^h) \tilde{p}_x^h (\mathcal{P}_h v)_x dx d\tau \right)^2$$

converges (along a subsequence) $\tilde{\mathbb{P}}$ -almost surely to

$$\widetilde{\mathcal{M}}_v^2(t) = \left(\int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(0)) v dx + \int_0^t \int_{\mathcal{O}} \tilde{u}^2 \tilde{p}_x v_x dx d\tau \right)^2$$

for all $t \in [0, T]$ with $\tilde{p} = -\tilde{u}_{xx} + \mathcal{W}'(\tilde{u})$.

The next step is to prove

$$(5.48) \quad \mathbb{E}(\widetilde{\mathcal{M}}_{h,v}^2(t)) \rightarrow \mathbb{E}(\widetilde{\mathcal{M}}_v^2(t)) \quad \forall t \in [0, T_{\max}].$$

For this, we show higher integrability starting from the representation

$$(5.49) \quad \widetilde{\mathcal{M}}_{h,v}(t) = - \sum_{\ell=1}^{N_h} \int_0^{t \wedge \tilde{T}_h} \lambda_\ell \int_{\mathcal{O}} \tilde{u}^h g_\ell (\mathcal{P}_h v)_x dx d\tilde{\beta}_\ell^h.$$

Combining the martingale moment inequality (see [44, Prop. 3.26])

$$\mathbb{E}(|\widetilde{\mathcal{M}}_{h,v}(t)|^{2q}) \leq C_q \mathbb{E}(\langle \langle \widetilde{\mathcal{M}}_{h,v} \rangle \rangle_t^q)$$

for any $q > 0$ with (5.28) formulated for $\widetilde{\mathcal{M}}_{h,v}$, we get—using Lemma 2.1 and (H3)—

$$(5.50) \quad \begin{aligned} \mathbb{E}(|\widetilde{\mathcal{M}}_{h,v}(t)|^{2q}) &\leq C \|v\|_{H_{per}^{1, 2q}}^{2q} \mathbb{E} \left(\left(\int_0^{t \wedge \tilde{T}_h} \|\tilde{u}^h(s)\|_{L^2(\mathcal{O})}^2 ds \right)^q \right) \\ &\leq C \|v\|_{H_{per}^{1, 2q}}^{2q} T_{\max}^q \mathbb{E} \left[\sup_{s \in [0, T_{\max}]} E_h(\tilde{u}^h(s))^q + \sup_{s \in [0, T_{\max}]} \left(\int_{\mathcal{O}} \tilde{u}^h(s) dx \right)^q \right]. \end{aligned}$$

Choosing $1 < q \leq \bar{p}$, we deduce the uniform integrability of $|\widetilde{\mathcal{M}}_{h,v}(t)|$ in $L^{2q}(\Omega)$. Using Vitali's theorem and the boundedness of Ψ , (5.48) is established.

Now, we discuss the convergence behavior of $\langle\langle \widetilde{\mathcal{M}}_{h,v} \rangle\rangle_t$ which—according to (5.35)—is given by

$$(5.51) \quad \int_0^{t \wedge \widetilde{T}_h} \sum_{\ell=1}^{N_h} \lambda_\ell^2 \left(\int_{\mathcal{O}} \tilde{u}^h g_\ell (\mathcal{P}_h v)_x dx \right)^2 ds.$$

Convergence $\tilde{\mathbb{P}}$ -almost surely is obvious, given the boundedness of the g_ℓ , $\ell \in \mathbb{N}$, as well as the strong convergence of $(\tilde{u}^h)_{h \rightarrow 0}$ and $((\mathcal{P}_h v)_x)_{h \rightarrow 0}$ in $L^2(\mathcal{O} \times [0, T_{max}])$ and $L^2(\mathcal{O})$, respectively. Higher integrability for (5.51)—which equals $\langle\langle \widetilde{\mathcal{M}}_{h,v} \rangle\rangle_t$ —has already been proven in (5.50); hence, we may conclude by Vitali’s theorem. \square

Furthermore, we have the following result on the cross variation of $\widetilde{\mathcal{M}}_v$ and $\tilde{\beta}_\ell$, the proof of which we omit as it is similar to the preceding proofs.

LEMMA 5.15. *For all $[0, 1]$ -valued continuous functions Ψ defined on $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, s]) \times C([0, s]; L^2(\mathcal{O}))$, we have*

$$(5.52) \quad \mathbb{E} \left(\left(\widetilde{\mathcal{M}}_v(t) \tilde{\beta}_\ell(t) - \widetilde{\mathcal{M}}_v(s) \tilde{\beta}_\ell(s) - \int_s^t \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tau \right) \Psi(r_s \tilde{u}, r_s \tilde{W}) \right) = 0$$

for all $\ell \in \mathbb{N}$ and all $s \leq t \in [0, T_{max})$.

LEMMA 5.16. *We have*

$$(5.53) \quad \widetilde{\mathcal{M}}_v(t) = \sum_{\ell=1}^{\infty} \int_0^t \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tilde{\beta}_\ell.$$

Proof. It is sufficient to show that the quadratic variation of

$$\widetilde{\mathcal{M}}_v(t) - \sum_{\ell=1}^{\infty} \int_0^t \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tilde{\beta}_\ell$$

vanishes (as a martingale with vanishing quadratic variation is almost surely constant). We get

$$(5.54) \quad \begin{aligned} & \left\langle \left\langle \widetilde{\mathcal{M}}_v(\cdot) - \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tilde{\beta}_\ell \right\rangle \right\rangle_T \\ &= \left\langle \left\langle \widetilde{\mathcal{M}}_v(\cdot) \right\rangle \right\rangle_T + \left\langle \left\langle \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tilde{\beta}_\ell \right\rangle \right\rangle_T \\ &- 2 \left\langle \left\langle \widetilde{\mathcal{M}}_v(\cdot), \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tilde{\beta}_\ell \right\rangle \right\rangle_T. \end{aligned}$$

For the third term on the right-hand side, we use the cross-variation formula (cf. [44, section 3.2]) to get

$$(5.55) \quad \begin{aligned} & \left\langle \left\langle \widetilde{\mathcal{M}}_v(\cdot), \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u} g_\ell)_x v dx d\tilde{\beta}_\ell \right\rangle \right\rangle_T \\ &= \sum_{\ell=1}^{\infty} \int_0^T \lambda_\ell \int_{\mathcal{O}} v (\tilde{u} g_\ell)_x dx d\langle\langle \widetilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_\ell(\cdot) \rangle\rangle_s. \end{aligned}$$

By the identity

$$\langle\langle \widetilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_\ell(\cdot) \rangle\rangle_T = \int_0^T \lambda_\ell \int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \, dt$$

(which follows directly from (5.52)) and $\tilde{u}_x \in L^\infty([0, T_{max}]; L^2(\mathcal{O}))$ $\tilde{\mathbb{P}}$ -almost surely, we observe that the process $s \mapsto \langle\langle \widetilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_\ell(\cdot) \rangle\rangle_s$ is absolutely continuous $\tilde{\mathbb{P}}$ -almost surely. As a consequence,

$$d\langle\langle \widetilde{\mathcal{M}}_v(\cdot), \tilde{\beta}_\ell(\cdot) \rangle\rangle_s = \lambda_\ell \int_{\mathcal{O}} (\tilde{u}(s)g_\ell)_x v \, dx \, ds.$$

Hence,

$$(5.56) \quad \left\langle\left\langle \widetilde{\mathcal{M}}_v(\cdot), \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \, d\tilde{\beta}_\ell \right\rangle\right\rangle_T = \int_0^T \sum_{\ell=1}^{\infty} \lambda_\ell^2 \left(\int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \right)^2 ds.$$

Together with the identities

$$\langle\langle \widetilde{\mathcal{M}}_v(\cdot) \rangle\rangle_T = \int_0^T \sum_{\ell=1}^{\infty} \lambda_\ell^2 \left(\int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \right)^2 ds$$

and

$$\left\langle\left\langle \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \, d\tilde{\beta}_\ell \right\rangle\right\rangle_T = \sum_{\ell=1}^{\infty} \int_0^T \lambda_\ell^2 \left(\int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \right)^2 ds,$$

we note

$$\left\langle\left\langle \widetilde{\mathcal{M}}_v(\cdot) - \sum_{\ell=1}^{\infty} \int_0^{(\cdot)} \lambda_\ell \int_{\mathcal{O}} (\tilde{u}g_\ell)_x v \, dx \, d\tilde{\beta}_\ell \right\rangle\right\rangle_T \equiv 0,$$

which gives the claim. □

It remains to establish Theorem 3.2.

Proof of Theorem 3.2. From Proposition 5.4, Lemma 5.7, and Lemma 5.9, we infer the existence of a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$, of a Wiener process $\tilde{W}(t) = \sum_{\ell=1}^{\infty} \lambda_\ell \tilde{\beta}_\ell(t)g_\ell$, and of random variables

$$\begin{aligned} \tilde{u} &\in L^2(\tilde{\Omega}; C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])), \\ \tilde{p} &\in L^{3/2}(\tilde{\Omega}; L^2([0, T_{max}]; H_{per}^1(\mathcal{O}))), \\ \tilde{J} &\in L^2(\tilde{\Omega} \times \mathcal{O} \times [0, T_{max}]) \end{aligned}$$

satisfying

$$\begin{aligned} \tilde{J} &= \tilde{u}\tilde{p}_x && \tilde{\mathbb{P}}\text{-almost surely in } L^2(\mathcal{O} \times [0, T_{max}]), \\ \tilde{p} &= -\tilde{u}_{xx} + \mathcal{W}'(\tilde{u}) && \tilde{\mathbb{P}}\text{-almost surely in } L^2(\mathcal{O} \times [0, T_{max}]), \\ \tilde{u} &\in L^2([0, T_{max}]; H_{per}^3(\mathcal{O})) && \tilde{\mathbb{P}}\text{-almost surely.} \end{aligned}$$

Furthermore, we have $\Lambda = \tilde{\mathbb{P}} \circ \tilde{u}_0^{-1}$ by construction.

Lemma 5.13 implies that for arbitrary $v \in H_{per}^2(\mathcal{O})$

$$\tilde{\mathcal{M}}_v(t) = \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(0))v \, dx + \int_0^t \int_{\mathcal{O}} \tilde{u}^2 \tilde{p}_x v_x \, dx \, d\tau$$

is an $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ -martingale. Due to Lemma 5.16, we conclude

$$\int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(0))v \, dx + \int_0^t \int_{\mathcal{O}} \tilde{u}^2 \tilde{p}_x v_x \, dx \, d\tau = \sum_{\ell=1}^{\infty} \int_0^t \lambda_{\ell} \int_{\mathcal{O}} (\tilde{u} g_{\ell})_x v \, dx \, d\tilde{\beta}_{\ell}.$$

Furthermore, Lemma 5.8 almost surely provides a positive lower bound for \tilde{u} .

Finally, by Fatou's lemma and Proposition 4.4, we have for any $\bar{p} \geq 1$

$$\begin{aligned} & \mathbb{E} \left[\liminf_{h \rightarrow 0} \left(\sup_{t \in [0, T_{max}]} E_h[\tilde{u}^h]^{\bar{p}} + \int_0^{T_{max}} \int_{\mathcal{O}} |\tilde{J}^h|^2 \, dx \, dt \right) \right] \\ & \leq \liminf_{h \rightarrow 0} \mathbb{E} \left[\sup_{t \in [0, T_{max}]} E_h[\tilde{u}^h]^{\bar{p}} + \int_0^{T_{max}} \int_{\mathcal{O}} |\tilde{J}^h|^2 \, dx \, dt \right] \\ & \leq C(\bar{p}, u_0, T_{max}). \end{aligned}$$

By the almost sure convergence of \tilde{u}^h in $C^{\gamma, \gamma/4}(\mathcal{O} \times [0, T_{max}])$ and the almost sure strict positivity of the limit \tilde{u} , we deduce $\int_{\mathcal{O}} \mathcal{I}_h \mathcal{W}(\tilde{u}^h) \, dx \rightarrow \int_{\mathcal{O}} \mathcal{W}(\tilde{u}) \, dx$ in $L^{\infty}([0, T_{max}])$ almost surely. By the lower semicontinuity of the $L^2(\mathcal{O} \times [0, T_{max}])$ norm with respect to weak convergence (for $\tilde{J}^h \rightarrow \tilde{J}$) and the lower semicontinuity of the $L^{\infty}([0, T_{max}]; H^1(\mathcal{O}))$ norm with respect to convergence in the sense of distributions (for $\tilde{u}^h \rightarrow \tilde{u}$), we finally get

$$\mathbb{E} \left[\sup_{t \in [0, T_{max}]} E[\tilde{u}]^{\bar{p}} + \int_0^{T_{max}} \int_{\mathcal{O}} |\tilde{J}|^2 \, dx \, dt \right] \leq C(\bar{p}, u_0, T_{max}). \quad \square$$

6. Concluding remarks. We have proved a first result on the existence of almost surely positive solutions to a stochastic thin-film equation. In a forthcoming paper, we wish to study pathwise uniqueness of solutions—this way establishing the existence of pathwise solutions. Even concerning existence of solutions, interesting questions remain open: The case of higher space dimensions and the case of general mobilities $m(u) = u^n$, $n > 0$, which refers to different flow boundary conditions at the liquid-solid interface. For both problems, we do not expect the techniques developed in this paper to carry over easily. For the thin-film equation in the multi-dimensional setting, this is due to the fact that already in the deterministic setting Hölder regularity of solutions is still an open problem.

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