

On the nonexistence of k -reptile simplices in \mathbb{R}^3 and \mathbb{R}^4 *

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Abstract

A d -dimensional simplex S is called a k -reptile (or a k -reptile simplex) if it can be tiled by k simplices with disjoint interiors that are all mutually congruent and similar to S . For $d = 2$, triangular k -reptiles exist for all k of the form $a^2, 3a^2$ or $a^2 + b^2$ and they have been completely characterized by Snover, Waiveris, and Williams. On the other hand, the only k -reptile simplices that are known for $d \geq 3$, have $k = m^d$, where m is a positive integer. We substantially simplify the proof by Matoušek and the second author that for $d = 3$, k -reptile tetrahedra can exist only for $k = m^3$. We then prove a weaker analogue of this result for $d = 4$ by showing that four-dimensional k -reptile simplices can exist only for $k = m^2$.

Keywords: k -reptile simplex; space-filling simplex; tiling; spherical triangle

1 Introduction

A *tiling* of a closed set X in \mathbb{R}^d (or in the unit sphere S^d) is a locally finite decomposition $X = \bigcup_{i \in I} X_i$ into closed sets with nonempty and pairwise disjoint interiors. The sets X_i

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are called *tiles*. If X has a tiling where all the tiles are congruent to a set T , we say that T *tiles* X , or, that X *can be tiled with* ($|I|$ *copies of*) T . We emphasize that congruence includes mirror symmetries.

A closed set $X \subset \mathbb{R}^d$ with nonempty interior is called a k -*reptile* (or a k -*reptile set*) if X can be tiled with k mutually congruent copies of a set similar to X .

It is easy to see that whenever S is a d -dimensional k -reptile set, then S is *space-filling*, that is, the space \mathbb{R}^d can be tiled with S : indeed, using the tiling of S by its smaller copies as a pattern, one can inductively tile larger and larger similar copies of S . On the other hand, it is a simple exercise to find space-filling polytopes or polygons that are not k -reptiles for any $k \geq 2$.

Clearly, every triangle tiles \mathbb{R}^2 . Moreover, every triangle T is a k -reptile for $k = m^2$, since T can be tiled in a regular way with m^2 congruent tiles, each positively or negatively homothetic to T . See Snover et al. [40] for an illustration.

In this paper we study the existence of k -reptile *simplices* in \mathbb{R}^d , especially for $d = 3$ and $d = 4$.

Space-filling simplices. The question of characterizing the tetrahedra that tile \mathbb{R}^3 is still open and apparently rather difficult. The first systematic study of space-filling tetrahedra was made by Sommerville. Sommerville [41] discovered a list of exactly four tilings (up to isometry and rescaling), but he assumed that all tiles are *properly congruent* (that is, congruent by an orientation-preserving isometry) and meet face-to-face. Edmonds [12] noticed a gap in Sommerville's proof and by completing the analysis, he confirmed that Sommerville's classification of proper, face-to-face tilings is complete. Baumgartner [5] found three of Sommerville's tetrahedra and one new tetrahedron that admits a non-proper face-to-face tiling (and also a proper non face-to-face tiling [17]). Goldberg [17] described three families of proper (generally not face-to-face) tilings, obtained by partitioning a triangular prism. In fact, Goldberg's first family was found by Sommerville [41] before, but he selected only special cases with a certain symmetry. Goldberg's first family also coincides with the family of simplices found by Hill [23], whose aim was to classify *rectifiable* simplices, that is, simplices that can be cut by straight cuts into finitely many pieces and rearranged to form a cube. The simplices in Goldberg's second and third families are obtained from the simplices in the first family by splitting into two congruent halves. According to Senechal's survey [38], no other space-filling tetrahedra are known.

For $d \geq 3$, Debrunner [9] constructed $\lfloor d/2 \rfloor + 2$ one-parameter families and several special types of d -dimensional simplices that tile \mathbb{R}^d . Smith [39] generalized Goldberg's construction and using Debrunner's ideas, he obtained $(\lfloor d/2 \rfloor + 2)\phi(d)/2$ one-parameter families of space-filling d -dimensional simplices; here $\phi(d)$ is the Euler's totient function. It is not known whether for some $d \geq 3$ there is a space-filling simplex with all dihedral angles acute or a two-parameter family of space-filling simplices [39].

Hilbert's problems. Two Hilbert's problems are related to tilings of the Euclidean space. The second part of Hilbert's 18th problem asks whether there exists a polyhedron that tiles the 3-dimensional Euclidean space but does not admit an isohedral (tile-transitive) tiling. The first such tile in three dimensions was found by Reinhardt [36].

Later Heesch [21] found a planar anisohedral nonconvex polygon and Kershner [26] found an anisohedral convex pentagon. Hilbert's 18th problem was discussed in detail by Milnor [34]. See also the survey by Grünbaum and Shepard [19] for a discussion of this problem and related questions. While iterated tilings of the space using tilings of some k -reptiles as a pattern may be highly irregular, it is an interesting question whether there is an anisohedral k -reptile polytope or polygon. Vince [44, Question 2] asked whether there is a k -reptile that admits no periodic tiling.

The third Hilbert's problem asks whether two tetrahedra with equal bases and altitudes are *equidecomposable*, that is, whether one can cut one tetrahedron into finitely many polytopes and reassemble them to form the second tetrahedron. A positive answer would provide an elementary proof of the formula for the volume of the tetrahedron. However, Dehn [10] answered the question in the negative, by introducing an algebraic invariant for equidecomposability. See [11] for an elementary exposition or [3, Chapter 9], [6] for alternative proofs. Debrunner [8] proved that every polytope that tiles \mathbb{R}^d has its *codimension 2 Dehn's invariant* equal to zero. Lagarias and Moews [27, 28] showed that, more generally, every polytope that tiles \mathbb{R}^d has its *classical total Euclidean Dehn's invariant* equal to zero. In particular, these properties are necessary for every k -reptile simplex. For $d = 3$ and $d = 4$, the results of Sydler [42] (see also [24]) and Jessen [25] imply that every polytope that tiles \mathbb{R}^d is equidecomposable with a cube [8, 27, 28].

Reptiles and other animals. Motivated by classical puzzles that require splitting a given figure into a given number of congruent *replicas* of the original figure, Langford [29] initiated a systematic study of planar k -reptiles. Golomb [18] introduced the term *replicating figure of order k* , shortly a *rep- k* , and described several more examples, including disconnected or totally disconnected fractal tiles. See also Gardner's [14] short survey. Extending the theory of self-similar sets and fractals, Bandt [4] described a general construction of infinitely many self-similar k -reptiles, including several species of dragons, which are examples of *disk-like* (that is, homeomorphic to a disk) reptiles. Gelbrich [15] proved that for every k , there are only finitely many planar disk-like crystallographic (isohedral) k -reptiles. See Gelbrich and Giesche [16] for illustrations of several such 7-reptiles, such as sea horses or salamanders. Vince [44] studied lattice reptiles and their connection with generalized number systems.

k -reptile simplices. In recent years the subject of tilings has received a certain impulse from computer graphics and other computer applications. In fact, our original motivation for studying simplices that are k -reptiles comes from a problem of probabilistic marking of Internet packets for IP traceback [1, 2]. See [32] for a brief summary of the ideas of this method. For this application, it would be interesting to find a d -dimensional simplex that is a k -reptile with k as small as possible.

For dimension 2 there are several possible types of k -reptile triangles, and they have been completely classified by Snover et al. [40]. In particular, k -reptile triangles exist for all k of the form $a^2 + b^2$, a^2 or $3a^2$ for arbitrary integers a, b . In contrast, for $d \geq 3$, reptile simplices seem to be much more rare. The only known constructions of higher-dimensional k -reptile simplices have $k = m^d$. The best known examples are the *Hill*

simplices (or the *Hadwiger–Hill simplices*) [9, 20, 23]. A d -dimensional Hill simplex is the convex hull of vectors $0, b_1, b_1 + b_2, \dots, b_1 + b_2 + \dots + b_d$, where b_1, b_2, \dots, b_d are vectors of equal length such that the angle between every two of them is the same and lies in the interval $(0, \frac{\pi}{2} + \arcsin \frac{1}{d-1})$.

Hertel [22] proved that a 3-dimensional simplex is an m^3 -reptile using a “standard” way of dissection (which we will not define here) if and only if it is a Hill simplex. He conjectured that Hill simplices are the only 3-dimensional reptile simplices. Herman Haverkort recently pointed us to an example of a k -reptile tetrahedron by Liu and Joe [30] which is not Hill, and thus contradicts Hertel’s conjecture. In fact, except for the one-parameter family of Hill tetrahedra, two other space-filling tetrahedra described by Sommerville [41] and Goldberg [17] are also k -reptiles for every $k = m^3$. Both these tetrahedra tile the right-angled Hill tetrahedron, and their tilings are based on the barycentric subdivision of the cube. Maehara [31] described a generalized construction of d distinct k -reptile simplices in \mathbb{R}^d for $k = 2^d$. It is easy to see that the lattice tiling of \mathbb{R}^d by barycentrically subdivided unit cubes can be obtained by cutting the space with hyperplanes $x_i = n/2, x_i + x_j = n, x_i - x_j = n$, for every $i, j \in [d], i \neq j$ and $n \in \mathbb{Z}$. Each tile in this tiling is congruent to the right-angled Hill simplex H_d^0 defined as the convex hull of the points $(0, 0, \dots, 0), (1/2, 0, \dots, 0), \dots, (1/2, 1/2, \dots, 1/2)$. For every m , this tiling contains a tiling of an m times scaled copy of H_d^0 . Similarly, by removing the hyperplanes $x_i = (2n + 1)/2$ from the cutting, we obtain a tiling of \mathbb{R}^d with tiles that are made of two copies of H_d^0 ; more precisely, each tile is congruent to the simplex H_d^1 defined as the convex hull of the points $(0, 0, \dots, 0), (1, 0, \dots, 0), (1/2, 1/2, 0, \dots, 0), \dots, (1/2, 1/2, \dots, 1/2)$. Again, for every m , this tiling contains a tiling of an m times scaled copy of H_d^1 .

Let H_d^2 be the convex hull of the points $(0, 0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0), (1/2, 1/2, 1/2, 0, \dots, 0), \dots, (1/2, 1/2, \dots, 1/2)$. The simplex H_d^2 can be tiled with two copies of H_d^1 or four copies of H_d^0 . Let m be a fixed positive integer and let $m \cdot H_d^2$ be the m times scaled copy of H_d^2 obtained from H_d^2 by multiplying all the coordinates of all its points by m . The tiling described in the previous paragraph provides a tiling of $m \cdot H_d^2$ by $2m^d$ copies of H_d^1 . To obtain a tiling of $m \cdot H_d^2$ by copies of H_d^2 , it is enough to join the copies of H_d^1 into m^d disjoint pairs so that each pair forms a copy of H_d^2 . Every tile H in the tiling of \mathbb{R}^d by copies of H_d^1 can be represented by the center $\mathbf{z} = (n_1 + 1/2, n_2 + 1/2, \dots, n_d + 1/2)$ of a unit cube it contains and by a signed partial permutation $(\varepsilon_1 i_1, \varepsilon_2 i_2, \dots, \varepsilon_{d-1} i_{d-1})$ where $\varepsilon_i \in \{-1, 1\}, i_j \in [d]$ and $i_j \neq i_k$ if $j \neq k$. Let $\{i_d\} = \{1, 2, \dots, d\} \setminus \{i_1, i_2, \dots, i_{d-1}\}$ and let \mathbf{e}_i be the i th unit vector of the canonical basis. The tile H is then the convex hull of the points

$$\begin{aligned} & \mathbf{z}, \\ & \mathbf{z} + \frac{1}{2} \varepsilon_1 \mathbf{e}_{i_1}, \dots, \\ & \mathbf{z} + \frac{1}{2} (\varepsilon_1 \mathbf{e}_{i_1} + \varepsilon_2 \mathbf{e}_{i_2} + \dots + \varepsilon_{d-2} \mathbf{e}_{i_{d-2}}), \\ & \mathbf{z} + \frac{1}{2} (\varepsilon_1 \mathbf{e}_{i_1} + \varepsilon_2 \mathbf{e}_{i_2} + \dots + \varepsilon_{d-2} \mathbf{e}_{i_{d-2}} + \varepsilon_{d-1} \mathbf{e}_{i_{d-1}} + \mathbf{e}_{i_d}), \\ & \mathbf{z} + \frac{1}{2} (\varepsilon_1 \mathbf{e}_{i_1} + \varepsilon_2 \mathbf{e}_{i_2} + \dots + \varepsilon_{d-2} \mathbf{e}_{i_{d-2}} + \varepsilon_{d-1} \mathbf{e}_{i_{d-1}} - \mathbf{e}_{i_d}). \end{aligned}$$

We say that a tile H is *compatible* with a tile H' if their union is a simplex congruent to H_d^2 . A tile H represented by \mathbf{z} and $(\varepsilon_1 i_1, \varepsilon_2 i_2, \dots, \varepsilon_{d-1} i_{d-1})$ and a tile H' represented by \mathbf{z}' and $(\varepsilon'_1 i'_1, \varepsilon'_2 i'_2, \dots, \varepsilon'_{d-1} i'_{d-1})$ are compatible if and only if $\mathbf{z} = \mathbf{z}'$, $(\varepsilon_1 i_1, \varepsilon_2 i_2, \dots, \varepsilon_{d-2} i_{d-2}) = (\varepsilon'_1 i'_1, \varepsilon'_2 i'_2, \dots, \varepsilon'_{d-2} i'_{d-2})$ and $i_{d-1} \neq i'_{d-1}$. In particular, each tile H is compatible with two other neighboring tiles, and every component in the corresponding *compatibility graph* \mathcal{G} is a four-cycle. Let H be a tile represented by \mathbf{z} and $(\varepsilon_1 i_1, \varepsilon_2 i_2, \dots, \varepsilon_{d-1} i_{d-1})$. The four tiles H, H', H', H'' forming a component of \mathcal{G} containing H are separated by hyperplanes orthogonal to the vectors $x_{i_{d-1}} + x_{i_d}$ and $x_{i_{d-1}} - x_{i_d}$. Since at most one of these hyperplanes determines a facet of $m \cdot H_d^2$, the simplex $m \cdot H_d^2$ contains an even number of the tiles H, H', H', H'' , which can be matched into zero, one or two compatible pairs.

Maehara [31] described d -dimensional simplices $\sigma_0, \sigma_1, \dots, \sigma_d$, which satisfy $\sigma_i = H_d^i$ for $i \in \{0, 1, 2\}$ and $\sigma_d = 2 \cdot H_d^0$. Thus, we may denote σ_i by H_d^i for all $i \in [d]$. In general, for each $i \in [d]$, the simplex H_d^i can be tiled with two copies of H_d^{i-1} . Since for every positive integer m the simplex H_d^0 is m^d -reptile, each of the simplices H_d^i is $(2m)^d$ -reptile. Except for $i \in \{0, 1, 2\}$, we do not know whether H_d^i is m^d -reptile for odd $m \geq 3$.

Problem 1.1. Let d and i be positive integers satisfying $3 \leq i \leq d - 1$ and let $m \geq 3$ be an odd integer. Is it true that the simplex H_d^i is m^d -reptile?

Matoušek [32] showed that there are no 2-reptile simplices of dimension 3 or larger. For dimension $d = 3$, Matoušek and the second author [33] proved the following theorem.

Theorem 1.2. [33] *In \mathbb{R}^3 , k -reptile simplices (tetrahedra) exist only for k of the form m^3 where m is a positive integer.*

We give a new simple proof of Theorem 1.2 in Section 3.

Matoušek and the second author [33] conjectured that for $d \geq 3$, a d -dimensional k -reptile simplex can exist only for k of the form m^d for some positive integer m . We prove a weaker version of this conjecture for four-dimensional simplices.

Theorem 1.3. *Four-dimensional k -reptile simplices can exist only for k of the form m^2 , where m is a positive integer.*

Four-dimensional Hill simplices are examples of k -reptile simplices for $k = m^4$. However, the following question remains open.

Problem 1.4. Is there a four-dimensional m^2 -reptile simplex for m non-square?

New ingredients. Debrunner's lemma [9] connects the symmetries of a d -simplex with the symmetries of its Coxeter diagram (which represents the "arrangement" of the dihedral angles), and is an important tool in our analysis. This lemma allows us to substantially simplify the proof of Theorem 1.2 and enables us to step one dimension up and prove Theorem 1.3, which seemed unmanageable before.

In the proof of Theorem 1.3 we encounter the problem of tiling spherical triangles by congruent triangular tiles, which might be of independent interest. A related question, a classification of edge-to-edge tilings of the sphere by congruent triangles, has been completely solved by Agaoka and Ueno [43].

2 Basic notions and facts about simplices and group actions

2.1 Angles in simplices and Coxeter diagrams

Given a d -dimensional simplex S with vertices v_1, v_2, \dots, v_{d+1} , let F_i be the facet opposite to v_i . If $\alpha_{i,j}$ is the angle between the normals of F_i and F_j pointing outward, then the *dihedral angle* $\beta_{i,j}$ is defined as $\pi - \alpha_{i,j}$. By an *internal angle* φ at the point x of S , where x is on the boundary of S , we mean the set $\mathbb{S}^{d-1}(x, \varepsilon) \cap S$, where $\mathbb{S}^{d-1}(x, \varepsilon)$ denotes the $(d-1)$ -dimensional sphere with radius ε centered at x , and $\varepsilon > 0$ is small enough so that $\mathbb{S}^{d-1}(x, \varepsilon)$ does not meet the facets not containing x . An *edge-angle* of S is the internal $(d-1)$ -dimensional angle at an interior point of an edge of S and can be represented by a $(d-2)$ -dimensional spherical simplex. Indeed, select an interior point x of the edge e and consider the hyperplane h orthogonal to e and containing x . The edge-angle incident to e can be represented as the intersection $h \cap S \cap \mathbb{S}^{d-1}(x, \varepsilon)$. This intersection is clearly $(d-2)$ -dimensional and forms a spherical simplex.

From now on we normalize all edge-angles, that is, we consider them as subsets of the $(d-2)$ -dimensional unit sphere.

The *Coxeter diagram* of S is a graph $c(S)$ with labeled edges such that the vertices of $c(S)$ represent the facets of S and for every pair of facets F_i and F_j , there is an edge $e_{i,j}$ labeled by the dihedral angle $\beta_{i,j}$. Note that our labeling differs from the traditional one, where the edge corresponding to a dihedral angle π/p is labeled by p and the label 3 is omitted. Debrunner [9] labels the edge corresponding to a dihedral angle $\beta_{i,j}$ by $\cos \beta_{i,j}$.

Observation 2.1. *The edge-angles of a four-dimensional simplex S can be represented by spherical triangles, whose angles are dihedral angles in S . Therefore, an edge-angle in S represented by a spherical triangle with angles α, β, γ corresponds to a triangle in the Coxeter diagram with edges labeled by α, β, γ . \square*

Debrunner [9, Lemma 1] proved the following important lemma. Here the *symmetries* of S are Euclidean isometries, and the *symmetries* of $c(S)$ are graph automorphisms preserving the labels of edges.

Lemma 2.2 (Debrunner's lemma [9]). *Let S be a d -dimensional simplex. The symmetries of S are in one-to-one correspondence with the symmetries of its Coxeter diagram $c(S)$, in the following sense: each symmetry φ of S induces a symmetry Φ of $c(S)$ so that $\varphi(v_i) = v_j \Leftrightarrow \Phi(F_i) = F_j$, and vice versa.*

2.2 Existence of simplices with given dihedral angles

Fiedler [13] proved the following elegant property of $\binom{d+1}{2}$ -tuples of dihedral angles. A proof in English can be found in [33].

Theorem 2.3 (Fiedler's theorem [13]). *Let $\beta_{i,j}$, $i, j = 1, 2, \dots, d+1$, be the dihedral angles of some d -dimensional simplex, let $\beta_{i,i} = \pi$ for convenience, and let A be the $(d+1) \times (d+1)$ matrix with $a_{i,j} := \cos \beta_{i,j}$ for all i, j . Then A is negative semidefinite of rank d , and the (1-dimensional) kernel of A is generated by a vector $z \in \mathbb{R}^{d+1}$ with all components strictly positive.*

In our proof of Theorem 1.3 we use only the fact that the matrix A defined in Theorem 2.3 is singular; indeed, it is a $(d + 1) \times (d + 1)$ matrix of rank d .

2.3 Group actions

An *action* φ of a group G on a set M is a homomorphism from G to the symmetric group $\text{Sym}(M)$ of M , where the symmetric group $\text{Sym}(M)$ is the group of all permutations of M . We say that an action φ of G on M is *faithful* if its kernel is trivial. In other words, φ is faithful if for every $g \neq 1$ there exists an element $m \in M$ with $\varphi(g)(m) \neq m$. It is usual to omit φ and write just gm instead of $\varphi(g)(m)$.

The set $Gm := \{gm : g \in G\}$ is called the *orbit* of the element m under the action of G . It is obvious that the set of orbits forms a partition of M . The following well-known lemma counts the number of orbits in the partition.

Lemma 2.4 (Burnside's lemma [7]). *Let M be a finite set and G a finite group acting on M via $m \mapsto gm$. Let X_g be the number of elements of M fixed by g , that is, those satisfying the identity $gm = m$. Then the action of G on M has exactly $\frac{1}{|G|} \sum_{g \in G} X_g$ orbits.*

We will need the following lemma:

Lemma 2.5. *Let M be a finite set and G a finite group acting on M nontrivially and faithfully via $m \mapsto gm$. Then G also acts on the (unordered) pairs $\{m, n\} \in \binom{M}{2}$ via $g\{m, n\} = \{gm, gn\}$ and the action of G on pairs has at most $\binom{|M|}{2} - |M| + 2$ orbits. Moreover, the bound is tight and it is achieved if the image of G under the action is generated by a single transposition.*

Proof. Let o_1 be the number of orbits of the action of G on M . Let o_2 be the number of orbits of the induced action of G on $\binom{M}{2}$. Since the action on M is nontrivial, we have $o_1 \leq |M| - 1$.

Let X_g be the number of elements of M fixed by g . We show that the number of elements of $\binom{M}{2}$ fixed by g is $\binom{X_g}{2} + \frac{1}{2}(X_{g^2} - X_g) = \frac{1}{2}(X_g^2 + X_{g^2}) - X_g$. Indeed, there are two possibilities for stabilizing the pair $\{m, n\}$:

1. $gm = m$ and $gn = n$; this gives $\binom{X_g}{2}$ fixed elements of $\binom{M}{2}$.
2. $gm = n$ and $gn = m$; this can be rewritten as $ggn = n$ and $gn \neq n$. Thus in this case we have $\frac{1}{2}(X_{g^2} - X_g)$ fixed elements of $\binom{M}{2}$.

By Burnside's lemma, we have

$$o_2 = \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{2}(X_g^2 + X_{g^2}) - X_g \right). \quad (1)$$

In order to bound (1), we need to bound $\sum X_g^2$ in terms of $\sum X_g$:

$$\sum_{g \in G} X_g^2 \leq (|M| - 2) \sum_{g \in G} X_g + 2|M|. \quad (2)$$

Indeed, the action is faithful and nontrivial, hence $X_1 = |M|$ and $X_g \leq |M| - 2$ otherwise. Using $\sum_{g \neq 1} X_g^2 \leq (|M| - 2) \sum_{g \neq 1} X_g$, the bound (2) follows.

Plugging (2) into (1) and using $X_{g^2} \leq |M|$ we have

$$\begin{aligned} o_2 &= \frac{1}{2|G|} \sum_{g \in G} X_g^2 + \frac{1}{2|G|} \sum_{g \in G} X_{g^2} - \frac{1}{|G|} \sum_{g \in G} X_g \\ &\leq \frac{|M|}{2|G|} \sum_{g \in G} X_g - \frac{2}{|G|} \sum_{g \in G} X_g + \frac{|M|}{|G|} + \frac{|M|}{2}. \end{aligned}$$

By Burnside's lemma for the action on M , we have $\frac{1}{|G|} \sum X_g = o_1 \leq |M| - 1$. Since $|G| \geq 2$, we get the desired bound:

$$o_2 \leq \frac{|M|(|M| - 1)}{2} - 2(|M| - 1) + |M| = \binom{|M|}{2} - |M| + 2.$$

It remains to show the last part of the statement. But this is clear, since a single transposition of elements of M induces a product of $|M| - 2$ transpositions of elements of $\binom{M}{2}$. \square

3 A simple proof of Theorem 1.2

We proceed as in the original proof [33], but instead of using the theory of scissors congruence, Jahnke's theorem about values of rational angles and Fiedler's theorem, we only use Debrunner's lemma (Lemma 2.2).

Assume for contradiction that S is a k -reptile tetrahedron where k is not a third power of a positive integer. A dihedral angle α is called *indivisible* if it cannot be written as a linear combination of other dihedral angles in S with nonnegative integer coefficients.

The following lemmas are proved in [33].

Lemma 3.1. [33, Lemma 3.1] *If α is an indivisible dihedral angle in S , then the edges of S with dihedral angle α have at least three different lengths.*

Lemma 3.1 is analogous to Lemma 4.1, which we prove in the next section.

Lemma 3.2. [33, Lemma 3.3] *One of the following two possibilities occur:*

- (i) *All the dihedral angles of S are integer multiples of the minimal dihedral angle α , which has the form $\frac{\pi}{n}$ for an integer $n \geq 3$.*
- (ii) *There are exactly two distinct dihedral angles β_1 and β_2 , each of them occurring three times in S .*

First we exclude case (ii) of Lemma 3.2. If S has two distinct dihedral angles $\beta_1 \neq \beta_2$, each occurring at three edges, then they can be placed in S in two essentially different ways; see Figure 1. In both cases, for each $i \in \{1, 2\}$, the Coxeter diagram of S has at

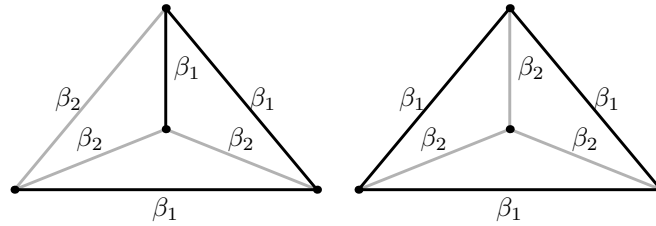


Figure 1: Two possible configurations of two dihedral angles.

least one nontrivial symmetry swapping two distinct edges with label β_i . By Debrunner's lemma, the corresponding symmetry of S swaps two distinct edges with dihedral angle β_i , which thus have the same length. But then the edges with dihedral angle β_i have at most two different lengths and this contradicts Lemma 3.1, since the smaller of the two angles β_1, β_2 is indivisible.

Now we exclude case (i) of Lemma 3.2. Call the edges of S (and of $c(S)$) with dihedral angle α the α -edges. Since there are at least three α -edges in S , there is a vertex v of S where two α -edges meet. Let β be the dihedral angle of the third edge incident to v (possibly β can be equal to α). In the proof of Lemma 3.5 in [33] it was shown that $\beta = \pi - \alpha$. For the Coxeter diagram of S , this implies that whenever two α -edges meet in $c(S)$, then the label of the edge forming a triangle with the two α -edges is β .

Now we distinguish several cases depending on the subgraph H_α of $c(S)$ formed by the α -edges.

- H_α contains three edges incident to a common vertex. Then all the other edges must be labeled with β and thus we get a configuration like in Figure 1, right, which we excluded earlier.
- H_α contains a triangle. Then $\beta = \alpha$, and thus $\alpha = \frac{\pi}{2}$, which contradicts the condition $n \geq 3$ from Lemma 3.2(i).
- H_α contains a path of length 3. Then two other edges have label β and the remaining edge has some label γ (possibly γ can be equal to α). See Figure 2, left. The symmetric group of the resulting Coxeter diagram always contains an involution swapping two α -edges. Unless $\gamma = \alpha$, there are, by Debrunner's lemma, only two α -edge lengths; a contradiction with Lemma 3.1. For $\gamma = \alpha$ the Coxeter diagram has a dihedral symmetry group, D_4 , acting transitively on the α -edges; see Figure 2, right. This again contradicts Lemma 3.1, since by Debrunner's lemma, all the α -edges have the same length.

We obtained a contradiction in each of the cases, hence the proof of Theorem 1.2 is finished.

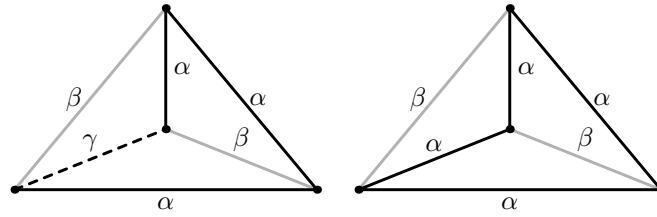


Figure 2: The α -edges form a path (left) or a four-cycle (right) in $c(S)$.

4 The proof of Theorem 1.3

The method of the proof is similar to the three-dimensional case [33].

Assume for contradiction that S is a four-dimensional k -reptile simplex where k is not a square of a positive integer. Let S_1, S_2, \dots, S_k be mutually congruent simplices similar to S that form a tiling of S . Then each S_i has volume k -times smaller than S , and thus S_i is scaled by the ratio $\rho := k^{-1/4}$ compared to S . For k non-square, ρ is an irrational number of algebraic degree 4 over \mathbb{Q} .

Similarly to [33] we define an *indivisible* edge-angle (spherical triangle) as a spherical triangle that cannot be tiled with smaller spherical triangles representing the other edge-angles of S or their mirror images. Clearly, the edge-angle with the smallest spherical area is indivisible. In this paper we always consider a spherical triangle and its mirror image as the same spherical triangle.

Lemma 4.1. *If \mathcal{T}_0 is an indivisible edge-angle in S , then the edges of S with edge-angle \mathcal{T}_0 have at least four different lengths (and in particular, there are at least four such edges).*

Proof. The proof is basically the same as for indivisible dihedral angles in tetrahedra [33, Lemma 3.1]. Let e be an edge with edge-angle \mathcal{T}_0 . Every point of e belongs to an edge of some of the smaller simplices S_i . Since \mathcal{T}_0 is indivisible, we get that e is tiled by edges of the simplices S_i and each of these edges has edge-angle \mathcal{T}_0 .

Assume for contradiction that there are at most three edges with edge-angle \mathcal{T}_0 , with lengths x_1, x_2, x_3 . Then the edge of length x_1 is tiled by edges with lengths $\rho x_1, \rho x_2$ and ρx_3 , and similarly for the edges of lengths x_2 and x_3 . In other words, there are nonnegative integers n_{ij} , $i, j = 1, 2, 3$, such that

$$\begin{aligned} n_{11}\rho x_1 + n_{12}\rho x_2 + n_{13}\rho x_3 &= x_1, \\ n_{21}\rho x_1 + n_{22}\rho x_2 + n_{23}\rho x_3 &= x_2, \\ n_{31}\rho x_1 + n_{32}\rho x_2 + n_{33}\rho x_3 &= x_3. \end{aligned}$$

This can be rewritten as $\rho \mathbf{A} \mathbf{x} = \mathbf{x}$, where $\mathbf{x} = (x_1, x_2, x_3)^T$ and \mathbf{A} is a 3×3 matrix with integer coefficients. Since \mathbf{x} is nonzero, we immediately see that $1/\rho$ is an eigenvalue of \mathbf{A} . Since the characteristic polynomial of \mathbf{A} has degree 3, we get a contradiction with $1/\rho$ (and hence also ρ) having algebraic degree 4. \square

Since S has 10 edges, Lemma 4.1 implies that there are at most two indivisible edge-angles.

The strategy of the proof is now the following. First we exclude the case of two indivisible edge-angles, using only elementary combinatorial arguments, Debrunner's lemma and Lemma 2.5. Then we consider the case of one indivisible edge-angle. Here we need more involved arguments: we study tilings of spherical triangles with copies of a single spherical triangle and use various observations from spherical geometry. We also use Fiedler's theorem (Theorem 2.3) to solve several cases.

4.1 Two indivisible edge-angles

First, we prove an elementary observation about symmetries of the simplex S and its Coxeter diagram.

Lemma 4.2. *If $c(S)$ has a nontrivial symmetry, then the edges of S have at most seven orbits under the action of the symmetry group of S .*

Proof. Let M be the set of vertices of S . By Debrunner's lemma, the symmetry groups of S and $c(S)$ are isomorphic. In particular, S has a nontrivial symmetry group $\Phi \subseteq \text{Sym}(M)$ acting faithfully on M . By Lemma 2.5, this action has at most seven orbits. \square

Corollary 4.3. *If S has two distinct indivisible edge-angles, then the symmetry group of S (and of $c(S)$) is trivial.*

Proof. By Lemma 4.1, S has at least four edges of different lengths for each of the two edge-angles. In particular, no symmetry can identify any two of these eight edges and so the symmetry group of S induces at least eight orbits. It is therefore trivial by Lemma 4.2. \square

Now assume for contradiction that S has two indivisible edge-angles \mathcal{T}_1 and \mathcal{T}_2 . Let T_1 and T_2 be the corresponding triangles in $c(S)$. By Lemma 4.1, each of T_1, T_2 occurs at least four times in $c(S)$.

We say that two edges of $c(S)$ are of the same *edge-type* if they have equal labels; that is, they represent equal dihedral angles. An edge of type α is also called an α -*edge*. A triangle T of $c(S)$ with edges of types α, β, γ is called an $(\alpha\beta\gamma)$ -*triangle* and we write $T = (\alpha\beta\gamma)$.

Observation 4.4. *Every edge of $c(S)$ belongs to a copy of the triangle T_1 or T_2 . Moreover, every edge-type of T_1 and T_2 occurs at least twice in $c(S)$.*

Proof. The first part follows from the fact that every edge of $c(S)$ is contained in three triangles and at least eight of the ten triangles of $c(S)$ are copies of T_1 or T_2 . The second claim follows again from the fact that every edge of $c(S)$ is common to only three triangles. \square

Observation 4.5. *An edge-type common to both triangles T_1, T_2 occurs at least four times in $c(S)$. Similarly, an edge-type occurring twice in T_1 (or T_2) occurs at least four times in $c(S)$.*

Case	type of T_1	type of T_2	α -edges	β -edges	γ -edges	δ -edges
(1)	$(\alpha\alpha\beta)$	$(\alpha\gamma\delta)$	4	2	2	2
(2)	$(\alpha\alpha\alpha)$	$(\alpha\beta\gamma)$	6	2	2	0
(3)	$(\alpha\alpha\beta)$	$(\alpha\alpha\gamma)$	6	2	2	0
(4)	$(\alpha\alpha\beta)$	$(\alpha\gamma\gamma)$	4	2	4	0
(5)	$(\alpha\alpha\beta)$	$(\alpha\beta\gamma)$	4	4	2	0

Table 1: Types of triangles T_1 and T_2 and the numbers of edges of each edge-type in $c(S)$.

Proof. Let α be an edge-type common to both T_1 and T_2 and suppose that each of T_1, T_2 has just one α -edge. There are at least eight triangles with an α -edge in $c(S)$, therefore $c(S)$ has at least three α -edges. But if there are just three α -edges, then some two of them share a vertex (and hence a triangle). Therefore there are at most seven triangles in $c(S)$ with exactly one α -edge.

If T_1 has at least two α -edges, then there are at least four pairs of α -edges in $c(S)$, hence at least four α -edges. \square

Observation 4.6. *An edge-type α occurring four times together in T_1 and T_2 occurs at least six times in $c(S)$.*

Proof. Since each of the triangles T_1, T_2 has at least four copies in $c(S)$, the number of incidences of α -edges with copies of triangles T_1 and T_2 in $c(S)$ is at least 16. Since every edge forms at most three incidences, the observation follows. \square

Observation 4.7. *The triangles T_1 and T_2 have at least one common edge-type.*

Proof. The observation follows from the fact that the union of four different triangles in $c(S)$ has always at least six edges. \square

By Observations 4.4, 4.5 and 4.7, the edges of $c(S)$ have at most four types in total, since the common edge-type of T_1 and T_2 occurs four times and every other edge-type occurs at least twice. From these observations it also follows that if there are four different edge-types, then three of them, β, γ, δ , appear just once in T_1 or T_2 and the remaining one, α , is common to T_1 and T_2 and appears twice in T_1 or twice in T_2 . Similarly if there are three different edge-types, then one of them appears at least three times together in T_1 and T_2 .

If there are just two different edge-types in $c(S)$, then $c(S)$ has a non-trivial symmetry, which follows from the fact that every graph on five vertices has a nontrivial automorphism. But this contradicts Corollary 4.3.

Thus there are three or four different edge-types in $c(S)$ and we have five essentially different cases for the types of T_1 and T_2 ; see Table 1. Here by $\alpha, \beta, \gamma, \delta$ we denote pairwise different angles. By Observations 4.4, 4.5 and 4.6, we can exactly determine the numbers of edges of each edge-type in $c(S)$. These are also shown in Table 1.

In case (1), since there are just two β -edges, some two $(\alpha\alpha\beta)$ -triangles in $c(S)$ share a β -edge. This means that the α -edges form a four-cycle. Further it follows that both

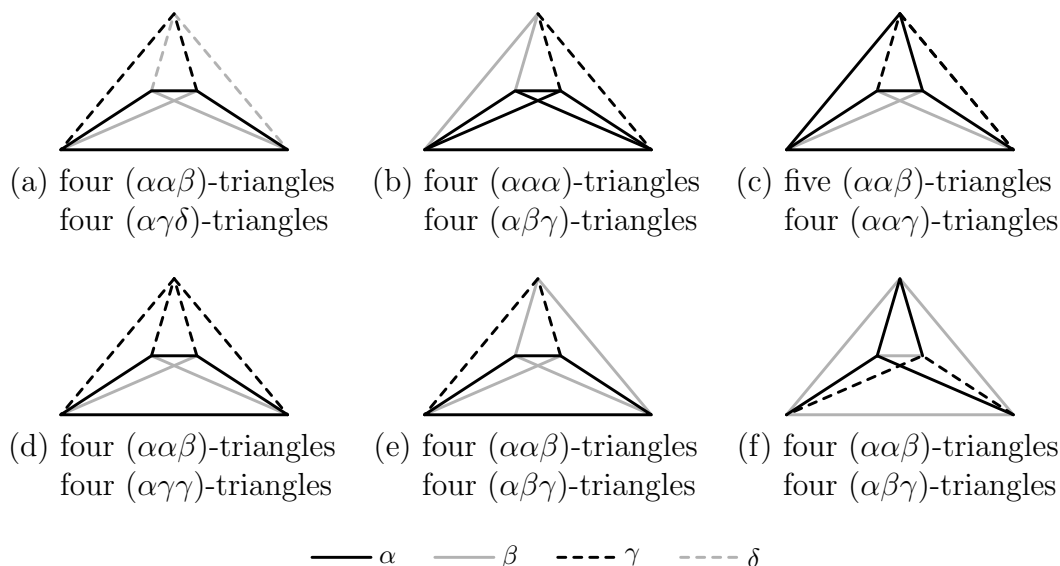


Figure 3: Coxeter diagrams for the case of two indivisible edge-angles.

diagonals of the four-cycle are β -edges and that the fifth vertex of $c(S)$ is joined by γ -edges to two opposite vertices of the four-cycle and by δ -edges to the other pair of opposite vertices; see Figure 3(a). This diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, which contradicts Corollary 4.3.

Now we consider case (2). Since K_4 is the only graph with six edges and four triangles, the α -edges form a K_4 subgraph in $c(S)$, with two vertices joined by a β -edge and two by a γ -edge to the remaining vertex of $c(S)$; see Figure 3(b). Again, this diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, in contradiction with Corollary 4.3.

In case (3), let H_α be the subgraph of $c(S)$ formed by the α -edges. Like in case (1), H_α contains a four-cycle. From the five possible extensions of the four-cycle by two edges, only $K_{2,3}$ has at least eight induced paths of length 2. Thus H_α is isomorphic to $K_{2,3}$. The remaining edges form a disjoint union of an edge and a triangle, so without loss of generality the two γ -edges are contained in the triangle; see Figure 3(c). This diagram has again a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, in contradiction with Corollary 4.3.

In case (4), just like in case (1), the α -edges form a four-cycle whose diagonals are the two β -edges. The remaining edges are then γ -edges; see Figure 3(d). This diagram has a D_4 symmetry, in contradiction with Corollary 4.3.

Now we consider case (5). Out of the six subgraphs of K_5 with four edges, only the following three have four induced paths of length 2: the star $K_{1,4}$, the four-cycle, and the *fork*, which is the tree with the degree sequence $(3, 2, 1, 1, 1)$. If the α -edges form $K_{1,4}$, there can be no $(\alpha\beta\gamma)$ -triangles in $c(S)$. Thus, there are just two possibilities for the subgraph H_α of $c(S)$ formed by the α -edges.

Suppose that H_α is a four-cycle. The diagonals of the four-cycle are then β -edges. In order to create four $(\alpha\beta\gamma)$ -triangles, the vertices of the four-cycle must be joined to the

remaining vertex by two β -edges and two γ -edges, in an alternating way; see Figure 3(e). This diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, in contradiction with Corollary 4.3.

If H_α is a fork, the β -edges are uniquely determined, since the α -edges form exactly four induced paths of length 2. The remaining two edges are γ -edges; see Figure 3(f). This diagram has a \mathbb{Z}_2 -symmetry, in contradiction with Corollary 4.3.

We have finished the proof of the following statement.

Proposition 4.8. *For $k \neq m^2$, every k -reptile four-dimensional simplex contains exactly one indivisible edge-angle. \square*

4.2 Basic facts and observations from spherical geometry

All spherical triangles are regarded as subsets of the 2-dimensional unit sphere. In this subsection we assume that \mathcal{T} is a spherical triangle with angles $\alpha \leq \beta \leq \gamma < \pi$ and corresponding opposite edges a, b, c . The lengths of the edges are measured in radians and again denoted by a, b, c , respectively.

The following lemma lists a few standard facts about spherical triangles (see, for example, [46]). The proof of part (a) can be found in [35, Chapter 41].

Lemma 4.9. *For a spherical triangle \mathcal{T} , we have*

- (a) $\alpha + \beta + \gamma > \pi$, and $\alpha + \beta + \gamma - \pi$ is equal to the spherical area $\Delta(\mathcal{T})$ of \mathcal{T} .
- (b) $\beta + \gamma < \pi + \alpha$; equivalently, $\Delta(\mathcal{T}) < 2\alpha$ (spherical triangle inequality).
- (c) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$ (spherical law of cosines for angles).
- (d) If $\alpha < \beta < \gamma$, then $a < b < c$. If $\alpha = \beta < \gamma$, then $a = b < c$. If $\alpha < \beta = \gamma$, then $a < b = c$.
- (e) $a < b + c$, $b < a + c$ and $c < a + b$ (triangle inequality for the spherical distance).
- (f) $a, b, c < \pi$.

The quantity $\alpha + \beta + \gamma - \pi$ is also called the *spherical excess* of \mathcal{T} .

A *spherical lune* \mathcal{L} with angle $\varphi < \pi$, which we shortly call the φ -lune, is a slice of the sphere bounded by two great half-circles whose supporting planes have dihedral angle φ . In other words, \mathcal{L} is a spherical 2-gon whose vertices are two antipodal points and both inner angles are equal to φ . The spherical area of \mathcal{L} is 2φ . Note that \mathcal{L} contains every spherical triangle with angle φ ; this implies the spherical triangle inequality (Lemma 4.9(b)).

Consider a tiling of a lune \mathcal{L} by spherical triangles. A tile \mathcal{T} is called a *corner tile* if \mathcal{T} shares a vertex v with \mathcal{L} . A tile \mathcal{T} is called *corner-filling* if the complement of \mathcal{T} in \mathcal{L} is a spherical triangle. In particular, \mathcal{T} shares a vertex v with \mathcal{L} and the two other vertices of \mathcal{T} are internal points of the edges of \mathcal{L} . See Figure 4.

Observation 4.10. *Let φ be the minimum angle of a spherical triangle \mathcal{T} and let f be the edge opposite to φ . Then in every tiling of the φ -lune by copies of \mathcal{T} there are two corner-filling tiles. Moreover, each of the corner-filling tiles is a neighbor of exactly one other tile, sharing the edge f .*

Proof. Since φ is the minimum angle of \mathcal{T} , every corner tile must be corner-filling. By Lemma 4.9(f), a corner-filling tile contains only one vertex of the lune, hence there are at least two such tiles. By Lemma 4.9(d), f is the shortest edge of \mathcal{T} . The rest of the observation follows. \square

4.3 One indivisible edge-angle

By Proposition 4.8, the simplex S has only one indivisible edge-angle \mathcal{T}_0 . This means that all the remaining edge-angles of S can be tiled with \mathcal{T}_0 . In particular, the spherical area of every spherical triangle representing an edge-angle of S is an integer multiple of the spherical area of \mathcal{T}_0 . Let T_0 be the triangle in $c(S)$ corresponding to the spherical triangle \mathcal{T}_0 .

We say that a spherical triangle \mathcal{T} has *type* $(\varphi\psi\chi)$ if its internal angles are φ, ψ and χ ; in this case we write $\mathcal{T} = (\varphi\psi\chi)$. We sometimes write the type of \mathcal{T} as (φ, ψ, χ) , to avoid confusion when substituting linear combinations of angles. We say that \mathcal{T} has type $(\varphi **)$ or $(\varphi\psi*)$ if it has type $(\varphi\psi\chi)$ for some angles ψ, χ , which may also be equal to φ or to each other. Note that a spherical triangle $\mathcal{T} = (\varphi\psi\chi)$ corresponds to a triangle $T = (\varphi\psi\chi)$ in $c(S)$.

To simplify the notation, we will label the vertices of the Coxeter diagram $c(S)$ by u, v, w, x, y instead of F_1, \dots, F_5 .

A Coxeter diagram $c(S)$ is *rich* if there is a triangle T such that under the action of the symmetry group of $c(S)$ on the set of triangles in $c(S)$, the copies of T form at least four different orbits. In this case we also say that the diagram $c(S)$ is *T-rich*. Lemma 4.1 and Debrunner's lemma imply the following important fact.

Fact 4.11. *The Coxeter diagram of S is T_0 -rich.*

A spherical triangle \mathcal{T} is *realizable* if \mathcal{T} can be tiled with \mathcal{T}_0 . A triangle T is *realizable* if its corresponding spherical triangle \mathcal{T} is realizable.

The strategy of the proof is the following.

- Find all possible types of \mathcal{T}_0 .
- For every such triangle \mathcal{T}_0 , let α be the minimal angle in \mathcal{T}_0 (and β the second minimal angle, if applicable). Investigate which spherical triangles of type $(\alpha **)$ (or $(\beta **)$, if needed) are realizable.
- Find all T_0 -rich Coxeter diagrams whose all $(\alpha **)$ -triangles (and $(\beta **)$ -triangles) are realizable.
- Verify that such diagrams do not satisfy Fiedler's theorem.

We start with a simple observation about the Coxeter diagram of S .

Observation 4.12. *The Coxeter diagram of S has at least two different types of triangles.*

Proof. Suppose for contrary that all triangles in $c(S)$ are of the same type $T = (\varphi_1\varphi_2\varphi_3)$. By double-counting, the numbers of occurrences of the edge types in $c(S)$ are in the same ratio as in T . Since the numbers 10 and 3 are relatively prime, it follows that $\varphi_1 = \varphi_2 = \varphi_3$ and thus all dihedral angles in S are equal. But then S is the regular simplex, which contradicts Lemma 4.1. \square

4.3.1 Conditions on dihedral angles

Here we prove several facts about the dihedral angles of S , which we use further to restrict the set of possible types of the indivisible triangle \mathcal{T}_0 .

Lemma 4.13. *Let $\varphi_1, \varphi_2, \varphi_3$ be the angles of \mathcal{T}_0 . Then for every $i \in \{1, 2, 3\}$, the spherical lune with angle φ_i can be tiled with \mathcal{T}_0 .*

Proof. Assume that $\varphi_1 \leq \varphi_2 \leq \varphi_3$. Fix $i \in \{1, 2, 3\}$. If S is a k -reptile simplex for some $k > 1$, then by induction, S is k^n -reptile for every $n \geq 1$. In particular, there is a tiling of S with simplices similar to S where some of the tiles, S' , has an edge e that is contained in the interior of a 2-face of S with dihedral angle φ_i . Select an interior point x_i of e that misses all vertices of all the tiles. Let h_i be the hyperplane orthogonal to e and containing x_i . In a small neighborhood of x_i in h_i , the tiles with x_i on their boundary induce a tiling of a wedge with angle φ_i by triangular cones originating in x_i and possibly by wedges with angles $\psi < \varphi_i$, where ψ is an internal angle of some realizable triangle \mathcal{T} .

This is analogous to the situation in a tiling of a three-dimensional simplex in the neighborhood of an internal point x of an edge such that x is also a vertex of some tile. In the intersection of h_i with a small sphere centered in x_i , we thus obtain a tiling of the spherical lune with angle φ_i by spherical triangles corresponding to edge-angles of the tiles and possibly by spherical lunes with angles $\psi < \varphi_i$ corresponding to dihedral angles of the tiles. Since \mathcal{T} can be tiled with \mathcal{T}_0 , it follows that ψ is a nonnegative integer combination of the angles φ_j where $1 \leq j < i$. This means that the ψ -lune can be tiled by lunes with angles φ_j where $1 \leq j < i$.

Observe that since x_i is an internal point of an edge of at least one tile, the tiling of the φ_i -lune contains at least one spherical triangle. Thus for $i = 1$, the tiling consists solely of realizable spherical triangles. For $i > 1$, the φ_i -lune can be tiled with realizable spherical triangles and possibly φ_j -lunes with $j < i$. The lemma follows by induction on i . \square

The following statement is a stronger variant of Bricard's condition for equidecomposable polyhedra [3, 6], [35, Chapter 15].

Lemma 4.14. *Let $\varphi_1, \varphi_2, \varphi_3$ be the angles of \mathcal{T}_0 . Then for every $i \in \{1, 2, 3\}$, there exist nonnegative integers $m_{i,1}, m_{i,2}, m_{i,3}$ such that $m_{i,i} > 0$ and $m_{i,1}\varphi_1 + m_{i,2}\varphi_2 + m_{i,3}\varphi_3 = \pi$.*

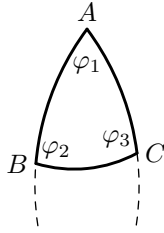


Figure 4: A corner-filling tile in the φ_1 -lune.

Proof. Assume that $\varphi_1 \leq \varphi_2 \leq \varphi_3$. By Lemma 4.13, the φ_1 -lune \mathcal{L}_1 is tiled with \mathcal{T}_0 . By Observation 4.10, there is a corner-filling tile \mathcal{T}_0^1 whose vertices with inner angles φ_2 and φ_3 are internal points of the edges of \mathcal{L}_1 ; see Figure 4. In a small neighborhood of each of these two points we observe a tiling of the straight angle by the angles of \mathcal{T}_0 , including φ_2 or φ_3 , respectively. This shows the lemma for $i = 2$ and $i = 3$.

To show the lemma for $i = 1$, we distinguish two cases. If φ_1 divides φ_2 , the claim follows from the case $i = 2$. Otherwise, we use Lemma 4.13 again, now for the φ_2 -lune \mathcal{L}_2 . The argument is analogous to the previous case since in the tiling of \mathcal{L}_2 with \mathcal{T}_0 each corner tile is corner-filling. \square

For the rest of this section, let α be the minimum angle of \mathcal{T}_0 .

Corollary 4.15. *We have $\alpha < \pi/2$.*

Proof. Lemma 4.14 implies that $\alpha \leq \pi/2$. By Lemma 4.13, the α -lune is tiled with at least two copies of \mathcal{T}_0 . Now suppose that $\alpha = \pi/2$. Since $\Delta(\mathcal{T}_0) \geq \pi/2$ and the spherical area of the $\pi/2$ -lune is π , the tiling of the $\pi/2$ -lune consists of precisely two copies of \mathcal{T}_0 and $\Delta(\mathcal{T}_0) = \pi/2$. This means that $\mathcal{T}_0 = (\pi/2, \pi/2, \pi/2)$. The angles of every spherical triangle tiled with \mathcal{T}_0 must be integer multiples of $\pi/2$, thus \mathcal{T}_0 is the only realizable triangle. This contradicts Observation 4.12. \square

Corollary 4.16. *We cannot have $\mathcal{T}_0 = (\alpha\alpha\alpha)$.*

Proof. If $\mathcal{T}_0 = (\alpha\alpha\alpha)$, then by Lemma 4.14, $\alpha = \pi/n$ for some positive integer n . But we have $\alpha > \pi/3$ by Lemma 4.9(a) and $\alpha < \pi/2$ by Corollary 4.15; a contradiction. \square

We are left with three main cases for the type of the indivisible triangle \mathcal{T}_0 , according to the symmetries and relative sizes of its angles.

- (A) Exactly two angles in \mathcal{T}_0 are equal, but not to α . We write $\mathcal{T}_0 = (\alpha\beta\beta)$.
- (B) Exactly two angles in \mathcal{T}_0 are equal to α . We write $\mathcal{T}_0 = (\alpha\alpha\beta)$.
- (C) All three angles in \mathcal{T}_0 are different. We write $\mathcal{T}_0 = (\alpha\beta\gamma)$.

For the rest of this section we assume that $\alpha < \beta < \gamma$.

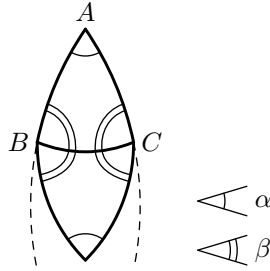


Figure 5: A corner-filling $\alpha\beta\beta$ -tile and its adjacent tile in the α -lune.

4.3.2 Case (A): $\mathcal{T}_0 = (\alpha\beta\beta)$.

By Lemma 4.13, the α -lune \mathcal{L}_α can be tiled with \mathcal{T}_0 . By Observation 4.10, there is a corner-filling tile \mathcal{T}_0^1 sharing its shortest edge with another tile \mathcal{T}_0^2 . See Figure 5. In the neighborhood of either common vertex of \mathcal{T}_0^1 and \mathcal{T}_0^2 , we see the straight angle tiled with two angles β and possibly other angles α or β . Since $\alpha + 2\beta > \pi$ by Lemma 4.9(a), the two angles β already tile the straight angle and hence $\beta = \pi/2$. It follows that two copies of \mathcal{T}_0 tile the whole α -lune and thus \mathcal{T}_0 is the only realizable spherical triangle of type $(\alpha **)$. (We recall that $(\alpha **)$ stands for $(\alpha\varphi\psi)$, where φ, ψ may also be equal to α or to each other.)

This implies that the Coxeter diagram of S has two vertex-disjoint α -edges, since it is $(\alpha\beta\beta)$ -rich and has no $(\alpha\alpha*)$ -triangle. Every remaining edge is adjacent to at least one α -edge, hence it is a β -edge. The resulting diagram has only two orbits of $(\alpha\beta\beta)$ -triangles and so it is not $(\alpha\beta\beta)$ -rich; a contradiction.

4.3.3 Case (B): $\mathcal{T}_0 = (\alpha\alpha\beta)$.

We start with an observation about $(\alpha\alpha\beta)$ -rich Coxeter diagrams.

Lemma 4.17. *There is at least one realizable spherical $(\alpha **)$ -triangle different from \mathcal{T}_0 .*

Proof. If $\mathcal{T}_0 = (\alpha\alpha\beta)$ is the only realizable $(\alpha **)$ triangle, then the α -edges form a spanning complete bipartite subgraph in the Coxeter diagram of S and all the remaining edges are β -edges. The $(\alpha\alpha\beta)$ -triangles then form at most two orbits; a contradiction. \square

By Lemma 4.14, there exist integers $m_1 \geq 0$ and $m_2 \geq 1$ such that $m_1\alpha + m_2\beta = \pi$. We distinguish two cases.

- (1) $m_1 \neq 0$. Since $2\alpha + \beta > \pi$ by Lemma 4.9(a), we have $m_1 = m_2 = 1$ and thus $\alpha + \beta = \pi$.
- (2) $m_1 = 0$. Then $\beta = \pi/m_2$. Since $3\beta > \pi$ and $\beta < \pi$, we have $m_2 = 2$ and so $\beta = \pi/2$. Now the inequality $2\alpha + \beta > \pi$ implies that $\alpha > \pi/4$. By Lemma 4.14, there exist integers $m'_1 \geq 1$ and $m'_2 \geq 0$ such that $m'_1\alpha + m'_2\pi/2 = \pi$. Since $m'_2 > 0$ leads to a contradiction, we have $\alpha = \pi/m'_1$. The only solution satisfying $\pi/4 < \alpha < \pi/2$ is $\alpha = \pi/3$.

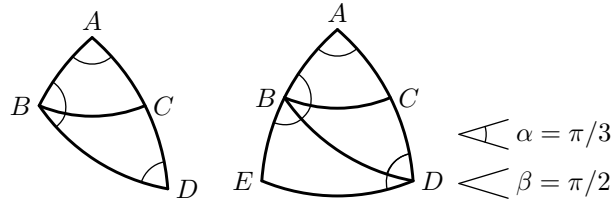


Figure 6: The $(\alpha\alpha2\alpha)$ -triangle and the $(\alpha, 2\alpha, \pi/2)$ -triangle composed of two and three $(\alpha\alpha\beta)$ -tiles, respectively.

*Case (1): $\alpha + \beta = \pi$. Let $\mathcal{T} = (\alpha\varphi\psi)$ be a realizable spherical $(\alpha **)$ -triangle different from \mathcal{T}_0 (with some of the angles possibly equal), whose existence is guaranteed by Lemma 4.17. The spherical area of \mathcal{T} satisfies $\Delta(\mathcal{T}) \geq 2\Delta(\mathcal{T}_0) = 2 \cdot (2\alpha + \beta - \pi) = 2\alpha$. But this contradicts the spherical triangle inequality $\Delta(\mathcal{T}) < 2\alpha$ (Lemma 4.9(b)).

*Case (2): $\alpha = \pi/3, \beta = \pi/2$.

The only nonnegative integer combinations of α and β that sum up to π are 3α and 2β . This implies that in every tiling of a spherical polygon \mathcal{P} by \mathcal{T}_0 , for every internal point x of an edge of \mathcal{P} , all incident tiles have the same angle at x . This somewhat restricts the set of possible tilings. Further restriction is obtained using the area argument.

The spherical area of \mathcal{T}_0 is $\beta + 2\alpha - \pi = \pi/6$. The α -lune, which contains every $(\alpha **)$ -triangle, has spherical area $2\alpha = 2\pi/3$. It follows that every $(\alpha **)$ -triangle is composed of at most three tiles.

When constructing a tiling of an $(\alpha **)$ -triangle, we always start with a corner-filling tile \mathcal{T}_0^1 of the α -lune and then try to place additional tiles. There is only one way of attaching a second tile to \mathcal{T}_0^1 , yielding the triangle of type $(\alpha, \alpha, 2\alpha)$; see Figure 6, left. Similarly, there is a unique way of attaching the third tile, which yields the triangle of type $(\alpha, 2\alpha, \pi/2)$; see Figure 6, right. These are the only realizable $(\alpha **)$ -triangles other than \mathcal{T}_0 .

Realizable $(\beta **)$ -triangles can be composed from at most five tiles, as their spherical area is smaller than $2\beta = \pi$. To construct a $(\beta **)$ -triangle, we start with a corner tile \mathcal{T}_0^1 in the β -lune. Since α does not divide β , the corner tile is corner-filling. By Lemma 4.9(e) and by the symmetry of \mathcal{T}_0 , the longest edge of \mathcal{T}_0 cannot be tiled with the shorter edges. This means that there is just one possible way of attaching another tile, \mathcal{T}_0^2 , to \mathcal{T}_0^1 ; see Figure 7, where \mathcal{T}_0^1 has vertices ABC and \mathcal{T}_0^2 has vertices BCF . The two tiles do not form a triangle, yet. Hence, there is at least one more tile \mathcal{T}_0^3 adjacent to, say, B . The orientation of \mathcal{T}_0^3 where \mathcal{T}_0^3 shares the edge BF with \mathcal{T}_0^2 gives an $(\alpha, 2\alpha, \pi/2)$ -triangle obtained earlier. The other orientation of \mathcal{T}_0^3 , where the longest edge of \mathcal{T}_0^3 partially coincides with the edge BF , forces a fourth tile sharing the edge CF with \mathcal{T}_0^2 , forming an $(\alpha, 2\alpha, \pi/2)$ -triangle ABE with \mathcal{T}_0^1 and \mathcal{T}_0^2 . The remaining uncovered part of the edge BE is shorter than all edges of \mathcal{T}_0 , thus such a tiling cannot be completed to a $(\beta **)$ -triangle.

To extend the $(\alpha, 2\alpha, \pi/2)$ -triangle ACD , at least two more tiles are needed. There is precisely one way of attaching two more tiles, giving a $(\beta, 2\alpha, 2\alpha)$ -triangle composed of

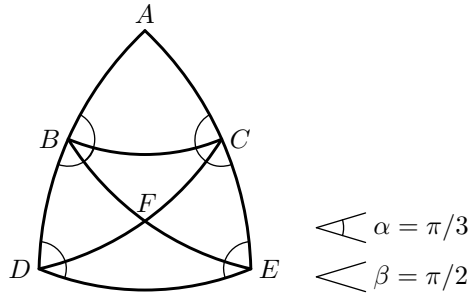


Figure 7: The $(\beta 2\alpha 2\alpha)$ -triangle composed of five $(\alpha\alpha\beta)$ -tiles with $\alpha = \pi/3, \beta = \pi/2$.

five pieces; see Figure 7. Therefore, the only $(\beta **)$ -triangles are $(\alpha\alpha\beta)$, $(\alpha, 2\alpha, \beta)$ and $(\beta, 2\alpha, 2\alpha)$. In particular, there is no $(\beta\beta*)$ -triangle.

This implies that the Coxeter diagram of S has exactly two vertex-disjoint β -edges. Let uv and xy be the two β -edges and let w be the fifth vertex of $c(S)$. If both triangles uvw and xyw are $(\alpha\alpha\beta)$ -triangles, then all edges incident with w are α -edges. No other edge can be an α -edge, since the $(\alpha\alpha\alpha)$ -triangle is not realizable. In other words, the triangles uvw and xyw are the only $(\alpha\alpha\beta)$ -triangles; a contradiction. Hence at least three of the triangles induced by the vertices u, v, x, y are $(\alpha\alpha\beta)$ -triangles. But then all four of the triangles are $(\alpha\alpha\beta)$ -triangles since the edges ux, uy, vx, vy must be α -edges. Every triangle containing one of these four α -edges and the vertex w must be of type $(\alpha, \alpha, 2\alpha)$. Therefore, without loss of generality, the edges uw and vw are α -edges, and the edges xw and yw are 2α -edges. A similar diagram is displayed in Figure 3(c), where instead of 2α -edges we have γ -edges. But in such a diagram, there are only three orbits of $(\alpha\alpha\beta)$ -triangles; a contradiction.

4.3.4 Case (C): $\mathcal{T}_0 = (\alpha\beta\gamma)$.

First we obtain some more information about the Coxeter diagram of S .

Lemma 4.18. *There is at least one realizable spherical $(\alpha **)$ -triangle different from \mathcal{T}_0 . The same is true for triangles of type $(\beta **)$ and $(\gamma **)$.*

Proof. Assume for contradiction that \mathcal{T}_0 is the only realizable spherical triangle of type $(\alpha **)$. It follows that there are exactly two vertex-disjoint α -edges in $c(S)$. Since every other edge in $c(S)$ is adjacent to an α -edge, all edges in $c(S)$ are β -edges or γ -edges, and $c(S)$ is isomorphic to one of the two diagrams in Figure 8. In both diagrams, the six $(\alpha\beta\gamma)$ -triangles form only three orbits; a contradiction. \square

By P_k we denote a path with k vertices and by $P_k + P_l$ a disjoint union of paths.

Lemma 4.19. *The Coxeter diagram of S has two or three α -edges and they form a subgraph isomorphic to $P_2 + P_2$ or $P_2 + P_3$.*

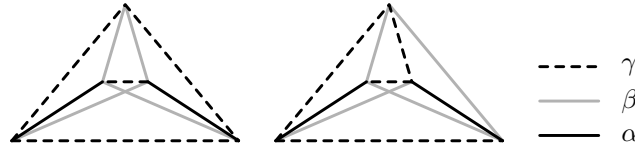


Figure 8: Two possible Coxeter diagrams if \mathcal{T}_0 is the only triangle of type $(\alpha **)$.

Proof. Let $H = c(S)$ be the Coxeter diagram of S . Let $V(H) = \{u, v, w, x, y\}$ and let $H_\alpha = (V(G), E_\alpha)$ be the subgraph of H formed by the α -edges. Since H is $(\alpha\beta\gamma)$ -rich, it has at least four $(\alpha\beta\gamma)$ -triangles, and hence it has at least two α -edges, at least two β -edges and at least two γ -edges.

Suppose that $|E_\alpha| = 2$. If the two α -edges are adjacent, say, $E_\alpha = \{uv, uw\}$, then all the triangles uvx, uvy, uwx, uwy are of type $(\alpha\beta\gamma)$. In particular, the edges vx and wx are of the same type, either β or γ , and the edges vy and wy are of the same type as well. Therefore, H has a symmetry switching v with w , and so there are at most two orbits of $(\alpha\beta\gamma)$ -triangles; a contradiction. It follows that H_α is a matching.

Suppose that $|E_\alpha| = 3$. If H_α is isomorphic to the star $K_{1,3}$, say, $E_\alpha = \{xu, xv, xw\}$, then every $(\alpha\beta\gamma)$ -triangle must contain the vertex y , so there can be at most three such triangles. If H_α is isomorphic to the path P_4 , say, $E_\alpha = \{xu, uv, vw\}$, then the edges xv and uw cannot have type β or γ , since the spherical triangles of type $(\alpha\alpha\beta)$ and $(\alpha\alpha\gamma)$ have smaller area than \mathcal{T}_0 and so they are not realizable. This again implies that every $(\alpha\beta\gamma)$ -triangle must contain the vertex y and so there are at most three of them. Also, H_α cannot form a triangle, since the spherical triangle of type $(\alpha\alpha\alpha)$ is not realizable. This leaves only one option: H_α forms a subgraph isomorphic to $P_2 + P_3$.

Suppose that $|E_\alpha| \geq 4$. If H_α contains a star $K_{1,4}$, then no other edge can be of type β or γ . If H_α contains a fork, say, $E_\alpha \supseteq \{uv, vw, wx, wy\}$, then only two edges, ux and uy , can be of type β or γ . If H_α contains a path P_5 , say, $E_\alpha \supseteq \{uv, vw, wx, xy\}$, then only three edges, ux, uy and vy , can be of type β or γ . Since H_α cannot contain triangles, the only remaining possibility is that H_α is isomorphic to the 4-cycle, say, $E_\alpha = \{uv, vx, xy, yu\}$. All edges of type β or γ must be incident with w , hence uw and xw are of the same type, and also vw and yw are of the same type. Regardless of the type of the diagonals ux and vy , this diagram has a symmetry group generated by the transpositions (u, x) and (v, y) , and so the $(\alpha\beta\gamma)$ -triangles form just one orbit; a contradiction. \square

In the following lemma we obtain some partial information about the angles of \mathcal{T}_0 and identify two basic cases.

Lemma 4.20. *If $\mathcal{T}_0 = (\alpha\beta\gamma)$ then $\gamma = \pi/2$. Furthermore,*

- a) $\alpha + 2\beta = \pi$, or
- b) $\beta = \pi/3$ and $\alpha > \pi/6$.

Proof. By Lemma 4.18, the spherical area of the α -lune \mathcal{L}_α is greater than $2\Delta(\mathcal{T}_0)$. By Lemma 4.13, there is a tiling of the lune \mathcal{L}_α by at least three copies of \mathcal{T}_0 .

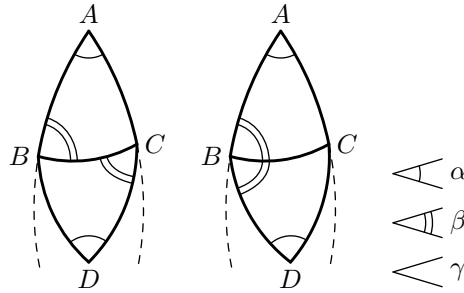


Figure 9: Two possibilities for the first two tiles in a tiling of \mathcal{L}_α by \mathcal{T}_0 .

Let \mathcal{T}_0^1 be a corner-filling tile with vertices A, B, C incident with angles α, β, γ , respectively. In particular, A is a vertex of \mathcal{L}_α . By Observation 4.10, \mathcal{T}_0^1 is adjacent to a tile \mathcal{T}_0^2 with vertices B, C, D , which can be placed in two possible orientations; see Figure 9. If \mathcal{T}_0^2 has the same orientation as \mathcal{T}_0^1 , the quadrilateral $ABDC$ is a spherical parallelogram with angles $\beta + \gamma$ at vertices B and C . By Lemma 4.18, two copies of \mathcal{T}_0 cannot tile \mathcal{L}_α and so $\beta + \gamma < \pi$. Since $\alpha + \beta + \gamma > \pi$ by Lemma 4.9(a), the parallelogram $ABDC$ cannot be completed to a tiling of \mathcal{L}_α . Therefore \mathcal{T}_0^2 and \mathcal{T}_0^1 have opposite orientations.

Since $\alpha + 2\gamma > \alpha + \beta + \gamma > \pi$, no other tile can be incident to C and so $\gamma = \pi/2$. That is, the two tiles \mathcal{T}_0^1 and \mathcal{T}_0^2 form a triangle ABD . The angles of the tiles incident to B include two angles β , and together they sum up to π . No tile can have angle γ at B since $2\beta + \gamma > \pi$. Therefore, there exist non-negative integers n_1 and $n_2 \geq 2$ such that $n_1\alpha + n_2\beta = \pi$. Since $\gamma = \pi/2$, we have $\alpha + \beta > \pi/2$, implying $n_1 \leq 1$ and $\beta > \pi/4$. There are only two cases:

- a) $n_1 = 1$: then $n_2 = 2$ and thus $\alpha + 2\beta = \pi$.
- b) $n_1 = 0$: then $\beta = \pi/3$ and consequently $\alpha > \pi/6$.

This concludes the proof. □

Now we deal separately with the two cases from Lemma 4.20.

Case a) $\alpha + 2\beta = \pi$.

By Observation 4.10 and by the fact that $\beta + \gamma < \pi < \alpha + \beta + \gamma$, the tiling of every realizable $(\alpha * *)$ -triangle other than \mathcal{T}_0 contains two tiles ABC and BCD with opposite orientations as in Figure 10, forming a triangle of type $(\alpha, \alpha, 2\beta)$. By the triangle inequality (Lemma 4.9(e)), the edge BD cannot be subdivided by the edge of \mathcal{T}_0 opposite to β , thus there is just one possible way of placing a third tile: the triangle BDE in Figure 10. These three tiles form a triangle of type $(\alpha, \alpha + \beta, \gamma)$. The fourth tile would fill the whole α -lune, therefore the only realizable $(\alpha * *)$ -triangles are $(\alpha\beta\gamma)$, $(\alpha, \alpha, 2\beta)$ and $(\alpha, \alpha + \beta, \gamma)$.

Lemma 4.21. *Every $(\alpha\beta\gamma)$ -rich Coxeter diagram with five vertices where all $(\alpha * *)$ -triangles are of type $(\alpha\beta\gamma)$, $(\alpha, \alpha, 2\beta)$ or $(\alpha, \alpha + \beta, \gamma)$, is isomorphic to one of the five diagrams in Figure 11.*

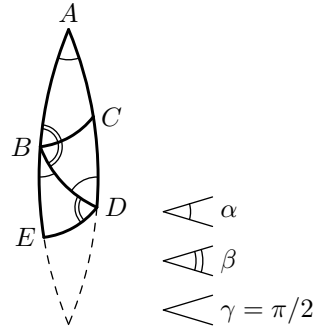


Figure 10: Tiling of spherical triangles for $\gamma = \pi/2$ and $\alpha + 2\beta = \pi$.

Proof. Let H be a Coxeter diagram satisfying the assumptions of the lemma. Let $V(H) = \{u, v, w, x, y\}$ and let E_α be the set of α -edges. By Lemma 4.19, we distinguish two cases, up to isomorphism.

1) $E_\alpha = \{ux, vy\}$. In this case, all triangles containing an α -edge are of type $(\alpha\beta\gamma)$ or $(\alpha, \alpha + \beta, \gamma)$. In particular, every such triangle contains exactly one γ -edge. By symmetry, we may assume that xw and yw are γ -edges. The other two γ -edges form a matching on vertices u, v, x, y ; there are two possibilities, $\{uv, xy\}$ and $\{uy, vx\}$. The remaining four edges are of type β or $\alpha + \beta$.

If all the remaining edges are β -edges, or if both uw and vw are of type $\alpha + \beta$, the diagram has a symmetry Φ exchanging simultaneously u with v and x with y , so the $(\alpha\beta\gamma)$ -triangles form at most three orbits. Since there are at least four $(\alpha\beta\gamma)$ -triangles in H , it follows that exactly one of the four remaining edges is of type $\alpha + \beta$ and the remaining three edges are β -edges. If uv or xy is the edge of type $\alpha + \beta$ (so uy and vx were chosen as γ -edges), then Φ is again a symmetry of the diagram and the $(\alpha\beta\gamma)$ -triangles form only two orbits. Thus, up to isomorphism, we have three possibilities for H ; see Figure 11(a)–(c).

2) $E_\alpha = \{uw, vw, xy\}$. In this case the edge uv must be of type 2β and it is the only edge of this type. Each of the seven $(\alpha * *)$ -triangles other than uvw has exactly one γ -edge. Due to symmetry, we may assume that xw is a γ -edge. Then yu and yv must be γ -edges as well. The remaining three edges, yw, xu and xv , are of type β or $\alpha + \beta$. If all these three edges are of type β , the diagram has a symmetry Ψ exchanging u and v , but the $(\alpha\beta\gamma)$ -triangles still form four orbits; see Figure 11(d). Otherwise, exactly one of the three edges yw, xu, xv is of type $\alpha + \beta$. If yw is of type $\alpha + \beta$, then due to the symmetry Ψ , the $(\alpha\beta\gamma)$ -triangles form only two orbits. The other two cases give isomorphic diagrams; see Figure 11(e). \square

We immediately notice that the diagram in Figure 11(e) cannot be a Coxeter diagram of S , since xuv is a $(\beta, \alpha + \beta, 2\beta)$ -triangle but there is no spherical triangle of type $(\beta, \alpha + \beta, 2\beta)$ by Lemma 4.9(b).

We are left with the diagrams in Figure 11(a)–(d). Since investigating realizable $(\beta **)$ -triangles does not seem to help much, we proceed to the next step and use Fiedler's

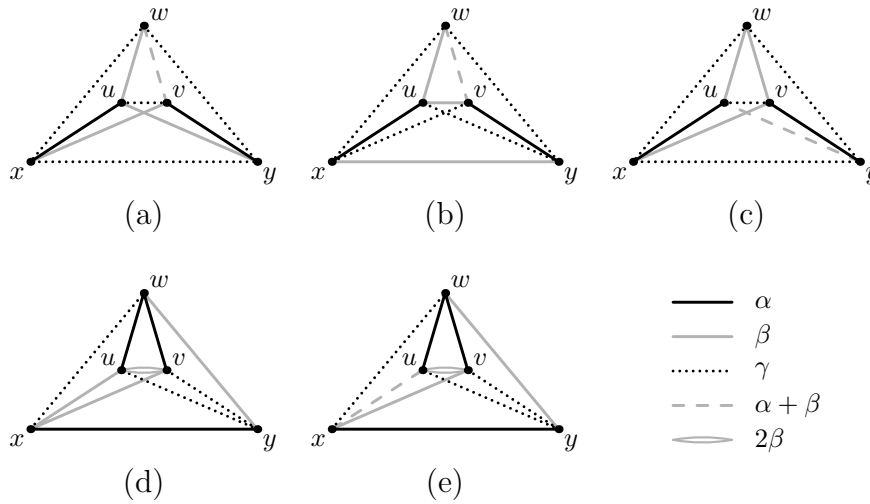


Figure 11: Coxeter diagrams for the case $\alpha + 2\beta = \pi$.

theorem (Theorem 2.3).

Recall that the matrix A associated to a simplex S satisfies $a_{i,i} = -1$ and $a_{ij} = \cos \beta_{ij}$ for $i \neq j$, where β_{ij} is the dihedral angle between facets F_i and F_j . In particular, the matrix A is completely determined by the Coxeter diagram $c(S)$.

Let $t := \cos \beta$ and $s := \cos \alpha$. Since $\alpha + 2\beta = \pi$ and $\alpha < \beta$, it follows that $s = 1 - 2t^2$ and $t \in (0, 1/2)$. Moreover, $\cos 2\beta = 2t^2 - 1$ and $\cos(\alpha + \beta) = \cos(\pi - \beta) = -t$.

The following matrices A_1, \dots, A_4 are associated to the simplices represented by the diagrams in Figure 11(a)–(d), respectively. The rows and columns of A_1, \dots, A_4 are indexed by u, v, w, x, y (in this order), where the vertices u, v, w, x, y of $c(S)$ represent facets F_1, \dots, F_5 in S .

$$\begin{aligned}
 A_1 &= \begin{pmatrix} -1 & 0 & t & 1 - 2t^2 & t \\ 0 & -1 & -t & t & 1 - 2t^2 \\ t & -t & -1 & 0 & 0 \\ 1 - 2t^2 & t & 0 & -1 & 0 \\ t & 1 - 2t^2 & 0 & 0 & -1 \end{pmatrix} \\
 A_2 &= \begin{pmatrix} -1 & t & t & 1 - 2t^2 & 0 \\ t & -1 & -t & 0 & 1 - 2t^2 \\ t & -t & -1 & 0 & 0 \\ 1 - 2t^2 & 0 & 0 & -1 & t \\ 0 & 1 - 2t^2 & 0 & t & -1 \end{pmatrix} \\
 A_3 &= \begin{pmatrix} -1 & 0 & t & 1 - 2t^2 & -t \\ 0 & -1 & t & t & 1 - 2t^2 \\ t & t & -1 & 0 & 0 \\ 1 - 2t^2 & t & 0 & -1 & 0 \\ -t & 1 - 2t^2 & 0 & 0 & -1 \end{pmatrix}
 \end{aligned}$$

$$A_4 = \begin{pmatrix} -1 & 2t^2 - 1 & 1 - 2t^2 & t & 0 \\ 2t^2 - 1 & -1 & 1 - 2t^2 & t & 0 \\ 1 - 2t^2 & 1 - 2t^2 & -1 & 0 & t \\ t & t & 0 & -1 & 1 - 2t^2 \\ 0 & 0 & t & 1 - 2t^2 & -1 \end{pmatrix}$$

Considering t as a variable, the determinants of the matrices A_1, \dots, A_4 are polynomials in t . Let $\Gamma(A_i)$ be the set of real roots of the determinant of A_i . Rounding the roots to two decimal places, we have:

A	$\det(A)$	$\Gamma(A)$
A_1	$-t^2(2t - 1)(2t^2 - t - 2)(4t^3 + 4t^2 - t - 2)$	$\{0, 0.5, -0.78, 1.28, 0.63\}$
A_2	$-t^2(2t - 1)(2t^2 + t - 2)(4t^3 + 2t^2 - 3t - 2)$	$\{0, 0.5, -1.28, 0.78, 0.92\}$
A_3	$-t^4(2t - 1)(2t + 1)(4t^2 - 3)$	$\{0, \pm 0.5, \pm 0.87\}$
A_4	$-8t^4(2t^2 - 1)(4t^4 - 7t^2 + 2)$	$\{0, \pm 0.71, \pm 1.18, \pm 0.60\}$

By Fiedler's theorem, the matrix associated to a simplex is singular. Therefore, the determinant of A_i must have a root in the interval $(0, 1/2)$. Since no A_i satisfies this condition, we have a contradiction and the lemma follows.

Case b) $\beta = \pi/3$ and $\alpha > \pi/6$.

Lemma 4.22. *If $\mathcal{T}_0 = (\alpha, \pi/3, \pi/2)$ with $\alpha > \pi/6$, then $\alpha \in \{\pi/4, 2\pi/9, \pi/5\}$.*

Proof. By Lemma 4.14, there are integers $m \geq 1$ and $n, p \geq 0$ such that $m\alpha + n\pi/3 + p\pi/2 = \pi$. If $p \geq 1$, then $p = 1$ and $n = 0$. Since $\pi/6 < \alpha < \pi/3$, we have $m = 2$ and thus $\alpha = \pi/4$. If $p = 0$ then $n \leq 1$. For $n = 0$ we have $\alpha = \pi/m$, thus $m = 4$ or 5 . For $n = 1$ we have $\alpha = 2\pi/(3m)$, which is in the interval $(\pi/6, \pi/3)$ only for $m = 3$. The lemma follows. \square

The following simple observation will be useful for determining all realizable $(\alpha **)$ - and $(\pi/3 **)$ -triangles.

Observation 4.23. *Let \mathcal{L} be a φ -lune that can be tiled with \mathcal{T}_0 . Let \mathcal{H} be a realizable spherical triangle whose two copies tile the lune \mathcal{L} and let $\mathcal{K} \subseteq \mathcal{H}$ be a realizable corner-filling spherical triangle in \mathcal{L} whose tiling by copies of \mathcal{T}_0 can be extended to a tiling of \mathcal{H} . Then the complement of \mathcal{K} in \mathcal{L} is also realizable; see Figure 12. \square*

In the following lemma we investigate all realizable $(\alpha **)$ - and $(\beta **)$ -triangles for $\mathcal{T}_0 = (\alpha, \pi/3, \pi/2)$. For better clarity we write α and β rather than their numerical values.

Lemma 4.24. *Let $\mathcal{T}_0 = (\alpha, \beta, \pi/2)$, where $\beta = \pi/3$. Depending on the value of α , the realizable $(\alpha **)$ -triangles and $(\beta **)$ -triangles other than \mathcal{T}_0 are the following.*

- 1) $\alpha = \pi/4$

- $(\alpha, \alpha, 2\beta), (\alpha, \pi/2, \pi/2), (\alpha, \beta, 3\alpha), (\alpha, \pi/2, 2\beta)$
- $(\beta, \beta, \pi/2), (\beta, \beta, 2\beta), (\beta, \pi/2, 2\beta), (\beta, \pi/2, 3\alpha)$

2) $\alpha = \pi/5$

- $(\alpha, \alpha, 2\beta), (\alpha, 2\alpha, \pi/2), (\alpha, \beta, 3\alpha), (\alpha, \beta, 2\beta), (\alpha, \alpha, 4\alpha), (\alpha, 2\alpha, 2\beta),$
 $(\alpha, \pi/2, 3\alpha), (\alpha, \beta, 4\alpha), (\alpha, \pi/2, 2\beta)$
- $(\beta, \beta, 2\alpha), (\beta, 2\alpha, \pi/2), (\beta, 2\alpha, 3\alpha), (\beta, \pi/2, 3\alpha), (\beta, \beta, 4\alpha), (\beta, 2\alpha, 4\alpha),$
 $(\beta, 3\alpha, 2\beta), (\beta, \pi/2, 4\alpha)$

3) $\alpha = 2\pi/9$

- $(\alpha, \alpha, 2\beta), (\alpha, 2\alpha, \pi/2), (\alpha, \beta, 2\beta), (\alpha, \pi/2, \alpha + \beta), (\alpha, \beta, 2\alpha + \beta), (\alpha, \alpha, 4\alpha),$
 $(\alpha, \pi/2, 2\beta)$
- $(\beta, \beta, 2\alpha), (\beta, \alpha + \beta, 2\beta), (\beta, \pi/2, 2\alpha + \beta).$

Proof. Let a, b, c denote the edges (and also their lengths) of \mathcal{T}_0 opposite to angles α, β, γ , respectively.

Now we describe a general procedure of finding all realizable $(\tau **)$ -triangles, which we then apply to specific values of τ . First we generate all spherical triangles whose angles and edges can be obtained as nonnegative integer combinations of angles and edges, respectively, of the basic tile \mathcal{T}_0 , and whose spherical area is a multiple of $\Delta(\mathcal{T}_0)$, the spherical area of \mathcal{T}_0 . After that we check whether they are realizable. Clearly, this involves going through only a finite number of triangles and a finite number of tilings. In fact, it will turn out that all the generated spherical triangles but one are realizable.

Here we provide more details about the procedure. Let $\mathcal{T} = (\tau\varphi\psi)$, $\mathcal{T} \neq \mathcal{T}_0$, be a realizable triangle. We may assume that \mathcal{T} is a corner-filling triangle in the τ -lune and $\varphi \leq \psi$.

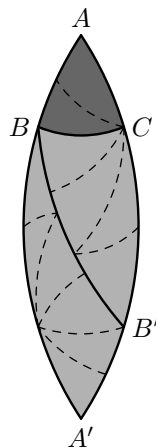


Figure 12: A tiling of a lune \mathcal{L} where $\mathcal{H} = ABB'$ and $\mathcal{K} = ABC$.

Since \mathcal{T} is realizable, it follows that $\Delta(\mathcal{T}) = n\Delta(\mathcal{T}_0)$, where n is the number of copies of \mathcal{T}_0 needed to tile \mathcal{T} . Moreover, the angles τ, φ, ψ can be expressed as nonnegative integer combinations of α, β, γ . By Lemma 4.9(a), we have

$$\begin{aligned} n\Delta(\mathcal{T}_0) &= \Delta(\mathcal{T}) = \tau + \varphi + \psi - \pi \\ n(\alpha + \beta + \gamma - \pi) + \pi - \tau &= \varphi + \psi \\ n(\alpha + \beta + \gamma - \pi) + \pi - \tau &= m_1\alpha + m_2\pi/3 + m_3\pi/2. \end{aligned} \tag{3}$$

Note that $2 \leq n < 2\tau/\Delta(\mathcal{T}_0)$, since $\mathcal{T} \neq \mathcal{T}_0$ and the volume of the τ -lune is 2τ . From the solutions of the equation (3) we get all possible values of $\varphi + \psi$ and hence also all possible pairs φ, ψ . We keep only those triples τ, φ, ψ that satisfy the spherical triangle inequality (Lemma 4.9(b)). Using the spherical law of cosines (Lemma 4.9(c)), we compute the length of the edge x of \mathcal{T} opposite to the angle τ and check whether it can be obtained as a nonnegative integer combination of a, b, c . If not, then \mathcal{T} is clearly not realizable.

Since the calculations are tedious and it is relatively easy to make a mistake or forget some case, we decided to write a computer program to go through all the possibilities. The program was written in Sage 5.4.1 [37] and can be found in Appendix A. The program takes five arguments $\alpha, \beta, \gamma, \tau, \rho$ and searches for all possible realizable $(\tau\varphi\psi)$ -triangles with $\rho < \varphi \leq \psi$ as follows. First, the program goes through all 4-tuples n, m_1, m_2, m_3 satisfying equation (3). Then it tries to split $m_1\alpha + m_2\beta + m_3\gamma$ between φ and ψ . If two splittings provide the same pair (φ, ψ) , we list it just once. Then we test the condition (b) from Lemma 4.9 for the $(\tau\varphi\psi)$ -triangle. As the last step, we use the spherical law of cosines (Lemma 4.9(c)) to approximate the length of the edge x opposite to the angle τ and test whether it can be obtained as an approximate nonnegative integer combination of a, b, c . If not, we list the nonnegative integer combinations of a, b, c that are closest to x . In the output we round the numerical values to three decimal places.

For every triangle $\mathcal{T} = (\tau\varphi\psi)$ not excluded by the program, we try to find some tiling of \mathcal{T} by hand. We use Observation 4.23 to simplify the search: in many cases, it will be enough to find a tiling of a corner-filling triangle whose two copies fill the whole τ -lune.

1) First we find realizable $(\alpha **)$ -triangles for $\alpha = \pi/4$. Table 2 lists the output of the program for $\alpha, \beta, \gamma, \alpha, 0$. The four highlighted pairs (φ, ψ) in Table 2 are the only candidates for realizable $(\alpha\varphi\psi)$ -triangles. Figure 13, left, shows that all four candidates from Table 2 are indeed realizable. The realizability of the triangles $A'BD$ and $A'CB$ in Figure 13, left, follows by Observation 4.23 since $A'DE$ is a copy of ADE .

Now we find all realizable $(\beta\varphi\psi)$ -triangles. Since we have characterized all realizable $(\alpha **)$ -triangles, we can assume that $\varphi, \psi > \alpha$. Table 3 lists the output of the program for $\alpha, \beta, \gamma, \beta, \alpha$. Figure 13, right, shows that all the candidates from Table 3 are realizable. Again, triangles $B'AD$ and $B'CA$ in Figure 13, right, are realizable by Observation 4.23 since $B'AG$ is a copy of BGA .

2) First we find realizable $(\alpha **)$ -triangles for $\alpha = \pi/5$. Table 4 lists the output of the program for $\alpha, \beta, \gamma, \alpha, 0$. Figure 14, left, shows that all the candidates from Table 4 are realizable. The triangles $A'DF, A'ED, A'BD, A'CB$ are realizable by Observation 4.23 since $A'HD$ is a copy of ADH .

n	(m_1, m_2, m_3)	(φ, ψ)	x	best approximation for x using a, b, c
2	(1,2,0)	$(\beta, \alpha + \beta)$	0.808	$0.785 \sim b < x < c \sim 0.955$
		$(\alpha, 2\beta)$	0.955	$x = c$
3	(0,0,2)	(γ, γ)	0.785	$x = b$
	(0,3,0)	$(\beta, 2\beta)$	0.915	$0.785 \sim b < x < c \sim 0.955$
	(2,0,1)	$(\alpha, \alpha + \gamma)$	1.144	$0.955 \sim c < x < 2a \sim 1.231$
4	(1,1,1)	$(\gamma, \alpha + \beta)$	0.749	$0.615 \sim a < x < b \sim 0.785$
		$(\beta, \alpha + \gamma)$	0.955	$x = c$
		$(\alpha, \beta + \gamma)$	1.300	$1.231 \sim 2a < x < a + b \sim 1.401$
5	(0,2,1)	$(\gamma, 2\beta)$	0.615	$x = a$
		$(\beta, \beta + \gamma)$	0.885	$0.785 \sim b < x < c \sim 0.95532$
	(2,2,0)	$(\alpha, \alpha + 2\beta)$	1.439	$1.401 \sim a + b < x < 2b \sim 1.571$
		$(\alpha + \beta, \alpha + \beta)$	0.592	$0.0 \sim 0 < x < a \sim 0.615$

Table 2: $(\alpha * *)$ -triangles for $\alpha = \pi/4$. Approximate values of a, b, c are $a \sim 0.615$, $b \sim 0.785$ and $c \sim 0.955$.

n	(m_1, m_2, m_3)	(φ, ψ)	x	best approximation for x using a, b, c
2	(0,1,1)	(β, γ)	0.955	$x = c$
3	(1,2,0)	$(\beta, \alpha + \beta)$	1.112	$0.955 \sim c < x < 2a \sim 1.231$
4	(0,0,2)	(γ, γ)	1.047	$0.955 \sim c < x < 2a \sim 1.231$
	(0,3,0)	$(\beta, 2\beta)$	1.231	$x = 2a$
5	(1,1,1)	$(\gamma, \alpha + \beta)$	1.027	$0.955 \sim c < x < 2a \sim 1.231$
		$(\beta, \alpha + \gamma)$	1.329	$1.231 \sim 2a < x < a + b \sim 1.401$
6	(0,2,1)	$(\gamma, 2\beta)$	0.955	$x = c$
		$(\beta, \beta + \gamma)$	1.415	$1.401 \sim a + b < x < 2b \sim 1.571$
	(2,2,0)	$(\alpha + \beta, \alpha + \beta)$	0.918	$0.785 \sim b < x < c \sim 0.955$
7	(1,0,2)	$(\gamma, \alpha + \gamma)$	0.785	$x = b$
	(1,3,0)	$(\beta, \alpha + 2\beta)$	1.495	$1.401 \sim a + b < x < 2b \sim 1.571$
$(\alpha + \beta, 2\beta)$		0.719	$0.615 \sim a < x < b \sim 0.785$	

Table 3: $(\beta\varphi\psi)$ -triangles for $\alpha = \pi/4$ and $\varphi, \psi > \alpha$. Approximate values of a, b, c are $a \sim 0.615$, $b \sim 0.785$ and $c \sim 0.955$.

Now we find all realizable $(\beta\varphi\psi)$ -triangles. We may again assume that $\varphi, \psi > \alpha$. Table 5 lists the output of the program for $\alpha, \beta, \gamma, \beta, \alpha$. Figure 14, right, shows that all the candidates from Table 5 are realizable. The triangles $B'GI, B'DH, B'AI, B'AD$ and $B'CA$ are realizable by Observation 4.23 since $B'IH$ is a copy of BHI .

3) First we find all realizable $(\alpha * *)$ -triangles. Table 6 lists the output of the program for $\alpha, \beta, \gamma, \alpha, 0$. Figure 15, left, shows that all the candidates from Table 6 are realizable. The triangles $A'ED, A'BD$ and $A'CB$ are realizable by Observation 4.23 since $A'DF$ is a copy of AFD .

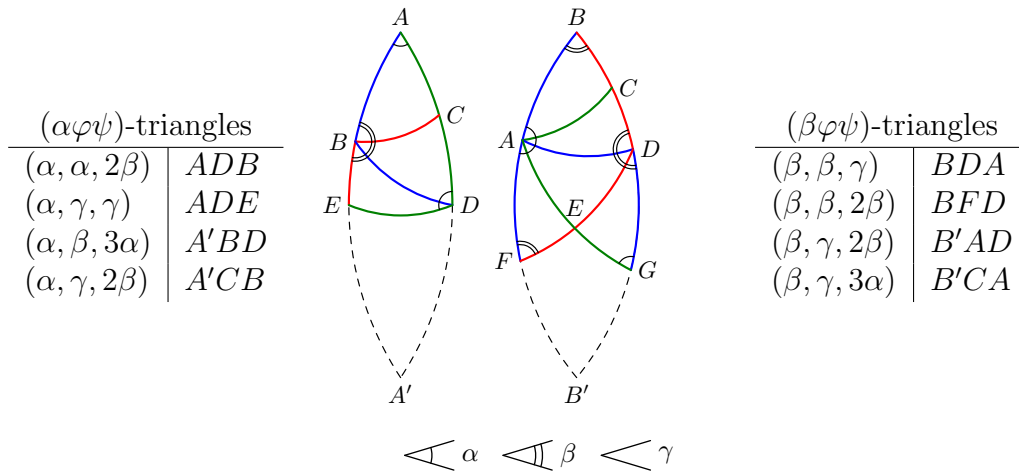


Figure 13: A tiling of $(\alpha **)$ -triangles (left) and $(\beta **)$ -triangles (right) for $\alpha = \frac{\pi}{4}$.

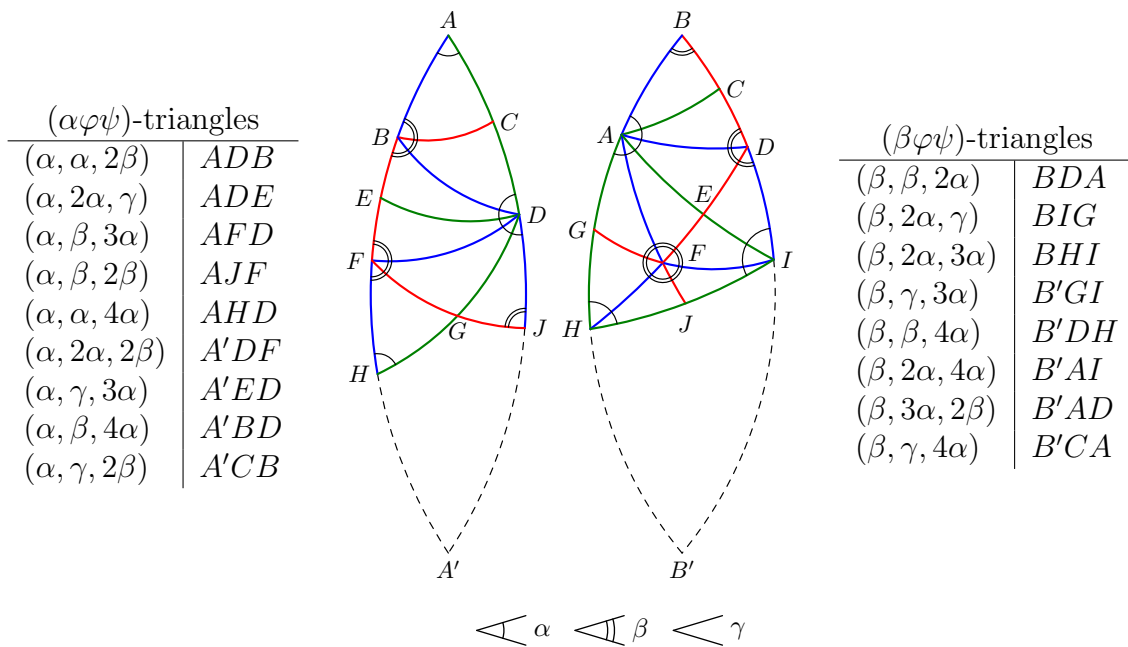


Figure 14: A tiling of $(\alpha **)$ -triangles (left) and $(\beta **)$ -triangles (right) for $\alpha = \frac{\pi}{5}$.

It remains to find all realizable $(\beta **)$ -triangles. Table 7 lists the output of the program for $\alpha, \beta, \gamma, \beta, \alpha$. We show that the $(\beta, \beta, 2\alpha + \beta)$ -triangle is not realizable. This might be a bit surprising, as the edges of this triangle have lengths $2b, 2b$ and $2a + 2b$. Suppose that $\mathcal{T} = (\beta, \beta, 2\alpha + \beta)$ is realizable. Let B, U, V be the vertices of \mathcal{T} so that BU and UV are edges of length $2b$. We may assume that \mathcal{T} is corner-filling in the β -lune with vertex B . Since \mathcal{T} is realizable and β is not a multiple of α , we may assume that in a tiling of \mathcal{T} by copies of \mathcal{T}_0 , some tile is also a corner-filling tile of the β -lune. It follows that the

n	(m_1, m_2, m_3)	(φ, ψ)	x	best approximation for x using a, b, c
2	(1,2,0)	$(\beta, \alpha + \beta)$	0.498	$0.365 \sim a < x < b \sim 0.554$
		$(\alpha, 2\beta)$	0.652	$x = c$
3	(2,0,1)	$(\alpha, \alpha + \gamma)$	0.794	$0.730 \sim 2a < x < a + b \sim 0.918$
		$(2\alpha, \gamma)$	0.554	$x = b$
4	(3,1,0)	$(\beta, 3\alpha)$	0.652	$x = c$
		$(\alpha, 2\alpha + \beta)$	0.911	$0.730 \sim 2a < x < a + b \sim 0.918$
		$(2\alpha, \alpha + \beta)$	0.607	$0.554 \sim b < x < c \sim 0.652$
6	(0,0,2)	(γ, γ)	0.628	$0.554 \sim b < x < c \sim 0.652$
	(0,3,0)	$(\beta, 2\beta)$	0.730	$x = 2a$
	(5,0,0)	$(\alpha, 4\alpha)$	1.107	$x = 2b$
7	(1,1,1)	$(2\alpha, 3\alpha)$	0.662	$0.652 \sim c < x < 2a \sim 0.730$
		$(\gamma, \alpha + \beta)$	0.620	$0.554 \sim b < x < c \sim 0.652$
		$(\beta, \alpha + \gamma)$	0.745	$0.730 \sim 2a < x < a + b \sim 0.918$
8	(2,2,0)	$(\alpha, \beta + \gamma)$	1.193	$1.107 \sim 2b < x < b + c \sim 1.206$
		$(\beta, 2\alpha + \beta)$	0.742	$0.730 \sim 2a < x < a + b \sim 0.918$
		$(\alpha, \alpha + 2\beta)$	1.274	$1.206 \sim b + c < x < 2a + b \sim 1.283$
9	(3,0,1)	$(\alpha + \beta, \alpha + \beta)$	0.593	$0.554 \sim b < x < c \sim 0.652$
		$(2\alpha, 2\beta)$	0.652	$x = c$
		$(\gamma, 3\alpha)$	0.554	$x = b$
10	(4,1,0)	$(\alpha, 2\alpha + \gamma)$	1.351	$1.305 \sim 2c < x < 2a + c \sim 1.382$
		$(2\alpha, \alpha + \gamma)$	0.617	$0.554 \sim b < x < c \sim 0.652$
		$(\beta, 4\alpha)$	0.652	$x = c$
11	(0,2,1)	$(\alpha, 3\alpha + \beta)$	1.426	$1.382 \sim 2a + c < x < 4a \sim 1.459$
		$(\alpha + \beta, 3\alpha)$	0.475	$0.365 \sim a < x < b \sim 0.554$
		$(2\alpha, 2\alpha + \beta)$	0.551	$0.365 \sim a < x < b \sim 0.554$
11	(0,2,1)	$(\gamma, 2\beta)$	0.365	$x = a$
		$(\beta, \beta + \gamma)$	0.519	$0.365 \sim a < x < b \sim 0.554$

Table 4: $(\alpha * *)$ -triangles for $\alpha = \pi/5$. Approximate values of a, b, c are $a \sim 0.365$, $b \sim 0.554$ and $c \sim 0.652$.

induced tiling of the edge BU of length $2b$ by edges a, b, c contains at least one edge a or c . However, it is easy to check that neither $2b - a \sim 0.874$ nor $2b - c \sim 0.547$ can be expressed as a nonnegative integer combination of a, b, c .

Figure 15, right, shows that all the other candidates are realizable. Triangles $B'AD$ and $B'CA$ are realizable by Observation 4.23 since $B'HA$ is a copy of BAH .

This finishes the proof of Lemma 4.24. □

n	(m_1, m_2, m_3)	(φ, ψ)	x	best approximation for x using a, b, c
2	(2,1,0)	$(\beta, 2\alpha)$	0.652	$x = c$
4	(4,0,0)	$(2\alpha, 2\alpha)$	0.852	$0.730 \sim 2a < x < a + b \sim 0.918$
5	(0,1,1)	(β, γ)	0.955	$0.918 \sim a + b < x < a + c \sim 1.017$
6	(1,2,0)	$(\beta, \alpha + \beta)$	1.024	$1.017 \sim a + c < x < 3a \sim 1.095$
7	(2,0,1)	$(2\alpha, \gamma)$	1.017	$x = a + c$
8	(3,1,0)	$(\beta, 3\alpha)$	1.138	$1.107 \sim 2b < x < b + c \sim 1.206$
		$(2\alpha, \alpha + \beta)$	1.054	$1.017 \sim a + c < x < 3a \sim 1.095$
10	(0,0,2)	(γ, γ)	1.047	$1.017 \sim a + c < x < 3a \sim 1.095$
	(0,3,0)	$(\beta, 2\beta)$	1.231	$1.206 \sim b + c < x < 2a + b \sim 1.283$
	(5,0,0)	$(2\alpha, 3\alpha)$	1.107	$x = 2b$
11	(1,1,1)	$(\gamma, \alpha + \beta)$	1.044	$1.017 \sim a + c < x < 3a \sim 1.095$
		$(\beta, \alpha + \gamma)$	1.272	$1.206 \sim b + c < x < 2a + b \sim 1.283$
12	(2,2,0)	$(\beta, 2\alpha + \beta)$	1.311	$1.305 \sim 2c < x < 2a + c \sim 1.382$
		$(\alpha + \beta, \alpha + \beta)$	1.028	$1.017 \sim a + c < x < 3a \sim 1.095$
		$(2\alpha, 2\beta)$	1.138	$1.107 \sim 2b < x < b + c \sim 1.206$
13	(3,0,1)	$(\gamma, 3\alpha)$	1.017	$x = a + c$
		$(2\alpha, \alpha + \gamma)$	1.144	$1.107 \sim 2b < x < b + c \sim 1.206$
14	(4,1,0)	$(\beta, 4\alpha)$	1.382	$x = 2a + c$
		$(\alpha + \beta, 3\alpha)$	0.973	$0.918 \sim a + b < x < a + c \sim 1.017$
		$(2\alpha, 2\alpha + \beta)$	1.143	$1.107 \sim 2b < x < b + c \sim 1.206$
15	(0,2,1)	$(\gamma, 2\beta)$	0.955	$0.918 \sim a + b < x < a + c \sim 1.017$
		$(\beta, \beta + \gamma)$	1.415	$1.382 \sim 2a + c < x < 4a \sim 1.459$
16	(1,0,2)	$(\gamma, \alpha + \gamma)$	0.905	$0.730 \sim 2a < x < a + b \sim 0.918$
	(1,3,0)	$(\beta, \alpha + 2\beta)$	1.448	$1.382 \sim 2a + c < x < 4a \sim 1.459$
		$(\alpha + \beta, 2\beta)$	0.875	$0.730 \sim 2a < x < a + b \sim 0.918$
	(6,0,0)	$(2\alpha, 4\alpha)$	1.107	$x = 2b$
		$(3\alpha, 3\alpha)$	0.852	$0.730 \sim 2a < x < a + b \sim 0.918$
17	(2,1,1)	$(\gamma, 2\alpha + \beta)$	0.833	$0.730 \sim 2a < x < a + b \sim 0.918$
		$(\beta, 2\alpha + \gamma)$	1.479	$1.472 \sim a + 2b < x < a + b + c \sim 1.571$
		$(\alpha + \beta, \alpha + \gamma)$	0.798	$0.730 \sim 2a < x < a + b \sim 0.918$
		$(2\alpha, \beta + \gamma)$	1.060	$1.017 \sim a + c < x < 3a \sim 1.095$
18	(3,2,0)	$(\beta, 3\alpha + \beta)$	1.510	$1.472 \sim a + 2b < x < a + b + c \sim 1.571$
		$(\alpha + \beta, 2\alpha + \beta)$	0.690	$0.652 \sim c < x < 2a \sim 0.730$
		$(2\alpha, \alpha + 2\beta)$	0.973	$0.918 \sim a + b < x < a + c \sim 1.017$
		$(3\alpha, 2\beta)$	0.652	$x = c$
19	(4,0,1)	$(\gamma, 4\alpha)$	0.554	$x = b$
		$(2\alpha, 2\alpha + \gamma)$	0.794	$0.730 \sim 2a < x < a + b \sim 0.918$
		$(3\alpha, \alpha + \gamma)$	0.482	$0.365 \sim a < x < b \sim 0.554$

Table 5: $(\beta\varphi\psi)$ -triangles for $\alpha = \pi/5$ and $\varphi, \psi > \alpha$. Approximate values of a, b, c are $a \sim 0.365$, $b \sim 0.554$ and $c \sim 0.652$.

n	(m_1, m_2, m_3)	(φ, ψ)	x	best approximation for x using a, b, c
2	(1,2,0)	$(\beta, \alpha + \beta)$	0.649	$0.485 \sim a < x < b \sim 0.680$
		$(\alpha, 2\beta)$	0.812	$x = c$
3	(4,0,0)	$(2\alpha, 2\alpha)$	0.608	$0.485 \sim a < x < b \sim 0.680$
	(2,0,1)	$(\alpha, \alpha + \gamma)$	0.982	$0.971 \sim 2a < x < a + b \sim 1.165$
	$(2\alpha, \gamma)$	0.680	$x = b$	
4	(0,0,2)	(γ, γ)	0.698	$0.680 \sim b < x < c \sim 0.812$
	(0,3,0)	$(\beta, 2\beta)$	0.812	$x = c$
	(3,1,0)	$(\alpha, 2\alpha + \beta)$	1.122	$0.971 \sim 2a < x < a + b \sim 1.165$
5	(1,1,1)	$(2\alpha, \alpha + \beta)$	0.709	$0.680 \sim b < x < c \sim 0.812$
		$(\gamma, \alpha + \beta)$	0.680	$x = b$
		$(\beta, \alpha + \gamma)$	0.836	$0.812 \sim c < x < 2a \sim 0.971$
6	(2,2,0)	$(\alpha, \beta + \gamma)$	1.246	$1.164 \sim a + b < x < a + c \sim 1.297$
		$(\beta, 2\alpha + \beta)$	0.812	$x = c$
		$(\alpha, \alpha + 2\beta)$	1.359	$x = 2b$
7	(0,2,1)	$(\alpha + \beta, \alpha + \beta)$	0.608	$0.485 \sim a < x < b \sim 0.680$
		$(2\alpha, 2\beta)$	0.649	$0.485 \sim a < x < b \sim 0.680$
		$(\gamma, 2\beta)$	0.485	$x = a$
		$(\beta, \beta + \gamma)$	0.693	$0.680 \sim b < x < c \sim 0.812$
(3,0,1)	$(\alpha, 2\alpha + \gamma)$	1.466	$1.456 \sim 3a < x < b + c \sim 1.492$	
	$(2\alpha, \alpha + \gamma)$	0.521	$0.485 \sim a < x < b \sim 0.680$	

Table 6: $(\alpha * *)$ -triangles for $\alpha = 2\pi/9$. Approximate values of a, b, c are $a \sim 0.485$, $b \sim 0.680$ and $c \sim 0.812$.

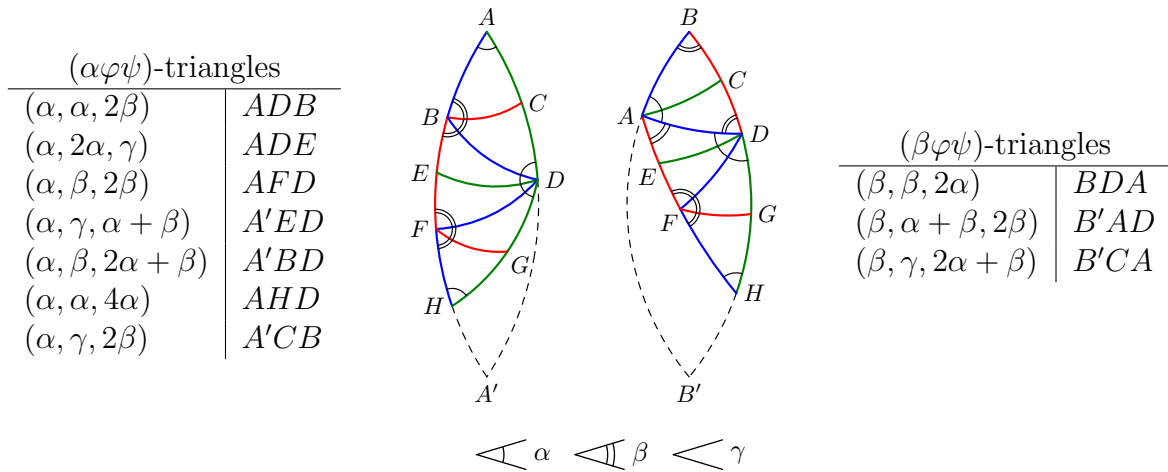


Figure 15: A tiling of $(\alpha * *)$ -triangles (left) and $(\beta * *)$ -triangles (right) for $\alpha = \frac{2\pi}{9}$.

n	(m_1, m_2, m_3)	(φ, ψ)	x	best approximation for x using a, b, c	
2	(2,1,0)	$(\beta, 2\alpha)$	0.812	$x = c$	
3	(0,1,1)	(β, γ)	0.955	$0.812 \sim c < x < 2a \sim 0.971$	
4	(1,2,0)	$(\beta, \alpha + \beta)$	1.065	$0.971 \sim 2a < x < a + b \sim 1.165$	
		$(2\alpha, 2\alpha)$	0.992	$0.971 \sim 2a < x < a + b \sim 1.165$	
5	(2,0,1)	$(2\alpha, \gamma)$	1.038	$0.971 \sim 2a < x < a + b \sim 1.165$	
6	(0,0,2)	(γ, γ)	1.047	$0.971 \sim 2a < x < a + b \sim 1.165$	
		$(\beta, 2\beta)$	1.231	$1.165 \sim a + b < x < a + c \sim 1.297$	
		$(3,1,0)$	$(2\alpha, \alpha + \beta)$	1.065	$0.971 \sim 2a < x < a + b \sim 1.165$
7	(1,1,1)	$(\gamma, \alpha + \beta)$	1.038	$0.971 \sim 2a < x < a + b \sim 1.165$	
		$(\beta, \alpha + \gamma)$	1.298	$1.297 \sim a + c < x < 2b \sim 1.359$	
8	(2,2,0)	$(\beta, 2\alpha + \beta)$	1.359	$x = 2b$	
		$(\alpha + \beta, \alpha + \beta)$	0.992	$0.971 \sim 2a < x < a + b \sim 1.165$	
		$(2\alpha, 2\beta)$	1.065	$0.971 \sim 2a < x < a + b \sim 1.165$	
9	(0,2,1)	$(\gamma, 2\beta)$	0.955	$0.812 \sim c < x < 2a \sim 0.971$	
		$(\beta, \beta + \gamma)$	1.415	$1.359 \sim 2b < x < 3a \sim 1.456$	
10	(3,0,1)	$(2\alpha, \alpha + \gamma)$	1.030	$0.971 \sim 2a < x < a + b \sim 1.165$	
		(1,0,2)	$(\gamma, \alpha + \gamma)$	0.860	$0.812 \sim c < x < 2a \sim 0.971$
		(1,3,0)	$(\beta, \alpha + 2\beta)$	1.469	$1.456 \sim 3a < x < b + c \sim 1.492$
11	(2,1,1)	$(\alpha + \beta, 2\beta)$	0.812	$x = c$	
		$(2\alpha, 2\alpha + \beta)$	0.952	$0.812 \sim c < x < 2a \sim 0.971$	
		$(\gamma, 2\alpha + \beta)$	0.680	$x = b$	
		$(\beta, 2\alpha + \gamma)$	1.520	$1.492 \sim b + c < x < 2c \sim 1.624$	
		$(\alpha + \beta, \alpha + \gamma)$	0.625	$0.485 \sim a < x < b \sim 0.680$	
		$(2\alpha, \beta + \gamma)$	0.781	$0.680 \sim b < x < c \sim 0.812$	

Table 7: $(\beta\varphi\psi)$ -triangles for $\alpha = 2\pi/9$ and $\varphi, \psi > \alpha$. Approximate values of a, b, c are $a \sim 0.485$, $b \sim 0.680$ and $c \sim 0.812$.

The following lemma extends Lemma 4.19.

Lemma 4.25. *Let $\mathcal{T}_0 = (\alpha\beta\gamma)$, where $\alpha < \beta < \gamma$ and $\beta = \pi/3$. Then the α -edges and β -edges of S together form a subgraph isomorphic to one of the six graphs in Figure 16.*

Proof. Let $H = c(S)$ be the Coxeter diagram of S . Let $V(H) = \{u, v, w, x, y\}$. Let $H_\alpha = (V(H), E_\alpha)$ and $H_\beta = (V(H), E_\beta)$ be the subgraphs of H formed by the α -edges and β -edges, respectively. Let $H_{\alpha\beta}$ be the edge-labeled subgraph of H formed by the α -edges and the β -edges. Since H is $(\alpha\beta\gamma)$ -rich, we have $|E_\beta| \geq 2$. Furthermore, $H_{\alpha\beta}$ contains at least four induced $\alpha\beta$ -paths, that is, induced paths of length 2 consisting of an α -edge and a β -edge. Moreover, $H_{\alpha\beta}$ is triangle-free: indeed, by Lemma 4.9(a), there are no spherical triangles of type $(\alpha\alpha\alpha)$, $(\alpha\alpha\beta)$, $(\alpha\beta\beta)$ or $(\beta\beta\beta)$ for $\alpha < \beta = \pi/3$. By Lemma 4.19, H_α is isomorphic to $P_2 + P_2$ or $P_2 + P_3$.

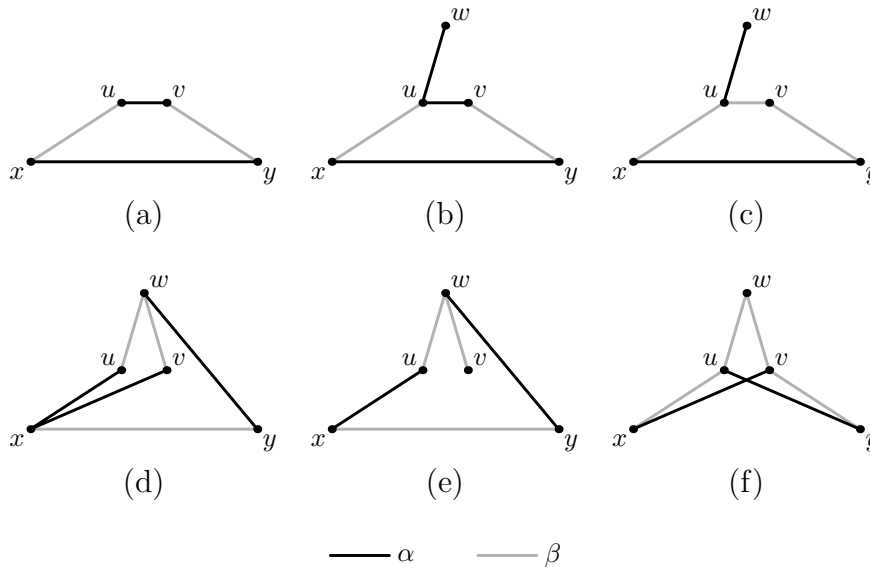


Figure 16: α -edges and β -edges of S for $\beta = \pi/3$.

Suppose that $|E_\beta| = 2$. Then H_β is a matching by the same argument as in the case $|E_\alpha| = 2$ in the proof of Lemma 4.19. Let $E_\beta = \{xu, yv\}$. If H_α also forms a matching, then to get four $\alpha\beta$ -paths, $H_{\alpha\beta}$ must be isomorphic to an alternating 4-cycle; see Figure 16(a). If H_α is isomorphic to $P_2 + P_3$, there are three possible non-isomorphic graphs $H_{\alpha\beta}$, determined by the choice of E_α :

1. $E_\alpha = \{xy, yu, vw\}$. In this case $H_{\alpha\beta}$ contains the triangle xyu ; a contradiction.
2. $E_\alpha = \{xy, uw, vw\}$. In this case there are exactly four $\alpha\beta$ -paths, forcing xw, yw, xv, yu to be γ -edges. Due to the symmetry simultaneously exchanging u with v and x with y , the $(\alpha\beta\gamma)$ -triangles form only two orbits; a contradiction.
3. $E_\alpha = \{xy, uv, uw\}$. The graph $H_{\alpha\beta}$ is drawn in Figure 16(b).

Suppose that $|E_\beta| = 3$. If H_β is isomorphic to the star $K_{1,3}$, then there are at most three $(\alpha\beta\gamma)$ -triangles. Also, H_β cannot form a triangle as $H_{\alpha\beta}$ is triangle-free.

Suppose that H_β is isomorphic to the path P_4 . Let $E_\beta = \{xu, uv, vy\}$. There is no triangle-free extension by α -edges where H_α is isomorphic to $P_2 + P_3$, and exactly one, up to isomorphism, where H_α is a matching; see Figure 16(c).

Suppose that H_β is isomorphic to $P_2 + P_3$. Let $E_\beta = \{uw, vw, xy\}$. Up to isomorphism, there are three triangle-free extensions of H_β by α -edges: $E_\alpha = \{xu, yv\}$, $E_\alpha = \{xu, yw\}$ and $E_\alpha = \{xu, xv, yw\}$. If $E_\alpha = \{xu, yv\}$, there are exactly four $\alpha\beta$ -paths, forcing xw, yw, xv, yu to be γ -edges. Due to the symmetry simultaneously exchanging u with v and x with y , the $(\alpha\beta\gamma)$ -triangles form only two orbits; a contradiction. The other two cases are displayed in Figure 16(d), (e).

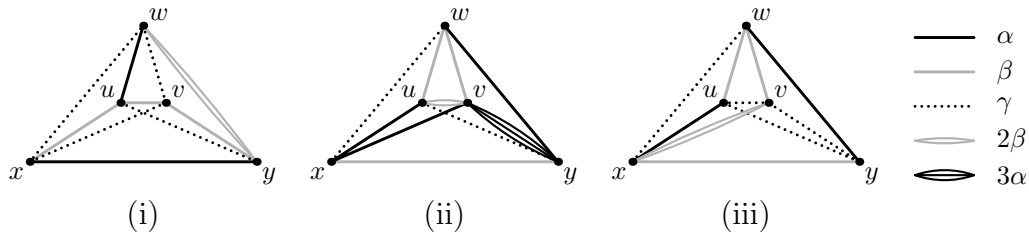


Figure 17: Coxeter diagrams for $\alpha = \frac{\pi}{4}$.

Suppose that $|E_\beta| \geq 4$. Then by Turán's theorem, $|E_\beta| = 4$, $|E_\alpha| = 2$ and $H_{\alpha\beta}$ is isomorphic to $K_{2,3}$. Since H_α is a matching, there is just one possibility for $H_{\alpha\beta}$ up to isomorphism; see Figure 16(f). \square

Now we finish the description of all possible Coxeter diagrams of S .

Lemma 4.26. *Let $\gamma = \pi/2$ and $\beta = \pi/3$. For $\alpha = \pi/4$ and $\alpha = \pi/5$, Figures 17 and 18, respectively, show all $(\alpha\beta\gamma)$ -rich Coxeter diagrams with five vertices whose all triangles of type $(\alpha **)$ and $(\beta **)$ are listed in Lemma 4.24. For $\alpha = 2\pi/9$, no such diagram exists.*

Proof. Let H be a Coxeter diagram satisfying the assumptions of the lemma. Let $V(H) = \{u, v, w, x, y\}$. Let E_α be the set of α -edges and E_β the set of β -edges of H . For each of the three values of α , we consider the six cases of diagrams given by Lemma 4.25. We refer to these cases by (a)–(f) according to Figure 16.

First we exclude case (f) for all three values of α . Suppose that $E_\alpha = \{xv, yu\}$ and $E_\beta = \{xu, uw, wv, vy\}$. Since H is $(\alpha\beta\gamma)$ -rich, the edges xy and uv must be γ -edges. This gives a $(\beta\beta\gamma)$ -triangle uvw , which is realizable only for $\alpha = \pi/4$. The remaining edges xw and yw are also γ -edges since $\varphi = \gamma$ is the only angle such that the $(\beta\beta\varphi)$ -triangle and the $(\alpha\beta\varphi)$ -triangle are both realizable for $\alpha = \pi/4$. But then due to the symmetry exchanging simultaneously u with v and x with y , the $(\alpha\beta\gamma)$ -triangles form at most three orbits; a contradiction.

Now we continue separately for each value of α .

1) $\alpha = \pi/4$. Recall that by the spherical law of cosines (Lemma 4.9(c)), we have, approximately, $a \sim 0.615$, $b = \pi/4 \sim 0.785$ and $c \sim 0.955$. The triangle of type $(\pi/2, \pi/2, 2\beta)$ is not realizable since its longest edge has length $2\pi/3 \sim 2.094$, which cannot be obtained as a nonnegative integer combination of a, b, c .

(a) $E_\alpha = \{uv, xy\}$, $E_\beta = \{xu, yv\}$. Since H is $(\alpha\beta\gamma)$ -rich, the diagonals xv and uy are γ -edges. Since $(\alpha\gamma\gamma)$ and $(\alpha, \gamma, 2\beta)$ are the only realizable $(\alpha\varphi\psi)$ -triangles for $\varphi, \psi \notin \{\alpha, \beta\}$, it follows that all edges incident with w are of type γ or 2β . Since the $(\beta\gamma\gamma)$ -triangle is not realizable, one of the edges uw, xv is of type 2β . We can assume without loss of generality that uw is of type 2β . Then, necessarily, xw and vw are of type γ and yw is of type 2β . The resulting diagram has a symmetry group generated by the product of the transpositions (x, v) and (u, y) , so it has at most two orbits of $(\alpha\beta\gamma)$ -triangles.

(b) $E_\alpha = \{uv, uw, xy\}$, $E_\beta = \{xu, yv\}$. Like in case (a), the edges xv and uy are of type γ . The types of the edges vw and yw are now uniquely determined: vw has type 2β and yw has type γ . If φ is the type of xw then the $(\alpha\beta\varphi)$ -triangle and the $(\alpha\gamma\varphi)$ -triangle are realizable, hence $\varphi = \gamma$. But then the triangle xvw is a $(\gamma, \gamma, 2\beta)$ -triangle, which is not realizable.

(c) $E_\alpha = \{uw, xy\}$, $E_\beta = \{xu, uv, vy\}$. Since H is $(\alpha\beta\gamma)$ -rich, the edges xw, wv, vx and uy have type γ . The type of the remaining edge wy is uniquely determined: the only value of $\varphi \neq \alpha, \beta$ for which the $(\alpha\gamma\varphi)$ -triangle and the $(\beta\gamma\varphi)$ -triangle are both realizable is 2β . The diagram is drawn in Figure 17(i).

(d) $E_\alpha = \{ux, xv, yw\}$, $E_\beta = \{uw, wv, xy\}$. The only possible type of uv is 2β . Since the $(\gamma, \gamma, 2\beta)$ -triangle is not realizable, it follows that at most one of the edges uy, vy has type γ . Since H contains at least two γ -edges, we can assume that uy and xw have type γ . The only possible type of vy is then 3α . See Figure 17(ii). We note that the $(\gamma, 2\beta, 3\alpha)$ -triangle is realizable.

(e) $E_\alpha = \{ux, yw\}$, $E_\beta = \{uw, wv, xy\}$. Since H is $(\alpha\beta\gamma)$ -rich, the edges xw and uy have type γ . Denote the types of uv, xv, yv by φ, ψ, ω , respectively. Since $\varphi, \psi, \omega \notin \{\alpha, \beta\}$, the triangles of type $(\beta\beta\varphi)$, $(\alpha\varphi\psi)$ and $(\beta\gamma\psi)$ are all realizable only if $\varphi = \gamma$ and $\psi = 2\beta$. The triangles of type $(\beta, 2\beta, \omega)$ and $(\alpha\beta\omega)$ are realizable only if $\omega = \gamma$. See Figure 17(iii).

Before continuing with $\alpha = \pi/5$, we exclude cases (c) and (e) for both $\alpha = \pi/5$ and $\alpha = 2\pi/9$ simultaneously, since the arguments are the same.

(c) $E_\alpha = \{uw, xy\}$, $E_\beta = \{xu, uv, vy\}$. Since the $(\beta\beta\gamma)$ -triangle is not realizable, there are at most two $(\alpha\beta\gamma)$ -triangles, uvw and xuw .

(e) $E_\alpha = \{ux, yw\}$, $E_\beta = \{uw, wv, xy\}$. Since H is $(\alpha\beta\gamma)$ -rich, the edges xw and uy have type γ . Denote the type of uv by φ and the type of xv by ψ . Since the $(\beta\beta\varphi)$ -triangle is realizable, we have $\varphi \in \{2\alpha, 4\alpha\}$. But there is no $\psi \notin \{\alpha, \beta\}$ such that the triangles of types $(\alpha\varphi\psi)$ and $(\beta\gamma\psi)$ are both realizable.

2) $\alpha = \pi/5$. Recall that by the spherical law of cosines (Lemma 4.9(c)), we have, approximately, $a \sim 0.365$, $b \sim 0.554$ and $c \sim 0.652$. The spherical triangles of type $(\pi/2, \pi/2, 4\alpha)$, $(\pi/2, \pi/2, 2\beta)$ and $(\pi/2, 3\alpha, 2\alpha)$ are not realizable since they have an edge of length $4\pi/5 \sim 2.513$, $2\pi/3 \sim 2.094$ or $\arccos((1-2/\sqrt{5})^{1/2}) \sim 1.240$, respectively, which cannot be obtained as nonnegative integer combinations of a, b, c . A spherical triangle of type $(\pi/2, 4\alpha, 4\alpha)$ does not exist by Lemma 4.9(b).

(a) $E_\alpha = \{uv, xy\}$, $E_\beta = \{xu, yv\}$. Since H is $(\alpha\beta\gamma)$ -rich, the diagonals xv and uy are γ -edges. Let $\varphi, \psi, \omega, \tau$ be the types of wx, wu, wv, wy , respectively. Since the $(\alpha\psi\omega)$ -triangle and the $(\alpha\varphi\tau)$ -triangle have to be realizable and since $\varphi, \psi, \omega, \tau \notin \{\alpha, \beta\}$, for each of the pairs $\{\psi, \omega\}$ and $\{\varphi, \tau\}$ we have the following four options: $\{2\alpha, \gamma\}$, $\{2\alpha, 2\beta\}$, $\{\gamma, 3\alpha\}$ or $\{\gamma, 2\beta\}$. Since the $(\beta\varphi\psi)$ -triangle and the $(\beta\omega\tau)$ -triangle have to be realizable and since $\varphi, \psi, \omega, \tau \neq 4\alpha$, for each of the pairs $\{\varphi, \psi\}$ and $\{\omega, \tau\}$ we have the following four options: $\{2\alpha, \gamma\}$, $\{2\alpha, 3\alpha\}$, $\{\gamma, 3\alpha\}$ or $\{3\alpha, 2\beta\}$.

Since there are just four $(\alpha\beta\gamma)$ -triangles in H , the only symmetry of H is the identity, and so $\{\varphi, \psi\} \neq \{\omega, \tau\}$. Hence, up to symmetry, the 4-tuple $(\varphi, \psi, \omega, \tau)$ is equal either to $(\pi/2, 2\alpha, \pi/2, 3\alpha)$ or $(\pi/2, 2\alpha, 2\beta, 3\alpha)$. In both cases, the triangle uyw has type

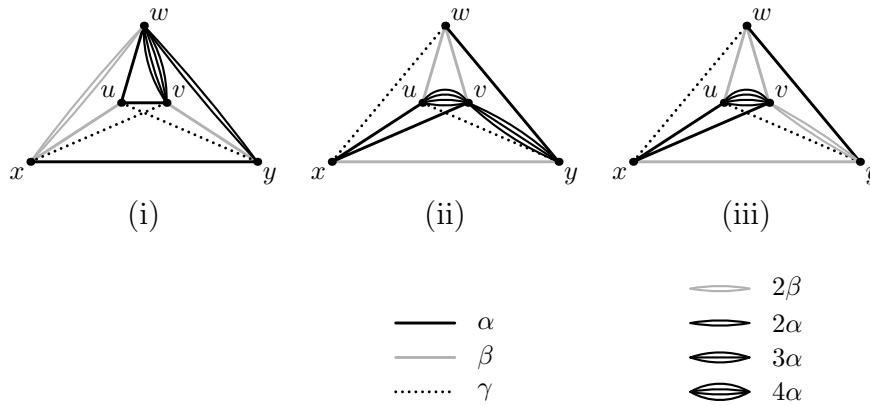


Figure 18: Coxeter diagrams for $\alpha = \frac{\pi}{5}$.

$(\pi/2, 2\alpha, 3\alpha)$, which is not realizable.

(b) $E_\alpha = \{uv, uw, xy\}$, $E_\beta = \{xu, yv\}$. Like in case (a), the edges xv and uy are of type γ . Let φ, ψ, ω be the types of wv, wy, wx , respectively. Since $\varphi, \psi, \omega \notin \{\alpha, \beta\}$ and the $(\alpha\alpha\varphi)$ -triangle must be realizable, we have $\varphi \in \{2\beta, 4\alpha\}$. Suppose that $\varphi = 2\beta$. Since the $(\beta\varphi\psi)$ -triangle must be realizable, we have $\psi = 3\alpha$. Since the $(\alpha\psi\omega)$ -triangle must be realizable, it follows that $\omega = \pi/2$. But now the triangle xvw is of type $(\pi/2, \pi/2, 2\beta)$, which is not realizable; a contradiction. Thus we are left with the option $\varphi = 4\alpha$. Since the $(\alpha\gamma\psi)$ -triangle and the $(\beta\varphi\psi)$ -triangle must be realizable, we have $\psi = 2\alpha$. Since the $(\alpha\psi\omega)$ -triangle must be realizable, it follows that $\omega \in \{\pi/2, 2\beta\}$. If $\omega = \pi/2$, the triangle xvw has type $(\pi/2, \pi/2, 4\alpha)$, which is not realizable. Therefore, $\omega = 2\beta$. See Figure 18(i).

(d) $E_\alpha = \{ux, xv, yw\}$, $E_\beta = \{uw, wv, xy\}$. The only possible type of uv is 4α . Since the $(\pi/2, \pi/2, 4\alpha)$ -triangle is not realizable, it follows that at most one of the edges uy, vy is of type γ . Since H contains at least two γ -edges, we can assume that uy and xw are of type γ . Since the $(\pi/2, 4\alpha, 4\alpha)$ -triangle does not exist, the type of vy is either 3α or 2β ; see Figure 18(ii), (iii).

3) $\alpha = 2\pi/9$. Recall that by the spherical law of cosines (Lemma 4.9(c)), we have, approximately, $a \sim 0.485$, $b \sim 0.680$ and $c \sim 0.812$. The spherical triangle of type $(\pi/2, \pi/2, 2\beta)$ is not realizable since it has an edge of length $2\pi/3 \sim 2.0944$, which cannot be obtained as a nonnegative integer combination of a, b, c . We show that there is no H satisfying the required conditions.

(a) $E_\alpha = \{uv, xy\}$, $E_\beta = \{xu, yv\}$. It is straightforward to check that there are no $\varphi, \psi, \omega, \tau \notin \{\alpha, \beta\}$ such that the triangles of types $(\alpha\psi\omega), (\alpha\varphi\tau), (\beta\varphi\psi), (\beta\omega\tau)$ are all realizable.

(b) $E_\alpha = \{uv, uw, xy\}$, $E_\beta = \{xu, yv\}$. Since H is $(\alpha\beta\gamma)$ -rich, the edges xv and uy are of type γ . Since there is no realizable $(\beta, 4\alpha, *)$ -triangle, the edge wv is of type 2β . Consequently, the edge yw is of type $\alpha + \beta$. Further it follows that xw is a γ -edge, hence the triangle xvw is of type $(\pi/2, \pi/2, 2\beta)$, which is not realizable.

(d) $E_\alpha = \{ux, xv, yw\}$, $E_\beta = \{uw, wv, xy\}$. There is no φ such that the $(\alpha\alpha\varphi)$ -triangle and the $(\beta\beta\varphi)$ -triangle are both realizable. \square

We finish the proof using Fiedler's theorem. Compared to Case a), it is now easier to compute the determinants of the matrices corresponding to the Coxeter diagrams from Figures 17 and 18, since we know precise values of all the entries.

We now list matrices B_1, B_2, B_3 corresponding to the diagrams in Figure 17 and matrices C_1, C_2, C_3 corresponding to the diagrams in Figure 18. Again, the rows and columns of the matrices are indexed by u, v, w, x, y , in this order.

$$\begin{aligned}
 B_1 &= \begin{pmatrix} -1 & 0.5 & \sqrt{2}/2 & 0.5 & 0 \\ 0.5 & -1 & 0 & 0 & 0.5 \\ \sqrt{2}/2 & 0 & -1 & 0 & -0.5 \\ 0.5 & 0 & 0 & -1 & \sqrt{2}/2 \\ 0 & 0.5 & -0.5 & \sqrt{2}/2 & -1 \end{pmatrix} \\
 B_2 &= \begin{pmatrix} -1 & -0.5 & 0.5 & \sqrt{2}/2 & 0 \\ -0.5 & -1 & 0.5 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0.5 & 0.5 & -1 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & -1 & 0.5 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 & 0.5 & -1 \end{pmatrix} \\
 B_3 &= \begin{pmatrix} -1 & 0 & 0.5 & \sqrt{2}/2 & 0 \\ 0 & -1 & 0.5 & -0.5 & 0 \\ 0.5 & 0.5 & -1 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & -0.5 & 0 & -1 & 0.5 \\ 0 & 0 & \sqrt{2}/2 & 0.5 & -1 \end{pmatrix} \\
 C_1 &= \begin{pmatrix} -1 & \frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0.5 & 0 \\ \frac{1}{4}(\sqrt{5}+1) & -1 & -\frac{1}{4}(\sqrt{5}+1) & 0 & 0.5 \\ \frac{1}{4}(\sqrt{5}+1) & -\frac{1}{4}(\sqrt{5}+1) & -1 & -0.5 & \frac{1}{4}(\sqrt{5}-1) \\ 0.5 & 0 & -0.5 & -1 & \frac{1}{4}(\sqrt{5}+1) \\ 0 & 0.5 & \frac{1}{4}(\sqrt{5}-1) & \frac{1}{4}(\sqrt{5}+1) & -1 \end{pmatrix} \\
 C_2 &= \begin{pmatrix} -1 & -\frac{1}{4}(\sqrt{5}+1) & 0.5 & \frac{1}{4}(\sqrt{5}+1) & 0 \\ -\frac{1}{4}(\sqrt{5}+1) & -1 & 0.5 & \frac{1}{4}(\sqrt{5}+1) & -\frac{1}{4}(\sqrt{5}-1) \\ 0.5 & 0.5 & -1 & 0 & \frac{1}{4}(\sqrt{5}+1) \\ \frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0 & -1 & 0.5 \\ 0 & -\frac{1}{4}(\sqrt{5}-1) & \frac{1}{4}(\sqrt{5}+1) & 0.5 & -1 \end{pmatrix} \\
 C_3 &= \begin{pmatrix} -1 & -\frac{1}{4}(\sqrt{5}+1) & 0.5 & \frac{1}{4}(\sqrt{5}+1) & 0 \\ -\frac{1}{4}(\sqrt{5}+1) & -1 & 0.5 & \frac{1}{4}(\sqrt{5}+1) & -0.5 \\ 0.5 & 0.5 & -1 & 0 & \frac{1}{4}(\sqrt{5}+1) \\ \frac{1}{4}(\sqrt{5}+1) & \frac{1}{4}(\sqrt{5}+1) & 0 & -1 & 0.5 \\ 0 & -0.5 & \frac{1}{4}(\sqrt{5}+1) & 0.5 & -1 \end{pmatrix}
 \end{aligned}$$

The matrices have the following determinants, rounded to two decimal places:

$$\begin{array}{lll} \det(B_1) = 1/16 \sim 0.06, & \det(B_2) = 1/8 \sim 0.13, & \det(B_3) \sim 0.21, \\ \det(C_1) \sim 0.16, & \det(C_2) \sim 0.16, & \det(C_3) \sim 0.12. \end{array}$$

Since all the determinants are nonzero, simplices corresponding to Coxeter diagrams from Figures 17 and 18 cannot exist. This finishes the case $\mathcal{T}_0 = (\alpha\beta\gamma)$ and also the whole proof of Theorem 1.3.

5 A few remarks about general dimension $d \geq 5$

The existence of k -reptile simplices in \mathbb{R}^d for $d \geq 5$, $k \geq 3$, $k \neq m^d$, is wide open. Our approach for dimension 4 does not seem to be powerful enough already in dimension 5. Let S be a k -reptile simplex in \mathbb{R}^d . Similarly as in Section 4, we may define an *indivisible* edge-angle of S as a spherical $(d-2)$ -simplex that cannot be tiled with smaller spherical $(d-2)$ -simplices representing the other edge-angles of S or their mirror images. Using the same arguments as in the proof of Lemma 4.1 one can prove a generalized statement saying that if \mathcal{T}_0 is an indivisible edge-angle in S , then the edges of S with edge-angle \mathcal{T}_0 have at least D different lengths, where D is the algebraic degree of $k^{-1/d}$ over \mathbb{Q} . In particular, if d is prime then $D = d$.

For $d \geq 5$, we can have up to $(d+1)/2 \geq 3$ indivisible edge-angles in S . If S has a nontrivial symmetry, Lemma 2.5 helps only a little: it still allows up to $(d+1)/2 - 1 \geq 2$ indivisible edge-angles.

For d prime, it might be beneficial to consider internal angles at inner points of k -faces for $k \geq 2$, and use the corresponding generalization of Lemma 4.1. However, the combinatorial explosion of possible cases still seems overwhelming. Therefore, we think that to attack the problem in higher dimensions, new ideas and possibly more advanced tools will be needed.

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A The program for generating realizable spherical triangles

```
def possible_tilings(u,v,w,lens,min_angle):

    def cos_law(a,b,c):
        z=(cos(c)+cos(a)*cos(b))/(sin(a)*sin(b));
        return numerical_approx(arccos(z));

    A=cos_law(w,v,u);
    B=cos_law(w,u,v);
    C=cos_law(u,v,w);

    def is_equal(x):
        var("a b c")
        min_error_below=10;meb_i=0;meb_j=0;meb_k=0;
        min_error_above=10;mea_i=0;mea_j=0;mea_k=0;
        for i in range(0,20):
            for j in range(0,20-i):
                for k in range (0,20-i-j):
                    t=x-i*A-j*B-k*C;
                    if (-0.00001<=t<=min_error_below):
                        min_error_below=t;meb_i=i;meb_j=j;meb_k=k;
                    if (-0.00001<=-t<=min_error_above):
                        min_error_above=-t;mea_i=i;mea_j=j;mea_k=k;
        if (min_error_below<0.00001):
            print round(x,5), "& $ x = ",
                meb_i*a +meb_j*b + meb_k*c,
                "$","!!!!\\\\"
            return;
        print round(x,5), "& $", round(meb_i*A + meb_j*B + meb_k*C,5),
            "=", meb_i * a~ + meb_j * b + meb_k * c, "<", "x", "<",
            mea_i*a + mea_j *b + mea_k*c, "=",
            round(mea_i*A + mea_j*B + mea_k*C,5), "$\\\\"

    def tries(x,y,z,tlens):
        visited = set();
        def split(n,x,y,z):
            var("alpha beta gamma");
            for i in range(0,x+1):
                for j in range(0,y+1):
                    for k in range (0,z+1):
                        psi=i*u + j*v + k*w
                        phi=(x-i)*u + (y-j)*v + (z-k)*w
```

```

L = sorted([psi,phi,lens]);
if (0 < psi <= phi < pi
    and L[1]+L[2] < pi+L[0]
    and min_angle<psi
    and (psi,phi) not in visited):
    print n, "&", (x,y,z) ," &";
    visited.add((psi,phi))
    print "$(",(i*alpha)+j*beta+k*gamma,",",
        (x-i)*alpha+(y-j)*beta+(z-k)*gamma,
        ")$&";
    is_equal(cos_law(psi,phi,lens));

S = (x + y + z - pi);
for d in range(2,2*tlens/S):
    for k in range(0,(d*S+pi-tlens)/x+1):
        for l in range(0,(d*S+pi-tlens-k*x)/y + 1):
            for m in range(0,(d*S+pi-tlens-k*x-l*y)/z + 1):
                if (d*S + pi == k*x + l*y + m*z + tlens):
                    split(d,k,l,m)

tries(u,v,w,lens)

```