SIMPLICES MODELLED ON SPACES OF CONSTANT CURVATURE*

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Abstract. We give non-degeneracy criteria for Riemannian simplices based on simplices in spaces of constant sectional curvature. It extends previous work on Riemannian simplices, where we developed Riemannian simplices with respect to Euclidean reference simplices. The criteria we give in this article are in terms of quality measures for spaces of constant curvature that we develop here.

We see that simplices in spaces that have nearly constant curvature, are already non-degenerate under very weak quality demands. This is of importance because it allows for sampling of Riemannian manifolds based on anisotropy of the manifold and not (absolute) curvature.

1 Introduction

Simplices and triangulations in ambient Euclidean space have a long history in the computational geometry community. In [4] we discussed the generalization of Euclidean simplices to non-degenerate Riemannian simplices on a Riemannian manifold, and triangulations of the manifold constructed with such simplices. These Riemannian simplices are defined using Riemannian centres of mass: suppose that we are given $(n+1)$ vertices $v_0, \dots, v_n$ on an n-dimensional Riemannian manifold M. Given $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$, we define the point $y \in M$ with barycentric coordinates λ by $y = \operatorname{argmin}_x \sum \lambda_i d_M(x, v_i)^2$, where d_M denotes the geodesic distance on M. This gives us a way to map the standard n-simplex, $\{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} \mid \sum \lambda_i = 1, \lambda_i \geq 0\}$, into the manifold. A simplex is called non-degenerate if this mapping is a diffeomorphism. The criteria we gave in [4] for non-degeneracy were based on Euclidean quality measures: we employed bounds on the quality of the simplex found by lifting the vertices by the exponential map to the tangent space of one of the vertices.

The comparison to Euclidean simplices leads to very stringent requirements on the size of the simplices on a manifold that is geometrically close to a small sphere, for example. However, we know that the non-degeneracy requirements for Riemannian simplices on

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spheres, for example, are very light. It suffices that the vertices be contained in an open hemisphere, and do not lie on a lower dimensional sphere of the same radius.

Our former approach was therefore poorly attuned to spaces of non-zero constant curvature. One also expects (and we shall prove) that Riemannian simplices are somewhat stable if one perturbs the metric of the sphere or any space of constant curvature. So in this text we focus on manifolds whose sectional curvatures are nearly constant. For simplices in these spaces we now give conditions for non-degeneracy similar to those in [4], but based on spaces of constant curvature and such that requirements for non-degeneracy are very mild if the metric deviates only slightly from the constant one.

More specifically, we are interested in Riemannian simplices on an n-manifold M whose sectional curvatures are very close to constant, meaning that the sectional curvatures K satisfy $\Lambda_\ell \leq K \leq \Lambda_u$ with $|\Lambda_\ell - \Lambda_u|$ small relative to $\min\{|\Lambda_\ell|, |\Lambda_u|\}$. We therefore suppose that $0 < \Lambda_\ell$ or $\Lambda_u < 0$, because the case where Λ_ℓ and Λ_u are nearly zero has been adequately treated in [4] by comparison to Euclidean space. Here we compare to spaces of constant curvature Λ_{mid}, where Λ_{mid} lies in the interval $[\Lambda_\ell, \Lambda_u]$. For positively curved spaces we take $\Lambda_{\text{mid}} = \frac{1}{2}(\Lambda_\ell + \Lambda_u)$; for negatively curved spaces the expression is more complicated.

To convey the main ideas of this paper in this introduction, we restrict ourselves here to manifolds of *positive* curvature embedded in Euclidean space. The embedding is not in any way essential but it clarifies the geometric picture.

In this setting we are given $n+1$ points $v_0, \dots, v_n$ in a small open set inside the n-dimensional manifold M, with nearly constant positive sectional curvatures. These points are the vertices of the Riemannian simplex. We pick a vertex v_r and we identify $T_{v_r}M$ with a tangent space of $\mathbb{H}(\Lambda_{\text{mid}})$, an n-sphere of curvature Λ_{mid}. Here we used that for both tangent spaces there exists a linear isometry to the Euclidean space of the same dimension and thus to each other.

In general, $\mathbb{H}(K_c)$ denotes the complete simply connected space of constant sectional curvature K_c, regardless of the sign of K_c. Since the sectional curvatures of $\mathbb{H}(K_c)$ are constant, this space is unique up to isometries [8, Theorem 2.1, Chapter 1].

We consider the geodesics on M emanating from v_r to the other vertices v_i. We can transplant these geodesics to $\mathbb{H}(\Lambda_{\text{mid}})$. By this we mean that we map geodesics emanating from v_r on M to geodesics emanating from the same point v_r in $\mathbb{H}(\Lambda_{\text{mid}})$, preserving both the lengths of the geodesics and the angles θ_{il} between any two geodesics emanating from v_r to v_i, and v_l.

The transplantation map is formally defined using the exponential map $\exp_{p,M}$. The exponential map maps straight line segments emanating from the origin in T_pM to geodesics emanating from $p \in M$, such that the length of the geodesic equals the length of the line segment in T_pM. We can now formally define the transplantation map as $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}$. Here we do not indicate a point on the sphere because any point is equivalent thanks to spherical symmetry, but one may think of the point where the sphere touches the tangent plane T_pM.

The transplantation is a diffeomorphism onto its image if restricted to a sufficiently small neighbourhood $U \subset M$. Given $x \in U$, we have a corresponding point in $\mathbb{H}(\Lambda_{\text{mid}})$:

$$x(v_r) := \exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(x).$$

The simplex we find on $\mathbb{H}(\Lambda_{\text{mid}})$ after the transplantation, with vertices $v_i(v_r)$, is denoted by $\sigma_{\mathbb{H}(\Lambda_{\text{mid}})}(v_r)$. Note that this simplex depends on all points $v_0, \dots, v_n$, even though we drop this dependence from our notation.

The majority of this paper is dedicated to proving that the transplantation map $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}$ has low distortion. Intuitively we can think of the manifold as a sphere with small metric distortion. The demonstrations of low distortion are very visual and are based on the Toponogov comparison theorem, but they require technical estimates, which are calculated in the appendices.

Because the transplantation map has low distortion, we expect that if the simplex $\sigma_{\mathbb{H}(\Lambda_{\text{mid}})}(v_r)$ on $\mathbb{H}(\Lambda_{\text{mid}})$ is well shaped, the Riemannian simplex is also well shaped and in particular is non-degenerate, meaning that it is homeomorphic to the standard simplex. We shall prove this. To quantify what we mean by well-shaped in a space of constant curvature, we introduce a new volume-based quality measure $Q_{\mathbb{H}(K_c)}$ for simplices on spaces of constant curvature $\mathbb{H}(K_c)$, combining ingredients from spherical geometry with Euclidean geometry for ease of computation.

Our quality measure $Q_{\mathbb{H}(K_c)}$ of an n-simplex on the n-sphere with vertices v_i is defined as follows: we think of the sphere as embedded isometrically in Euclidean space $\mathbb{R}^{n+1}$. For each point x on the sphere we consider the $n+1$ (abstract) n-simplices indexed by j whose vertices are x and the n vertices v_i with $i \neq j$. We project the vertices orthogonally onto T_xM and consider the volume of the $n+1$ sub-simplices. We now choose x as the point where all the volumes are equal and define $Q_{\mathbb{H}(K_c)}$ proportional to these equal volumes. The precise definition is detailed in Section 1.1.

We can now state the main result for manifolds of positive curvature:

Theorem 1 *Let M be a manifold whose sectional curvatures K satisfy $0 < \Lambda_\ell \leq K \leq \Lambda_u$. Suppose that $v_0, \dots, v_n$ are vertices on M. Assume*[1] *that all vertices lie within a convex geodesic ball of radius $\frac{1}{2}\tilde{D}$ with centre v_r, where $\tilde{D} \leq 1/(2\sqrt{\Lambda_u})$ and $r \in \{0, \dots, n\}$. Under these assumptions the Riemannian simplex with vertices $v_0, \dots, v_n$ on M is non-degenerate if*

$$Q_{\mathbb{H}(\Lambda_{\text{mid}})}(\sigma_{\mathbb{H}(\Lambda_{\text{mid}})}(v_r)) > n2^{2n}|\Lambda_\ell - \Lambda_u|\tilde{D}^{2n+2},$$

with $Q_{\mathbb{H}(\Lambda_{\text{mid}})}$ the simplex quality, $\sigma_{\mathbb{H}(\Lambda_{\text{mid}})}(v_r)$ the simplex on $\mathbb{H}(\Lambda_{\text{mid}})$ with vertices $v_i(v_r)$ defined by $v_i(v_r) = \exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(v_i)$ and $\Lambda_{\text{mid}} = \frac{1}{2}(\Lambda_\ell + \Lambda_u)$. The quality of a simplex $W = \{w_i\} \subset \mathbb{H}(\Lambda_{\text{mid}})$ is given by

$$Q_{\mathbb{H}(\Lambda_{\text{mid}})}(W) = \min_{y \in \mathbb{H}(\Lambda_{\text{mid}})} \max_j \left\{ \det \left(\frac{1}{\Lambda_{\text{mid}}} \sin\left(\sqrt{\Lambda_{\text{mid}}}\, d_{\mathbb{H}(\Lambda_{\text{mid}})}(y, w_i)\right) \cdot \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{\text{mid}})}(y, w_l)\right) \cos\theta_{il} \right)_{i,l \neq j} \right\}, \quad (1)$$

[1]This bound is stronger than necessary. In fact it suffices for the lengths of geodesics in the proof to be bounded by $\tilde{D}$. We have chosen this formulation because it is natural in the Delaunay setting, which motivates this work, see Section 5.

with θ_{il} the angle between the geodesics on $\mathbb{H}(\Lambda_{\mathrm{mid}})$ from y to w_i and w_l.

We have a similar result (Theorem 14) for spaces of negative curvature, however for negative curvature the bounds are significantly more involved. In our exposition here we will focus on the positive curvature case and only mention some results for negative curvature; the arguments for negatively curved spaces are almost identical but the calculations are more involved.

1.1 Quality for spaces of constant curvature

In Theorem 1 we introduced qualities for simplices of constant positive curvature. This quality has a nice geometric interpretation (the hyperbolic -negative sectional curvature-case is slightly more involved).

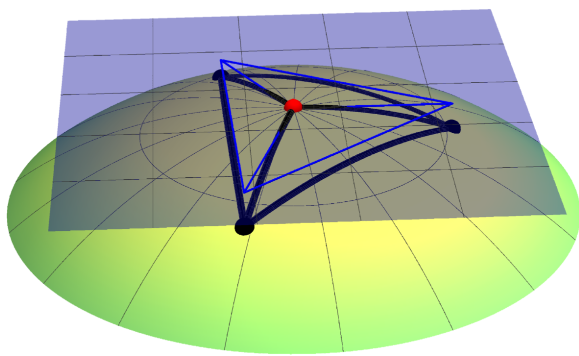

Figure 1: Geometric interpretation of the quality of a simplex on a space of positive constant curvature. The parallelepipeds whose volume are central in (2) are spanned by pairs of blue straight lines emanating from the red point.

The determinant

$$\det\left(\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}\,d_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(y,v_i)\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(y,v_l)\right)\cos\theta_{il}\right)_{i,l\neq j}, \qquad (2)$$

with θ_{il} the angle between the geodesics from y to v_i and v_l, in Theorem 1 gives the volume of a parallelepiped. This parallelepiped can be found as follows: we embed the sphere in Euclidean space in the standard manner. We take the tangent space at y and project the vertices v_i on the tangent space via the normal of the tangent space, see Figure 1. The parallelepiped is given by the vectors from y to the projected vertices. Formula (2) is like the determinant of a Gram matrix, because the entries of the matrix are inner products after the orthogonal projection on the tangent space (of the sphere). We will therefore say that it is the determinant of a pseudo Gram matrix.

The search for the point where the maximum volume is minimized in (1), is reminiscent of the Euclidean context. More precisely, the barycentre of a Euclidean simplex with vertices $v_0,\ldots,v_n \in \mathbb{R}^n$ is the point x for which the volumes of all $n+1$ full-dimensional simplices with vertices $x, v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$ are equal. This means that maximum volume of the simplices is minimized.

A more extensive discussion of the quality will follow in Section 3.

Bounds on determinants Estimates on (2) are central to our analysis. Because (2) involves a determinant, the following result by Friedland [5] will be of use:

$$|\det(A+E) - \det(A)| \leq n \max\{\|A\|_p, \|A+E\|_p\}^{n-1} \|E\|_p, \tag{3}$$

where A and E are $n \times n$-matrices and $\|\cdot\|_p$ is the p-norm, with $1 \leq p \leq \infty$, for linear operators: $\|A\|_p = \max_{x\in\mathbb{R}^n} |Ax|_p/|x|_p$, with $|\cdot|_p$ the p-norm on $\mathbb{R}^n$. In our context A will be a pseudo Gram matrix for a space of constant curvature. The matrix E is the matrix of small angle deviations from the constant curvature case (or rather the deviations of their cosines). These angle deviations are due to the local geometry, and because of the bounds on the geometry, each entry of this matrix is bounded (which we prove using the Toponogov comparison theorem).

1.2 Quality constraints based on Euclidean simplices

Using the geometric intuition we just developed, it is easy to compare Theorem 1 to the previous result of [4] (or rather to [9], which contains an improved version of the result of [4]), where we compare to simplices in Euclidean space:

Theorem 2 (Theorem 3.6.6 of [9]) *Let $v_0, \ldots, v_n$ be a set of vertices lying in a Riemannian manifold M, whose sectional curvatures are bounded in absolute value by Λ, within a convex geodesic ball of radius D centred at one of the vertices (v_r) and such that $\sqrt{\Lambda}D < 1/2$. If $\sigma_{\mathbb{E}}(v_r)$, the convex hull of $(\exp^{-1}_{v_r,M}(v_i))_{i=0}^n = (v_i(v_r))_{i=0}^n$ in $T_{v_r}M$, satisfies*

$$vol(\sigma_{\mathbb{E}}(v_r))^2 > \frac{25}{24}\frac{n(n+1)^2}{(n!)^2} 2^{2n} \Lambda D^{2n+2}, \tag{4}$$

then the Riemannian simplex with vertices $v_0, \ldots, v_n$ is non-degenerate, that is diffeomorphic to the standard n-simplex.

We stress that in [9] we compared to Euclidean space and therefore the vertices $(v_i(v_r))_{i=0}^n$, were assumed to lie in a linear space. When comparing Theorems 1 and 2 note that because Theorem 2 is formulated in terms of the volume of a simplex and not a quality measure that is given in terms of a determinant the occurrence of the n-factorial terms is to be expected.

To compare the result of this paper to our previous work, we note that (2) tends to

$$(\det(v_i(v_r) - x(v_r))_{i\neq j})^2$$

so that $Q_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(\sigma_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(v_r))$ tends to $(\frac{n!}{n+1}\mathrm{vol}(\sigma_{\mathbb{E}}(v_r)))^2$, as the curvature tends to zero. This means conditions in Theorem 2 and Theorem 1 coincide in the limit except that the prefactor is slightly better for Theorem 1 and Λ is replaced by $|\Lambda_\ell - \Lambda_u|$. This replacement by $|\Lambda_\ell - \Lambda_u|$ is the significant step we make in this paper. In the hyperbolic setting we find a similar proportionality to $|\Lambda_\ell - \Lambda_u|$.

2 Preliminaries to Riemannian Simplices and comparison theorems

2.1 Riemannian simplices

We now remind the reader of the results needed to define Riemannian simplices, this partly coincides with the introduction of [4]. We need to define convex neighbourhoods that will provide the stage for most of the comparison results. By convex we always mean geodesically convex, to be precise: a set $A \subset M$ is *convex* if for any two distinct points $a, b \in A$ there is a unique minimizing geodesic in M between a and b, this geodesic is contained in A, and no other geodesic between a and b is contained in A. A function on a convex set is convex if its restriction to any geodesic in the set is convex. We have [3, Thm. IX.6.1]:

Lemma 3 *Suppose the sectional curvatures of M are bounded by $K \leq \Lambda_u$, and ι_M is the injectivity radius. If*

$$r < \min\left\{\frac{\iota_M}{2}, \frac{\pi}{2\sqrt{\Lambda_u}}\right\},$$

then $\overline{B}_M(x, r)$ is convex. (If $\Lambda_u \leq 0$, we take $1/\sqrt{\Lambda_u}$ to be infinite.)

In our context, we are interested in finding a weighted centre of mass of a finite set $\{p_0, \ldots, p_j\} \subset B \subset M$, where the containing set B is open, and its closure $\overline{B}$ is convex. The centre of mass construction is based on minimising the function $\mathcal{E}_\lambda : \overline{B} \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x, p_i)^2, \tag{5}$$

where the $\lambda_i \geq 0$ are non-negative weights that sum to 1, and d_M is the geodesic distance function on M.

We have the following result [6, Thm 1.2]:

Lemma 4 (Unique centre of mass) *Suppose the sectional curvatures of M are bounded by $K \leq \Lambda_u$, and ι_M is the injectivity radius. If $\{p_0, \ldots, p_j\} \subset B_\rho \subset M$, and B_ρ is an open ball of radius ρ with*

$$\rho < \rho_0 = \min\left\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{\Lambda_u}}\right\}, \tag{6}$$

then $\mathcal{E}_\lambda$ is convex and has a unique minimum in B_ρ. (If $\Lambda_u \leq 0$, we take $1/\sqrt{\Lambda_u}$ to be infinite.)

Using this result we can now define the Riemannian simplices:

Definition 5 (Riemannian simplex) If a finite set $\sigma^j = \{p_0, \ldots, p_j\} \subset M$ in an n-manifold is contained in an open geodesic ball B_ρ whose radius, ρ, satisfies (6), then σ^j is the set of vertices of a geometric *Riemannian simplex*, denoted σ^j_M, and defined to be the image of the map

$$\mathcal{B}_{\sigma^j} : \Delta^j \to M, \quad \lambda \mapsto \operatorname*{argmin}_{x \in \overline{B}_\rho} \mathcal{E}_\lambda(x),$$

where Δ^j denotes the j-dimensional standard simplex. We say that σ^j_M is *non-degenerate* if $\mathcal{B}_{\sigma^j}$ is a smooth embedding; otherwise it is *degenerate*.

We will focus on full dimensional Riemannian simplices, i.e., unless otherwise specified, σ_M will refer to a Riemannian simplex specified by $n+1$ vertices in M. The following result from [4, 9] will be often used:

Lemma 6 (Lemma 3.6.4 of [9]) *If for all x in σ_M (the image of the map given in Definition 5) there are n tangents to geodesics connecting this point x to a subset of n vertices $v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n$ (this choice does depend on x) that are linearly independent then:*

- *The map $\Delta^n \to \sigma_M$ is bijective.*
- *The inverse of $\Delta^n \to \sigma_M$ is smooth.*

2.2 The Toponogov Comparison Theorem

Our analysis is based on the Toponogov Comparison Theorem, which we shall discuss now as well as the definitions that go with it. Our exposition will follow Karcher [7].

We use the notation $\mathbb{H}^n(K_c)$, or simply $\mathbb{H}(K_c)$, for the complete, simply connected space of dimension n with constant sectional curvature K_c. A complete simply connected space with constant sectional curvature is also called a *space form*. Often when we mention a space of constant curvature we shall tacitly assume that it is a space form.

We are now ready to make the following definitions, that have been illustrated in Figure 2.

Definition 7 A *geodesic triangle* T in a Riemannian manifold consists of three minimizing geodesics connecting three points, sometimes also referred to as vertices.[2] Assume lower curvature bounds $\Lambda_\ell \leq K$ (or upper bounds $K \leq \Lambda_u$). A triangle with the same edge lengths as T in $\mathbb{H}^n(\Lambda_\ell)$ (or $\mathbb{H}^n(\Lambda_u)$), is called an *Alexandrov triangle* T_{Λ_ℓ} (or T_{Λ_u}) associated with T.

Definition 8 Two edges of a geodesic triangle and the enclosed angle form a *hinge*; a *Rauch hinge* in $\mathbb{H}^n(\Lambda_\ell)$ (or $\mathbb{H}^n(\Lambda_u)$) of a given hinge, consists of two geodesics (legs) emanating from a single point with the same lengths and enclosed angles as the original hinge. The edge closing the Rauch hinge in $\mathbb{H}^n(\Lambda_\ell)$ (or $\mathbb{H}^n(\Lambda_u)$), that is the minimizing geodesic connecting the two endpoints of the legs, will be called the *Rauch edge* of the hinge.

Note that because space forms are homogeneous, the Alexandrov triangles and Rauch hinges are uniquely defined, up to isometry of $\mathbb{H}(\Lambda_{l,u})$.

Theorem 9 (Toponogov Comparison Theorem) *Let T be a geodesic triangle in M such that for each of its vertices T lies within a geodesic ball of radius less than the injectivity radius centred at the vertex and assume that the sectional curvatures K of M satisfy the bounds $\Lambda_\ell \leq K \leq \Lambda_u$. If $\Lambda_u > 0$, assume also that the triangle circumference is less then $2\pi {\Lambda_u}^{-1/2}$. Than Alexandrov triangles T_{Λ_ℓ} and T_{Λ_u} exist. Moreover, any angle α of T satisfies*

$$\alpha_{\Lambda_\ell} \leq \alpha \leq \alpha_{\Lambda_u},$$

[2]We stress that a geodesic triangle does not include an interior.

where α_{Λ_ℓ} and α_{Λ_u} are the corresponding angles in T_{Λ_ℓ} and T_{Λ_u} respectively. The length c of the third edge closing a hinge is bounded in length by the lengths of the Rauch edges, c_{Λ_ℓ} and c_{Λ_u}, closing the Rauch hinges on $\mathbb{H}(\Lambda_\ell)$ and $\mathbb{H}(\Lambda_u)$ respectively;

$$c_{\Lambda_\ell} \geq c \geq c_{\Lambda_u}.$$

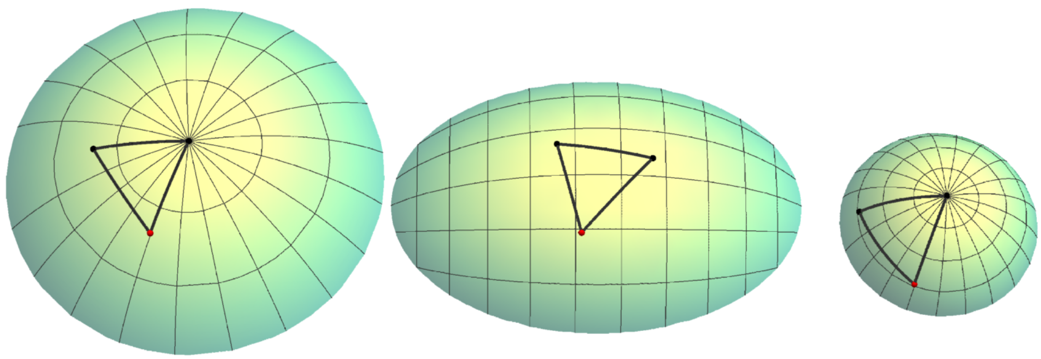

Figure 2: An ellipsoid (centre) with a hinge with closing geodesic and corresponding Rauch hinges with closing geodesic on the spaces of constant curvatures (on both sides of the ellipsoid), in this case both spheres. For the version of the Toponogov comparison theorem using Rauch hinges the interpretation of the figure is the following: the legs emanating from the red vertex have the same length and the same enclosed angle on all three spaces, and the length of the third edge (closing the Rauch hinge) on the ellipse is bounded by the lengths of the corresponding edges on the spheres. For the version of the comparison theorem involving Alexandrov triangles the interpretation is the following: the edge lengths of the triangles on all three spaces are the same and the angle between two edges on the ellipse (lets say at the red vertex) is bounded by the corresponding angles on the spheres (at the other red vertices).

We also give the cosine rule which is of use in explicit calculations involving the Toponogov comparison theorem. The cosine rule [1, Section 18.6, Section 19.3] for curved spaces of sectional constant curvature is

$$\cos\frac{a}{k} = \cos\frac{b}{k}\cos\frac{c}{k} + \sin\frac{b}{k}\sin\frac{c}{k}\cos\alpha, \qquad \text{with curvature } 1/k^2, \text{ here } k > 0 \tag{7}$$

$$\cosh\frac{a}{k} = \cosh\frac{b}{k}\cosh\frac{c}{k} - \sinh\frac{b}{k}\sinh\frac{c}{k}\cos\alpha, \qquad \text{with curvature } -1/k^2, \text{ here } k > 0. \tag{8}$$

See Figure 3 for a figure with the edge lengths and angles indicated.

3 Simplex quality on constant curvature spaces

In this section we shall introduce an alternative for the Gram matrix that is specific to spaces of non-trivial constant curvature. In this section $\mathbb{H}(1/r^2)$, the n-sphere with radius r

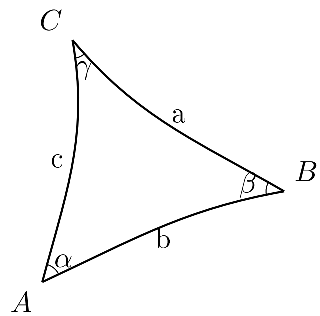

Figure 3: Triangle with the standard symbols for angles and lengths.

and therefore sectional curvature $1/r^2$, will be assumed to be embedded in Euclidean space in the canonical way with co-dimension one.

The hyperbolic sphere $\mathbb{H}(-1/r^2)$, the hyperbolic n-sphere with imaginary radius r and therefore sectional curvature $-1/r^2$, is often viewed as embedded using the Minkowski or Hyperboloid model. This is an embedding as the 'upper' connected component of a two sheeted hyperboloid, given by $-x_0^2 + x_1^2 + \cdots + x_n^2 = -r^2$, in Minkowski space with metric[3] $\mathrm{d}s^2 = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$.

We shall mostly denote distances as $d_M(x, y)$. M is often a space of constant curvature. The exception will be when we are in Euclidean space and we want to emphasize that it is a vector space. If so, we shall write $|x - y|$. So $|x - y|$ is used interchangeably with $d_{\mathbb{R}^{n+1}}(x, y)$, but in the one case x and y are thought of as vectors and the other as points in Euclidean space.

We discuss the elliptic case first, because the geometric interpretation is easier. Our first lemma helps to establish that the pseudo Gram matrices, which we define below, for spaces of constant curvature are indeed a measure of quality. By which we mean that the pseudo Gram matrix is zero if and only if degeneracy occurs.

Lemma 10 *Suppose that $x, v_0, \ldots, v_{n-1}$ are the vertices of a simplex in $L \subset \mathbb{R}^{n+1}$, where L is an n-dimensional linear subspace. Also suppose that $d_{\mathbb{R}^{n+1}}(x, v_i) \leq d$. Furthermore let $(\mathbb{H}(1/r^2))(y, r)$ be a sphere in $\mathbb{R}^{n+1}$ with centre y, such that $x \in (\mathbb{H}(1/r^2))(y, r)$ and the tangent space $T_x(\mathbb{H}(1/r^2))(y, r)$ coincides with L and $d < r$. We emphasize that $x \in L$. Denote by π the projection from L to $(\mathbb{H}(1/r^2))(y, r)$ via the normal of L. The domain of this map is $B^n_L(x, r)$ the ball in L centred at x with radius r, and the image the 'upper' hemisphere defined by taking x as the 'north pole'. Then*

$$\det \left(d_{(\mathbb{H}(1/r^2))}(x, \pi(v_i)) d_{(\mathbb{H}(1/r^2))}(x, \pi(v_l)) \cos\theta_{il} \right)_{0 \leq i,l \leq n-1} = 0,$$

where $\theta_{ij} = \angle v_i x v_j$, if and only if

$$\det(|x - v_i||x - v_l| \cos\theta_{il})_{0 \leq i,l \leq n-1}$$
$$= \det \left(r^2 \sin\left(\frac{d_{(\mathbb{H}(1/r^2))}(x, \pi(v_i))}{r} \right) \sin\left(\frac{d_{(\mathbb{H}(1/r^2))}(x, \pi(v_l))}{r} \right) \cos\theta_{il} \right)_{0 \leq i,l \leq n-1} = 0.$$

[3]The other choice would yield a negatively definite metric.

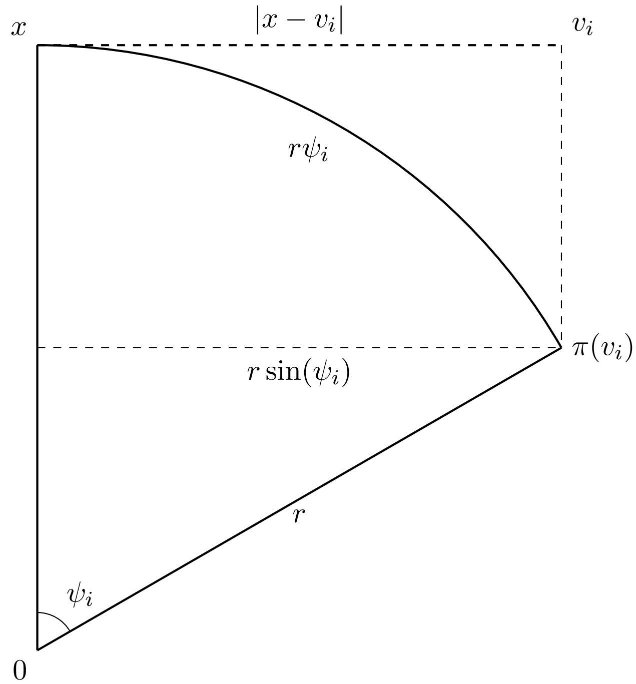

Figure 4: Intersection of the sphere with x, v_i and $\pi(v_i)$. Because of lack of space we have introduced the notation $\psi_i = d_{\mathbb{H}(1/r^2)}(x, \pi(v_i))/r$.

Proof We can assume that $x \neq v_i$ for all i, because if $x = v_i$ for some i there is nothing to prove. Due to linearity of the determinant we have

$$\begin{aligned}
&\det(d_{(\mathbb{H}(1/r^2))}(x, \pi(v_i)) d_{\mathbb{H}(1/r^2)}(x, \pi(v_l)) \cos\theta_{il}) \\
&\qquad = \left(\prod_i d_{\mathbb{H}(1/r^2)}(x, \pi(v_i))\right) \det\left(d_{\mathbb{H}(1/r^2)}(x, \pi(v_l)) \cos\theta_{il}\right) \\
&\qquad = \left(\prod_i d_{\mathbb{H}(1/r^2)}(x, \pi(v_i))\right) \left(\prod_l d_{\mathbb{H}(1/r^2)}(x, \pi(v_l))\right) \det\left(\cos\theta_{il}\right) \\
&\qquad = \left(\prod_i d_{\mathbb{H}(1/r^2)}(x, \pi(v_i))^2\right) \det\left(\cos\theta_{il}\right),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\det(|x - v_i||x - v_l| \cos\theta_{il}) \\
&\qquad = \det\left(r \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, \pi(v_i))}{r}\right) r \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, \pi(v_l))}{r}\right) \cos\theta_{il}\right) \\
&\qquad = \left(\prod_i r \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, \pi(v_i))}{r}\right)\right)^2 \det\left(\cos\theta_{il}\right).
\end{aligned} \tag{9}$$

Given that by assumption $|x - v_i| \neq 0$ and

$$\sin\left(\frac{d_{(\mathbb{H}(1/r^2))}(x, \pi(v_i))}{r}\right) \neq 0,$$

for all i the claim follows. $\square$

We refer to a matrix of the form

$$\left(r^2 \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, v_i)}{r}\right) \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, v_l)}{r}\right) \cos\theta_{il}\right)_{i,l}$$

as a spherical pseudo Gram matrix.

We use the pseudo Gram matrix to introduce a quality measure $Q_{\mathbb{H}(1/r^2)}$ for simplices on the sphere with vertices $\sigma = \{v_0, \ldots, v_n\}$:

$$\begin{aligned} &Q_{\mathbb{H}(1/r^2)}(\sigma) \\ &\quad = \min_{x\in\mathbb{H}(1/r^2)} \max_j \left\{\det\left(r^2 \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, v_i)}{r}\right) \sin\left(\frac{d_{\mathbb{H}(1/r^2)}(x, v_l)}{r}\right) \cos\theta_{il}\right)_{i,l\neq j}\right\}. \end{aligned} \tag{10}$$

By the notation $i, l \neq j$ we mean to imply that $i, l \in \{0, \ldots, j-1, j+1, \ldots, n\}$. If we view the sphere $\mathbb{H}(1/r^2)$ as being embedded in the Euclidean space $\mathbb{R}^{n+1}$ this has the interpretation

$$Q_{\mathbb{H}(1/r^2)}(\sigma) = \min_{x\in\mathbb{H}(1/r^2)} \max_j\{\det(\tilde{\pi}_x v_i \cdot \tilde{\pi}_x v_l)_{i,l\neq j}\} = \min_{x\in\mathbb{H}(1/r^2)} \max_j\{(\det(\tilde{\pi}_x v_i)_{i\neq j})^2\}, \tag{11}$$

where $\tilde{\pi}_x$ is the projection onto the hyperplane characterized by the normal x, that is tangent to $\mathbb{H}(1/r^2)$ and $(w_i)_{i\neq j}$ denotes the matrix whose columns are all vectors w_i with $i \neq j$. This interpretation follows from (9).

In our definition we have chosen specifically to let x run over the entire sphere. In particular we include the case where all vertices are equally distributed on the equator. In this case the quality is zero. This is in accordance with our intuition because we would not know on which hemisphere to draw the simplex.

Now we shall give some lower bounds on the quality $Q_{\mathbb{H}(1/r^2)}$. These bounds should strengthen our intuition. Let us consider the simplex $\{0, v_0, \ldots, v_n\}$ in $\mathbb{R}^{n+1}$, with $v_0, \ldots v_n$ lying on the sphere $\mathbb{H}(1/r^2)$. If we project $\{0, v_0, \ldots, v_n\}$ to $T_x\mathbb{H}(1/r^2)$ we find $\{x, \pi(v_0), \ldots, \pi(v_n)\}$. Suppose that $B(y, \rho)$ is a ball that lies inside the simplex with vertices $\{0, v_0, \ldots, v_n\} \subset \mathbb{R}^{n+1}$. Now denote by $\pi_x(B(y, \rho))$ the projection of $B(y, \rho)$ onto $T_x\mathbb{H}(1/r^2)$. We see that $\pi_x(B(y, \rho)) \subset \cup_j(\pi\sigma)_j$, with $(\pi\sigma)_j$ the simplex whose vertices are $\{x, \pi(v_0), \ldots, \pi(v_{j-1}), \pi(v_{j+1}), \ldots, \pi(v_n)\}$. The choice of hyperplane onto which one orthogonally projects this ball does not influence the volume of the projected ball. Let us denote the volume of the projected ball by $\mathrm{vol}(\pi_x(B(y, \rho)))$. The inclusion of $\pi_x(B(y, \rho))$ now gives $\mathrm{vol}(\pi_x(B(y, \rho))) < \sum_j \mathrm{vol}(\pi\sigma)_j$ and thus $\mathrm{vol}(\pi_x(B(y, \rho))) < \frac{1}{(n+1)^2} Q_{\mathbb{H}(1/r^2)}(\sigma)$.

This completes our discussion of quality in the elliptic setting and we continue to the hyperbolic case.

The direct analogue of Lemma 10 holds for the Minkowski or hyperboloid model of hyperbolic spaces of constant curvature. For this setting we therefore introduce the hyperbolic pseudo Gram matrix

$$\left(r^2 \sinh\left(\frac{d_{\mathbb{H}(-1/r^2)}(x, v_i)}{r}\right) \sinh\left(\frac{d_{\mathbb{H}(-1/r^2)}(x, v_l)}{r}\right) \cos\theta_{il}\right)_{i,l\neq j},$$

as well as the quality measure for simplices on the hyperbolic sphere with vertices $\sigma = \{v_0, \dots, v_n\}$ for some (large enough convex) neighbourhood N

$$\begin{aligned} &Q_{\mathbb{H}(-1/r^2)}(\sigma) \\ &\quad = \min_{x\in N} \max_j \left\{\det\left(r^2 \sinh\left(\frac{d_{\mathbb{H}(-1/r^2)}(x, v_i)}{r}\right) \sinh\left(\frac{d_{\mathbb{H}(-1/r^2)}(x, v_l)}{r}\right) \cos\theta_{il}\right)_{i,l\neq j}\right\}. \end{aligned}$$

Remark 11 Unfortunately the geometric interpretation is more difficult in this hyperbolic setting and we have not (yet) been able to provide a discussion similar to the one for spaces of positive curvature above. The authors also think that it should be possible to replace the (convex) neighbourhood N in definition of hyperbolic quality by the entire space, because we believe that the minimum is always attained in a (convex) geodesic ball containing all points. We have not been able to prove this, however for the paper the definition above suffices, because we know that a Riemannian simplex is contained in any convex ball that contains all vertices.

4 Non-degeneracy criteria

In this section we present our main results. We first give an overview of the proof and then give the details where we distinguish the positive (elliptic) and negative (hyperbolic) curvature cases. We rely on technical results concerning the approximation of the cosine rule in spaces of constant curvature. Although these results may seem believable at first glance the proofs are rather technical and presented in the appendix.

The steps involved in the proof are similar to the steps presented in the appendix of [4] and Section 3.6 of [9] (where a slightly improved version can be found) for simplices based on Euclidean simplices. A precise comparison between the steps of the proof presented in this paper and the ones for Euclidean simplices can be found in Section 3.10.2 of [9].

4.1 Overview of the proof method

We are interested in Riemannian simplices on manifolds whose sectional curvatures are very close to constant, meaning that the sectional curvature K satisfies $\Lambda_\ell \leq K \leq \Lambda_u$ with $|\Lambda_\ell - \Lambda_u|$ small relative to $|\Lambda_\ell|$. This means that we always suppose that $0 < \Lambda_\ell$ or $\Lambda_u < 0$.

We have to distinguish between positive sectional curvature (elliptic) and negative sectional curvature (hyperbolic). Because the sectional curvatures are very close to constant, comparing the manifold to Euclidean space is unnatural. Instead we compare to spaces of

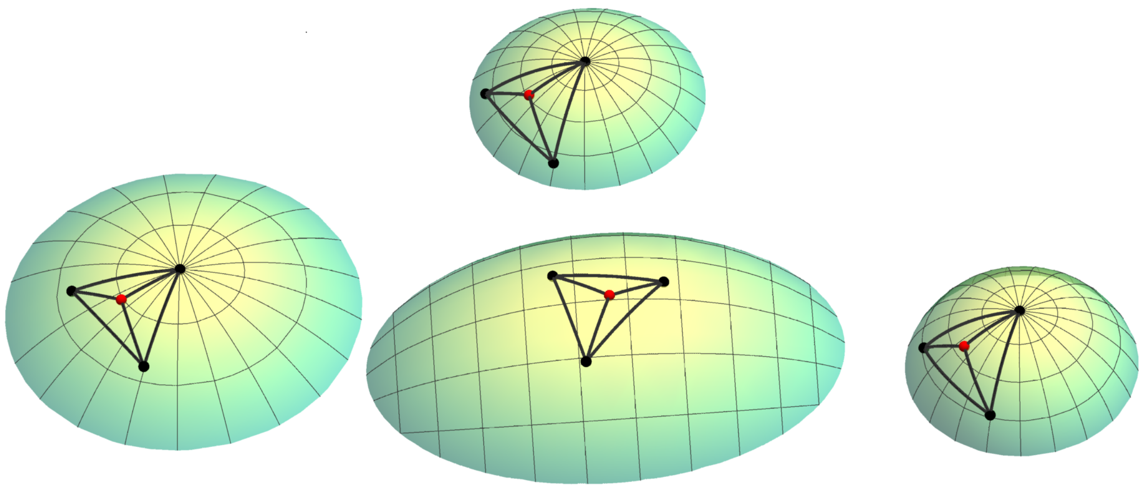

Figure 5: Pictorial overview of our approach in the case of positive curvature, the negative curvature case is similar but difficult to depict.
Above we see a sphere of mediocre size. Below we see from left to right a small sphere, an ellipsoid and a large sphere. The small sphere is the example of $\mathbb{H}(\Lambda_u)$, the large sphere of $\mathbb{H}(\Lambda_\ell)$ and the sphere of mediocre size of $\mathbb{H}(\Lambda_{\text{mid}})$. The ellipsoid is the manifold M.
The vertices on M are depicted in black, as are the vertices on the spaces of constant curvature left, right and below. The vertices on M are transplanted on spaces of constant curvature by the maps $\exp_{\mathbb{H}(\Lambda_u)} \circ \exp^{-1}_{v_r,M}$, $\exp_{\mathbb{H}(\Lambda_\ell)} \circ \exp^{-1}_{v_r,M}$ and $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}$, respectively. The same holds for the arbitrary point x (red).
The angles between the tangents to the geodesics emanating from the red point on M (the ellipsoid in the middle) are by the Topogonov comparison theorem close to the corresponding angles in the spaces of constant curvature (left and right). In turn these spaces of constant curvature are similar to the space with curvature Λ_{mid} (middle bottom).

constant curvature Λ_{mid} if the curvature is positive and $\Lambda^{\mathbb{H}}_{\text{mid}}$ if the curvature is negative, whose sectional curvature lies in the interval $[\Lambda_\ell, \Lambda_u]$.

In the elliptic case we shall compare to the space with constant curvature $\Lambda_{\text{mid}} = \frac{1}{2}(\Lambda_\ell + \Lambda_u)$. The hyperbolic case will be more involved. In particular the non-degeneracy conditions will be in terms of the quality of the simplex $\sigma_{\mathbb{H}(\Lambda^{(\mathbb{H})}_{\text{mid}})}(v_r)$ with vertices $\exp_{\mathbb{H}(\Lambda^{(\mathbb{H})}_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(v_i)$, where $\exp_{v_r,M}$ denotes the exponential function of M at v_r. Here $\Lambda^{(\mathbb{H})}_{\text{mid}}$ stands for one of the alternatives, namely Λ_{mid} or $\Lambda^{\mathbb{H}}_{\text{mid}}$. Simplex quality measures for constant curvature spaces are not common and no such measure suiting our need existed previously. We introduce a quality measure in Section 3 that is suited.

Our steps to provide quality bounds that guarantee non-degeneracy of the simplex shall be the following:

1. Use the Toponogov comparison theorem to bound the difference between the lengths of the geodesics connecting vertices v_i and v_j and the lengths the geodesics connecting vertices x and v_i on the one hand and the corresponding (via the map $\exp_{\mathbb{H}(\Lambda_u)} \circ \exp^{-1}_{v_r,M}$ and $\exp_{\mathbb{H}(\Lambda_\ell)} \circ \exp^{-1}_{v_r,M}$, respectively) lengths of geodesics on spaces of constant curvature $\mathbb{H}(\Lambda_\ell)$ and $\mathbb{H}(\Lambda_u)$.

   This step relies heavily on calculations that are presented in the first part of Appendix A.

2. Prove that the lengths of these geodesics are not far from what you would expect on the space $\mathbb{H}(\Lambda^{(\mathbb{H})}_{\text{mid}})$ (via the map $\exp_{\mathbb{H}(\Lambda^{(\mathbb{H})}_{\text{mid}})} \circ \exp^{-1}_{v_r,M}$) if the vertices and x lie close together relative to the bounds on the sectional curvature on M. It is at this point where the analysis in the elliptic and hyperbolic cases really start to differ.

3. Given these approximate lengths of the geodesics we can again use the Toponogov comparison theorem and explicit calculations on spaces of constant curvature to give estimates on difference between the following 'inner products':

$$\frac{1}{\Lambda_{\text{mid}}} \sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x, v_i)\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x, v_j)\right) \cos\theta_{ij,M} \qquad \text{(elliptic)},$$
$$\frac{1}{|\Lambda^{\mathbb{H}}_{\text{mid}}|} \sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\text{mid}}|} d_M(x, v_i)\right) \sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\text{mid}}|} d_M(x, v_j)\right) \cos\theta_{ij,M} \quad \text{(hyperbolic)},$$

   with $\theta_{ij,M}$ the angle between the geodesics from x to v_i and v_j, and the expectation of these 'inner products' in $\mathbb{H}(\Lambda^{(\mathbb{H})}_{\text{mid}})$. As before elliptic refers to positive sectional curvature, here $\Lambda_{\text{mid}} > 0$, and hyperbolic refers to negative sectional curvature, here $\Lambda^{\mathbb{H}}_{\text{mid}} < 0$.

   This step relies heavily on calculations that are presented in the second half of Appendix A.

4. $n \times n$ of these 'inner products' are put into a pseudo Gram matrix. We shall introduce this pseudo Gram matrix below. The determinant of this matrix is non-zero if and only if the tangents to the geodesics emanating from x are linearly independent. A

result by Friedland describes the behaviour of the determinant under perturbations of the entries. This means that the determinant of the Gram matrix is close to the determinant of the Gram matrix one expects for $\mathbb{H}(\Lambda^{(\mathbb{H})}_{\text{mid}})$. This allows us to give conditions that guarantee that there are n tangents to the geodesics emanating from x that are linearly independent, based on the quality of the simplex you would expect in the constant curvature case.

5. If for every x in a sufficiently large convex neighbourhood some n tangents to the geodesics emanating from x and going to the vertices are linearly independent, the simplex is non-degenerate, see Lemma 6.

The main building blocks for this approach where we model our intrinsic simplices on simplices on spaces of constant curvature are the pseudo Gram matrices which we encountered in Sections 1.1 and 3. The calculations necessary to compare different spaces of constant curvature are given in Appendix A. The comparison of spaces of different constant curvature will mainly focus on the cosine rule.

4.2 Degeneracy criteria for simplices on spaces of constant positive curvature

Let us now give the steps in Section 4.1 in more detail.

Step 1 We define $\Lambda_{\text{mid}} = \frac{1}{2}(\Lambda_\ell + \Lambda_u)$. Note that this is the arithmetic mean of the upper and lower bounds on the sectional curvature and has nothing to do with the mean curvature. Let us further assume that all distances involved are bounded from above by some maximum distance $\tilde{D}$, where we also assume that $\tilde{D} \leq 1/(2\sqrt{\Lambda_\ell})$. By the Toponogov comparison theorem we have that $d_M(y,z)$ the distances between $y, z \in \{x, v_0, \dots, v_n\}$ in M are bounded by those in $\mathbb{H}(\Lambda_\ell)$ and $\mathbb{H}(\Lambda_u)$. Or in other words we have that $d_{\mathbb{H}(\Lambda_u)}(y_{\Lambda_u}, z_{\Lambda_u}) \leq d_M(y,z) \leq d_{\mathbb{H}(\Lambda_\ell)}(y_{\Lambda_\ell}, z_{\Lambda_\ell})$ where $y_K = \exp_{\mathbb{H}(K)} \circ \exp^{-1}_{v_r,M}(y)$. Because we use the exponential map at v_r, regard v_r as a fixed point and we shall not use the notation $(v_r)_K$, but write v_r regardless.

Step 2 The distances $d_{\mathbb{H}(\Lambda_\ell)}(y_{\Lambda_\ell}, z_{\Lambda_\ell})$ and $d_{\mathbb{H}(\Lambda_u)}(y_{\Lambda_u}, z_{\Lambda_u})$ satisfy

$$\begin{aligned}\frac{1}{\Lambda_\ell}\cos\left(\sqrt{\Lambda_\ell} d_{\mathbb{H}(\Lambda_\ell)}(y,z)\right) &= \frac{1}{\Lambda_\ell}\cos\left(\sqrt{\Lambda_\ell} d_M(y,v_r)\right)\cos\left(\sqrt{\Lambda_\ell} d_M(z,v_r)\right) \\ &\quad + \frac{1}{\Lambda_\ell}\sin\left(\sqrt{\Lambda_\ell} d_M(y,v_r)\right)\sin\left(\sqrt{\Lambda_\ell} d_M(z,v_r)\right)\cos\theta_{\angle_M y v_r z},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\Lambda_u}\cos\left(\sqrt{\Lambda_u} d_{\mathbb{H}(\Lambda_u)}(y,z)\right) &= \frac{1}{\Lambda_u}\cos\left(\sqrt{\Lambda_u} d_M(y,v_r)\right)\cos\left(\sqrt{\Lambda_u} d_M(z,v_r)\right) \\ &\quad + \frac{1}{\Lambda_u}\sin\left(\sqrt{\Lambda_u} d_M(y,v_r)\right)\sin\left(\sqrt{\Lambda_u} d_M(z,v_r)\right)\cos\theta_{\angle_M y v_r z},\end{aligned}$$

where $\theta_{\angle_M y v_r z}$ denotes the angle between the tangents to the geodesics on M from v_r to y

and v_r to z. One can prove (see Lemma 16 in the appendix) that

$$\begin{aligned}
&\frac{1}{\Lambda_{\mathrm{mid}}}\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_\ell)}(y,z)\right)\\
&\qquad=\frac{1}{\Lambda_{\mathrm{mid}}}\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,v_r)\right)\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(z,v_r)\right)\\
&\qquad\qquad+\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,v_r)\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(z,v_r)\right)\cos\theta_{\angle_M yv_rz}+R_{T_1},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Lambda_{\mathrm{mid}}}\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_u)}(y,z)\right)\\
&\qquad=\frac{1}{\Lambda_{\mathrm{mid}}}\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,v_r)\right)\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(z,v_r)\right)\\
&\qquad\qquad+\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,v_r)\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(z,v_r)\right)\cos\theta_{\angle_M yv_rz}+R_{T_2},
\end{aligned}$$

with $|R_{T_1}|,|R_{T_2}|\leq\frac{1}{2}\left|\Lambda_\ell-\Lambda_u\right|\frac{11\tilde{D}^4}{4!}$, so that

$$\begin{aligned}
&\frac{1}{\Lambda_{\mathrm{mid}}}\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,z)\right)\\
&\qquad=\frac{1}{\Lambda_{\mathrm{mid}}}\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,v_r)\right)\cos\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(z,v_r)\right)\\
&\qquad\qquad+\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(y,v_r)\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(z,v_r)\right)\cos\theta_{\angle_M yv_rz}+R_T,
\end{aligned}\tag{12}$$

with the same bound on $|R_T|$.

Step 3 We are now going to study the $\sin a\sin b\cos\gamma$ terms we discussed in Section 3. These generalize the $ab\cos\gamma$ terms we use in the Gram-matrix in the Euclidean case. The point we focus on is x.

Thanks to (12) we have the (approximate) lengths of all geodesics. So we can now apply the Toponogov comparison theorem for a second time. The Toponogov comparison theorem gives us that

$$d_{\mathbb{H}(\Lambda_\ell)}(x_{\Lambda_\ell},y_{\Lambda_\ell})\geq d_M(x,y)\geq d_{\mathbb{H}(\Lambda_u)}(x_{\Lambda_u},y_{\Lambda_u}),\quad \cos\theta_{\mathbb{H}(\Lambda_\ell)}\geq\cos\theta_M\geq\cos\theta_{\mathbb{H}(\Lambda_u)},$$

where we identify angles in the obvious manner using $\exp_{\mathbb{H}(\Lambda_{u,\ell})}\circ\exp_M^{-1}$, in the same way as we do the points, see Figure 5. From these inequalities we infer that

$$\begin{aligned}
&\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_\ell)}(x_{\Lambda_\ell},y_{\Lambda_\ell})\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_\ell)}(x_{\Lambda_\ell},z_{\Lambda_\ell})\right)\cos\theta_{\angle_{\Lambda_\ell^x}yxz}\\
&\qquad\geq\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(x,y)\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(x,z)\right)\cos\theta_{\angle_M yxz}\\
&\qquad\geq\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_u)}(x_{\Lambda_u},y_{\Lambda_u})\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_u)}(x_{\Lambda_u},z_{\Lambda_u})\right)\cos\theta_{\angle_{\Lambda_u^x}yxz},
\end{aligned}$$

where $\angle_{\Lambda^x_{u,\ell}} yxz$ denotes the angle between the tangents of the geodesics from x to $\exp_{\mathbb{H}(\Lambda_{u,\ell})} \circ \exp^{-1}_{x,M}(y)$ and from x to $\exp_{\mathbb{H}(\Lambda_{u,\ell})} \circ \exp^{-1}_{x,M}(z)$. Here $\Lambda_{u,\ell}$ should be interpreted either Λ_ℓ or Λ_u. We can prove (see (19) of Lemma 20 in the appendix, while we also note that by definition of Λ_{mid}, one has $2|\Lambda_{\text{mid}} - \Lambda_\ell| = 2|\Lambda_{\text{mid}} - \Lambda_u| = |\Lambda_\ell - \Lambda_u|$) that

$$\begin{aligned}&\bigg|\frac{1}{\Lambda_{\text{mid}}} \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{u,\ell})}(x_{\Lambda_{u,\ell}}, y_{\Lambda_{u,\ell}})\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{u,\ell})}(x_{\Lambda_{u,\ell}}, z_{\Lambda_{u,\ell}})\right) \cos\theta_{\angle_{\Lambda^x_{u,\ell}} yxz} \\ &-\frac{1}{\Lambda_{\text{mid}}} \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{\text{mid}})}(x_{\Lambda_{\text{mid}}}, y_{\Lambda_{\text{mid}}})\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{\text{mid}})}(x_{\Lambda_{\text{mid}}}, z_{\Lambda_{\text{mid}}})\right) \cos\theta_{\angle_{\Lambda_{\text{mid}}} yxz}\bigg| \\ &\qquad\qquad \leq |\Lambda_\ell - \Lambda_u|\, \tilde{D}^4,\end{aligned}$$

where $\angle_{\Lambda_{\text{mid}}} yxz$ is the angle between the tangents to the geodesics from $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(x)$ to $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(y)$ and from $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(x)$ to $\exp_{\mathbb{H}(\Lambda_{\text{mid}})} \circ \exp^{-1}_{v_r,M}(z)$. Here we went from angles determined by the exponential map at x to those determined by the exponential map at v_r, this is possible because Lemma 20 only takes lengths of geodesics as input.

This means

$$\begin{aligned}&\bigg|\frac{1}{\Lambda_{\text{mid}}} \sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x,y)\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x,z)\right) \cos\theta_{\angle_M yxz} \\ &-\frac{1}{\Lambda_{\text{mid}}} \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{\text{mid}})}(x_{\Lambda_{\text{mid}}}, y_{\Lambda_{\text{mid}}})\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_{\mathbb{H}(\Lambda_{\text{mid}})}(x_{\Lambda_{\text{mid}}}, z_{\Lambda_{\text{mid}}})\right) \cos\theta_{\angle_{\Lambda_{\text{mid}}} yxz}\bigg| \\ &\qquad\qquad \leq |\Lambda_\ell - \Lambda_u|\, \tilde{D}^4. \qquad (13)\end{aligned}$$

Step 4 We can now exploit (13) to prove that the (rescaled) pseudo Gram matrix

$$\left(\frac{\sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x, v_i)\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x, v_l)\right) \cos\theta_{il}}{(2\tilde{D})^2 \Lambda_{\text{mid}}}\right)_{i,l \neq j}, \qquad (14)$$

with $\theta_{\angle_M v_i x v_l} = \theta_{il}$ is non-degenerate. Here we use Lemma 10 which says that the determinant of (14) is zero if and only if the vectors $\exp^{-1}_{x,M}(v_i)$ with $i \neq j$ are linearly independent. We use the result by Friedland, see Equation (3) in Section 1.1, to give condition that guarantee that the determinant of (14) is non-zero. Because of the role of $\max\{\|A\|_\infty, \|A+E\|_\infty\}$, where in this case A is (14) and $A+E$ is the approximate Gram matrix, it is important to note that

$$\left|\frac{\sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x, v_i)\right) \sin\left(\sqrt{\Lambda_{\text{mid}}} d_M(x, v_l)\right) \cos\theta_{il}}{(2\tilde{D})^2 \Lambda_{\text{mid}}}\right| \leq 1. \qquad (15)$$

This follows from the observation that if $y_{\max} < \pi/2$ then $\sup_{y \in [0, y_{\max}]} \frac{\sin(y)}{y_{\max}} \leq 1$. We are now able to conclude that

$$\left|\det\left(\frac{\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(x,v_i)\right)\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_M(x,v_l)\right)\cos\theta_{il}}{(2\tilde{D})^2\Lambda_{\mathrm{mid}}}\right)_{i,l\neq j}\right|$$
$$\geq \left|\det\Bigg(\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(x_{\Lambda_{\mathrm{mid}}},(v_i)_{\Lambda_{\mathrm{mid}}})\right)\right.$$
$$\left.\cdot\frac{\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(x_{\Lambda_{\mathrm{mid}}},(v_l)_{\Lambda_{\mathrm{mid}}})\right)\cos\theta_{\angle_{\Lambda_{\mathrm{mid}}}v_ixv_l}}{(2\tilde{D})^2\Lambda_{\mathrm{mid}}}\Bigg)_{i,l\neq j}\right| - n\left|\Lambda_\ell-\Lambda_u\right|\tilde{D}^2. \quad (16)$$

Step 5 Combining Lemma 10 and Lemma 6 and using that if for all x there is a j such that the pseudo Gram matrix above is non-degenerate then the simplex is non-degenerate, we have:

Theorem 1 Let M be a manifold whose sectional curvatures K satisfy $0<\Lambda_\ell\leq K\leq\Lambda_u$. Suppose that $v_0,\dots,v_n$ are vertices on M. Assume[4] that all vertices lie within a convex geodesic ball of radius $\frac{1}{2}\tilde{D}$ with centre v_r, where $\tilde{D}\leq 1/(2\sqrt{\Lambda_u})$ and $r\in\{0,\dots,n\}$. Under these assumptions the Riemannian simplex with vertices $v_0,\dots,v_n$ on M is non-degenerate if

$$Q_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(\sigma_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(v_r)) > n2^{2n}|\Lambda_\ell-\Lambda_u|\tilde{D}^{2n+2},$$

with $Q_{\mathbb{H}(\Lambda_{\mathrm{mid}})}$ the simplex quality, $\sigma_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(v_r)$ the simplex on $\mathbb{H}(\Lambda_{\mathrm{mid}})$ with vertices $v_i(v_r)$ defined by $v_i(v_r)=\exp_{\mathbb{H}(\Lambda_{\mathrm{mid}})}\circ\exp^{-1}_{v_r,M}(v_i)$ and $\Lambda_{\mathrm{mid}}=\frac{1}{2}(\Lambda_\ell+\Lambda_u)$. The quality of a simplex $W=\{w_i\}\subset\mathbb{H}(\Lambda_{\mathrm{mid}})$ is given by

$$Q_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(W)=\min_{y\in\mathbb{H}(\Lambda_{\mathrm{mid}})}\max_j\Bigg\{\det\Bigg(\frac{1}{\Lambda_{\mathrm{mid}}}\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}\,d_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(y,w_i)\right)$$
$$\cdot\sin\left(\sqrt{\Lambda_{\mathrm{mid}}}d_{\mathbb{H}(\Lambda_{\mathrm{mid}})}(y,w_l)\right)\cos\theta_{il}\Bigg)_{i,l\neq j}\Bigg\}, \quad (1)$$

with θ_{il} the angle between the geodesics on $\mathbb{H}(\Lambda_{\mathrm{mid}})$ from y to w_i and w_l.

Remark 12 If one would choose $\Lambda_{\mathrm{mid}}\in[\Lambda_\ell,\Lambda_u]$ not equal to $\frac{1}{2}(\Lambda_\ell+\Lambda_u)$ one would need to replace $\frac{1}{2}|\Lambda_\ell-\Lambda_u|$ in the quality bound by

$$\max\{|\Lambda_\ell-\Lambda_{\mathrm{mid}}|,|\Lambda_u-\Lambda_{\mathrm{mid}}|\}.$$

For a Λ_{mid} outside the interval $[\Lambda_\ell,\Lambda_u]$, one would further need to adjust the bound on $\tilde{D}$ by replacing Λ_u by the maximum of Λ_u and Λ_{mid}. The quality bound directly follows from the error in the cosine rule, see Lemma 20 and the overview of the proof as given in Section 4.1.

[4] This bound is stronger than necessary. In fact it suffices for the lengths of geodesics in the proof to be bounded by $\tilde{D}$. We have chosen this formulation because it is natural in the Delaunay setting, which motivates this work, see Section 5.

4.3 Degeneracy criteria for simplices on spaces of constant negative curvature

We can perform similar calculations for the hyperbolic case but the result is significantly more complicated due to the fact that the hyperbolic cosine is not bounded by one. Moreover we will not impose a bound on $\tilde{D}$ in the negative curvature setting.

Let us start by defining $\Lambda^{\mathbb{H}}_{\mathrm{mid}}$ as the negative solution to the following equation:

$$\begin{aligned}\left| |\Lambda^{\mathbb{H}}_{\mathrm{mid}}| \cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right) - |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)\right| \\ = \left| |\Lambda^{\mathbb{H}}_{\mathrm{mid}}| \cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right)\right|.\end{aligned}$$

Note that by construction

$$\begin{aligned}\left| |\Lambda^{\mathbb{H}}_{\mathrm{mid}}| \cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right) - |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)\right| \\ = \left| |\Lambda^{\mathbb{H}}_{\mathrm{mid}}| \cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right)\right| \\ = \frac{1}{2}\left| |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right)\right|.\end{aligned}$$

Remark 13 This definition of $\Lambda^{\mathbb{H}}_{\mathrm{mid}}$ is made to make the final expression of (17) as simple as possible. For an arbitrary $\Lambda^{\mathbb{H}}_{\mathrm{mid}} \in [\Lambda_\ell, \Lambda_u]$ one would need to replace

$$\frac{1}{2}\left| |\Lambda_\ell| \cosh^2(\sqrt{|\Lambda_\ell|}\tilde{D}) - |\Lambda_u| \cosh^2(\sqrt{|\Lambda_u|}\tilde{D})\right|$$

in (17) by

$$\begin{aligned}\max\left\{\left| |\Lambda^{\mathbb{H}}_{\mathrm{mid}}| \cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right) - |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)\right|,\right. \\ \left.\left| |\Lambda^{\mathbb{H}}_{\mathrm{mid}}| \cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right)\right|\right\}.\end{aligned}$$

If one would further want to consider $\Lambda^{\mathbb{H}}_{\mathrm{mid}}$ outside $[\Lambda_\ell, \Lambda_u]$ one would also need to replace $|\Lambda_\ell|$ by $\max\{|\Lambda_\ell|, |\Lambda^{\mathbb{H}}_{\mathrm{mid}}|\}$ in the first two lines of (17).

We shall once more employ the bound of Friedland, see Section 1.1. In order to be able to do so we need, similarly to (15), that

$$\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}a\right)\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}b\right)\cos\gamma \leq \sinh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right).$$

So that similarly to (16), we have

$$
\begin{aligned}
&\left|\det\left(\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}d_M(x,v_i)\right)\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}d_M(x,v_l)\right)\cos\theta_{ij}\right)_{i,l\neq j}\right| \\
&\quad\geq \Bigg|\det\Bigg(\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}d_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(x_{\Lambda^{\mathbb{H}}_{\mathrm{mid}}},(v_i)_{\Lambda^{\mathbb{H}}_{\mathrm{mid}}})\right) \\
&\qquad\cdot\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}d_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(x_{\Lambda^{\mathbb{H}}_{\mathrm{mid}}},(v_l)_{\Lambda^{\mathbb{H}}_{\mathrm{mid}}})\right)\cos\theta_{\angle_{\Lambda^{\mathbb{H}}_{\mathrm{mid}}}v_ixv_l}\Bigg)_{i,l\neq j}\Bigg| \\
&\qquad- n(\sinh^2\sqrt{|\Lambda_\ell|}\tilde{D})^{n-1}\left(2+2\cosh\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)+|\Lambda_\ell|^2\cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)\frac{11\tilde{D}^4}{4!}\right) \\
&\qquad\cdot\left||\Lambda_\ell|\cosh^2\sqrt{|\Lambda_\ell|}\tilde{D}-|\Lambda_u|\cosh^2\sqrt{|\Lambda_u|}\tilde{D}\right||\Lambda^{\mathbb{H}}_{\mathrm{mid}}|\frac{11\tilde{D}^4}{2\cdot 4!}.
\end{aligned}
$$

From which we infer that

Theorem 14 *Let M be a manifold with bounded negative sectional curvatures K, that is $\Lambda_\ell \leq K \leq \Lambda_u < 0$. Suppose that $v_0,\dots,v_n$ are vertices on M. Let us assume that all vertices lie within a convex geodesic ball of radius $\frac{1}{2}\tilde{D}$ with centre v_r. Under these assumptions the Riemannian simplex with vertices $v_0,\dots,v_n$ on M is non-degenerate if*

$$
\begin{aligned}
&Q_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(\sigma_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(v_r)) \\
&> n|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|^{-n}(\sinh\sqrt{|\Lambda_\ell|}\tilde{D})^{2(n-1)}\cdot\left(2+2\cosh\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)+|\Lambda_\ell|^2\cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)\frac{11\tilde{D}^4}{4!}\right) \\
&\qquad\cdot\left||\Lambda_\ell|\cosh^2(\sqrt{|\Lambda_\ell|}\tilde{D})-|\Lambda_u|\cosh^2(\sqrt{|\Lambda_u|}\tilde{D})\right||\Lambda^{\mathbb{H}}_{\mathrm{mid}}|\frac{11\tilde{D}^4}{2\cdot 4!}. \qquad (17)
\end{aligned}
$$

with $\sigma_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(v_r)$ the simplex on $\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})$ with vertices $v_i(v_r) = \exp_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}\circ\exp^{-1}_{v_r,M}(v_i)$, $Q_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(\sigma_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(v_r))$ the simplex quality

$$
\begin{aligned}
Q_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(\sigma_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(v_r)) = \min_{x\in N}\max_j\Bigg\{\det\Bigg(&\frac{1}{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}d_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(x,v_i(v_r))\right) \\
&\sinh\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}d_{\mathbb{H}(\Lambda^{\mathbb{H}}_{\mathrm{mid}})}(x,v_l(v_r))\right)\cos\theta_{il}\Bigg)_{i,l\neq j}\Bigg\},
\end{aligned}
$$

where N is a geodesic ball with radius $2\tilde{D}$ centred at any one of the vertices and $\Lambda^{\mathbb{H}}_{\mathrm{mid}}$ satisfies

$$
\begin{aligned}
&\left||\Lambda^{\mathbb{H}}_{\mathrm{mid}}|\cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right)-|\Lambda_\ell|\cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)\right| \\
&\qquad= \left||\Lambda^{\mathbb{H}}_{\mathrm{mid}}|\cosh^2\left(\sqrt{|\Lambda^{\mathbb{H}}_{\mathrm{mid}}|}\tilde{D}\right)-|\Lambda_u|\cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right)\right| \\
&\qquad= \frac{1}{2}\left||\Lambda_\ell|\cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right)-|\Lambda_u|\cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right)\right|.
\end{aligned}
$$

Remark 15 The bounds are far more complicated than the elliptic case and thus more difficult to interpret. However if $|\Lambda_\ell - \Lambda_u|$ tends to zero, so does

$$\left| |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right) \right|.$$

This can be made more precise using the mean value theorem which states that for a function f there exists a $c \in [a, b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Applying this to $x \cosh^2 \sqrt{x}\tilde{D}$ we see that

$$\begin{aligned}
&\left| |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right) \right| \\
&\qquad = |\Lambda_u - \Lambda_\ell| \left| \cosh^2\left(\sqrt{|\tilde{\Lambda}|}\tilde{D}\right) + \tilde{D}\sqrt{|\tilde{\Lambda}|} \sinh\left(\sqrt{|\tilde{\Lambda}|}\tilde{D}\right) \cosh\left(\sqrt{|\tilde{\Lambda}|}\tilde{D}\right) \right| \\
&\qquad \leq |\Lambda_u - \Lambda_\ell| \left| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) + \tilde{D}\sqrt{|\Lambda_\ell|} \sinh\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) \cosh\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) \right|,
\end{aligned}$$

with $\tilde{\Lambda} \in [\Lambda_\ell, \Lambda_u]$, where we used that $\cosh^2\left(\sqrt{x}\tilde{D}\right) + \tilde{D}\sqrt{x} \sinh\left(\sqrt{x}\tilde{D}\right) \cosh\left(\sqrt{x}\tilde{D}\right)$ is monotone increasing for $x \geq 0$.

This implies that if one has a set of points on a manifold which is close to a space of constant curvature, in the sense of the sectional curvature, then very small quality is required to guarantee non-degeneracy, where we regard the lower bound on the sectional curvature and the distance between the vertices as fixed.

One further has that

$$\left| |\Lambda_\ell| \cosh^2\left(\sqrt{|\Lambda_\ell|}\tilde{D}\right) - |\Lambda_u| \cosh^2\left(\sqrt{|\Lambda_u|}\tilde{D}\right) \right| \geq |\Lambda_u - \Lambda_\ell|.$$

This can be seen by considering $f(x) = -x \cosh^2(\sqrt{-x})$ on the domain $(-\infty, 0]$ and checking that $f'(x) \leq -1$. From this we conclude that the quality requirements are in a certain sense stricter in the negative curvature setting than in the positive curvature setting.

5 Delaunay triangulations and future research

In [4] we assumed that we were given an abstract simplicial complex $\mathcal{A}$, whose vertices were identified with points on a given Riemannian manifold M, and the non-degeneracy criterion for simplices was the main ingredient for criteria that ensure that $\mathcal{A}$ is homeomorphic to M (i.e., that it is a triangulation). The construction of $\mathcal{A}$ itself was addressed in earlier work which is based on the stability of Euclidean Delaunay triangulations [2]. By considering simplices in space forms, the current work relaxes the non-degeneracy criterion, but a further relaxation of the triangulation criteria could be obtained by extending the stability results of [2] to space forms.

Although there has been some attention paid to questions related to the quality of simplices in spaces of constant curvature, see for example [8], the topic is ripe for further investigation.

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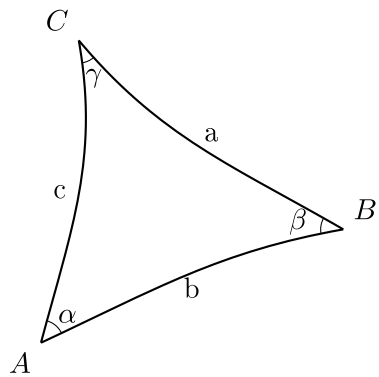
A Approximating cosine rules

In this section we give estimates on how much the cosine rules on spaces of constant curvature differ if the two curvatures are close.

In Lemmas 16 and 18 one assumes that a, b and c are the lengths of the edges of a geodesic triangle in a space of constant curvature $\pm 1/k^2$ and γ the enclosed angle, see the sketch below. This means that a, b, c and γ satisfy the cosine rule for a space of curvature $\pm 1/k^2$, that is (7) or (8). If $\pm 1/l^2$ is close to $\pm 1/k^2$, then a, b, c and γ 'nearly satisfy' the cosine rule for a space of curvature $\pm 1/l^2$. We will quantify 'nearly satisfy' in Lemmas 16 and 18. This result is essential to compare hinges.



In Lemmas 20 and 21 we assume that a, b and c are given up to some error. Once more a, b and c are the lengths of the edges of a geodesic triangle in a space of constant curvature $\pm 1/k^2$. In Lemmas 20 and 21 we derive some bounds on

$$\sin\frac{b}{l}\sin\frac{c}{l}\cos\alpha,$$

and

$$\sinh\frac{b}{l}\sinh\frac{c}{l}\cos\alpha,$$

respectively.

We shall first discuss the elliptic and then the hyperbolic case.

A.1 Cosine rule for spaces of positive curvature

We want to compare the cosine rule in two spaces of positive curvature. To be precise we will prove the following

Lemma 16 *Let*

$$0 < l \leq k \qquad a,b,c \leq \frac{1}{2}l \qquad a,b,c \leq d_{max}.$$

If a, b, c and γ satisfy

$$k^2\cos\frac{c}{k} - k^2\cos\frac{a}{k}\cos\frac{b}{k} = k^2\sin\frac{a}{k}\sin\frac{b}{k}\cos\gamma, \tag{18}$$

a, b, c and γ also satisfy

$$l^2\cos\frac{c}{l} - l^2\cos\frac{a}{l}\cos\frac{b}{l} = l^2\sin\frac{a}{l}\sin\frac{b}{l}\cos\gamma + R_T(a,b,c),$$

with

$$R_T(a,b,c) \leq \left|\frac{1}{l^2} - \frac{1}{k^2}\right|\frac{11 d_{max}^4}{4!}.$$

Here R_T stands for the total remainder, because it is found by studying remainders in the Taylor series of the constituents in (18).

The assumption $k \geq l$ is only used to streamline the calculations, in the sense that it can be replaced by $k > 0$ and $a, b, c \leq \frac{1}{2}k$. We shall prove this statement by examining the individual terms in (18). Our estimates are based on Taylor's theorem with remainder in one and multiple variables.

We start with the first term on the left hand side of the cosine rule, we multiply by k and l respectively to ensure that the quadratic terms cancel

$$\begin{aligned} l^2 \cos\frac{c}{l} - k^2 \cos\frac{c}{k} = l^2 - k^2 - R_{pc}(c) \\ |R_{pc}(c)| &\leq \frac{c^4}{4!} \sup_{c\in[0,l/2]} \left| l^2 \partial_c^4 \cos\frac{c}{l} - k^2 \partial_c^4 \cos\frac{c}{k} \right| \\ &= \frac{c^4}{4!} \sup_{c\in[0,l/2]} \left| \frac{1}{l^2} \cos\frac{c}{l} - \frac{1}{k^2} \cos\frac{c}{k} \right| \\ &\leq \left| \frac{1}{l^2} - \frac{1}{k^2} \right| \frac{c^4}{4!}, \end{aligned}$$

the notation R_{pc} is used to remind us of the fact that it is a remainder, in the sense of Taylor, and we consider perturbations of the cosine, hence pc. The supremum is assumed in $c = 0$ because

$$\frac{1}{l^2} \cos\frac{c}{l} - \frac{1}{k^2} \cos\frac{c}{k}$$

is monotone in c, which can be seen by taking the derivative and noting that

$$\frac{k^3}{l^3} > 1 \qquad\qquad \frac{\sin\frac{c}{k}}{\sin\frac{c}{l}} < 1.$$

For the other terms we need the following estimates

Lemma 17 *Provided $k \geq l > 0$ and $a/l, b/l < 1/2$, we have that*

$$\frac{1}{l^2} \cos\frac{a}{l} \cos\frac{b}{l} - \frac{1}{k^2} \cos\frac{a}{k} \cos\frac{b}{k} \leq \frac{1}{l^2} - \frac{1}{k^2},$$

and

$$\frac{1}{l^2} \sin\frac{a}{l} \sin\frac{b}{l} - \frac{1}{k^2} \sin\frac{a}{k} \sin\frac{b}{k} \leq \frac{1}{l^2} - \frac{1}{k^2}$$

Proof The first inequality is equivalent to

$$\frac{1}{k^2} - \frac{1}{k^2} \cos\frac{a}{k} \cos\frac{b}{k} \leq \frac{1}{l^2} - \frac{1}{l^2} \cos\frac{a}{l} \cos\frac{b}{l},$$

it therefore suffices to prove that

$$x^2 - x^2 \cos ax \cos bx$$

is monotone increasing in x on the given domain, which can be seen by differentiating and equating to zero and noting that there is no solution given the conditions on a and b:

$$\begin{aligned}
&\partial_x(x^2 - x^2\cos ax\cos bx) \\
&\quad = 2x - 2x\cos ax\cos bx + x^2 a\sin ax\cos bx + x^2 b\cos ax\sin bx = 0 \\
&\quad 2 + xa\sin ax\cos bx + xb\cos ax\sin bx = 2\cos ax\cos bx,
\end{aligned}$$

and considering the Taylor series at $x = 0$. The last equality has no solutions except $x = 0$ under the assumptions because $0 \leq ax, bx \leq 1/2$ implies that

$$xa\sin ax\cos bx + xb\cos ax\sin bx \geq 0 \qquad\qquad \cos ax\cos bx \leq 1$$

with the equality only achieved if $x = 0$ or trivial a, b. Similarly the second inequality can be proven

$$\frac{1}{l^2}\sin\frac{a}{l}\sin\frac{b}{l} - \frac{1}{k^2}\sin\frac{a}{k}\sin\frac{b}{k} \leq \frac{1}{l^2} - \frac{1}{k^2}$$

is equivalent to

$$\frac{1}{k^2} - \frac{1}{k^2}\sin\frac{a}{k}\sin\frac{b}{k} \leq \frac{1}{l^2} - \frac{1}{l^2}\sin\frac{a}{l}\sin\frac{b}{l},$$

which follows from the monotonicity of $x^2(1 - \sin ax\sin bx)$ in x, which is again established by taking the derivative and equating to zero

$$\begin{aligned}
&\partial_x(x^2(1 - \sin ax\sin bx)) \\
&\quad = 2x(1 - \sin ax\sin bx) - x^2(b\sin ax\cos bx + a\cos ax\sin bx) = 0,
\end{aligned}$$

which has no solutions except $x = 0$ because in the given domain

$$2\sin ax\sin bx + bx\sin ax\cos bx + ax\cos ax\sin bx \leq \frac{3}{2},$$

which completes the proof of the second inequality. Note that this is the only place where we really use the assumption $a, b, c \leq \frac{1}{2}l$. $\square$

With this intermediate result we can return to the proof of Lemma 16, where we apply the result of Lemma 17 almost immediately.

Using Taylor's theorem for multiple variables we have

$$k^2\cos\frac{a}{k}\cos\frac{b}{k} - l^2\cos\frac{a}{l}\cos\frac{b}{l} = k^2 - l^2 + R_{pcc}(a, b),$$

with

$$|R_{pcc}(a, b)| \leq \left|\frac{1}{l^2} - \frac{1}{k^2}\right| \frac{5d_{\max}^4}{4!}.$$

This follows from the fact that

$$\partial_a^i \partial_b^j (k^2 \cos\frac{a}{k}\cos\frac{b}{k} - l^2\cos\frac{a}{l}\cos\frac{b}{l} - (k^2 - l^2)) = 0,$$

for $0 \le i \le 3 - j$, $0 \le j \le 3$ and

$$\begin{aligned} &|\partial_a^i \partial_b^j (k^2 \cos\frac{a}{k}\cos\frac{b}{k} - l^2\cos\frac{a}{l}\cos\frac{b}{l} - (k^2 - l^2))| \\ &= \begin{cases} |\frac{1}{k^2}\cos\frac{a}{k}\cos\frac{b}{k} - \frac{1}{l^2}\cos\frac{a}{l}\cos\frac{b}{l}| & \text{if } i+j=4 \text{ and } i,j \text{ even}, \\ |\frac{1}{l^2}\sin\frac{a}{l}\sin\frac{b}{l} - \frac{1}{k^2}\sin\frac{a}{k}\sin\frac{b}{k}|, & \text{if } i+j=4 \text{ and } i,j \text{ odd}, \end{cases} \\ &\le \left|\frac{1}{l^2} - \frac{1}{k^2}\right|, \end{aligned}$$

where in the last line we used the result of Lemma 17.

Likewise we have that

$$\begin{aligned} l^2\sin\frac{a}{l}\sin\frac{b}{l} - k^2\sin\frac{a}{k}\sin\frac{b}{k} &= R_{pss}(a,b) \\ |R_{pss}(a,b)| &\le \left|\frac{1}{l^2} - \frac{1}{k^2}\right| \frac{5d_{\max}^4}{4!}. \end{aligned}$$

This follows from exactly the same reasoning

$$\partial_a^i \partial_b^j (l^2\sin\frac{a}{l}\sin\frac{b}{l} - k^2\sin\frac{a}{k}\sin\frac{b}{k}) = 0,$$

for $0 \le i \le 3 - j$, $0 \le j \le 3$ and

$$\begin{aligned} &|\partial_a^i \partial_b^j (l^2\sin\frac{a}{l}\sin\frac{b}{l} - k^2\sin\frac{a}{k}\sin\frac{b}{k})| \\ &= \begin{cases} |\frac{1}{l^2}\sin\frac{a}{l}\sin\frac{b}{l} - \frac{1}{k^2}\sin\frac{a}{k}\sin\frac{b}{k}| & \text{if } i+j=4 \text{ and } i,j \text{ even}, \\ |\frac{1}{k^2}\cos\frac{a}{k}\cos\frac{b}{k} - \frac{1}{l^2}\cos\frac{a}{l}\cos\frac{b}{l}|, & \text{if } i+j=4 \text{ and } i,j \text{ odd}, \end{cases} \\ &\le \left|\frac{1}{l^2} - \frac{1}{k^2}\right|, \end{aligned}$$

where again in the last line we used the result of Lemma 17.

We can now combine these estimates and apply them to the cosine rule, which we multiply by k^2 and write down in different order for convenience. Assuming that $k > l > 0$, $a/l, b/l, c/l \le \frac{1}{2}$ and $a, b, c \le d_{\max}$, we see

$$\begin{aligned} k^2\cos\frac{c}{k} - k^2\cos\frac{a}{k}\cos\frac{b}{k} &= k^2\sin\frac{a}{k}\sin\frac{b}{k}\cos\gamma \\ l^2\cos\frac{c}{l} - l^2\cos\frac{a}{l}\cos\frac{b}{l} - R_{pc}(c) + R_{pcc}(a,b) &= l^2\sin\frac{a}{l}\sin\frac{b}{l}\cos\gamma \\ &\quad - R_{pss}(a,b)\cos\gamma \\ l^2\cos\frac{c}{l} - l^2\cos\frac{a}{l}\cos\frac{b}{l} &= l^2\sin\frac{a}{l}\sin\frac{b}{l}\cos\gamma \\ &\quad + R_T(a,b,c), \end{aligned}$$

with

$$R_T(a,b,c) \leq \left|\frac{1}{l^2} - \frac{1}{k^2}\right| \frac{11d_{\max}^4}{4!}.$$

This completes the proof of Lemma 16.

A.2 Cosine rule for spaces of negative curvature

We now want to compare the cosine rule in two spaces of negative curvature, like we have done for spaces of positive curvature in Lemma 16. To be precise we prove the following lemma

Lemma 18 *Assuming that $k \geq l > 0$, and $a,b,c \leq d_{max}$, we have that if a,b,c and γ satisfy*

$$k^2 \cosh\frac{c}{k} - k^2 \cosh\frac{a}{k}\cosh\frac{b}{k} = -k^2 \sinh\frac{a}{k}\sinh\frac{b}{k}\cos\gamma$$

they also satisfy

$$l^2 \cosh\frac{c}{l} - l^2 \cosh\frac{a}{l}\cosh\frac{b}{l} = -l^2 \sinh\frac{a}{l}\sinh\frac{b}{l}\cos\gamma + R_T(a,b,c),$$

with

$$R_T(a,b,c) \leq \left|\frac{1}{l^2}\cosh^2\frac{d_{max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{max}}{k}\right| \frac{11d_{max}^4}{4!}.$$

Again the assumption $k \geq l$ is only used to streamline the calculations, in the sense that it can be replaced by $k > 0$. We shall follow the same procedure as in the elliptic case.

By Taylor's we see

$$l^2 \cosh\frac{c}{l} - k^2 \cosh\frac{c}{k} = l^2 - k^2 - R_{pch}(c),$$

with

$$\begin{aligned}
|R_{pch}(c)| &\leq \frac{c^4}{4!} \sup_{c\in[0,d_{\max}]} \left| l^2 \partial_c^4 \cosh\frac{c}{l} - k^2 \partial_c^4 \cosh\frac{c}{k}\right| \\
&= \frac{c^4}{4!} \sup_{c\in[0,d_{\max}]} \left|\frac{1}{l^2}\cosh\frac{c}{l} - \frac{1}{k^2}\cosh\frac{c}{k}\right| \\
&\leq \left|\frac{1}{l^2}\cosh\frac{d_{\max}}{l} - \frac{1}{k^2}\cosh\frac{d_{\max}}{k}\right| \frac{c^4}{4!}.
\end{aligned}$$

The supremum is assumed in $c = d_{\max}$ because

$$\frac{1}{l^2}\cosh\frac{c}{l} - \frac{1}{k^2}\cosh\frac{c}{k}$$

is monotone in c, which can be seen by taking the derivative

$$\frac{1}{l^3}\cosh\frac{c}{l} - \frac{1}{k^3}\cosh\frac{c}{k}.$$

and observing that $\frac{1}{l} \geq \frac{1}{k}$ and $\cosh\frac{c}{l} \geq \cosh\frac{c}{k}$, because the hyperbolic cosine seen as a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ is monotone.

For the remaining terms we again need a sub-lemma:

Lemma 19 *Provided $k \geq l \geq 0$ and $a, b, c \leq d_m$, we have that*

$$\frac{1}{l^2}\cosh\frac{a}{l}\cosh\frac{b}{l} - \frac{1}{k^2}\cosh\frac{a}{k}\cosh\frac{b}{k} \leq \frac{1}{l^2}\cosh^2\frac{d_{max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{max}}{k},$$

and

$$\begin{aligned}\frac{1}{l^2}\sinh\frac{a}{l}\sinh\frac{b}{l} - \frac{1}{k^2}\sinh\frac{a}{k}\sinh\frac{b}{k} &\leq \frac{1}{l^2}\sinh^2\frac{d_{max}}{l} - \frac{1}{k^2}\sinh^2\frac{d_{max}}{k}\\ &\leq \frac{1}{l^2}\cosh^2\frac{d_{max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{max}}{k}\end{aligned}$$

Proof The first two inequalities follow from the fact that both functions are monotone if one leaves one of the variables fixed. Monotonicity is proven by noting that the derivative is of the form

$$\frac{1}{l^3}\cosh\frac{y}{l}\sinh\frac{z}{l} - \frac{1}{k^3}\cosh\frac{y}{k}\sinh\frac{z}{k} = 0,$$

which has no non-trivial solutions because cosh and sinh are monotone. The final inequality follows from

$$\frac{1}{k^2} = \frac{1}{k^2}\left(\cosh^2\frac{d_{\max}}{k} - \sinh^2\frac{d_{\max}}{k}\right) \leq \frac{1}{l^2}\left(\cosh^2\frac{d_{\max}}{l} - \sinh^2\frac{d_{\max}}{l}\right) = \frac{1}{l^2}.$$

$\square$

Using Taylors theorem for multiple variables we have

$$k^2\cosh\frac{a}{k}\cosh\frac{b}{k} - l^2\cosh\frac{a}{l}\cosh\frac{b}{l} = k^2 - l^2 + R_{pchch}(a,b),$$

with

$$|R_{pchch}(a,b)| \leq \left|\frac{1}{l^2}\cosh^2\frac{d_{\max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{\max}}{k}\right|\frac{5d_{\max}^4}{4!}.$$

This follows from the fact that

$$\partial_a^i\partial_b^j\left(k^2\cosh\frac{a}{k}\cosh\frac{b}{k} - l^2\cosh\frac{a}{l}\cosh\frac{b}{l} - (k^2-l^2)\right)\Big|_{a=b=0} = 0,$$

for $0 \leq i \leq 3-j$, $0 \leq j \leq 3$ and

$$\begin{aligned}&|\partial_a^i\partial_b^j(k^2\cosh\frac{a}{k}\cosh\frac{b}{k} - l^2\cosh\frac{a}{l}\cosh\frac{b}{l})|\\ &= \begin{cases}|\frac{1}{k^2}\cosh\frac{a}{k}\cosh\frac{b}{k} - \frac{1}{l^2}\cosh\frac{a}{l}\cosh\frac{b}{l}| & \text{if } i+j=4 \text{ and } i,j \text{ even},\\ |\frac{1}{l^2}\sinh\frac{a}{l}\sinh\frac{b}{l} - \frac{1}{k^2}\sinh\frac{a}{k}\sinh\frac{b}{k}|, & \text{if } i+j=4 \text{ and } i,j \text{ odd},\end{cases}\\ &\leq \left|\frac{1}{l^2}\cosh^2\frac{d_{\max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{\max}}{k}\right|,\end{aligned}$$

where in the last line we used the result of Lemma 19.

Likewise we have that

$$l^2 \sinh\frac{a}{l}\sinh\frac{b}{l} - k^2\sinh\frac{a}{k}\sinh\frac{b}{k} = R_{pshsh}(x)$$
$$|R_{pshsh}(x)| \leq \left|\left(\frac{1}{l^2}\cosh^2\frac{d_{\max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{\max}}{k}\right)\frac{5d_{\max}^4}{4!}\right|.$$

This follows from exactly the same reasoning

$$\partial_a^i\partial_b^j(l^2\sinh\frac{a}{l}\sinh\frac{b}{l} - k^2\sinh\frac{a}{k}\sinh\frac{b}{k}) = 0,$$

for $0 \leq i \leq 3-j$, $0 \leq j \leq 3$ and

$$\begin{aligned} &|\partial_a^i\partial_b^j(l^2\sinh\frac{a}{l}\sinh\frac{b}{l} - k^2\sinh\frac{a}{k}\sinh\frac{b}{k})| \\ &= \begin{cases} |\frac{1}{l^2}\sinh\frac{a}{l}\sinh\frac{b}{l} - \frac{1}{k^2}\sinh\frac{a}{k}\sinh\frac{b}{k}| & \text{if } i+j=4 \text{ and } i,j \text{ even}, \\ |\frac{1}{k^2}\cosh\frac{a}{k}\cosh\frac{b}{k} - \frac{1}{l^2}\cosh\frac{a}{l}\cosh\frac{b}{l}|, & \text{if } i+j=4 \text{ and } i,j \text{ odd}, \end{cases} \\ &\leq \left|\frac{1}{l^2}\cosh^2\frac{d_{\max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{\max}}{k}\right|, \end{aligned}$$

where again in the last line we used the result of Lemma 19.

We can now combine these estimates and apply them to the hyperbolic cosine rule, which we multiply by k^2 and write down in different order for convenience. Assuming that $k > l > 0$ and $a, b, c \leq d_{\max}$, we see

$$\begin{aligned} k^2\cosh\frac{c}{k} - k^2\cosh\frac{a}{k}\cosh\frac{b}{k} &= -k^2\sinh\frac{a}{k}\sinh\frac{b}{k}\cos\gamma \\ l^2\cosh\frac{c}{l} - l^2\cosh\frac{a}{l}\cosh\frac{b}{l} - R_{pch}(c) + R_{pchch}(a,b) &= -l^2\sinh\frac{a}{l}\sinh\frac{b}{l}\cos\gamma \\ &\quad + R_{pshsh}(a,b)\cos\gamma \\ l^2\cosh\frac{c}{l} - l^2\cosh\frac{a}{l}\cosh\frac{b}{l} &= -l^2\sin\frac{a}{l}\sin\frac{b}{l}\cos\gamma \\ &\quad + R_{TH}(a,b,c), \end{aligned}$$

with

$$R_{TH}(a,b,c) \leq \left|\frac{1}{l^2}\cosh^2\frac{d_{\max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{\max}}{k}\right|\frac{11d_{\max}^4}{4!}.$$

This completes the proof of Lemma 18 and therefore our discussion of the cosine rule.

A.3 The cosine with 'errors' in lengths for spaces of positive constant curvature

We can now give the estimates on the 'inner products' of the form $\sin b \sin c \cos\alpha$.

In the following lemma we start with a geodesic triangle on $\mathbb{H}(1/k^2)$ of which the lengths of the edges are approximately known. By approximately known we mean that the lengths of the edges a, b and c are close to the lengths a_l, b_l and c_l, the lengths of the edges

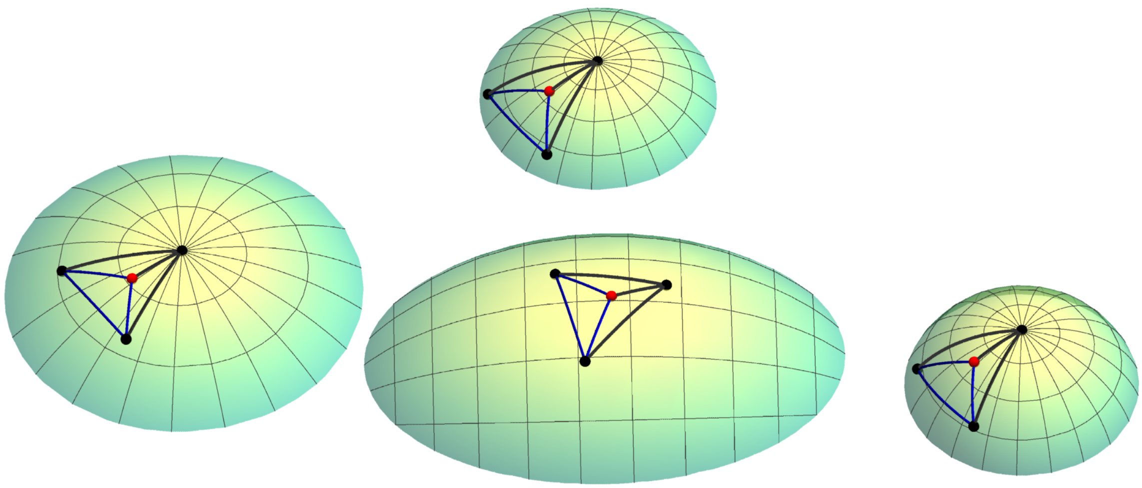


Figure 6: This figure illustrates Lemma 20.
Above we see part of a sphere of mediocre size. Below we see from left to right parts of a large sphere, an ellipsoid and a small sphere. The small sphere is the example of $\mathbb{H}(\Lambda_u)$, the large sphere of $\mathbb{H}(\Lambda_\ell)$ and the sphere of mediocre size of $\mathbb{H}(\Lambda_{\mathrm{mid}})$. The ellipsoid is the manifold M.
The vertices on M are depicted in black, as are the vertices on the spaces of constant curvature left, right and below. The vertices on M are transplanted on spaces of constant curvature by the maps $\exp_{\mathbb{H}(\Lambda_u)} \circ \exp^{-1}_{v_r,M}$, $\exp_{\mathbb{H}(\Lambda_\ell)} \circ \exp^{-1}_{v_r,M}$ and $\exp_{\mathbb{H}(\Lambda_{\mathrm{mid}})} \circ \exp^{-1}_{v_r,M}$, respectively. The same holds for the arbitrary point x (red).
v_r is the point from which only black geodesics emanate. Our criteria for non-degeneracy will be in terms of the simplex with vertices $\exp_{\mathbb{H}(\Lambda_{\mathrm{mid}})} \circ \exp^{-1}_{v_r,M}(v_i)$. This means that we think of the black edges as 'known'. The blue edges are only 'approximately known'. Lemma 20 gives us bounds on the 'difference' between the 'inner products' (of the form $\sin \frac{a}{l} \sin \frac{b}{l} \cos \gamma$) of edges on $\mathbb{H}(\Lambda_u)$ or $\mathbb{H}(\Lambda_\ell)$ and $\mathbb{H}(\Lambda_{\mathrm{mid}})$.
The worst case scenario are the 'inner products' for geodesic triangles of which all edges are blue, that is all edge lengths are 'approximately known'. Lemma 20 focusses on this.

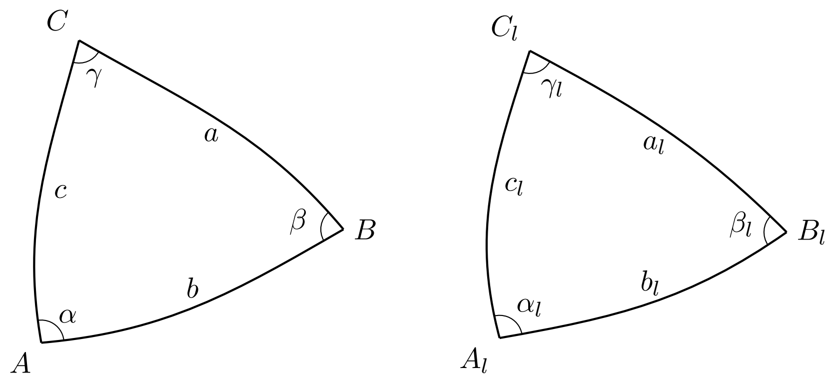


Figure 7: Two geodesic triangles on spaces of different constant curvature with the angles and edge lengths as used in Lemma 20 indicated.

of a geodesic triangle on $\mathbb{H}(1/l^2)$. The edges with length a_l, b_l and c_l will themselves be given as hinges, for example a_l satisfies

$$l^2 \cos\frac{a_l}{l} = l^2 \cos\frac{a_1}{l}\cos\frac{a_2}{l} + l^2 \sin\frac{a_1}{l}\sin\frac{a_2}{l}\cos\gamma_a.$$

The role of this lemma in Section 4 is to give us approximate values of the 'inner products' (of the form $\sin\frac{a}{l}\sin\frac{b}{l}\cos\gamma$), see Figure 6 for an overview. These 'inner products' are the entries in the pseudo Gram matrix.

Lemma 20 *Let $\mathbb{H}(+1/k^2)$ and $\mathbb{H}(+1/l^2)$ be spaces of positive constant sectional curvature, where for convenience we assume that $k > l > 0$. Moreover let the edge-lengths (a, b, c) of a geodesic triangle on $\mathbb{H}(+1/k^2)$ satisfy*

$$\begin{aligned} l^2 \cos\frac{a}{l} &= l^2 \cos\frac{a_l}{l} + R_{T_a}, \\ l^2 \cos\frac{b}{l} &= l^2 \cos\frac{b_l}{l} + R_{T_b} \\ l^2 \cos\frac{c}{l} &= l^2 \cos\frac{b_l}{l} + R_{T_c}, \end{aligned}$$

with

$$\begin{aligned} l^2 \cos\frac{a_l}{l} &= l^2 \cos\frac{a_1}{l}\cos\frac{a_2}{l} + l^2 \sin\frac{a_1}{l}\sin\frac{a_2}{l}\cos\gamma_a \\ l^2 \cos\frac{b_l}{l} &= l^2 \cos\frac{b_1}{l}\cos\frac{b_2}{l} + l^2 \sin\frac{b_1}{l}\sin\frac{b_2}{l}\cos\gamma_b \\ l^2 \cos\frac{c_l}{l} &= l^2 \cos\frac{c_1}{l}\cos\frac{c_2}{l} + l^2 \sin\frac{c_1}{l}\sin\frac{c_2}{l}\cos\gamma_c, \end{aligned}$$

and

$$|R_{T_a}|, |R_{T_b}|, |R_{T_c}| \leq \left|\frac{1}{l^2} - \frac{1}{k^2}\right| \frac{11 d_{max}^4}{4!},$$

and

$$a, a_1, a_2, b, b_1, b_2, c, c_1, c_2 \leq \frac{1}{2}l \qquad\qquad a, a_1, a_2, b, b_1, b_2, c, c_1, c_2 < d_{max},$$

then

$$|l^2 \sin\frac{b}{l}\sin\frac{c}{l}\cos\alpha - l^2 \sin\frac{b_l}{l}\sin\frac{c_l}{l}\cos\alpha_l| \leq 2\left|\frac{1}{l^2} - \frac{1}{k^2}\right| d_{max}^4, \tag{19}$$

with

$$l^2 \sin\frac{b_l}{l}\sin\frac{c_l}{l}\cos\alpha_l = l^2\cos\frac{a_l}{l} - l^2\cos\frac{b_l}{l}\cos\frac{c_l}{l}.$$

Here the notation for the lengths and angles of a geodesic triangle is as in Figure 7.

Clearly we can formulate Lemma 20 for each of the angles α, β and γ, as in Figure 7. We have chosen α over γ and β. The reason for this is that γ is used in Lemma 16 (elliptic) and Lemma 18 (hyperbolic) as a given quantity, while in Lemma 20 the angle is (approximately) determined based on (approximate) lengths of edges.

Proof Because a, b, c are the edge lengths of a geodesic triangle on $\mathbb{H}(+1/k^2)$ we have, by Lemma 16, that

$$l^2\cos\frac{a}{l} - l^2\cos\frac{b}{l}\cos\frac{c}{l} = l^2\sin\frac{b}{l}\sin\frac{c}{l}\cos\alpha + R_T(a, b, c),$$

with

$$R_T(a, b, c) \leq \left|\frac{1}{l^2} - \frac{1}{k^2}\right| \frac{11 d_{\max}^4}{4!}.$$

Filling in our assumptions we see that

$$\begin{aligned}
l^2\cos\frac{a_l}{l} + R_{T_a} - (l^2\cos\frac{b_l}{l} + R_{T_b})(\cos\frac{c_l}{l} + \frac{R_{T_c}}{l^2}) - R_T& \\
= l^2\sin\frac{b}{l}\sin\frac{c}{l}\cos\alpha& \\
l^2\cos\frac{a_l}{l} - l^2\cos\frac{b_l}{l}\cos\frac{c_l}{l} + R_{T_b}\cos\frac{c_l}{l} + R_{T_c}\cos\frac{b_l}{l} + R_{T_b}\frac{R_{T_c}}{l^2} + R_{T_a} - R_T& \\
= l^2\sin\frac{b}{l}\sin\frac{c}{l}\cos\alpha& \\
l^2\sin\frac{b_l}{l}\sin\frac{c_l}{l}\cos\alpha_l + R_{T_b}\cos\frac{c_l}{l} + R_{T_c}\cos\frac{b_l}{l} + R_{T_b}\frac{R_{T_c}}{l^2} + R_{T_a} - R_T& \\
= l^2\sin\frac{b}{l}\sin\frac{c}{l}\cos\alpha.&
\end{aligned}$$

Because

$$\begin{aligned}
\left|\frac{R_{T_c}}{l^2}\right| &\leq \frac{11}{4!}\left|\frac{1}{l^4} - \frac{1}{k^2 l^2}\right| d_{\max}^4 \\
&\leq \frac{11}{4!}\left|\frac{1}{l^4} - \frac{1}{k^2 l^2}\right| \left(\frac{l}{2}\right)^4 \\
&= \frac{11}{2^4 4!},
\end{aligned}$$

we have that

$$\begin{aligned}
|R_{T_b}\cos\frac{c_l}{l} + R_{T_c}\cos\frac{b_l}{l} + & R_{T_b}\frac{R_{T_c}}{l^2} + R_{T_a} - R_T| \\
& \leq |R_{T_b}| + |R_{T_c}| + \frac{11}{2^4 4!}|R_{T_b}| + |R_{T_a}| + |R_T| \\
& \leq (4 + \frac{11}{2^4 4!})\frac{11}{4!}\left|\frac{1}{l^2} - \frac{1}{k^2}\right| d_{\max}^4 \\
& \leq 2\left|\frac{1}{l^2} - \frac{1}{k^2}\right| d_{\max}^4 .
\end{aligned}$$

□

A.4 The cosine with 'errors' in lengths for spaces of negative constant curvature

Similarly, for hyperbolic spaces we have

Lemma 21 *Let $\mathbb{H}(-1/k^2)$ and $\mathbb{H}(-1/l^2)$ be spaces of negative constant sectional curvature, where for convenience we assume that $k > l > 0$. Moreover let the edge-lengths $(a,\ b,\ c)$ of a geodesic triangle on $\mathbb{H}(-1/k^2)$ satisfy*

$$\begin{aligned}
l^2\cosh\frac{a}{l} &= l^2\cosh\frac{a_l}{l} + R_{T_a}, \\
l^2\cosh\frac{b}{l} &= l^2\cosh\frac{b_l}{l} + R_{T_b} \\
l^2\cosh\frac{c}{l} &= l^2\cosh\frac{c_l}{l} + R_{T_c},
\end{aligned}$$

with

$$\begin{aligned}
l^2\cosh\frac{a_l}{l} &= l^2\cosh\frac{a_1}{l}\cosh\frac{a_2}{l} - l^2\sinh\frac{a_1}{l}\sinh\frac{a_2}{l}\cos\gamma_a \\
l^2\cosh\frac{b_l}{l} &= l^2\cosh\frac{b_1}{l}\cosh\frac{b_2}{l} - l^2\sinh\frac{b_1}{l}\sinh\frac{b_2}{l}\cos\gamma_b \\
l^2\cosh\frac{c_l}{l} &= l^2\cosh\frac{c_1}{l}\cosh\frac{c_2}{l} - l^2\sinh\frac{c_1}{l}\sinh\frac{c_2}{l}\cos\gamma_c,
\end{aligned}$$

and

$$|R_{T_a}|, |R_{T_b}|, |R_{T_c}| \leq \left|\frac{1}{l^2}\cosh^2\frac{d_{max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{max}}{k}\right| \frac{11 d_{max}^4}{4!},$$

and

$$a, a_1, a_2, b, b_1, b_2, c, c_1, c_2 < d_{max},$$

then

$$\begin{aligned}
|l^2\sinh\frac{b}{l}\sinh\frac{c}{l}\cos\alpha - & l^2\sinh\frac{b_l}{l}\sinh\frac{c_l}{l}\cos\alpha_l| \\
& \leq \left(2 + 2\cosh\frac{d_{max}}{l} + \frac{1}{l^4}\cosh^2\left(\frac{d_{max}}{l}\right)\frac{11 d_{max}^4}{4!}\right) \\
& \quad \cdot \left|\frac{1}{l^2}\cosh^2\frac{d_{max}}{l} - \frac{1}{k^2}\cosh^2\frac{d_{max}}{k}\right| \frac{11 d_{max}^4}{4!},
\end{aligned}$$

with

$$l^2 \sinh \frac{b_l}{l} \sinh \frac{c_l}{l} \cos \alpha_l = l^2 \cosh \frac{b_l}{l} \cosh \frac{c_l}{l} - l^2 \cosh \frac{a_l}{l}.$$

Note that we no longer impose bound on the lengths with respect to l or k as we did in the elliptic case.

Proof Because a, b, c are the edge lengths of a geodesic triangle on $\mathbb{H}(-1/k^2)$ Lemma 18 yields

$$l^2 \cosh \frac{c}{l} - l^2 \cosh \frac{a}{l} \cosh \frac{b}{l} = -l^2 \sinh \frac{a}{l} \sinh \frac{b}{l} \cos \gamma + R_T(a, b, c),$$

with

$$R_T(a, b, c) \leq \left| \frac{1}{l^2} \cosh^2 \frac{d_{\max}}{l} - \frac{1}{k^2} \cosh^2 \frac{d_{\max}}{k} \right| \frac{11 d_{\max}^4}{4!}.$$

Filling in our assumptions we see that

$$\begin{aligned} l^2 \cosh \frac{a_l}{l} + R_{T_a} - (l^2 \cosh \frac{b_l}{l} + R_{T_b})(\cosh \frac{c_l}{l} + \frac{R_{T_c}}{l^2}) - R_T \\ = -l^2 \sinh \frac{b}{l} \sinh \frac{c}{l} \cos \alpha \\ l^2 \cosh \frac{a_l}{l} - l^2 \cosh \frac{b_l}{l} \cosh \frac{c_l}{l} + R_{T_b} \cosh \frac{c_l}{l} + R_{T_c} \cosh \frac{b_l}{l} + R_{T_b} \frac{R_{T_c}}{l^2} + R_{T_a} - R_T \\ = -l^2 \sinh \frac{b}{l} \sinh \frac{c}{l} \cos \alpha \\ l^2 \sinh \frac{b_l}{l} \sinh \frac{c_l}{l} \cos \alpha_l - R_{T_b} \cosh \frac{c_l}{l} - R_{T_c} \cosh \frac{b_l}{l} - R_{T_b} \frac{R_{T_c}}{l^2} - R_{T_a} + R_T \\ = l^2 \sinh \frac{b}{l} \sinh \frac{c}{l} \cos \alpha. \end{aligned}$$

We now have that

$$\begin{aligned} &| - R_{T_b} \cosh \frac{c_l}{l} - R_{T_c} \cosh \frac{b_l}{l} - R_{T_b} \frac{R_{T_c}}{l^2} - R_{T_a} + R_T| \\ &\leq |R_{T_b}| \cosh \frac{d_{\max}}{l} + |R_{T_c}| \cosh \frac{d_{\max}}{l} + |R_{T_b}| \left| \frac{R_{T_c}}{l^2} \right| + |R_{T_a}| + |R_T| \\ &\leq \left(2 + 2 \cosh \frac{d_{\max}}{l} + \frac{1}{l^4} \cosh^2 \left(\frac{d_{\max}}{l} \right) \frac{11 d_{\max}^4}{4!} \right) \\ &\quad \cdot \left| \frac{1}{l^2} \cosh^2 \frac{d_{\max}}{l} - \frac{1}{k^2} \cosh^2 \frac{d_{\max}}{k} \right| \frac{11 d_{\max}^4}{4!}. \end{aligned}$$

□