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Periodicity and Invertibility of Lattice Gas Cellular Automata

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Abstract

A cellular automaton is a type of mathematical system that models the behavior of a set of cells with discrete values in progressing time steps. The often complicated behaviors of cellular automata are studied in computer science, mathematics, biology, and other science related fields. Lattice gas cellular automata are used to simulate the movements of particles. This thesis aims to discuss the properties of lattice gas models, including periodicity and invertibility, and to examine their accuracy in reflecting the physics of particles in real life. Analysis of elementary cellular automata is presented to introduce the concept of cellular automata and construct foundations for the analysis of properties of lattice gas.

1 Introduction

A cellular automaton is a type of discrete mathematical model that describes the states in a set of cells in progressing time steps by setting a set of simple rules that defines the ways each cell interacts with its neighbors. An automaton is a self-operating machine[1]. Cellular automata were originally introduced by Stanislaw Ulam and John von Neumann in 1940[4]. However, the concept was popularized by *Game of Life*, a two-dimensional cellular automaton introduced by the British mathematician John Horton Conway in 1970[2]. *Game of Life* consists of a lattice of cells with binary values that represent "alive" or "dead" states of each cell. A set of rules constrains the survival of each cell according to the states of its neighbors. Once the initial condition of the population is set, the game will start playing and simulating the evolution of the population by itself. *Game of Life* introduced cellular automata to the world by showing its capability of modeling complex systems with extreme simplicity and runtime efficiency. Since then, cellular automata have been widely studied and used in a variety of fields and concepts, including modeling physical and biological systems, generating random numbers, theory of computation, and artificial intelligence.

A lattice gas cellular automaton is one type of cellular automaton that is used to model and simulate the movements of particles in real life. The advantages of using a cellular automaton as the basic model for a lattice gas is the ease of setting local rules, which can be made invertible and follow a conservation law, to model physical constrains of particles in real life. This paper aims to give an overview of the basic concepts of lattice gas, by discussing a few properties related to this model. In order to have a better understanding of the lattice gas model, a large portion of this paper is going to be focused on the discussion of the concepts and properties of cellular automata in general and specifically the elementary cellular automaton, which is a one-dimensional cellular automaton model with simple and basic rules.

The elementary cellular automaton was introduced by Stephen Wolfram in 1983[7]. It is one of the most basic models of cellular automata. However, despite its simplicity in implementation, an elementary cellular automaton can create a variety of patterns, some of which are extremely complex or show interesting and mathematically valuable patterns. For example, the rule 30 elementary cellular automaton has been used as random number generator due to its aperiodic, chaotic and pseudo-random behavior[5]. This paper's focus is on the periodic properties and inverses of cellular automata. Some global periodic patterns of cellular automata can be derived and predicted with the analysis of local rules and initial conditions. Invertible cellular automata set an important foundation for constructing lattice gas models that reflect particles' natural behaviors. Expanding and developing upon the discovery of properties of elementary cellular automata to more general cases, the properties of lattice gases are studied, constructed, and then examined in terms of how much the model represents the behavior of particles in real life.

1.1 Cellular Automaton

Every cellular automaton starts with an initial condition, which consists of a lattice of cells whose values would evolve among finite number of states. The state of each cell in the next time step depends on the values and relationships between the current cell and its neighbors. Such relationship is constrained by a universal local rule that is given to the cellular automaton.

A cellular automaton could be in any number of dimensions. An elementary cellular automaton is one type of cellular automaton that contains a one-dimensional set of cells that grows in time. Some two-dimensional cellular automata, such as *Game of Life*, are also commonly studied. The lattices of cells can be in different shapes. For example, the lattice of *Game of Life* consists of a rectangular grid of cells. A two-dimensional cellular automaton can also have a lattice consisting of hexagon shaped cells.

1.2 Elementary Cellular Automaton

An elementary cellular automaton contains a one-dimensional set of cells that evolve in time. Each cell has one of two possible values 1 and 0. The value of each cell in each successive time step depends on the value of the cell itself and its direct left and right neighbors during the current time step (Figure 1). Such relationships of the cell and its neighbors are defined by a local rule.

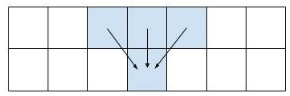

Figure 1: Neighbors of a cell in Elementary CA

Notation[3] Consider any one-dimensional cellular automaton with a set of cells x_i, each of which assumes any of the values $V = \{0, ..., k-1\}$. Then the general form of a rule defining a particular automaton is given by $x_i^{t+1} = f(x_{i-r}^t, ..., x_i^t, ...x_{i+r}^t)$, $f : V^{2r+1} \to V$.

The rules for elementary cellular automata can be written as: $x_i^{t+1} = f(x_{i-1}^t, x_i^t, x_{i+1}^t)$.

Numbering System for Rules A local rule for an elementary cellular automaton can be shown in a table like Table 1, which lists all the possible configurations of a cell and its two neighbors and their corresponding results of the cell in the successive time step. There are $2^3 = 8$ configurations of a cell and its neighbors.

111	110	101	100	011	010	001	000
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7

, $a_i = \{0, 1\}, i = 0..7$

Table 1: Local rule defining the relationship of a cell and its neighbors

The rule number is the decimal conversion of an eight-digit binary number that consists of digits a_0 to a_7 from left to right. For example, the rule shown in Table 2 is numbered as rule 30, which is equivalent to the binary number 00011110 on the bottom row of the table. There are two possible values for each a_i. Therefore, there are $2^8 = 256$ rules for total in elementary cellular automata.

111	110	101	100	011	010	001	000
0	0	0	1	1	1	1	0

Table 2: Rule 30

2 Deterministic Structures

As defined previously, the general form of rules for an elementary cellular automaton can be written as

$$x_i^{t+1} = f(x_{i-1}^t, x_i^t, x_{i+1}^t)$$

Since there are only two possible values for each cell, and there are only eight configurations of neighbors, some of the configurations in a specific rule have to be assigned the same value. In some cases, the rules can be defined in a "deterministic structure". For example, a specific rule can assign the same value to x_i^{t+1} from configurations $0x_i^t x_{i+1}^t$ and $1x_i^t x_{i+1}^t$. In this case, the value of the cell is only determined by the value of itself and its right-hand-side neighbor (case (d) in Figure 2).

There are eight cases of deterministic structures for an elementary cellular automaton.[3]

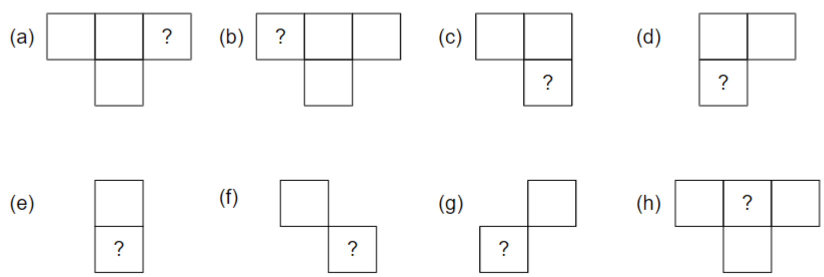

Figure 2: Deterministic Structures in Elementary Cellular Automata

3 Periodicity of Sequences in Elementary Cellular Automata

Stephen Wolfram is a pioneer in the study of elementary cellular automata. Since 1983, he has published numerous articles related to this topic. In his research, he suggests that all automata belong to at least one of four classes:[7]

"Class 1: evolution leads to a homogeneous state in which, for example, all sites have values 0;
Class 2: evolution leads to a set of stable or periodic structure that are separated and simple;
Class 3: evolution leads to a chaotic pattern;
Class 4: evolution leads to complex structures, often long-lived."

Among these four classes, this thesis will mainly be focusing on the study of cellular automata in class 2; more specifically, the ones that show periodic structures. To introduce the basic concept and properties of periodicity in cellular automata, this section will show and prove some theorems related to periodicity in elementary cellular automata.

Definition An initial condition $\{x_i^{t_0},\ -\infty < i < \infty\}$ such that for some $-\infty < M \leq N < \infty$, $x_i^{t_0} = 0$ for $i < M$, $i > N$, and $x_M^{t_0} = x_N^{t_0} = 1$, will be called an *arbitrary finite initial condition.*

Definition The sequence $\{x_i^t, t_0 \leq t < \infty\}$ is *periodic* if there exist T_i, p_i such that $x_i^{t+p_i} = x_i^t$ for all $t \geq T$. p_i is called the *period* of the sequence.

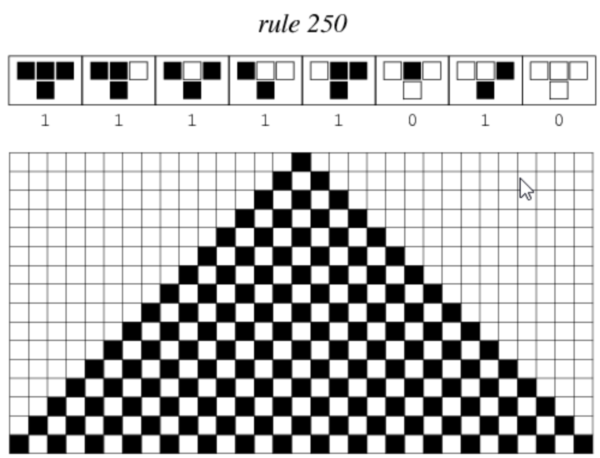

Figure 3: Weisstein, Eric W. "Rule 250." From Wolfram MathWorld.

Example Rule 250 with arbitrary finite initial condition "...000010000...", as shown in Figure 3, generates periodic behavior on every sequence. For example, the sequence on the center of the triangle pattern, $\{x_i^t, t \geq t_0\}$, is periodic with $p_i = 2, T = t_0$. The sequence $\{x_{i+2}^t, t \geq t_0\}$ is periodic with $p_{i+2} = 2, T = t_1$.

Theorem 3.1 (Theorem 1 of [3]) Let $\{x_i^t\}$, $\{x_j^t\}$ be two periodic sequences with $i < j$. Then for $i < k < j$, the sequence $\{x_k^t\}$ must be periodic.

Proof Let $\{x_k^t,\ i < k < j\}$ be a spatial sequence at any time t. Each cell can contain one of two values $\{0, 1\}$. Therefore, there are totally 2^{j-i-1} possibilities of the string of values. According to the pigeonhole principle, the string must repeat itself after at most 2^{j-i-1} time steps. Let the periodic sequences $\{x_i^t\}$ and $\{x_j^t\}$ have periods p_i and p_j correspondingly. The two sequences together have period less than or equal to $\mathrm{lcm}(p_i, p_j)$. Therefore, the entire "block" of temporal sequences $\{x_k^t,\ i \le k \le j\}$ must be periodic with period $P \le \mathrm{lcm}(p_i, p_j) \cdot 2^{j-i-1}$; i.e., after a certain amount of time, this block of sequences will start to repeat its pattern. Therefore, the sequence $\{x_k^t\}$ must be periodic with period less than or equal to P.

Lemma 3.1 (Symmetry of Lemma 1 of [3]) Let R be a rule belonging to class (b) of deterministic structure. Then given any two adjacent periodic sequences $\{x_i^t\}$,$\{x_{i-1}^t\}$, every sequence $\{x_{i-j}^t\}$ must be periodic with $T_{i-j} = \max(T_i, T_{i-1})$ and $P_{i-j} \le \mathrm{lcm}(p_i, p_{i-1})$ for all $j \ge 2$.

Theorem 3.2 (Symmetry of Theorem 2a of [3]) Let R be a rule belonging to class (b) of deterministic structure with $001 \to a_1 = 1$ and $000 \to a_0 = 0$. Then, with arbitrary finite initial conditions, there can exist at most one periodic sequence.

Proof Suppose there exist two periodic sequences x_i^t, x_j^t with $i < j$. By Theorem 3.1, x_{j-1}^t will also be periodic. Then according to Lemma 3.1, all sequences x_{j-l}^t, $l > 0$ will also be periodic with the same $T = \max(T_{j-1}, T_j)$ and $p \le \mathrm{lcm}(p_{j-1}, p_j)$.

Since $a_0 = 0$, $a_1 = 1$, and we have an arbitrary finite initial condition, the first 1 to the left would keep propagating to the left one position at a time in each successive row. Therefore, there will be some J such that $x_{j-J}^t = 0$, for all $T \le t \le T + p$, but $x_{j-J}^{T+q} = 1$ for some $q \ge p$, and thus the sequence cannot be periodic. This gives a contradiction. Therefore, there cannot exist more than one periodic sequences in an automaton generated with rule R, which belongs to class (b).

Definition A sequence $\{x_{i+t}^t, t \ge t_0\}$ will be called a *right diagonal* of $x_i^{t_0}$. A sequence $\{x_{i-t}^t, t \ge t_0\}$ will be called a *left diagonal* of $x_i^{t_0}$.

Definition A right diagonal $\{x_{i+t}^t\}$ of x_i is *periodic* if there exist some J_i, p_i such that $x_{i+t}^t = x_{i+t+p_i}^{t+p_i}$ for $t \ge J_i$. A left diagonal $\{x_{i-t}^t\}$ of x_i is *periodic* if there exist some J_i, p_i such that $x_{i-t}^t = x_{i-(t+p_i)}^{t+p_i}$ for $t \ge J_i$.

Theorem 3.3 (Theorem 4 of [3]) Let R be a rule in any elementary cellular automaton. Then the right and left diagonal sequences of the automaton generated by R are periodic.

Theorem 3 is cited and included here because the concept of diagonal sequences will be used in the analysis of lattice gas models in later section. Diagonal periodicity shows the behavior of cells across the lattice of the automaton, instead of only among individual sequences. Such technique is useful when analyzing the behavior of moving particles in lattice gas models.

4 Inverses of Rules in Elementary Cellular Automata

An important property in cellular automata is invertibility. Every stage in an invertible cellular automaton has an unique predecessor. Therefore, the process of an invertible cellular automaton can be reversed and traced backwards by using the unique inverse of its local rule. Such property is necessary for lattice gas cellular automaton in order to simulate particles in motion that are naturally invertible.

To obtain an invertible cellular automaton, several methods could be used. Partitioning cellular automata, which will be discussed in the next section, ensure the invertibility of the entire cellular automaton by making its local rule invertible. There are some rules (in fact only six rules) for elementary cellular automata that have inverses and can be used to construct invertible elementary cellular automata.

Definition Let $R\{x_i^t\}$ denote the sequence that results from applying rule R to $\{x_i^t, -\infty < i < \infty\}$. Rule R^{-1} is the inverse of R if and only if $R^{-1}\{R\{x_i^t\}\} = \{x_i^t\}$.

Before finding the inverse of a rule, it is necessary to determine whether the rule is invertible or not. Since every stage in an invertible cellular automaton has to have a unique predecessor, it can be shown that elementary cellular automata with deterministic structures (e), (f), and (g), which have one-to-one deterministic structures, are the only elementary cellular automata that have inverses.[3]

Take rule 15 as an example. The configurations of rule 15 are shown in the table below:

111	110	101	100	011	010	001	000
0	0	0	0	1	1	1	1

Table 3: Rule 15

Rule 15 belongs to class (f) in deterministic structure; i.e. $(x_{i-1}^t) \to x_i^{t+1}$. Therefore, it has an inverse. Suppose we have an arbitrary finite initial condition $S_1 = ...1100010111....$ After apply rule 15 to this spatial sequence S_1, the next time step generates $S_2 = ...100111010001....$ To find the inverse of rule 15, make S_2 the initial condition and S_1 the successive spatial sequence to S_2. Find the output of every three neighborhood and fill in the rule table for the inverse of rule 15. Then convert the binary numbers into decimal to get the rule number. As shown in the table below, the inverse of rule 15 is rule 85.

111	110	101	100	011	010	001	000
0	1	0	1	0	1	0	1

Table 4: Rule 85

5 Partitioning Cellular Automata

A partitioning cellular automaton contains a lattice of cells that evolve based on their interactions with each other in groups. Unlike other types of cellular automata, such as elementary cellular automata and Conway's *Game of Life*, where each single cell evolves based on its neighbors, a partitioning cellular automaton divides all its cells into groups with identical shape, and has a set of local rules that allow its cells to evolve based on the values of all the cells within its group. The grouping systematically changes in every time step and thus allows the interactions among cells to span across the entire lattice, instead of being separated as isolated subsystems.

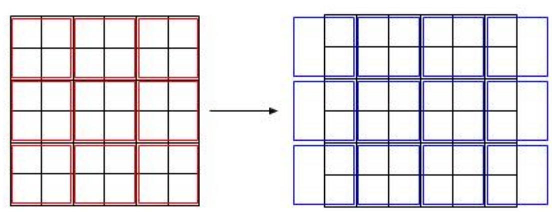

Figure 4: Change of groupings in a partitioning CA

Partitioning cellular automata have three important properties:

1. If all the local rules are invertible, then the entire automaton is invertible;
2. if all the local rules follow a conservation law, which means that the sum of all the values in a group of cells stays the same before and after each time step, then the entire cellular automaton follows a conservation law;
3. the system is an interacting whole rather than a collection of isolated subsystems.[6]

The technique of a partitioning cellular automaton is useful for modeling a system of moving particles, specifically the lattice gas cellular automaton model. The local rules and groupings in a partitioning cellular automaton give the opportunity to easily model the physical constrains among the moving particles.

In such an application, the automaton should follow a conservation law simply due to the equivalent law from physics.

6 Hexagon Lattice Gas

A lattice gas is a type of cellular automaton that is used to model the behavior of moving particles with discrete locations and speed. It uses the same techniques as partitioning cellular automata. A lattice gas consists of a lattice of cells that represents the locations of particles. Each cell has one of two possible values $\{0, 1\}$. 0 means that there is no particle and 1 means that a particle is present in that location. There is an invertible local rule for a lattice gas model to constrain the ways that particles interact with each other. Due to the conservation law of particles, a lattice gas has to follow a conservation law in order to correctly model the behavior of the particles. One of the most famous lattice gas models is the HPP lattice gas, where particles move at unit speed on a two-dimensional orthogonal lattice, in one of the four possible directions.[6]

In this hexagon lattice gas model that is going to be discussed in the rest of the paper, the lattice consists of a parallelogram-shaped lattice of location cells, which are divided into hexagon-shaped groups of seven. There are different ways that the hexagon groups can be laid out, and also different ways that the grouping changes in every time step.

The lattice gas model that is constructed in this paper aims to reflect the behavior of particles in real life as much as possible. However, this model might not be completely accurate in reflecting physics in real life. The local rule and changing of groupings are set up in a way so that the model is invertible and is interesting and valuable for mathematical research. For example, in the set-up of the local rule, particles are made to have the tendency to rotate clockwise for 60 degrees every time they collide. This might not be completely realistic, but the consistency of such behavior ensures the invertibility of this model.

In this particular model, the groups are divided in a pattern shown in Figure 5 as the red hexagons. The green hexagons represents the groupings in the succeeding time step. In other words, the groupings move one cell to the right in each time step.

There are $2^7 = 128$ configurations of cells in a hexagon group and $2^{(2^7)}$ number of possible rules for a hexagon model. Figure 6 shows the local rule defined in this study by visually demonstrating the behaviors of particles in some example cases. For example, if there are two particles located on the diagonal of a hexagon as in case (3), then they are going to collide and bounce off each other while rotating 60 degrees clockwise. If there is one particle in the center of a hexagon, and another somewhere else in the same hexagon, as in case (4), then the particle on the outside of the hexagon is going to move to the center and the one that was in the center will move to the diagonal corner of the original position of the other particle. Figure 6 shows all the cases for when there are 1 to 3 particles in a hexagon. The cases for when there are 4 to 6 particles in a hexagon are the same cases respectively as 1 to 3 particles except

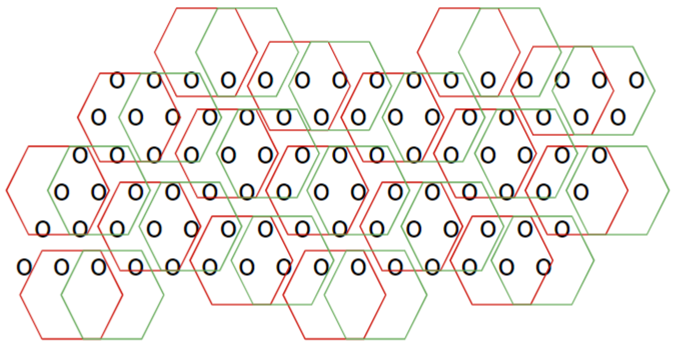

Figure 5: Hexagon lattice gas grouping

with the values 1s and 0s reversed. For example, in case (13), where there are 4 particles, if we reverse the value of 1s and 0s, the behaviors of the cells are the same as those in case (8).

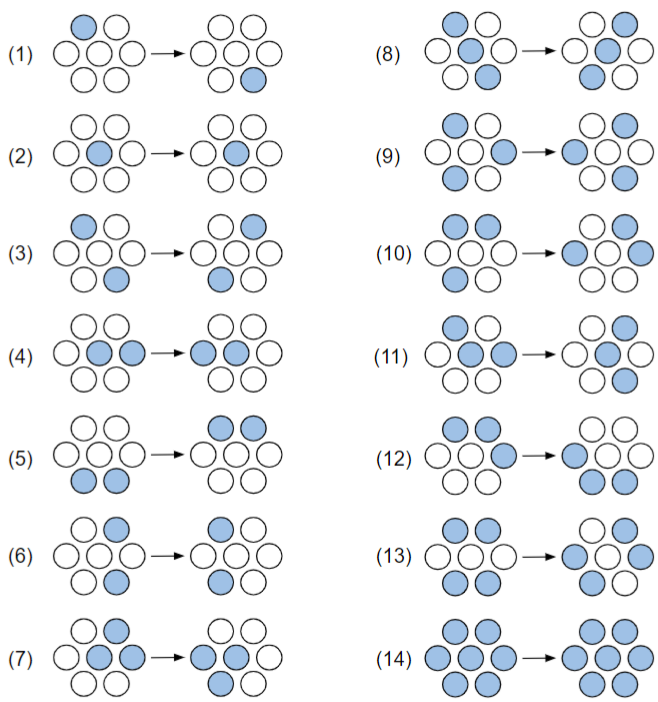

Figure 6: Example cases of the local rule

6.1 Terminologies and Definitions

Consider a hexagon lattice gas cellular automaton with parallelogram shaped lattice and the location of each cell is numbered as (i, j), as shown in Figure 2 below. The value of a cell at a specific location at time t is therefore $x_{i,j}^t \in \{0, 1\}$. This model is going to simulate gas particles in an open system. For the purpose of simplicity, we assume that the lattice is an infinitely expanding surface, and that any cell whose initial value is not assigned 1 will have value 0.

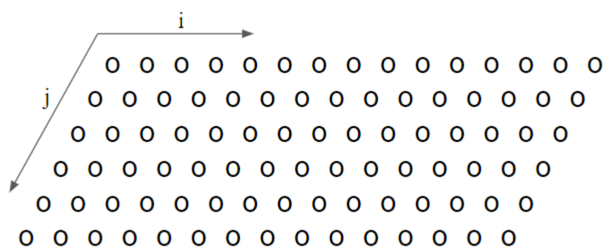

Figure 7: Coordinates on a lattice gas at any time t

Temporal Sequence A sequence $\{x_{i,j}^t, t \geq t_0\}$ will be called the *temporal sequence* at location (i, j).

Spatial Horizontal Rows and Columns A sequence $\{x_{i,j}^t, i \in \mathbb{Z}\}$ will be called the jth *spatial horizontal row* at time t. A sequence $\{x_{i,j}^t, j \in \mathbb{Z}\}$ will be called the ith *spatial horizontal column* at time t.

Spatial Diagonal Rows and Columns A sequence $\{x_{i+t,j}^t, t \geq t_0\}$ will be called the jth *spatial diagonal row*. A sequence $\{x_{i,j+t}^t, t \geq t_0\}$ will be called the ith *spatial diagonal column*.

General Spatial Diagonal Sequence A sequence $\{x_{i+k_1t,j+k_2t}^t, t \geq t_0\}$ is called a *general spatial diagonal sequence*.

Temporal Periodicity A temporal sequence is *periodic* if there exist some positive integers p and T such that $x_{i,j}^t = x_{i,j}^{t+p}$ for $t \geq T$. p is called the *period* of the sequence.

General Spatial Periodicity A lattice gas model has *general spatial periodicity*, if there exist some integer k_1, k_2, and positive integers p and T such that $x_{i+k_1t,j+k_2t}^t = x_{i+k_1(t+p),j+k_2(t+p)}^{t+p}$ for $t \geq T$. p is called the period of the sequence.

6.2 Periodicity Properties of Lattice Gases

Theorem 6.1 (Based on Theorem 3.1) Consider a spatial area A on a lattice that is closed and bounded at t_0 by a set of temporally periodic hexagon groups, $\{G_r\}$, which is not contained by A. In other words, the neighbors, which are the six cells surrounding any cell, of all cells within A belong to cells in $\{G_r\}$ or A. Let the periods of hexagon groups in $\{G_r\}$ be $\{p_r\}$. Suppose A contains k cells. Then the temporal sequences in A are periodic with period less than or equal to $\operatorname{lcm}\{p_r\} \cdot 2^k \cdot 7$.

Proof According to the pigeonhole principle, the cells within A will repeat their values after at most 2^k time steps. The hexagon groups in $\{G_r\}$ together have period less than or equal to $\operatorname{lcm}\{p_r\}$. Therefore, the temporal sequences inside A in addition to the ones in $\{G_r\}$, have period less than or equal to $\operatorname{lcm}\{p_r\} \cdot 2^k$. The groupings of this lattice gas model repeats every seven time steps. For a lattice gas cellular automaton to start to be periodic, the groupings and values of all cells in every group have to repeat. Therefore, temporal sequences in A are periodic with period $\operatorname{lcm}\{p_r\} \cdot 2^k \cdot 7$.

Theorem 6.2 If there is only one particle in a lattice gas model, the model has general spatial periodicity.

Proof This proof is going to be achieved by discussing 7 separate cases, where the particle is initially located in each of the seven possible locations in a hexagon group.

Case 1: if at initial time t_0, the particle is located at the upper-left corner of a hexagon group, i.e. when its coordinates satisfy $i \equiv 3 \cdot j \pmod 7$. The particle moves in a pattern as shown in Figure 8. The groupings are color coded as red, orange, and then green to indicate time t_0, t_1 and t_2 respectively.

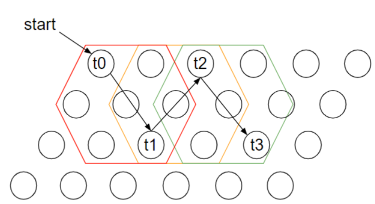

Figure 8: Behavior of a single particle in case 1

At time t_2, the particle is now located at the upper-left corner of the green hexagon, meaning that the pattern will start to repeat from this time step. Therefore, in this case, the spatial horizontal rows are periodic with period 2 after time t_0. In other words, $x^t_{i+t,j} = x^{t+2}_{i+t+2,j}$, for $t \geq t_0$.

Case 2: if the particle's initial location is at the bottom-left corner of any hexagon group, i.e. when its coordinates satisfy $i \equiv 3 \cdot j - 5 \pmod 7$, then, with the same proving process as case 1, the spatial horizontal rows are also periodic with period 2 after time t_0.

Case 3: if the particle is initially located at the center of a hexagon group, i.e. when its coordinates satisfy $i \equiv 3 \cdot j - 2 \pmod 7$, the particle moves in a pattern as shown in Figure 9. The color-coding is the same as in case 1.

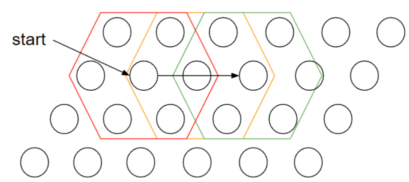

Figure 9: Behavior of a single particle in case 3

In this case, the particle does not move in every time step. Instead, it stays at a cell for one time step and moves two cells to the right in the following time step. At time t_2, after two time steps since t_0, the particle is now at the center of the green hexagon, and thus the pattern repeats every two time steps. The spatial horizontal rows in this automaton are periodic with period 2 after time t_0. In other words, $x^t_{i+t,j} = x^{t+2}_{i+t+2,j}$, for $t \geq t_0$.

Case 4: if the particle starts out at the left of a hexagon, i.e. $i \equiv 3 \cdot j - 3 \pmod 7$, its initial condition is the same as case 3 at time t_1, when the particle is located at the left corner of the orange hexagon.

Case 5: if the particle's initial position is at the bottom-right corner of a hexagon group, i.e. $i \equiv 3 \cdot j - 4 \pmod 7$, then its moving behavior is shown in Figure 10.

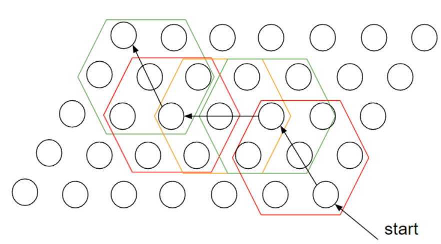

Figure 10: Behavior of a single particle in case 5

Notice after two time steps since t_0, at t_2, the particle is at the bottom-right corner of the green hexagon. Therefore, the pattern repeats every two time steps. The pattern follows general spatial periodicity, with period 2 after time t_0. In other words, $x^t_{i-2t,j-t} = x^{t+2}_{i-2(t+2),j-(t+2)}$, for $t \geq t_0$.

Case 6: if the particle starts out the right corner of a hexagon, i.e. $i \equiv 3 \cdot j - 1 \pmod 7$, it is in the same case as case 5 during time t_1.

Case 7: if the particle's initial location is at the upper-right corner of a hexagon group, i.e. $i \equiv 3 \cdot j + 1 \pmod 7$, then its behavior is shown in Figure 11.

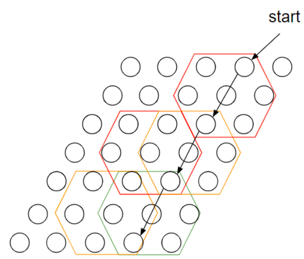

Figure 11: Behavior of single particle in case 7

The particle starts at the upper-right corner of the red hexagon at time t_0. At t_2, it goes to the upper-right corner of the green hexagon, and the pattern continues to repeat. The spatial horizontal columns in this case are periodic with period 2 after t_0. In other words, $x^t_{i,j-t} = x^{t+2}_{i,j-(t+2)}$, for $t \geq t_0$.

The result from this theorem for this specific lattice gas model is mostly consistent with the natural behavior of a single particle in real life, in terms of the periodic patterns in the movements of particles. For example, both case 1 and 5 shows that the particle moves in one broad direction and oscillates periodically while moving forward. However, the results shown in cases 3 and 4, where the particle pauses every other time step, are not as realistic. This might be caused by the discrete nature of this model. If the movement of the particle in case 3 is averaged out into every time step, it shows a particle with continuous but slower moving speed compared to particles in other cases.

7 Conclusion

By building up from a knowledge of elementary cellular automata and expanding and applying it to the lattice gas cellular automata, this paper has given an introductory overview of some properties related to lattice gas automata. With this basic skill of analyzing the lattice gas model, further and deeper research related to this topic can be conducted. This could include discovering the periodicity of lattice gas models with more than one particle, or with complex structures of particles in a large area of lattice.

During the research of the lattice gas model, a Python program was partially implemented to help visualize the behavior of gas particles in the model. However, due to the large number of rules and configurations, and the limitation of time for this thesis, the program was not fully implemented with a complete local rule. A computer simulation could potentially assist the study of lattice gas in a larger scale with more particles in future research.

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