

A new full-NT step interior-point method for circular cone optimization

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Abstract. We present a full step feasible interior-point algorithm for circular cone optimization using Euclidean Jordan algebras. The specificity of our method is to use a transformation similar to that introduced by Darvay and Takács for the centering equations of the central path of the linear optimization. The Nesterov and Todd symmetrization scheme is used to derive the search directions. The theoretical complexity bound of the algorithm coincides with the best-known iteration bound for small-update methods.

Keywords: circular cone, Euclidean Jordan algebra, interior-point methods

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1. Introduction

Interior-point methods (IPMs) that initiated by the landmark paper of Karmarkar [9] play a fundamental role in moder optimization. They attracted the attention of many researchers due to the polynomial complexity and their practical effect. For instance, Illés and Terlaky [8] presented a comparison between the IPMs and pivot methods from the practical and theoretical point of view. Monteiro [13] introduced primal-dual path-following algorithms for solving semidefinite optimization (SDO). Alizadeh and Goldfarb [2] proposed primal-dual path-following algorithms for second-order cone optimization (SOCO). The primal-dual path-following algorithms have been extended by Schmieta and Alizadeh [15] to symmetric cone optimization (SCO) using Euclidean Jordan algebras.

The full-Newton step primal-dual path-following IPM was first introduced by Roos et al. [14]. Later on, De Klerk [6], Wang et al. [20], Wang and Lesaja [16] and Kheirfam [11] extended the results for LO to SDO and $P_*(\kappa)$ -linear complementarity problem, shortly $P_*(\kappa)$ -LCP, the Cartesian $P_*(\kappa)$ -LCP over symmetric cone $P_*(\kappa)$ -SCLCP and circular cone optimization CCO. Based on a new proximity measure, Zhang et al. [22] and Kheirfam [10] respectively generalized Roos et al.'s algorithm to SDO and $P_*(\kappa)$ -HLCP. In the 2003s, Darvay [4] introduces a modification in the centering equations $xs = \mu e$ of the central path by considering $\varphi(\frac{xs}{\mu}) = \varphi(e)$, where $\varphi : R_+^n \rightarrow R_+^n$ is assumed to be a smooth function such that $\varphi(0) = 0$. This direction has become an active area in the past few years for the case $\varphi(t) = \sqrt{t}$. For example, the Darvay's results for LO is extended to convex quadratic optimization (CQO) [1], SDO [18], $P_*(\kappa)$ -LCP [21], SOCO [17] and SCO [19]. Very recently, Darvay and Takács [5] introduce another method for characterizing search directions for LO. They replace the centering equations

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$xs = \mu e$ by $\varphi(\frac{xs}{\mu}) = \varphi(\sqrt{\frac{xs}{\mu}})$, and then apply Newton's method in order to get the new search directions for the case $\varphi(t) = t^2$.

As mentioned above, the extension of the algorithms proposed for LO to more general optimization problems such as LCP, $P_*(\kappa)$ -LCP, SDO, SOCO, SCO is a very active area of research during the past years. The applications of the circular cone appear in many real-world engineering problems such as grasp optimization and the optimal grasping manipulation problems for multi-fingered robots [12]. However, there is little work on the algorithms for solving the CCO. These motivate us to generalize Darvay and Takács's technique [5] for LO to CCO. We prove that the proposed algorithm is well-defined and derive worst-case complexity bound for our algorithm.

The paper is organized as follows. In Section 2, we present the state of the art of CCO and related Euclidean Jordan algebra. In Section 3, we study the central path and the new search directions for CCO. Then we present a feasible full-NT step algorithm for CCO. In Section 4, we show the polynomial complexity of our algorithm. Finally, some conclusions are given in Section 5.

The following notations are used throughout this paper. Let R^n denotes the space of n -dimensional real column vectors. We use $x = (x_0; \bar{x})$ for the column vector $x = (x_0, \bar{x}^T)^T \in R \times R^{n-1}$, and use $(x; y; s)$ for adjoining vectors x, y, s in a column $(x^T, y^T, s^T)^T$. The symbols $\|\cdot\|$ and $\|\cdot\|_{\theta, F}$ respectively denote the standard Euclidean norm and the Frobenius norm. If $A \subseteq R^k$ and $B \subseteq R^l$, then

$$A \times B = \{(x; y) : x \in A, y \in B\}$$

is their Cartesian product. The $\text{int}A$ denotes the interior of A .

2. CCO and related Jordan algebra

Consider the CCO in the standard form

$$\min \left\{ \sum_{j=1}^N \langle c^j, x^j \rangle_{\theta_j} : \sum_{j=1}^N (A_j x^j)_{\theta_j} = b, x^j \in \mathcal{Q}_{\theta_j}^{n_j} \right\}, \quad (P)$$

where $b \in R^m$, $A_j \in R^{m \times n_j}$, $c^j \in R^{n_j}$, and $\mathcal{Q}_{\theta_j}^{n_j}$ is the circular cone of dimension n_j defined by

$$\mathcal{Q}_{\theta_j}^{n_j} := \{(x_0^j; \bar{x}^j) : x_0^j \geq \cot(\theta_j) \|\bar{x}^j\|\}$$

for $\theta_j \in (0, \frac{\pi}{2})$, $j = 1, \dots, N$, and $\|\cdot\|$ refers to the Euclidean norm. Moreover, $\langle c^j, x^j \rangle_{\theta_j} := (c^j)^T I_{\theta_j, n_j}^2 x^j$, $(A_j x^j)_{\theta_j} := A_j I_{\theta_j, n_j}^2 x^j$ with

$$I_{\theta_j, n_j} := \begin{bmatrix} 1 & 0^T \\ 0 & \cot(\theta_j) I_{n_j-1} \end{bmatrix} \in R^{n_j \times n_j}, \quad j = 1, \dots, N,$$

which respectively denote the circular inner product between c^j and x^j and the circular matrix-vector product between A_j and x^j with the circular identity matrix I_{θ_j, n_j} . Let us consider

$$A = [A_1, A_2, \dots, A_N] \in R^{m \times n}, c = (c^1; c^2; \dots; c^N) \in R^{n_1} \times \dots \times R^{n_N} = R^n,$$

$$I_{\theta, n} = \begin{bmatrix} I_{\theta_1, n_1} & & \\ & \ddots & \\ & & I_{\theta_N, n_N} \end{bmatrix}_{n \times n} \quad \text{and } x = (x^1; x^2; \dots; x^N) \in \mathcal{Q}_{\theta_1}^{n_1} \times \dots \times \mathcal{Q}_{\theta_N}^{n_N} := \mathcal{Q}_{\theta}^n,$$

where $n = \sum_{j=1}^N n_j$. Then, the problem (P) can be simply written as

$$\min \{ \langle c, x \rangle_\theta : (Ax)_\theta = b, x \in \mathcal{Q}_\theta^n \}. \tag{P}$$

The dual of (P) is defined by

$$\max \{ b^T y : A^T y + s = c, y \in R^m, s \in \mathcal{Q}_\theta^n \}. \tag{D}$$

We assume that A has full row rank; i.e., $\text{rank}(A) = m$, and both primal and dual problems satisfy the interior-point condition (IPC); i.e., there exists a primal-feasible vector x with $x \in \text{int}\mathcal{Q}_\theta^n$ and a dual-feasible y and s such that $s \in \text{int}\mathcal{Q}_\theta^n$. It is also worth noting that CCO includes second-order cone optimization as a special case. Here, for the sake of simplicity, we assume that $N = 1$, hence $\mathcal{Q}_{\theta_1}^n = \mathcal{Q}_\theta^n$. Let \mathcal{E}^n denote the n dimensional real vector space $R \times R^{n-1}$ whose elements x are indexed with 0. Note that for two vectors $x, y \in \mathcal{E}^n$, we can rewrite their circular inner product as follows

$$\langle x, y \rangle_\theta = x^T I_{\theta,n}^2 y = x_0 y_0 + \cot^2(\theta) \bar{x}^T \bar{y}.$$

Let $x \in \mathcal{E}^n$. The circular spectral decomposition of x with respect to the angle $\theta \in (0, \frac{\pi}{2})$ is defined as follows [3]

$$x = \lambda_{\theta, \max}(x) c_{\theta,1}(x) + \lambda_{\theta, \min}(x) c_{\theta,2}(x),$$

where $\lambda_{\theta, \max} = x_0 + \cot(\theta) \|\bar{x}\|, \lambda_{\theta, \min} = x_0 - \cot(\theta) \|\bar{x}\|$ are the eigenvalues of x and their associated eigenvectors given by

$$c_{\theta,1}(x) = \frac{1}{2} \left(1; \frac{\tan(\theta) \bar{x}}{\|\bar{x}\|} \right), c_{\theta,2}(x) = \frac{1}{2} \left(1; \frac{-\tan(\theta) \bar{x}}{\|\bar{x}\|} \right).$$

Under Spectral Decomposition, we have

$$\text{tr}(x) = \lambda_{\theta, \min}(x) + \lambda_{\theta, \max}(x) = 2x_0, \det_\theta(x) = \lambda_{\theta, \min}(x) \lambda_{\theta, \max}(x) = x_0^2 - \cot^2(\theta) \|\bar{x}\|^2.$$

As a result it can be easily seen that $c_{\theta,1}(x) + c_{\theta,2}(x) = e := (1; 0)$, which is the identity element of \mathcal{E}^n , $\text{trace}(e) = 2$ and $\det(e) = 1$. For any real-valued continuous function f , we define the image of x under f with respect to θ as

$$f(x) := f(\lambda_{\theta, \max}(x)) c_{\theta,1}(x) + f(\lambda_{\theta, \min}(x)) c_{\theta,2}(x),$$

in particular, we get $x^{-1} = \lambda_{\theta, \max}^{-1}(x) c_{\theta,1}(x) + \lambda_{\theta, \min}^{-1}(x) c_{\theta,2}(x)$. The Frobenius norm with respect to θ of x is defined as $\|x\|_{\theta, F} := \sqrt{\lambda_{\theta, \min}^2(x) + \lambda_{\theta, \max}^2(x)}$. The arrow-shaped matrix $\text{Arw}_\theta(x)$ associated with x in \mathcal{E}^n is defined as [3]

$$\text{Arw}_\theta(x) := \begin{bmatrix} x_0 \cot^2(\theta) \bar{x}^T \\ \bar{x} & x_0 I_{n-1} \end{bmatrix}. \tag{1}$$

Observe that $x \in \mathcal{Q}_\theta^n, (x \in \text{int}\mathcal{Q}_\theta^n)$ if and only if $\text{Arw}_\theta(x)$ is positive semidefinite (positive definite); i.e., $\text{Arw}_\theta(x) \succcurlyeq 0 (\text{Arw}_\theta(x) \succ 0)$. For any $x, s \in \mathcal{E}^n$, the Jordan multiplication with respect to θ is defined as

$$(x \circ s)_\theta = (\langle x, s \rangle_\theta; x_0 \bar{s} + s_0 \bar{x}) = \text{Arw}_\theta(x) s = \text{Arw}_\theta(x) \text{Arw}_\theta(s) e. \tag{2}$$

One can easily verify that $(\mathcal{E}^n, \theta, \circ)$ is a Euclidean Jordan algebra under the circular inner product $\langle \cdot, \cdot \rangle_\theta$ with $e = (1; 0) \in \mathcal{E}^n$; i.e., for $x, s, y \in \mathcal{E}^n, (x \circ s)_\theta = (s \circ x)_\theta, (x \circ (x^2 \circ s)_\theta)_\theta = (x^2 \circ (x \circ s)_\theta)_\theta$ where $x^2 = (x \circ x)_\theta$, and $\langle (x \circ s)_\theta, y \rangle_\theta = \langle x, (s \circ y)_\theta \rangle_\theta$ and $(x \circ e)_\theta = (e \circ x)_\theta = x$.

Moreover, we have $c_{\theta,1}^2 = (c_{\theta,1} \circ c_{\theta,1})_{\theta} = c_{\theta,1}$ and $c_{\theta,2}^2 = (c_{\theta,2} \circ c_{\theta,2})_{\theta} = c_{\theta,2}$ and $(c_{\theta,1} \circ c_{\theta,2})_{\theta} = 0$. Therefore, $\{c_{\theta,1}, c_{\theta,2}\}$ is a Jordan frame. We say that two vectors x and s in \mathcal{E}^n are similar, denoted by $x \sim s$, if they share a Jordan frame. The cone of squares of the Euclidean Jordan algebra $(\mathcal{E}^n, \theta, \circ)$ is the circular cone \mathcal{Q}_{θ}^n [3, Theorem 4]. The quadratic representation $P_{\theta}(x)$ associated with $x \in \mathcal{E}^n$ with respect to θ is given as $P_{\theta}(x) = 2\text{Arw}_{\theta}^2(x) - \text{Arw}_{\theta}(x^2)$. In particular, one has $P_{\theta}(x)e = x^2, P_{\theta}(x)x^{-1} = x$ and $P_{\theta}(x^{-1})P_{\theta}(x) = P_{\theta}(x)P_{\theta}(x^{-1}) = I_n$, i.e., $P_{\theta}(x^{-1}) = P_{\theta}(x)^{-1}$ [11]. In the sequel, we generalize the above notions and concepts to the case where $N > 1$, when $\mathcal{E}^n = \mathcal{E}^{n_1} \times \dots \times \mathcal{E}^{n_N}$ and the circular cone underlying \mathcal{Q}_{θ}^n is the Cartesian product on N circular cones; i.e., $\mathcal{Q}_{\theta}^n = \mathcal{Q}_{\theta_1}^{n_1} \times \dots \times \mathcal{Q}_{\theta_N}^{n_N}$. The Jordan algebra associated with the circular cone \mathcal{Q}_{θ}^n is given by

$$(x \circ s)_{\theta} = ((x^1 \circ s^1)_{\theta_1}; \dots; (x^N \circ s^N)_{\theta_N}),$$

with $e = (e^1; \dots; e^N)$ being its identity element. Note that $\text{tr}(e) = 2N$. The arrow-shaped matrix $\text{Arw}_{\theta}(x)$ and the quadratic representation $P_{\theta}(x)$ of the Jordan algebra $(\mathcal{E}^n, \theta, \circ)$ with respect to θ can be respectively adjusted to

$$\text{Arw}_{\theta}(x) := \text{diag}(\text{Arw}_{\theta_1}(x^1), \dots, \text{Arw}_{\theta_N}(x^N)), \quad P_{\theta}(x) := \text{diag}(P_{\theta_1}(x^1), \dots, P_{\theta_N}(x^N)).$$

Furthermore

$$\lambda_{\theta, \max}(x) = \max_{1 \leq j \leq N} \lambda_{\theta_j, \max}(x^j), \quad \lambda_{\theta, \min}(x) = \min_{1 \leq j \leq N} \lambda_{\theta_j, \min}(x^j),$$

and

$$\|x\|_{\theta, F}^2 = \sum_{j=1}^N \|x^j\|_{\theta_j, F}^2, \quad \text{tr}(x) = \sum_{j=1}^N \text{tr}(x^j).$$

3. Feasible full-NT step IPM

In this section, we first introduce the concept of the central path for CCO, and then generalize Darvay and Takács's technique [5] for LO to CCO.

3.1. Central path

The Karush-Kuhn-Tucker (KKT) optimality conditions for problems (P) and (D) is stated as follows:

$$\begin{aligned} (Ax)_{\theta} &= b, \quad x \in \mathcal{Q}_{\theta}^n, \\ A^T y + s &= c, \quad s \in \mathcal{Q}_{\theta}^n, \\ (x \circ s)_{\theta} &= 0. \end{aligned} \tag{3}$$

The first two equations represent primal and dual feasibility and the third equation is called the complementarity condition for (P) and (D). According to the key idea of IPMs, we replace the complementarity condition $(x \circ s)_{\theta} = 0$ by the parameterized equation $(x \circ s)_{\theta} = \mu_{\theta} e$ with $\mu_{\theta} > 0$. This leads to the following system

$$\begin{aligned} (Ax)_{\theta} &= b, \quad x \in \text{int} \mathcal{Q}_{\theta}^n, \\ A^T y + s &= c, \quad s \in \text{int} \mathcal{Q}_{\theta}^n, \\ (x \circ s)_{\theta} &= \mu_{\theta} e. \end{aligned} \tag{4}$$

Since the IPC holds and A has full rank, the system (4) has a unique solution, denoted by $(x(\mu_{\theta}), y(\mu_{\theta}), s(\mu_{\theta}))$ for each $\mu_{\theta} > 0$. We call $x(\mu_{\theta})$ the μ_{θ} -center of (P) and $(y(\mu_{\theta}), s(\mu_{\theta}))$ the

μ_θ -center of (D). The set of all μ_θ -centers is called the central path. It is worth noting that at the μ_θ -center, due to the definition of the trace function, together with (2), we get

$$\langle x(\mu_\theta), s(\mu_\theta) \rangle_\theta = \frac{1}{2} \text{tr}((x(\mu_\theta) \circ s(\mu_\theta))_\theta) = \frac{1}{2} \mu_\theta \text{tr}(e) = N \mu_\theta.$$

It is clear to see that if μ_θ goes to zero, then the limit points satisfy the complementarity condition $(x \circ s)_\theta = 0$, and yields an optimal solution for (P) and (D). Let us consider $u \in \text{int} \mathcal{Q}_\theta^n$. It is a well known fact that $(x \circ s)_\theta = \mu_\theta e$ if and only if $(P_\theta(u)x \circ P_\theta(u^{-1})s)_\theta = \mu_\theta e$ [15, cf. Lemma 28]. In this way, we see that system (4) can be rewritten in the following equivalent form:

$$\begin{aligned} (Ax)_\theta &= b, \quad x \in \text{int} \mathcal{Q}_\theta^n, \\ A^T y + s &= c, \quad s \in \text{int} \mathcal{Q}_\theta^n, \\ (P_\theta(u)x \circ P_\theta(u^{-1})s)_\theta &= \mu_\theta e. \end{aligned} \tag{5}$$

For each choice of u and applying Newton’s method to system (5) one gets a different direction. In this article, we will pay particular attention to the case of Nesterov-Todd (NT) direction; i.e., $u = w^{-\frac{1}{2}}$ with

$$w = P_\theta(x^{\frac{1}{2}})(P_\theta(x^{\frac{1}{2}})s)^{-\frac{1}{2}} \left[= P_\theta(s^{-\frac{1}{2}})(P_\theta(s^{\frac{1}{2}})x)^{\frac{1}{2}} \right],$$

which is called the scaling point of x and s . Let us define

$$v_\theta = \frac{P_\theta(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \left[= \frac{P_\theta(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}} \right].$$

Then, by using the third equation of (5), we have

$$v_\theta^2 = e \Leftrightarrow v_\theta = e \Leftrightarrow v_\theta^2 = v_\theta. \tag{6}$$

3.2. Search directions

Here, we present a class of search directions for CCO based on Darvay and Takács’s technique for LO [5]. Consider the univariate function φ continuously differentiable on (σ^2, ∞) with $0 \leq \sigma < 1$, such that $2t\varphi'(t^2) - \varphi'(t) > 0$ for each $t > \sigma^2$. Due to (6), we replace the standard centering equation $(P_\theta(w^{-\frac{1}{2}})x \circ P_\theta(w^{\frac{1}{2}})s)_\theta = \mu_\theta e$ by

$$\varphi \left(\left(\left(\frac{P_\theta(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{P_\theta(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}} \right) \right)_\theta \right) = \varphi \left(\left(\left(\frac{P_\theta(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{P_\theta(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}} \right) \right)_\theta^{\frac{1}{2}} \right).$$

Therefore, the system (5) can be rewritten in the following scaled form:

$$\begin{aligned} AI_{\theta,n}^2 P_\theta(w^{\frac{1}{2}}) P_\theta(w^{-\frac{1}{2}}) x &= b, \\ I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}}) A^T y + I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}}) s &= I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}}) c, \\ \varphi \left(\left(\left(\frac{P_\theta(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{P_\theta(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}} \right) \right)_\theta \right) &= \varphi \left(\left(\left(\frac{P_\theta(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{P_\theta(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}} \right) \right)_\theta^{\frac{1}{2}} \right), \end{aligned} \tag{7}$$

or simplicity,

$$\begin{aligned} \tilde{A} \tilde{x} &= b, \\ \tilde{A}^T y + I_{\theta,n}^2 \tilde{s} &= \tilde{c}, \\ \varphi \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta \right) &= \varphi \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta^{\frac{1}{2}} \right), \end{aligned} \tag{8}$$

where $\tilde{A} = AI_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})$, $\tilde{x} = P_\theta(w^{-\frac{1}{2}})x$, $\tilde{s} = P_\theta(w^{\frac{1}{2}})s$, $\tilde{c} = I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})c$. If we define

$$f(\tilde{x}, y, \tilde{s}) := \begin{bmatrix} \tilde{A}\tilde{x} - b \\ \tilde{A}^T y + I_{\theta,n}^2 \tilde{s} - \tilde{c} \\ \varphi\left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}}\right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}}\right)\right)_\theta\right) - \varphi\left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}}\right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}}\right)\right)_\theta\right)^{\frac{1}{2}} \end{bmatrix},$$

then system (8) can be written as follows: $f(\tilde{x}, y, \tilde{s}) = 0$. A Newton method applied to this system leads to the following linear system:

$$\begin{aligned} \tilde{A}\Delta\tilde{x} &= b - \tilde{A}\tilde{x} \\ \tilde{A}^T \Delta y + I_{\theta,n}^2 \Delta\tilde{s} &= \tilde{c} - \tilde{A}^T y - I_{\theta,n}^2 \tilde{s} \\ \varphi\left(\left(\left(\frac{\tilde{x} + \Delta\tilde{x}}{\sqrt{\mu_\theta}}\right) \circ \left(\frac{\tilde{s} + \Delta\tilde{s}}{\sqrt{\mu_\theta}}\right)\right)_\theta\right) - \varphi\left(\left(\left(\frac{\tilde{x} + \Delta\tilde{x}}{\sqrt{\mu_\theta}}\right) \circ \left(\frac{\tilde{s} + \Delta\tilde{s}}{\sqrt{\mu_\theta}}\right)\right)_\theta\right)^{\frac{1}{2}} &= 0, \end{aligned} \quad (9)$$

where $\Delta\tilde{x} = P_\theta(w^{-\frac{1}{2}})\Delta x$ and $\Delta\tilde{s} = P_\theta(w^{\frac{1}{2}})\Delta s$. From Corollary 2.14 in [17] and the fact that \tilde{x} and (y, \tilde{s}) are feasible, we get

$$\begin{aligned} \tilde{A}\Delta\tilde{x} &= 0 \\ \tilde{A}^T \Delta y + I_{\theta,n}^2 \Delta\tilde{s} &= 0 \\ \frac{1}{\mu_\theta} ((\tilde{x} \circ \Delta\tilde{s})_\theta + (\Delta\tilde{x} \circ \tilde{s})_\theta) &= (\tilde{z} \circ \underline{z})_\theta, \end{aligned} \quad (10)$$

where

$$\tilde{z} = \left[\varphi' \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta \right) - \frac{1}{2} \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta \right)^{-\frac{1}{2}} \circ \varphi' \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta \right)^{\frac{1}{2}} \right]^{-1},$$

and

$$\underline{z} = \varphi \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta \right)^{\frac{1}{2}} - \varphi \left(\left(\left(\frac{\tilde{x}}{\sqrt{\mu_\theta}} \right) \circ \left(\frac{\tilde{s}}{\sqrt{\mu_\theta}} \right) \right)_\theta \right).$$

If we define

$$d_x = \frac{\Delta\tilde{x}}{\sqrt{\mu_\theta}} = \frac{P_\theta(w^{-\frac{1}{2}})\Delta x}{\sqrt{\mu_\theta}}, \quad d_s = \frac{\Delta\tilde{s}}{\sqrt{\mu_\theta}} = \frac{P_\theta(w^{\frac{1}{2}})\Delta s}{\sqrt{\mu_\theta}}, \quad (11)$$

then system (9) can be simply written in the following form:

$$\begin{aligned} \sqrt{\mu_\theta} \tilde{A} d_x &= 0 \\ \sqrt{\mu_\theta} \tilde{A}^T \Delta y + I_{\theta,n}^2 d_s &= 0 \\ d_x + d_s &= p_{v_\theta}, \end{aligned} \quad (12)$$

where

$$p_{v_\theta} = \left(v_\theta^{-1} \circ \left(\left[\varphi'(v_\theta^2) - \frac{1}{2} (v_\theta^{-1} \circ \varphi'(v_\theta)) \right]^{-1} \circ (\varphi(v_\theta) - \varphi(v_\theta^2)) \right) \right)_\theta.$$

Here, we restrict our attention to the case where $\varphi(t) = t^2$ based on Darvay and Takács's method for LO [5], so we have

$$p_{v_\theta} = \left((2v_\theta^2 - e)^{-1} \circ (v_\theta - (v_\theta^2 \circ v_\theta)_\theta) \right)_\theta.$$

We obtain the search directions d_x and d_s by solving (12) and then compute Δx and Δs via (11). The new iterates are obtained by taking a full-NT step as follows:

$$x_+ := x + \Delta x, \quad y_+ := y + \Delta y, \quad s_+ := s + \Delta s. \quad (13)$$

3.3. The generic primal-dual algorithm for CCO

In this section, we will show that the algorithm can solve the CCO in polynomial time. In order to we need to a quantity that measure the distance between the given triple (x, y, s) and the μ_θ -center $(x(\mu_\theta), y(\mu_\theta), s(\mu_\theta))$. We use a norm-based proximity measure as follows:

$$\delta(x, s; \mu_\theta) := \delta(v_\theta) := \frac{1}{2} \|p_{v_\theta}\|_{\theta, F} = \frac{1}{2} \left\| \left((2v_\theta^2 - e)^{-1} \circ (v_\theta - (v_\theta^2 \circ v_\theta)_\theta) \right)_\theta \right\|_{\theta, F}. \quad (14)$$

From the first two equations of the system (12) it follows that

$$d_x I_{\theta, n}^2 d_s = \langle d_x, d_s \rangle_\theta = 0,$$

that is d_x and d_s with respect to θ are orthogonal. Let $q_{v_\theta} = d_x - d_s$. Then, we have

$$(d_x \circ d_s)_\theta = \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta - (q_{v_\theta} \circ q_{v_\theta})_\theta}{4}. \quad (15)$$

Noting that d_x and d_s are orthogonal, we get $\|q_{v_\theta}\|_{\theta, F} = \|p_{v_\theta}\|_{\theta, F} = 2\delta(v_\theta)$. The full-NT step algorithm is given in Figure 1.

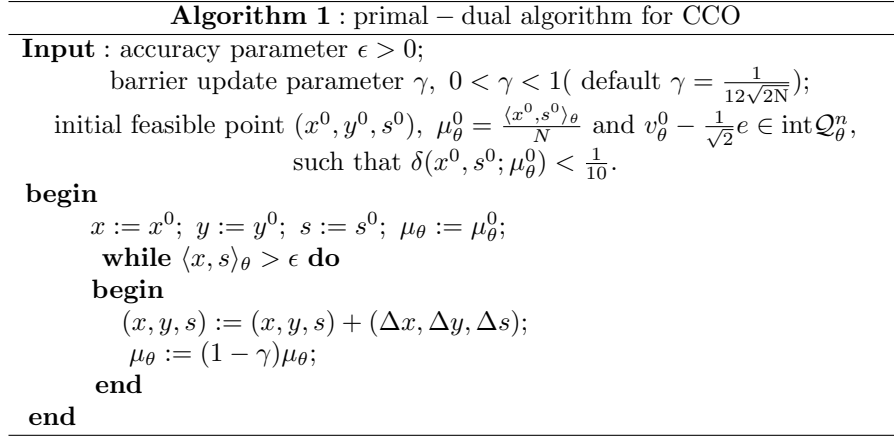


Figure 1: *Primal–dual algorithm*

4. Analysis of the algorithm

Let $x, s \in \text{int } \mathcal{Q}_\theta^n$, $\mu_\theta > 0$ and w be the NT-scaling point of x and s . Then, by the definition of v_θ , (11) and (13) we get

$$x_+ = \sqrt{\mu_\theta} P_\theta(w^{\frac{1}{2}})(v_\theta + d_x), \quad s_+ = \sqrt{\mu_\theta} P_\theta(w^{-\frac{1}{2}})(v_\theta + d_s). \quad (16)$$

Due to [2, Theorem 9] and the above relations, x_+ and s_+ belong to $\text{int } \mathcal{Q}_\theta^n$ if and only if $v_\theta + d_x$ and $v_\theta + d_s$ belong to $\text{int } \mathcal{Q}_\theta^n$. From now on, we denote $x - y \in \text{int } \mathcal{Q}_\theta^n$ by $x \succ_{\mathcal{Q}_\theta^n} y$ ($x - y \in \mathcal{Q}_\theta^n$ by $x \succ_{\mathcal{Q}_\theta^n} y$). Moreover, we have

$$\begin{aligned} v_\theta^2 + (v_\theta \circ p_{v_\theta})_\theta &= v_\theta^2 + \left(v_\theta \circ \left((2v_\theta^2 - e)^{-1} \circ (v_\theta - (v_\theta^2 \circ v_\theta)_\theta) \right)_\theta \right)_\theta \\ &= \left((2v_\theta^2 - e)^{-1} \circ \left(((2v_\theta^2 - e) \circ v_\theta^2)_\theta + \left(v_\theta \circ (v_\theta - (v_\theta^2 \circ v_\theta)_\theta) \right)_\theta \right) \right)_\theta \\ &= \left((2v_\theta^2 - e)^{-1} \circ (v_\theta^2 \circ v_\theta^2)_\theta \right)_\theta - \left((2v_\theta^2 - e)^{-1} \circ (2v_\theta^2 - e) \right)_\theta + e \\ &= \left((2v_\theta^2 - e)^{-1} \circ (v_\theta^2 - e)^2 \right)_\theta + e. \end{aligned} \quad (17)$$

The following lemma states that, under some mild conditions on the proximity measure, the full-NT steps lie in $\text{int } \mathcal{Q}_\theta^n$.

Lemma 1. *Let $\delta_\theta := \delta(v_\theta) < 1$ and $v_\theta \succ_{\mathcal{Q}_\theta^n} \frac{1}{\sqrt{2}}e$. Then the full-NT step is strictly feasible.*

Proof. Let $0 \leq \alpha \leq 1$. Define $v_x(\alpha) = v_\theta + \alpha d_x$ and $v_s(\alpha) = v_\theta + \alpha d_s$. Then, using (15) and (17), we obtain

$$\begin{aligned} (v_x(\alpha) \circ v_s(\alpha))_\theta &= ((v_\theta + \alpha d_x) \circ (v_\theta + \alpha d_s))_\theta \\ &= v_\theta^2 + \alpha(v_\theta \circ (d_x + d_s))_\theta + \alpha^2(d_x \circ d_s)_\theta \\ &= (1 - \alpha)v_\theta^2 + \alpha(v_\theta^2 + (v_\theta \circ p_{v_\theta})_\theta) + \alpha^2(d_x \circ d_s)_\theta \\ &\succ_{\mathcal{Q}_\theta^n} (1 - \alpha)v_\theta^2 + \alpha e + \alpha^2(d_x \circ d_s)_\theta \\ &\succ_{\mathcal{Q}_\theta^n} (1 - \alpha)v_\theta^2 + \alpha \left(e - (1 - \alpha) \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta}{4} - \alpha \frac{(q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \right). \end{aligned}$$

Furthermore, since $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} \left\| (1 - \alpha) \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta}{4} + \alpha \frac{(q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \right\|_{\theta, F} &\leq (1 - \alpha) \left\| \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta}{4} \right\|_{\theta, F} \\ &\quad + \alpha \left\| \frac{(q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \right\|_{\theta, F} \\ &\leq (1 - \alpha) \frac{\|p_{v_\theta}\|_{\theta, F}^2}{4} + \alpha \frac{\|q_{v_\theta}\|_{\theta, F}^2}{4} \\ &= \delta_\theta^2 < 1. \end{aligned}$$

From the last inequality, we deduce that

$$e - (1 - \alpha) \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta}{4} - \alpha \frac{(q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \succ_{\mathcal{Q}_\theta^n} 0.$$

Therefore, since $v_\theta^2 \succ_{\mathcal{Q}_\theta^n} 0$, we have $(v_x(\alpha) \circ v_s(\alpha))_\theta \succ_{\mathcal{Q}_\theta^n} 0$. Hence,

$$0 < \det_\theta \left((v_x(\alpha) \circ v_s(\alpha))_\theta \right) \leq \det_\theta(v_x(\alpha)) \det_\theta(v_s(\alpha)), \text{ (see Lemma 6 in [11]).}$$

From this inequality, it follows that $\det_\theta(v_x(\alpha)) \neq 0$ and $\det_\theta(v_s(\alpha)) \neq 0$. From an argument similar to the proof of [11, Lemma 9], it follows that $v_\theta + d_x \succ_{\mathcal{Q}_\theta^n} 0$ and $v_\theta + d_s \succ_{\mathcal{Q}_\theta^n} 0$. Therefore, the proof is complete. \square

Let us define

$$v_\theta^+ = \frac{P_\theta(w_+^{-\frac{1}{2}})x_+}{\sqrt{\mu_\theta}} \left[= \frac{P_\theta(w_+^{\frac{1}{2}})s_+}{\sqrt{\mu_\theta}} \right],$$

where w_+ is the NT-scaling point of x_+ and s_+ . Then, we have $(v_\theta^+)^2 \sim P_\theta((v_\theta + d_x)^{\frac{1}{2}})(v_\theta + d_s)$ ([11, Lemma 11]).

Lemma 2. *Let $\delta_\theta := \delta(x, s, \mu_\theta)$. Then, we have*

$$\lambda_{\theta, \min}(v_\theta^+) \geq \sqrt{1 - \delta_\theta^2}.$$

Proof. We have

$$\begin{aligned}
\lambda_{\theta, \min}(v_{\theta}^+)^2 &= \lambda_{\theta, \min}\left(P_{\theta}\left((v_{\theta} + d_x)^{\frac{1}{2}}\right)(v_{\theta} + d_s)\right) \\
&\geq \lambda_{\theta, \min}\left(\left((v_{\theta} + d_x) \circ (v_{\theta} + d_s)\right)_{\theta}\right) \\
&\geq \lambda_{\theta, \min}\left(e - \frac{(q_{v_{\theta}} \circ q_{v_{\theta}})_{\theta}}{4}\right) \geq 1 - \lambda_{\theta, \max}\left(\frac{(q_{v_{\theta}} \circ q_{v_{\theta}})_{\theta}}{4}\right) \\
&\geq 1 - \frac{\|(q_{v_{\theta}} \circ q_{v_{\theta}})_{\theta}\|_{\theta, F}}{4} \geq 1 - \frac{\|q_{v_{\theta}}\|_{\theta, F}^2}{4} = 1 - \delta_{\theta}^2,
\end{aligned}$$

where the first inequality is due to Lemma 4.58 [7], the second inequality follows from the proof of Lemma 1 with $\alpha = 1$ and the fourth inequality follows from the fact that $\lambda_{\max}(x) \leq \|x\|_F$. The proof of the lemma is complete. \square

We will often use the following inequality on the circular inner product of two elements of \mathcal{Q}_{θ}^n .

$$\begin{aligned}
\|(x \circ y)_{\theta}\|_{\theta, F}^2 &= \lambda_{\max, \theta}^2(x)\lambda_{\max, \theta}^2(y) + \lambda_{\min, \theta}^2(x)\lambda_{\min, \theta}^2(y) \\
&\leq \lambda_{\max, \theta}^2(x)\left(\lambda_{\max, \theta}^2(y) + \lambda_{\min, \theta}^2(y)\right) \\
&= \lambda_{\max, \theta}^2(x)\|y\|_{\theta, F}^2.
\end{aligned}$$

Lemma 3. Let $\delta_{\theta} := \delta(x, s, \mu_{\theta}) < \frac{1}{\sqrt{2}}$ and $v_{\theta} \succ_{\mathcal{Q}_{\theta}^n} \frac{1}{\sqrt{2}}e$. Then $v_{\theta}^+ \succ_{\mathcal{Q}_{\theta}^n} \frac{1}{\sqrt{2}}e$ and

$$\delta(v_{\theta}^+) := \delta(x_+, s_+; \mu_{\theta}) \leq \frac{5\delta_{\theta}^2}{1 - 2\delta_{\theta}^2} \sqrt{1 - \delta_{\theta}^2}.$$

Proof. In view of Lemma 2, we deduce that

$$\lambda_{\theta, \min}(v_{\theta}^+) \geq \sqrt{1 - \delta_{\theta}^2} > \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

From this inequality, it follows that $v_{\theta}^+ \succ_{\mathcal{Q}_{\theta}^n} \frac{1}{\sqrt{2}}e$. To prove the second claim, by the definition of δ_{θ} , we have

$$\begin{aligned}
2\delta(v_{\theta}^+) &= \|p_{v_{\theta}^+}\|_{\theta, F} = \left\| \left((2(v_{\theta}^+)^2 - e)^{-1} \circ (v_{\theta}^+ - (v_{\theta}^+ \circ (v_{\theta}^+)^2)_{\theta}) \right)_{\theta} \right\|_{\theta, F} \\
&= \left\| \left(((2(v_{\theta}^+)^2 - e)^{-1} \circ v_{\theta}^+)_{\theta} \circ (e - (v_{\theta}^+)^2) \right)_{\theta} \right\|_{\theta, F} \\
&\leq \lambda_{\max, \theta} \left(((2(v_{\theta}^+)^2 - e)^{-1} \circ v_{\theta}^+)_{\theta} \right) \|e - (v_{\theta}^+)^2\|_{\theta, F}.
\end{aligned}$$

Let us consider the function $f(t) = \frac{t}{2t^2 - 1}$ on $t > \frac{1}{\sqrt{2}}$. It is easy to see that $f'(t) < 0$, so the function $f(t)$ is decreasing. Thus, using Lemma 2, we get

$$2\delta(v_{\theta}^+) \leq \frac{\lambda_{\min, \theta}(v_{\theta}^+)}{2\lambda_{\min, \theta}^2(v_{\theta}^+) - 1} \|e - (v_{\theta}^+)^2\|_{\theta, F} \leq \frac{\sqrt{1 - \delta_{\theta}^2}}{1 - 2\delta_{\theta}^2} \|e - (v_{\theta}^+)^2\|_{\theta, F}. \quad (18)$$

On the other hand, using [15, Lemma 30], we get

$$\begin{aligned}
\|(v_\theta^+)^2 - e\|_{\theta, F} &= \|P_\theta((v_\theta + d_x)^{\frac{1}{2}})(v_\theta + d_s) - e\|_{\theta, F} \\
&\leq \|((v_\theta + d_x) \circ (v_\theta + d_s))_\theta - e\|_{\theta, F} \\
&= \|v_\theta^2 + (v_\theta \circ p_{v_\theta})_\theta + (d_x \circ d_s)_\theta - e\|_{\theta, F} \\
&= \left\| \left((2v_\theta^2 - e)^{-1} \circ (v_\theta^2 - e)^2 \right)_\theta + \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta - (q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \right\|_{\theta, F} \\
&= \left\| \left((v_\theta^{-2} \circ (9v_\theta^2 - 4e))_\theta \circ \left(\frac{(p_{v_\theta} \circ p_{v_\theta})_\theta}{4} \right)_\theta \right)_\theta - \frac{(q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \right\|_{\theta, F} \\
&\leq \lambda_{\max, \theta} \left((v_\theta^{-2} \circ (9v_\theta^2 - 4e))_\theta \right) \left\| \frac{(p_{v_\theta} \circ p_{v_\theta})_\theta}{4} \right\|_{\theta, F} + \left\| \frac{(q_{v_\theta} \circ q_{v_\theta})_\theta}{4} \right\|_{\theta, F} \\
&\leq 9 \frac{\|p_{v_\theta}\|_{\theta, F}^2}{4} + \frac{\|q_{v_\theta}\|_{\theta, F}^2}{4} = 10\delta_\theta^2,
\end{aligned}$$

where the second equality is due to the proof of Lemma 1 with $\alpha = 1$, the third equality follows from (15) and (17). Substituting this bound into (19) yields the inequality in lemma, and the proof is complete. \square

The next lemma gives an upper bound of the duality gap after taking a full-NT step.

Lemma 4. *Let $\delta_\theta := \delta(x, s; \mu_\theta)$. Then, we have*

$$\langle x_+, s_+ \rangle_\theta \leq \mu_\theta (N + 4\delta_\theta^2).$$

Proof. By the circular inner product and (16) it follows that

$$\begin{aligned}
\langle x_+, s_+ \rangle_\theta &= x_+^T I_{\theta, n}^2 s_+ = \mu_\theta (P_\theta(w^{\frac{1}{2}})(v_\theta + d_x))^T I_{\theta, n}^2 P_\theta(w^{-\frac{1}{2}})(v_\theta + d_s) \\
&= \mu_\theta (v_\theta + d_x)^T P_\theta(w^{\frac{1}{2}})^T I_{\theta, n}^2 P_\theta(w^{-\frac{1}{2}})(v_\theta + d_s) \\
&= \mu_\theta (v_\theta + d_x)^T I_{\theta, n}^2 P_\theta(w^{\frac{1}{2}}) P_\theta(w^{-\frac{1}{2}})(v_\theta + d_s) \\
&= \mu_\theta \langle v_\theta + d_x, v_\theta + d_s \rangle_\theta \\
&= \mu_\theta (\langle v_\theta, v_\theta \rangle_\theta + \langle v_\theta, d_x + d_s \rangle_\theta + \langle d_x, d_s \rangle_\theta) \\
&= \mu_\theta (\langle v_\theta, v_\theta \rangle_\theta + \langle v_\theta, p_{v_\theta} \rangle_\theta) = \frac{\mu_\theta}{2} \text{tr}(v_\theta^2 + (v_\theta \circ p_{v_\theta})_\theta) \\
&= \frac{\mu_\theta}{2} \text{tr} \left(\left((2v_\theta^2 - e)^{-1} \circ (v_\theta^2 - e)^2 \right)_\theta + e \right) \\
&= \frac{\mu_\theta}{2} \text{tr} \left(\left((v_\theta^{-2} \circ (2v_\theta^2 - e))_\theta \circ (p_{v_\theta} \circ p_{v_\theta})_\theta \right)_\theta + e \right) \\
&\leq \frac{\mu_\theta}{2} \left(\lambda_{\max, \theta} \left((v_\theta^{-2} \circ (2v_\theta^2 - e))_\theta \right) \text{tr}((p_{v_\theta} \circ p_{v_\theta})_\theta) + 2N \right) \\
&\leq \frac{\mu_\theta}{2} \left(\frac{8\|p_{v_\theta}\|_{\theta, F}^2}{4} + 2N \right) = \mu_\theta (N + 4\delta_\theta^2),
\end{aligned}$$

where the third equality follows from $(I_{\theta, n}^2 P_\theta(w^{\frac{1}{2}}))^T = I_{\theta, n}^2 P_\theta(w^{\frac{1}{2}})$ (see [11]), the sixth equality is due to the third equation of (12) and the fact that d_x and d_s are orthogonal. The proof of the lemma is complete. \square

Lemma 5. *Let $\delta_\theta := \delta(x, s, \mu_\theta) < \frac{1}{\sqrt{2}}$, $v_\theta \succ_{\mathcal{Q}_\theta^n} \frac{1}{\sqrt{2}}e$ and $\mu_\theta^+ := (1 - \gamma)\mu_\theta$ with $0 < \gamma < 1$. Furthermore, let $v_\theta^\# = \frac{v_\theta^+}{\sqrt{1-\gamma}}$. Then, $v_\theta^\# \succ_{\mathcal{Q}_\theta^n} \frac{1}{2}e$ and*

$$\delta(x_+, s_+; \mu_\theta^+) \leq \frac{\sqrt{1 - \delta_\theta^2}}{2\sqrt{1 - \gamma}(1 - 2\delta_\theta^2 + \gamma)} \left(\gamma\sqrt{2N} + 10\delta_\theta^2 \right).$$

Proof. Under the assumptions given in lemma, we conclude from Lemma 3 that $v_\theta^+ \succ_{\mathcal{Q}_\theta^n} \frac{1}{\sqrt{2}}e$. This, together with $\frac{1}{\sqrt{1-\gamma}} > 1$, implies that $v_\theta^\sharp \succ_{\mathcal{Q}_\theta^n} \frac{1}{\sqrt{2}}e$, and we have proved the first statement. To prove the second claim, by the definition of δ_θ , we have

$$\begin{aligned} \delta(x_+, s_+; \mu_\theta^+) &= \frac{1}{2} \left\| \left((2(v_\theta^\sharp)^2 - e)^{-1} \circ (v_\theta^\sharp - (v_\theta^\sharp \circ (v_\theta^\sharp)^2)_\theta) \right)_\theta \right\|_{\theta, F} \\ &= \frac{1}{2\sqrt{1-\gamma}} \left\| \left((2(v_\theta^+)^2 - e(1-\gamma))^{-1} \circ (v_\theta^+ \circ ((1-\gamma)e - (v_\theta^+)^2)_\theta) \right)_\theta \right\|_{\theta, F} \\ &\leq \frac{1}{2\sqrt{1-\gamma}} \lambda_{\max, \theta} \left(\left((2(v_\theta^+)^2 - e(1-\gamma))^{-1} \circ v_\theta^+ \right)_\theta \right) \|(1-\gamma)e - (v_\theta^+)^2\|_{\theta, F} \\ &\leq \frac{\sqrt{1-\delta_\theta^2}}{2\sqrt{1-\gamma}(1-2\delta_\theta^2+\gamma)} \left(\gamma\sqrt{2N} + \|e - (v_\theta^+)^2\|_{\theta, F} \right) \\ &\leq \frac{\sqrt{1-\delta_\theta^2}}{2\sqrt{1-\gamma}(1-2\delta_\theta^2+\gamma)} \left(\gamma\sqrt{2N} + 10\delta_\theta^2 \right), \end{aligned}$$

which concludes the proof of lemma. □

Corollary 1. Let $\delta_\theta := \delta(x, s, \mu_\theta) < \frac{1}{10}$ and $\gamma = \frac{1}{12\sqrt{2N}}$. Then $\delta(x_+, s_+; \mu_\theta^+) < \frac{1}{10}$.

The following lemma gives an upper bound for the number of iterations produced by Algorithm.

Lemma 6. Suppose that x^0 and s^0 are strictly feasible, $\mu_\theta^0 = \frac{\langle x^0, s^0 \rangle_\theta}{N}$ and $\delta(x^0, s^0; \mu_\theta^0) < \frac{1}{10}$. Let x^k and s^k be the iterates obtained after k iterations. Then, $\langle x^k, s^k \rangle_\theta \leq \varepsilon$ for any

$$k \geq \left\lceil \frac{1}{\gamma} \log \frac{\mu_\theta^0(N + \frac{1}{25})}{\varepsilon} \right\rceil.$$

Proof. From Lemma 4 and the assumption $\delta(x, s; \mu_\theta) < \frac{1}{10}$, we conclude that

$$\langle x^k, s^k \rangle_\theta \leq \mu_\theta^k(N + \frac{1}{25}) = (1-\gamma)^k \mu_\theta^0(N + \frac{1}{25}).$$

Then, the inequality $\langle x^k, s^k \rangle_\theta \leq \varepsilon$ holds if

$$(1-\gamma)^k \mu_\theta^0(N + \frac{1}{25}) \leq \varepsilon. \tag{19}$$

By taking the natural logarithm of both sides of (19), we get

$$k \log(1-\gamma) + \log \mu_\theta^0(N + \frac{1}{25}) \leq \log \varepsilon.$$

Using the inequality $-\log(1-\gamma) > \gamma$, we conclude that the above inequality is satisfied if

$$-k\gamma + \log \mu_\theta^0(N + \frac{1}{25}) \leq \log \varepsilon,$$

which implies the desired result. □

We are now in a position to state the main result of this paper.

Theorem 1. Let $\gamma = \frac{1}{12\sqrt{2N}}$. Then the algorithm requires at most

$$\mathcal{O}\left(\sqrt{N} \log \frac{\mu_\theta^0(N + \frac{1}{25})}{\varepsilon}\right)$$

iterations. The output gives an ε -approximate solution (x, s) satisfying $\langle x, s \rangle_\theta \leq \varepsilon$.

Proof. The proof is trivial by Lemma 6. □

5. Conclusion

In this paper, we proposed an IPM for circular cone optimization based on Darvay and Takács's technique for LO [5] and using the Nesterov-Todd (NT) symmetrization scheme. We used Newton's method to get the search directions and showed that the obtained complexity coincides with the now best-known iteration bound for small-update methods. As a further research one can consider the possible extension of the proposed method to the elliptic cone optimization.

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