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BOUNDS ON THE NUMBER OF IRREDUCIBLE SEMIGROUPS OF FIXED FROBENIUS NUMBER

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Abstract. In 2011, Blanco and Rosales gave an algorithm for constructing a directed tree graph whose vertices are the irreducible numerical semigroups with a fixed Frobenius number. Laird and Martinez in 2013 studied the levels of these trees and conjectured what their heights might be. In this paper, we give an exposition on irreducible numerical semigroups. We also present some data supporting the conjecture of Laird and Martinez, and give a lower and upper bound on the number of irreducible numerical semigroups with fixed Frobenius number.

1 Introduction

This paper concerns itself with the topic of numerical semigroups, which crop up in many areas of mathematics. Before introducing specifics, we will begin by going over a simple example of a numerical semigroup.

Example 1.1. Suppose you live on planet Zort. On the planet Zort, there is a sport, Zortball, that is incredibly popular. In Zortball, teams can score touchdowns, goals, or homeruns. These are respectively worth 13, 19, and 20 points. On an unsuspecting day your annoying neighbor, B'lachla, tells you his team won their game last night by a huge 74-20 split. You, being exceptionally talented at mathematics, are suspicious of his claim, which is confounded by the fact that you have never heard of a team scoring 74 points before. In order to verify his point totals, you begin combining possible scores, creating a set S of all elements of the form

$$S = \{13a + 19b + 20c \mid a, b, c \text{ are nonnegative integers}\}.$$

After some calculations, you realize that, indeed, there are no nonnegative integer solutions for the equation $13a + 19b + 20c = 74$. Having mathematically demonstrated that 74 points is impossible to score, you reveal B'lachla for the liar he is.

Whether or not 74 points can be scored in this game is not the only question we could ask. We also might ask: "Is there a maximum number of points someone cannot score?" If there is, we could then ask: "Is there a simple relationship between the numbers 13, 19, 20 and the unscorable maximum?" We will answer some of these questions. In particular, Section 2 will cover a class of objects called *numerical semigroups*. The set S of possible scores is an example of a numerical semigroup. A numerical semigroup is, in simple terms, a subset of the natural numbers where only finitely many numbers are missing and where adding makes sense. Since we are removing only finitely many elements, it makes sense to talk about the maximal element which is *not* in our subset, which is called the *Frobenius number*.

We can look at more than just the properties of a given semigroup, however. We can also look for semigroups which have certain properties. For example, we can investigate how many numerical semigroups exist with a given Frobenius number. In this paper we do just that, except instead of considering all numerical semigroups, we look for semigroups of a certain subclass, which we call *irreducible numerical semigroups*. Section 2 will also introduce irreducible numerical semigroups and their properties.

After the introduction to numerical semigroups in Section 2, Section 3 introduces original results. In particular, it gives formulas which are upper and lower bounds for the number of irreducible numerical semigroups that have a certain Frobenius number. By letting F be the Frobenius number, $I(F)$ the set of all irreducible semigroups with Frobenius number F, and $\#I(F)$ the number of irreducible semigroups, we can state our main theorem.

Theorem 1.1. *For a positive integer F,*

$$2^{\lceil \frac{F}{2} \rceil - \lfloor \frac{F}{3} \rfloor - 1} \leq \#I(F) \leq 1 + \sum_{\substack{2 \leq m \leq \lceil \frac{F}{3} \rceil, \\ m \nmid F}} 2^{\lfloor \frac{F-2m}{2} \rfloor - \lfloor \frac{F-2m-1}{m} \rfloor} + \sum_{\substack{\lceil \frac{F}{3} \rceil < m \leq \lfloor \frac{F}{2} \rfloor, \\ m \nmid F}} 2^{\lfloor \frac{F-2m}{2} \rfloor}.$$

Section 3 also contains a plot showing a comparison of computer-generated data to the bounds. Proofs of the bounds follow in Subsections 3.1 and 3.2. Section 4 concludes with questions for further research.

2 Numerical Semigroups

Throughout the paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Addition on $\mathbb{N}_0$ is associative, and since 0 is an additive identity in $\mathbb{N}_0$, $\mathbb{N}_0$ is also a *monoid*. A *submonoid* S of $\mathbb{N}_0$ is a subset of $\mathbb{N}_0$ that contains 0 and is closed under addition. Given any $A \subseteq \mathbb{N}_0$, let $\langle A \rangle$ denote the submonoid of $\mathbb{N}_0$ defined as

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_k a_k \mid \lambda_i \in \mathbb{N}_0, a_j \in A\}.$$

If, for a submonoid S of $\mathbb{N}_0$, $S = \langle A \rangle$ for some $A \subseteq \mathbb{N}_0$, then A *generates* S. In this case, the elements of A are referred to as a *system of generators* for S. Furthermore, if $|A| < \infty$, then S is *finitely generated*. A system of generators A for a submonoid S of $\mathbb{N}_0$ is called *minimal* if every proper subset of A generates a proper submonoid of S. If $A = \{n_1, \ldots, n_k\}$ for some $n_j \in \mathbb{N}$, we may also denote $\langle A \rangle$ as $\langle n_1, \ldots, n_k \rangle$.

Remark 2.1. The first example featured the semigroup S which is generated by the system of generators $A = \{13, 19, 20\}$. Thus $S = \langle A \rangle$ and S is finitely generated.

We will now consider another example before formally defining a numerical semigroup.

Example 2.1. Let S be the submonoid of $\mathbb{N}_0$ generated by $\{6, 7, 10, 11, 16\}$. Note that since $16 = 10 + 6$, we have $S = \langle 6, 7, 10, 11, 16 \rangle = \langle 6, 7, 10, 11 \rangle$, so that $\{6, 7, 10, 11, 16\}$ is *not* a minimal generating set for S. On the other hand, $\{6, 7, 10, 11\}$ *is* a minimal generating set for S. We have

$$\begin{aligned} S = \langle 6, 7, 10, 11 \rangle &= \{\lambda_1 6 + \lambda_2 7 + \lambda_3 10 + \lambda_4 11 \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{N}_0\} \\ &= \{0, 6, 7, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, \ldots\} \\ &= \{0, 6, 7, 10, 11, 12, 13, 14, 16, \rightarrow\} \end{aligned},$$

where the arrow means that all integers larger than 16 are in S.

Now we can give a formal definition. A *numerical semigroup* is a submonoid S of $\mathbb{N}_0$ whose complement $\mathbb{N}_0 \backslash S$ is finite. In other words, S is a submonoid of $\mathbb{N}_0$ that contains all but finitely many positive integers.

Example 2.2. Let $S = \{0, 2, 4, 6, 7, 8, 9, 10, 11, \rightarrow\}$. Then S is closed under addition, and since $\mathbb{N}_0 \backslash S = \{1, 3, 5\}$, S is a numerical semigroup.

Since the complement of a numerical semigroup is finite, there is a largest integer lying outside S, which is called the *Frobenius number* of S and is denoted by $F(S)$. That is,

$$F(S) = \max\{i \in \mathbb{N}_0 \mid i \notin S\}.$$

The *multiplicity* of a numerical semigroup S is the smallest nonzero integer that lies inside S, denoted by $m(S)$. In other words,

$$m(S) = \min\{S - \{0\}\}.$$

Example 2.3. Consider again $S = \langle 6, 7, 10, 11\rangle = \{0, 6, 7, 10, 11, 12, 13, 14, 16, \rightarrow\}$. Then $F(S) = 15$ and $m(S) = 6$.

The *greatest common divisor* of a nonempty set $B \subseteq \mathbb{N}_0$ is the unique integer $d \in \mathbb{N}_0$ that satisfies the following conditions:

1. $d \mid b$ for any $b \in B$.

2. If $d' \in \mathbb{N}_0$ and $d' \mid b$ for all $b \in B$, then $d' \mid d$.

This integer d is denoted by $\gcd(B)$.

Theorem 2.1. *The following are true:*

1. *If S is a numerical semigroup, then S has a unique minimal system of generators A. Furthermore, A is finite.*

2. *Let B be a nonempty subset of $\mathbb{N}_0$. Then $\langle B\rangle$ is a numerical semigroup if and only if* $\gcd(B) = 1$.

Proof. See Rosales and García-Sánchez [1, Lemma 2.1, Theorem 2.7]. □

Example 2.4. Let $S = \langle 6, 10\rangle$. Then

$$\begin{aligned} S = \langle 6, 10\rangle &= \{\lambda_1 6 + \lambda_2 10 \mid \lambda_1, \lambda_2 \in \mathbb{N}_0\} \\ &= \{6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, \ldots\}. \end{aligned}$$

Moreover $\mathbb{N}_0 \backslash S$ is infinite, since, for example, it contains no odd numbers. Therefore, S is not a numerical semigroup. We can see this as well by using the above theorem: since $\gcd(6, 10) = 2 \neq 1$, S is not a numerical semigroup.

Example 2.5. Let $S = \langle 6, 7, 10\rangle$. Since $\gcd(6, 7, 10) = 1$, S is a numerical semigroup.

Surprisingly, despite the seemingly simple definition of a numerical semigroup, there are many problems related to numerical semigroups that are easy to state yet difficult to solve. For example, the problem of determining the Frobenius number $F(S)$ of a numerical semigroup S is still largely unsolved. This is sometimes called the *Frobenius problem*. In 1884, J.J. Sylvester [2] showed that if $S = \langle n, m\rangle$ with $gcd(n, m) = 1$, then

$$F(S) = nm - n - m.$$

When $A = \{n_1, \ldots, n_k\}$, $\gcd(n_1, \ldots, n_k) = 1$, and $k \geq 3$, a closed-form formula of the Frobenius number of $S = \langle A\rangle$ is still elusive (see Ramírez Alfonsín [6]). We instead focus on the inverse problem to the Frobenius problem: Find all the numerical semigroups with a fixed Frobenius number. Rather than try and find *all* numerical semigroups with the same Frobenius number, we restrict ourselves to finding a special class of numerical semigroups, namely the irreducible ones.

A numerical semigroup S is *irreducible* if it is not the intersection of two numerical semigroups which properly contain S. In other words, if S is an irreducible numerical semigroup and $S = S_1 \cap S_2$ for numerical semigroups S_1, S_2, then $S_1 = S$ or $S_2 = S$.

Example 2.6. Let $S = \langle 5, 7, 8\rangle, S_1 = \langle 4, 5, 7, 8\rangle$ and $S_2 = \langle 5, 6, 7, 8\rangle$. Then $S = S_1 \cap S_2$ and both S_1 and S_2 properly contain S. Thus S is reducible.

Theorem 2.2. *Let S be a numerical semigroup. Then:*

1. *If $F(S)$ is odd, S is irreducible if and only if $x \in \mathbb{Z}\backslash S$ implies $F(S) - x \in S$.*
2. *If $F(S)$ is even, S is irreducible if and only if $x \in \mathbb{Z}\backslash S$ implies $F(S) - x \in S$ or $x = \dfrac{F(S)}{2}$.*

Proof. See Rosales and García-Sánchez [1, Proposition 4.4]. □

Irreducible numerical semigroups with odd Frobenius number are *symmetric* numerical semigroups, while those with even Frobenius number are *pseudo-symmetric* numerical semigroups.

Remark 2.2. If S is a numerical semigroup, observe that for any $x \in \mathbb{Z}$, *at most* one of x or $F(S)-x$ lies in S. Indeed, if both x and $F(S)-x$ were in S, then $F(S) = x+(F(S)-x) \in S$, which is a contradiction. On the other hand, Theorem 2.2 states that S is irreducible when, for any $x \in \mathbb{Z}$, *precisely* one of x or $F(S) - x$ lies in S.

Example 2.7. Let $S = \langle 3, 7\rangle$. Then $F(S) = 11$. To see that S is symmetric, and hence irreducible, consider Figure 2.1 below. This image establishes a pairing between x and $F - x$ for values of x between 0 and 11. These are the only values of interest, since we know which values outside this interval lie in S. By the above remark, since each pairing has exactly one filled in circle (indicating that value lies in S), we see that S is symmetric.

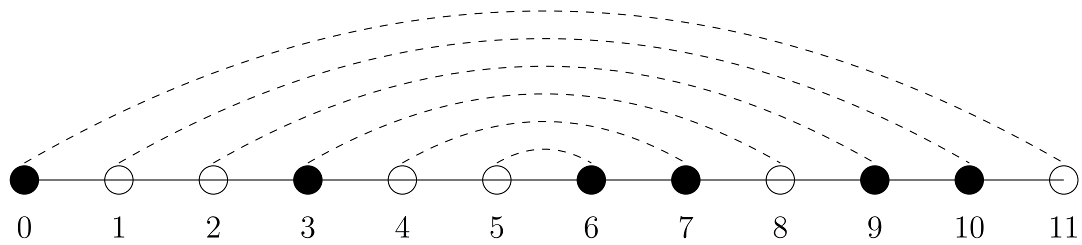

Figure 2.1 A symmetric semigroup

Example 2.8. Consider again the semigroup $S = \langle 5, 7, 8\rangle$. Then $F(S) = 11$. We have seen in Example 2.6 that S is not irreducible; but, we can also see this using Theorem 2.2. Indeed, observe that in Figure 2.2 below, the pairing $(2, 9)$ has no filled-in circle, meaning that neither 2 nor $11 - 2 = 9$ lie in S.

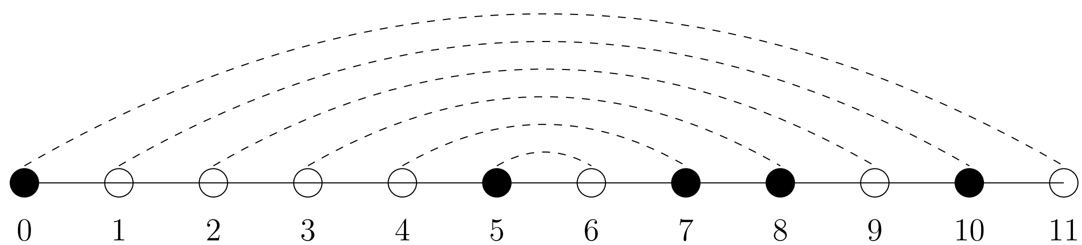

Figure 2.2 A non-symmetric semigroup

Example 2.9. Let $S = \langle 3, 8, 13\rangle$. Then $F(S) = 10$. To see that S is pseudo-symmetric, observe that Figure 2.3 below establishes a pairing between x and $F - x$ for values of x between 0 and 10, ignoring $F/2 = 5$. In each such pairing, only one circle is filled, so that we see S must be pseudo-symmetric, and hence irreducible.

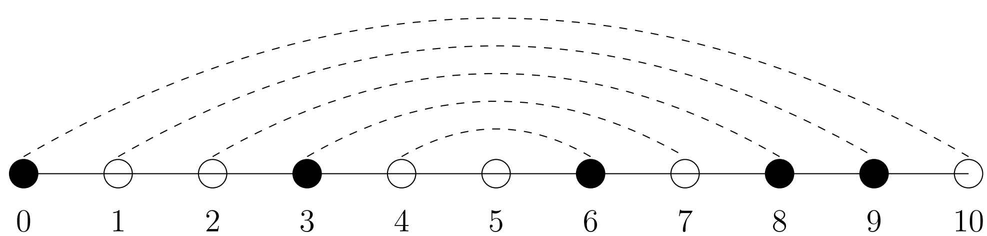

Figure 2.3 A pseudo-symmetric semigroup

We will now introduce some terms from graph theory. A *(directed) graph* G is an ordered pair (V, E) of sets V and E, where V is any set and E is a collection of ordered pairs of elements of V. V is the *vertex set* and E is the *edge set* of G. Elements $v, w \in V$ are *adjacent* vertices of G if $(v, w) \in E$. In this case, v is a *child* of w. A *tree* is a type of graph where for any two vertices $v, w \in V$, there exists a unique sequence of edges connecting v and w (see Figure 2.4).

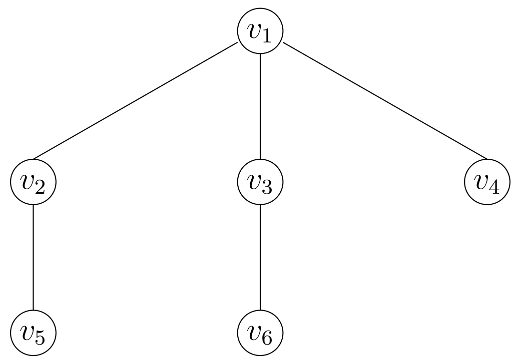

Figure 2.4 A tree

It is often useful to designate a specific vertex in the tree as the *root* of the graph. In general, there is nothing special about the vertex we choose to be the root. The *level* of a vertex $v \in V$ is the number of edges between v and the root. The *height* (with respect to a root) of a tree is the maximum level of a vertex in V. In Figure 2.4, if v_1 is the root, then the tree has height two. The vertex v_3 is on level one and has one child, v_6.

In a paper by Blanco and Rosales [3], they give an algorithm which constructs all irreducible numerical semigroups with a given Frobenius number in a tree structure. They begin with a numerical semigroup $C(F)$ that has the property that its elements are all larger than $\lfloor \frac{F}{2} \rfloor$:

$$C(F) := \{0, \lfloor F/2 \rfloor + 1, \rightarrow\} \backslash \{F\},$$

which is irreducible by Theorem 2.2.

Example 2.10. Let $F = 11$. Then $C(F) = \langle 6, 7, 8, 9, 10 \rangle$. From the figure below, we can see that each pairing has one filled in circle, and so $C(F)$ is symmetric.

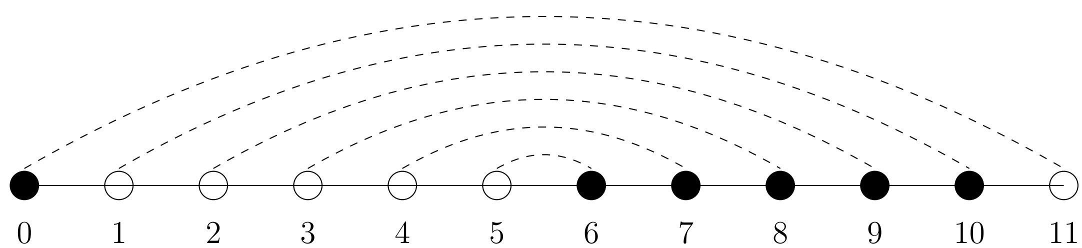

Figure 2.5 The semigroup $C(F) = \langle 6, 7, 8, 9, 10 \rangle$

Let $I(F)$ denote the set of all numerical semigroups with Frobenius number F. The algorithm of Blanco and Rosales below recursively defines a tree graph $G(I(F))$, starting with the root $C(F)$, whose vertices are precisely the elements of $I(F)$.

Theorem 2.3. *Let $F \in \mathbb{N}$. The elements of $I(F)$ comprise the vertices of a directed tree graph, denoted $G(I(F))$, with root $C(F)$. If $S \in I(F)$, then the children of S in $G(I(F))$ are the semigroups*

$$(S \backslash \{x_1\}) \cup \{F - x_1\}, \ldots, (S \backslash \{x_r\}) \cup \{F - x_r\},$$

where $\{x_1, \ldots, x_r\}$ *are the minimal generators of* S *that satisfy the following conditions for each* $x \in \{x_1, \ldots, x_r\}$*:*

1. $F/2 < x < F$
2. $2x - F \notin S$
3. $3x \neq 2F$
4. $4x \neq 3F$
5. $F - x < m(S)$.

Proof. See Blanco and Rosales [3, Theorem 9]. □

Example 2.11. Figure 2.6 is an image of $G(I(11))$. The six vertices of this tree are the irreducible numerical semigroups with Frobenius number 11.

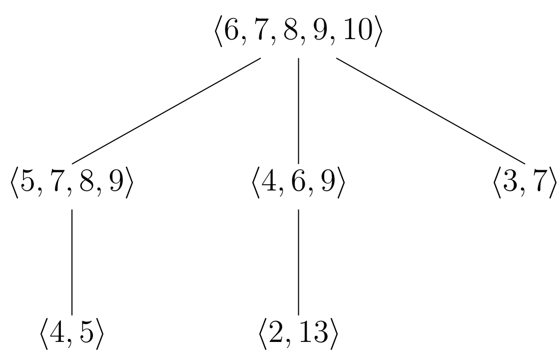

Figure 2.6 The tree of semigroups with Frobenius number 11

We have introduced Frobenius numbers, numerical semigroups, irreducible numerical semigroups, and their properties. We have discussed the graph theory properties of the algorithm we used to generate data. Now we will introduce our original results: upper and lower bounds on the number of irreducible numerical semigroups of a fixed Frobenius number.

3 Results

Laird and Martinez [4] investigate the graph-theoretic properties of $G(I(F))$ such as the height and the number of vertices of $G(I(F))$. They produced a table of the number of vertices and levels in $G(I(F))$ up to $F = 52$ (see Laird and Martinez [4, Figure 9]). They have shown the following about $G(I(F))$ [5]:

Theorem 3.1. *Let* $k > 6$ *be a positive integer. Then the height of* $G(I(F))$ *is* $\lfloor \frac{k}{2} \rfloor$ *for* $F = 2k + 1$ *and* $\lfloor \frac{k-1}{3} \rfloor$ *for* $F = 2k$*. Moreover, in the case* F *is odd, there is a unique branch in* $G(I(F))$ *of this length.*

To further the research they have started, we generated data for the number of vertices and levels of $G(I(F))$ up to $F = 174$. This data is located in the appendix. To create the data, we implemented the tree generating algorithm into a C++ program, which is available upon request, or online at

https://sites.google.com/site/ucrundergradmathresearch/about/numerical-semigroups.

Rather than looking at the graph-theoretic properties of $G(I(F))$, our effort went to finding estimates for the number of vertices of $G(I(F))$, that is, the number of irreducible numerical semigroups with Frobenius number F. Our result is a lower and upper bound on $\#I(F)$, where $\#I(F)$ denotes the number of irreducible numerical semigroups of Frobenius number F. We now restate Theorem 1.1:

Theorem 1.1 Let $F \in \mathbb{N}$. Then

$$2^{\lceil \frac{F}{2} \rceil - \lfloor \frac{F}{3} \rfloor - 1} \leq \#I(F) \leq 1 + \sum_{\substack{2 \leq m \leq \lceil \frac{F}{3} \rceil, \\ m \nmid F}} 2^{\lfloor \frac{F-2m}{2} \rfloor - \lfloor \frac{F-2m-1}{m} \rfloor} + \sum_{\substack{\lceil \frac{F}{3} \rceil < m \leq \lfloor \frac{F}{2} \rfloor, \\ m \nmid F}} 2^{\lfloor \frac{F-2m}{2} \rfloor}.$$

Figure 3.1 below gives a plot of these inequalities.

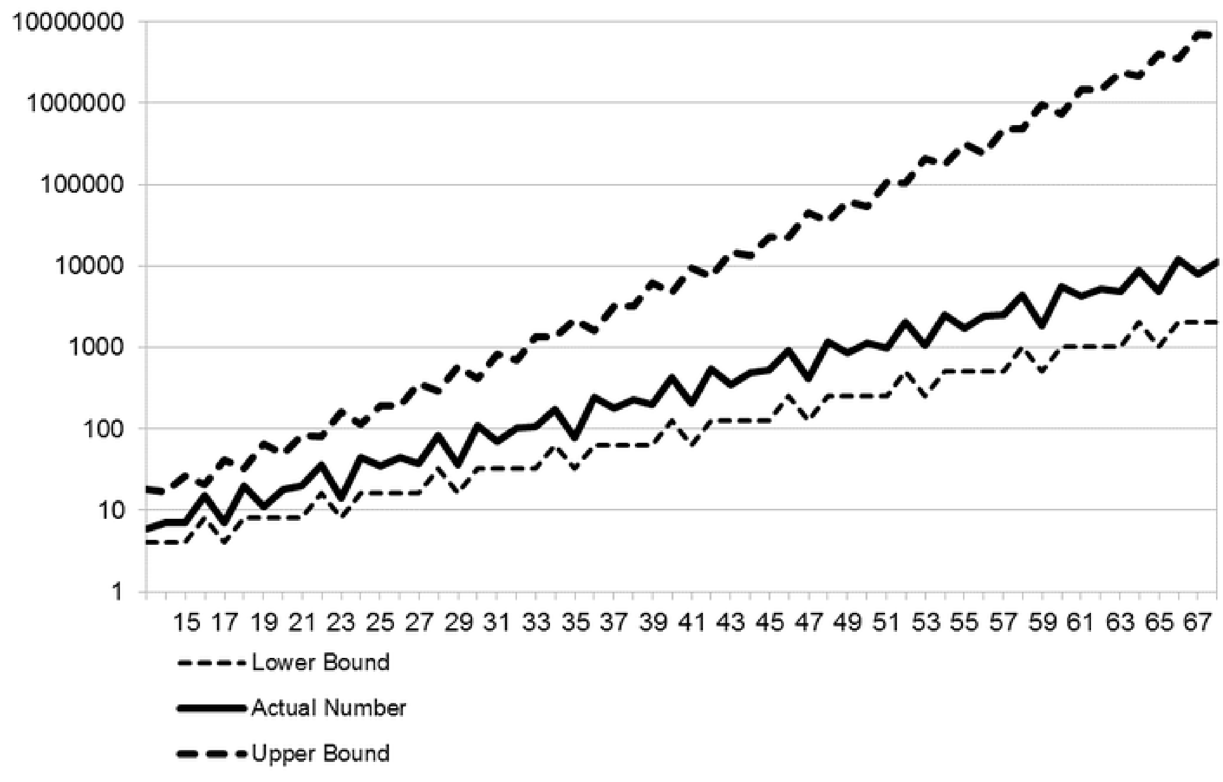

Figure 3.1 Plots of the inequalities and actual data

We split our proof into two subsections: the lower bound and upper bound. The lower bound is established by giving a collection of irreducible numerical semigroups with Frobenius number F whose size is easy to count. For the proof of the upper bound, the general idea is that an irreducible numerical semigroup is determined completely by the elements in a certain finite subset of $\mathbb{N}_0$.

3.1 Lower Bound

In this subsection we will prove Lemma 3.1, which shows how we can construct unique semigroups given a Frobenius number, and Corollary 3.1, which counts the number of these semigroups. Together these proofs give us the lower bound of Theorem 1.1. Let F be any natural number and let $\mathbb{N}_{>F}$ denote the set of natural numbers strictly larger than F.

Lemma 3.1. *Fix $F \in \mathbb{N}$, and let $A \subseteq \{x \in \mathbb{N} \mid \frac{F}{3} < x < \frac{F}{2}\}$. Then define the two sets*

$$B := \left\{F - x \mid \frac{F}{3} < x < \frac{F}{2} \text{ and } x \notin A\right\} \text{ and } C := \left\{x \in \mathbb{N} \mid \frac{2F}{3} \leq x < F\right\}.$$

If S is the set defined by

$$S = \{0\} \cup A \cup B \cup C \cup \mathbb{N}_{>F},$$

then S is an irreducible numerical semigroup with Frobenius number F.

Proof. Observe that each element of S is in only one of $\{0\}, A, B, C$, or $\mathbb{N}_{>F}$. We also note that, if $x, y \in S$, then $x + y \neq F$. Indeed, if on the other hand $x + y = F$, since $\frac{F}{2} \notin S$, without loss of generality we can say $x < \frac{F}{2}$. If $x \in S$, then x can only lie in A, and so $\frac{F}{2} < y = F - x < \frac{2F}{3}$. This forces $y \in B$. However this implies $y = F - x'$ for some $x' \notin A$. This would mean that $F - x = y = F - x' \Rightarrow x = x'$, a contradiction.

We will now show that S is closed under addition. That is, we claim that if $x, y \in S$, then $x + y \in S$. To do this, we look at several cases. If at least one of x or y is 0, then clearly $x + y \in S$, so that we can assume x and y are both nonzero. If x and y are both nonzero, then they are both larger than $\frac{F}{3}$, so that $x + y > \frac{2F}{3}$. Since $x + y \neq F$ by the above remark, we conclude $x + y \in C \cup \mathbb{N}_{>F} \subseteq S$. Thus S is closed under addition.

Since $\mathbb{N}_{>F} \subseteq S$, it is clear that S has a finite complement in $\mathbb{N}$ and hence is a numerical semigroup. By construction, it is apparent the Frobenius number of S is F. Therefore, the last thing to prove is that S is irreducible. Since $\frac{F}{2} \notin S$, by Theorem 2.2, it is enough to show that if $x \in \mathbb{Z} \backslash S$, then $F - x \in S$. We thus have the following cases for $x \notin S$:

- If $x < 0$, then since the Frobenius number of S is F, $F - x \in S$.
- If $0 < x \leq \frac{F}{3}$, then $\frac{2F}{3} \leq F - x < F$, so that $F - x \in C \subseteq S$.
- If $\frac{F}{3} < x < \frac{F}{2}$, then $F - x \in B \subseteq S$.
- If $\frac{F}{2} < x < \frac{2F}{3}$, then $\frac{F}{3} < F - x < \frac{F}{2}$. Now if $F - x \notin A$, then $F - (F - x) = x \in B \subseteq S$, a contradiction. Thus $F - x \in A \subseteq S$.

In every case, we conclude $F-x \in S$, and so S is symmetric or pseudo-symmetric depending on the parity of F. Hence S is irreducible. $\square$

Corollary 3.1. *If F is a natural number, then the number of irreducible numerical semigroups with Frobenius number F is at least $2^{\lceil \frac{F}{2} \rceil - \lfloor \frac{F}{3} \rfloor - 1}$.*

Proof. By the above lemma, for each $A \subseteq \{x \in \mathbb{N} \mid \frac{F}{3} < x < \frac{F}{2}\}$ there is an irreducible numerical semigroup S with Frobenius number F such that $S \cap \{x \in \mathbb{N} \mid \frac{F}{3} < x < \frac{F}{2}\} = A$. Since $S \in I(F)$, the set of all such S is contained in $I(F)$; therefore, the cardinality $|S| \leq |I(F)|$. Thus there are at least as many irreducible numerical semigroups with Frobenius number F as there are subsets of $\{x \in \mathbb{N} \mid \frac{F}{3} < x < \frac{F}{2}\}$. $\square$

3.2 Upper Bound

In this section, Lemmas 3.2, 3.3 and 3.4 show how we can construct sets with the symmetric property. Corollary 3.2 and Lemma 3.5 show how these lemmas can be used to prove the upper bound for Theorem 1.1. For $F \in \mathbb{N}$, we define Δ to be an *F-irreducible set* if $\Delta = S \cap [0, \frac{F}{2}]$ for some irreducible numerical semigroup S with Frobenius number F. We make a simple observation:

Lemma 3.2. *Let F be a natural number. There is a bijection between the collection of F-irreducible sets and the collection of irreducible numerical semigroups with Frobenius number F.*

Proof. If Δ is F-irreducible, then $\Delta = S \cap [0, \frac{F}{2}]$ for some irreducible numerical semigroup S with Frobenius number F. From Theorem 2.2, we then see that

$$S = \Delta \cup \{F - x \mid x \in \mathbb{Z} \cap [0, F/2] \text{ and } x \notin \Delta\} \cup \mathbb{N}_{>F},$$

and so S and Δ uniquely determine each other. $\square$

By the above lemma, if we establish an upper bound on the number of F-irreducible sets (for fixed F), this will be the same as finding an upper bound for the number of irreducible numerical semigroups with Frobenius number F. With this in mind, we shall frequently take advantage of this one-to-one correspondence.

Lemma 3.3. *If S is an irreducible numerical semigroup with Frobenius number F and multiplicity m, then $m \leq \frac{F}{2}$ or $S = C(F)$.*

Proof. Suppose $m > \frac{F}{2}$ and $x \in \mathbb{Z} \setminus S$ for some $F > x > \frac{F}{2}$. Then $F - x \in S$ since S is irreducible. Furthermore, $0 < F - x < F - \frac{F}{2} = \frac{F}{2} < m$, which contradicts that m is the multiplicity of S. Thus S contains $\lfloor \frac{F}{2} \rfloor + 1, \lfloor \frac{F}{2} \rfloor + 2, \ldots, F - 1$, and so $S = C(F)$. $\square$

Lemma 3.4. *Suppose $m \nmid F$. Let*

$$A(F, m) = \{x \in \mathbb{N} \mid m < x < F/2 \text{ and } m \mid x\}$$

and

$$B(F,m) = \{F - x \in \mathbb{N} \mid F/2 < x < F - m \text{ and } m \mid x\}.$$

Then $A(F,m) \cap B(F,m) = \varnothing$. Moreover, if Δ is an F-irreducible set with $m = \min\{\Delta - \{0\}\}$, then $A(F,m) \subseteq \Delta$ and $B(F,m) \cap \Delta = \varnothing$.

Proof. Suppose $y \in A(F,m) \cap B(F,m)$. Then $y = x$ where $m < x < F/2$, and $y = F - x'$ where $F/2 < x' < F - m$. Then $F = x + x'$, but since both $m \mid x$ and $m \mid x'$, we get $m \mid F$, which is a contradiction to our initial assumptions. So $A(F,m) \cap B(F,m) = \varnothing$. Let S be the irreducible numerical semigroup corresponding to Δ. Now if $x \in A(F,m)$, then

$$x = mt = \underbrace{m + m + \cdots + m}_{t \text{ times}} \in S$$

for some $t \in \mathbb{N}$. Since $x < F/2$, $x \in \Delta$. On the other hand, if $y \in B(F,m)$, then $y = F - x$ for some x where $m \mid x$. By the same reasoning as above, $x \in S$, so that $y \notin S$, otherwise $F = x + y \in S$. As well, $y \notin \Delta$. □

Corollary 3.2. *If $m \nmid F$ and $m \leq \lceil \frac{F}{3} \rceil$, then there are at most $2^{\lfloor \frac{F-2m}{2} \rfloor - \lfloor \frac{F-2m-1}{m} \rfloor}$ irreducible numerical semigroups with Frobenius number F and multiplicity m.*

Proof. Suppose that Δ is any F-irreducible subset with $m = \min\{\Delta - \{0\}\}$. In the notation of the above lemma, we know that $A(F,m) \cap B(F,m) = \varnothing$, $A(F,m) \subseteq \Delta$, and $B(F,m) \cap \Delta = \varnothing$. Now $\Delta, A(F,m)$, and $B(F,m)$ are contained in $\{0, 1, \ldots, \lfloor \frac{F}{2} \rfloor\}$. We know that $0, m \in \Delta$ and $1, \ldots, m-1 \notin \Delta$, and since there are $\lfloor \frac{F}{2} \rfloor + 1$ elements in the set $\{0, 1, \ldots, \lfloor \frac{F}{2} \rfloor\}$, we have $\lfloor \frac{F-2m}{2} \rfloor$ choices for the remaining elements of Δ. Also, we know every element of $A(F,m)$ lies in Δ, and no element of $B(F,m)$ lies in Δ. The set $B(F,m)$ is in bijection with the set

$$\widetilde{B}(F,m) = \{x \in \mathbb{N} \mid F/2 < x < F - m \text{ and } m \mid x\}.$$

The sets $A(F,m)$ and $B(F,m)$ are disjoint, so we have

$$\begin{aligned} \#A(F,m) \cup B(F,m) = \#A(F,m) \cup \widetilde{B}(F,m) &= \#\{x \in \mathbb{N} \mid m < x < F - m \text{ and } m \mid x\} \\ &= \left\lfloor \frac{F - 2m - 1}{m} \right\rfloor \end{aligned}$$

Thus there are at most $\lfloor \frac{F-2m}{2} \rfloor - \lfloor \frac{F-2m-1}{m} \rfloor$ choices for elements we can add to Δ. □

Remark 3.1. We require $m \leq \lceil \frac{F}{3} \rceil$ in the above corollary to ensure a multiple of m lies in the set $\{x \in \mathbb{N} \mid m < x < F - m\}$, otherwise the above bound is trivial. Indeed, if $m \leq \lceil \frac{F}{3} \rceil$, then $m < 2m < F - m$.

Lemma 3.5. *If $F \in \mathbb{N}$, then the number of irreducible numerical semigroups with Frobenius number F is at most*

$$1 + \sum_{\substack{2 \leq m \leq \lceil \frac{F}{3} \rceil, \\ m \nmid F}} 2^{\lfloor \frac{F-2m}{2} \rfloor - \lfloor \frac{F-2m-1}{m} \rfloor} + \sum_{\substack{\lceil \frac{F}{3} \rceil < m \leq \lfloor \frac{F}{2} \rfloor, \\ m \nmid F}} 2^{\lfloor \frac{F-2m}{2} \rfloor}.$$

Proof. There is only one numerical semigroup with Frobenius number F and multiplicity $m > \frac{F}{2}$ by Lemma 3.3. Furthermore, $m > 1$, since, if $m = 1$, then $F = 0$. This can be observed since, for any $G \geq 1$, then $G = \sum_1^G 1$, implying the largest number not in the set is less than 1. Suppose S is an irreducible numerical semigroup with Frobenius number F and multiplicity $2 \leq m \leq \frac{F}{2}$ and corresponding F-irreducible set Δ. Notice that $m \nmid F$, otherwise we would have $F \in S$. If

$$\left\lceil \frac{F}{3} \right\rceil < m \leq \left\lfloor \frac{F}{2} \right\rfloor,$$

then since $1, 2, \ldots, m-1 \notin \Delta$ and $0, m \in \Delta$ with $\Delta \subseteq \{0, 1, \ldots, \lfloor \frac{F}{2} \rfloor\}$, then we have at most $2^{\lfloor \frac{F-2m}{2} \rfloor}$ possibilities for Δ in this case. On the other hand, if $m \leq \lceil \frac{F}{3} \rceil$, then by Corollary 3.2, there are only $2^{\lfloor \frac{F-2m}{2} \rfloor - \lfloor \frac{F-2m-1}{m} \rfloor}$ possibilities for Δ. Therefore we have proved the lemma. $\square$

4 Further Research

The proof for the upper bound has us ignore sets where the multiplicity does not divide the Frobenius number. This suggests the number of divisors of F plays a role in the size of $\#I(F)$; the data as well seems to confirm this. Indeed, if p is prime, it seems we can expect $\#I(p)$ to be larger than $\#I(F)$ for values of F close to p. For example, compare $p = 173$, which is a prime number, to the numbers around it:

$$\#I(172) = 2489384088$$
$$\#I(173) = 4295034112$$
$$\#I(174) = 2501329283.$$

From our graph depicted in Figure 3.1, it appears that our upper bound for $\#I(F)$ is significantly less tight than our lower bound, so this leaves room for further investigation. Our lower bound worked by essentially building irreducible numerical semigroups with Frobenius number F from scratch. However, our construction only produced irreducible numerical semigroups where the multiplicity was larger than $F/3$. By modifying our proof, it may be possible to build more irreducible numerical semigroups with even smaller multiplicity, thus improving our lower bound.

Data Appendix

This is the data we generated with our code. It is sorted into columns by Frobenius number F, number of semigroups $\#I(F)$, and height of the tree H.

F	$\#I(F)$	H
50	844	8
51	1121	12
52	981	8
53	2015	13
54	1039	8
55	2496	13
56	1715	9
57	2436	14
58	2499	9
59	4350	14
60	1857	9
61	5602	15
62	4173	10
63	5317	15
64	4866	10
65	8925	16
66	4839	10
67	11971	16
68	7826	11
69	11276	17
70	10977	11
71	19812	17
72	9667	11
73	25405	18
74	19020	12
75	23297	18
76	21564	12
77	41642	19
78	22178	12
79	53629	19
80	35886	13
81	51367	20
82	51572	13
83	88093	20
84	41858	13
85	109693	21
86	84770	14
87	106526	21
88	100439	14
89	184466	22
90	98332	14
91	233557	22

F	$\#I(F)$	H
92	159833	15
93	222990	23
94	227059	15
95	375582	23
96	195266	15
97	490585	24
98	369450	16
99	471036	24
100	419865	16
101	799237	25
102	439443	16
103	1018271	25
104	721159	17
105	947145	26
106	984242	17
107	1655267	26
108	839515	17
109	2106583	27
110	1570954	18
111	2001431	27
112	1907837	18
113	3417576	28
114	1904303	18
115	4273853	28
116	3035371	19
117	4162984	29
118	4213006	19
119	7023282	29
120	3669716	19
121	8945931	30
122	6829161	20
123	8512341	30
124	8001407	20
125	14243981	31
126	8191764	20
127	18354729	31
128	13300505	21
129	17488308	32
130	17564858	21
131	29633619	32
132	15515628	21
133	37519331	33

F	$\#I(F)$	H
134	28837942	22
135	35684844	33
136	34851905	22
137	60616978	34
138	34528656	22
139	76862606	34
140	54324644	23
141	73350497	35
142	74940935	23
143	123578768	35
144	67741811	23
145	155117402	36
146	120786427	24
147	149694073	36
148	143249418	24
149	252298397	37
150	143390078	24
151	319562578	37
152	236185377	25
153	307429383	38
154	312030392	25
155	508565162	38
156	276966778	25
157	650111516	39
158	502427320	26
159	621579732	39
160	605352079	26
161	1043060844	40
162	606264503	26
163	1320746892	40
164	964432914	27
165	1253705942	41
166	1294345668	27
167	2118472893	41
168	1183417468	27
169	2676151556	42
170	2061857849	28
171	2577806875	42
172	2489384088	28
173	4295034112	43
174	2501329283	28

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