Rose-Hulman Undergraduate Mathematics Journal

Volume 18
Issue 1
Article 19

Analysis of Don't Break the Ice

Amy Hung
Doane University

Austin Uden
Doane University

Rose-Hulman Undergraduate Mathematics Journal

Analysis of Don't Break the Ice

Amy Hung[a] Austin Uden[b]

Volume 18, No. 1, Spring 2017

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
mathjournal@rose-hulman.edu
scholar.rose-hulman.edu/rhumj

[a]Doane University
[b]Doane University

Rose-Hulman Undergraduate Mathematics Journal
Volume 18, No. 1, Spring 2017

analysis of don't break the ice

Amy Hung Austin Uden

Abstract. The game Don't Break the Ice is a classic children's game that involves players taking turns hitting ice blocks out of a grid until a block containing a bear falls. We present Don't Break the Ice as a combinatorial game, and analyze various versions with an eye towards both normal and misére play. We present different winning strategies, some applying to specific games and some generalized for all versions of the game.

Acknowledgements: This project was funded by the Doane University Summer Research Program and Advised by Professor Kristopher Williams. The author wishes to thank the anonymous referee for a careful reading of the manuscript and many useful suggestions.

1 Introduction

Don't Break the Ice is a board game, typically played by children, in which players take turns hitting blocks of ice. The game usually consists of a 6-by-6 grid surrounded by walls, with a 2-by-2 block somewhere within the grid holding a bear. The rest of the grid is filled with 1-by-1 blocks of ice that the players are allowed to hit. The 2-by-2 block holding the bear is off limits to players. The objective of the original game is to force your opponent to remove a block that causes the 2-by-2 block holding the bear to fall.

The 1-by-1 blocks of ice will fall when ice on adjacent sides has been removed. Removing blocks on opposite sides of the ice block will not cause it to fall, as there will be opposing forces from the other opposite sides still holding that piece of ice in place.

The block containing the bear will fall when full supports on adjacent sides of the bear have been removed. If any parts of opposing supports remain in the game, the bear block will also remain in the game, since there will be opposing forces acting on the bear block to keep it suspended in the board. Figure 1 illustrates the support necessary to keep the bear from falling. The reflections and rotations of the diagrams are also possible.

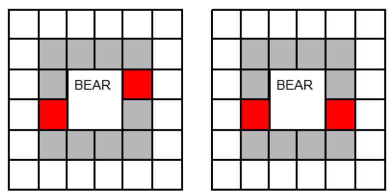

Figure 1: The blocks shaded red indicate blocks that have not been removed and are the least amount needed to keep the bear in play. The gray blocks represent blocks that may or may not still be in play.

In this paper, we discuss four unique games. Each game has a misére version and a normal version. Normal play is when the last person to make a legal move wins and misére play is when the last person to make a legal move loses. For more information, see Lessons in Play: An Introduction to Combinatorial Game Theory by Michael H. Albert, Richard J. Nowakowski, and David Wolfe [2]. Note that Don't Break the Ice is a misére game, although in this paper we will focus on the normal version, in which the player to drop the bear wins. We will refer to this game as Drop the Bear (DB).

We study an abstract version of Drop the Bear where the grid is not necessarily 6-by-6. Specifically, we prove the following.

Theorem 1.1. *On a 4-by-4 grid, Drop the Bear can always be won by the second player.*

Theorem 1.2. *On an even-by-even grid, Drop the Bear can always be won by the second player if the grid is divided into 2-by-2 sections and the bear is in one of those sections.*

In Section 2, we describe how we made the game a mathematical game instead of a game possibly determined by physics and misalignment. In Section 3, we prove a lemma for a related game, Clear the Ice, that will help us in later sections. In Section 4, we prove Theorem 1. In Section 5, we prove Theorem 2. In Section 6 we conclude with some conjectures related to other variations of the game.

2 Making Don't Break the Ice a Combinatorial Game

When looking at the original game of Don't Break the Ice, physics and misalignment play a key factor in the outcome of the game. With misaligned blocks, blocks that should fall might be held in place by a slight overlap of blocks. Physics can also play a role in blocks falling because if someone hits even a full game board hard enough, blocks surrounding the hit block might also fall.

To eliminate the effects of such anomalies, we analyzed the combinatorial analogue of the game on an abstract grid rather than attempting to study the physics of the actual game. When playing Don't Break the Ice, two players take turns hitting blocks of ice until the block holding the bear falls. Throughout the game, there are two scenarios that will cause a block to fall. If one of the players hits a block, the block will fall. The second way a block will fall is if adjacent supports for a block have been removed, as illustrated in Figure 2.

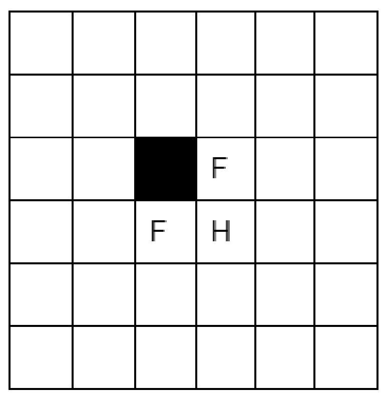

Figure 2: In this figure the black shaded block is a block that was previously removed from the game, the block label H is the block that has been hit, and the blocks labeled F are the two blocks that will fall since they are no longer supported on opposite sides.

The mathematical version of this game is considered a combinatorial game because it meets the following four criteria of a combinatorial game (see, for example, Bogomolny [1]). Criterion one is that there are two players moving alternately, which is satisfied directly by the rules of the game. Criterion two is that the rules are such that the game must eventually end. With a finite number of blocks, eventually all blocks will be eliminated, thus bringing the game to an end. Criterion three is that there are no draws and the winner is determined by who moves last. Whether playing a miseré version or normal version of the game, the winner is determined by whomever plays last; therefore, criterion three is satisfied. Criterion four is that there are no chance devices and both players have perfect information. Since we have eliminated both physics and possible misalignment, chance has been eliminated from the game. For both players to have perfect information means that both players can see the entire game board at the same time and nothing about the game is hidden by the other player. In Don't Break the Ice, both players can always see everything that is left in play that can be hit on their next turn; so, both players have perfect information. Therefore, criterion four is met.

3 Even-by-Even Clear the Ice

To help analyze the even-by-even games of Drop the Bear, we first simplify the game so that we do not have to worry about how the bear affects the outcome. In the simplified game, Clear the Ice (CI), there is no two-by-two block holding up a bear and instead the entire game board is made up of one-by-one blocks of ice. Clear the Ice is the normal version, where the goal is to drop the last block of ice, which wins the game. In this section, we will look at the even-by-even CI game and show that the second player can always win.

Lemma 3.1. *Let k and m be positive integers, and divide a $2k$-by-$2m$ board into km 2-by-2 sections. Then in Clear the Ice on such a board, the second player can always prevent the first player from removing the final block in any 2-by-2 section.*

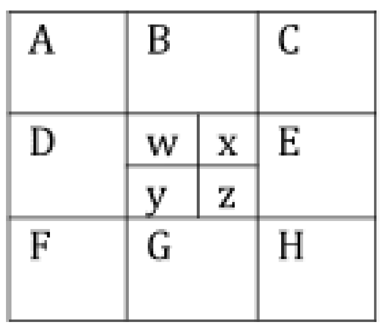

Figure 3: Two-by-two

Proof. First note that whenever a 2-by-2 section is removed from an even-by-even board in which other 2-by-2 sections are either totally present or completely removed, then the resulting board still has all 2-by-2 sections totally present or completely removed. We know this is true because if the 2-by-2 section that is removed is part of the only set of supporting forces that support an adjacent 2-by-2 section, then the adjacent 2-by-2 section will be completely removed as it will no longer be supported. If the removed 2-by-2 section is either not supporting the adjacent 2-by-2 section or if the adjacent 2-by-2 section is supported from all sides, then the adjacent 2-by-2 section will completely remain as it will still be supported by other 2-by-2 sections.

Now, the second player's strategy is as follows. Once the first player removes a block, the second player removes the brick diagonally opposite in the same 2-by-2 section. Consider

Figure 3. For the center 2-by-2 section to be present any combination of sections B, D, E, and G, that contains sections on opposite sides of the center section, must be present. When the first player removes block w, the most that will fall is either block x or block y, but not both. In order for blocks x and y to both fall after the first player removes block w, sections B and D would both need to be previously removed so the center 2-by-2 section would not be present. The second player can then remove block z which will remove any blocks in the center 2-by-2 section that remained after the first player's turn since both block x and block y are no longer supported on opposite sides.

Therefore, the second player is the only player that can remove the final block in any 2-by-2 section.

□

4 4-by-4 Drop the Bear

In this section, we will look at all unique 4-by-4 DB configurations and show that the second player can always win. In DB, recall that the goal of the game is to drop the bear in order to win. The three distinct positions (up to symmetry) for the bear in a 4-by-4 game are: the bear in the corner, the bear on the side, and the bear in the middle. We will use these to prove Theorem 1.1 from the introduction.

Lemma 4.1. *The 4-by-4 DB game with the bear in the corner can always be won by the second player.*

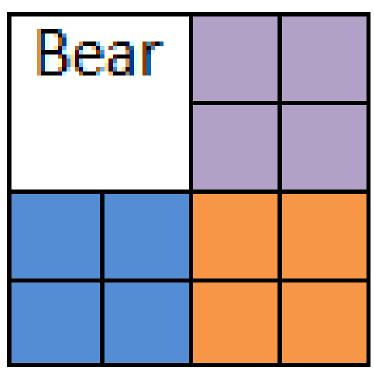

Figure 4: Corner bear

Proof. See Figure 4 for the setup of the game. The bear only falls when the entire blue section and the entire purple section have been eliminated. By Lemma 3.1, the second player can always be the player to eliminate a 2-by-2 section; thus, the second player will be the one to eliminate the final support, dropping the bear and winning the game. □

Lemma 4.2. *The 4-by-4 DB game with the bear on the side can always be won by the second player.*

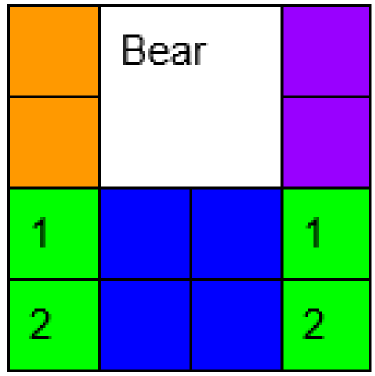

Figure 5: Side bear

Proof. See Figure 5 for the setup of the game. Wherever the first player goes, the second player responds by eliminating the same colored section. The only exception is the green section. If the first player hits a block in the green section, the second player responds by hitting the block that contains the same number as the one the first player hit. Note that the green section on the left is held on top and bottom by the orange section and the wall and on the left and right by the blue section and the wall. Therefore, the left green section will remain unless the orange and blue sections have been eliminated, in which case the bear would have dropped. Similarly, the green section on the right will always remain unless the bear has already dropped. This ensures that the move that the second player needs to respond with is always available if the bear is still in play.

Since the orange and purple sections are supported by the bear block and the wall, each of these sections will take two moves to remove. For example, if first player hits in either of these sections, then second player can respond by taking the second move in the same section and removing one of the bear supports.

In the blue section the top two blocks are supported above by the bear and below by the bottom two blocks. The bottom two blocks in the blue section are supported above by the top two blocks and below by the wall. Thus, the columns within the blue section are self supporting. If the first player hits in the left column, the most that can fall is the entire left column of the blue section since the right column will still be supported above and below. The second player should respond as they would to eliminate a 2-by-2 section to ensure the blue section is completely removed. Thus, the second player will always eliminate sections, eventually dropping the bear and winning the game.

□

Lemma 4.3. *The game 4-by-4 DB with the bear in the center is a second player win.*

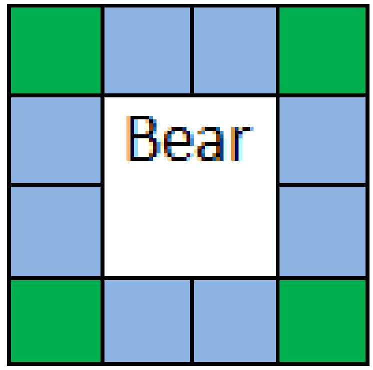

Figure 6: Center bear

Proof. See Figure 6 for the setup of the game. The second player's strategy is to move in the same blue section as the first player or remove a green block. The green blocks do not affect the bear falling since they are not part of a support. Also, if the first player were to hit a green block, then the second player would respond by hitting another one of the green blocks, leaving 0 or 2 green blocks in play and full blue sections. The only way for the first player to eliminate a green block without hitting the block is if the first player plays in a blue section that is adjacent to a blue section that is already eliminated. This scenario will be covered next.

Consider if the first player and second player both use their first move in the same blue section, eliminating that section. Then, if the first player plays in an adjacent blue section, the second player would respond by eliminating the blue section, which would cause the bear to fall since adjacent supports would be eliminated.

If the first player uses their second move to play in the blue section opposite of the one that is already eliminated, then second player would respond by eliminating the same section. This would leave the game in a state where opposite blue sections are holding the bear in place. If the first player were to play in either of these sections, then second player could respond by eliminating the other block in the section leaving the game with adjacent supports missing and dropping the bear.

Thus, second player will always be the last person to eliminate each section, eventually dropping the bear and winning the game. □

We now recall Theorem 1.1:

Theorem 1.1: *On a 4-by-4 grid, Drop the Bear can always be won by the second player.*

Proof. Once symmetries are eliminated, there are only three distinct bear positions in a 4-by-4 game. Our three lemmas showed that for each unique position, the second player can win the game. Therefore, the 4-by-4 DB game is always a second player win. □

At this time, we cannot extend Lemma 4.2 and Lemma 4.3 to larger boards because of the loss of symmetry when using these strategies on larger boards. These lemmas are currently only for the 4-by-4 game where the blocks are guaranteed to fall certain ways due to walls being present. Lemma 4.1 does extend, however, as we will see in the next section.

5 Even-by-Even Drop the Bear

In this section, we prove Theorem 1.2 from the introduction.

Theorem 1.2 *On an even-by-even grid, Drop the Bear can always be won by the second player if the grid is divided into 2-by-2 sections and the bear is in one of those sections.*

Proof. Consider a game board that is even-by-even, divided into 2-by-2 sections, with a 2-by-2 bear in one of the sections, as illustrated in Figure 7. The smallest this game board can be is 4-by-4, since a 2-by-2 game would consist only of the block holding the bear. From the previous section, we know that the 4-by-4 game with the bear in the corner is a second player win. We will now prove that any even-by-even game divided into 2-by-2 sections, with the bear in one of the sections, is a second player win.

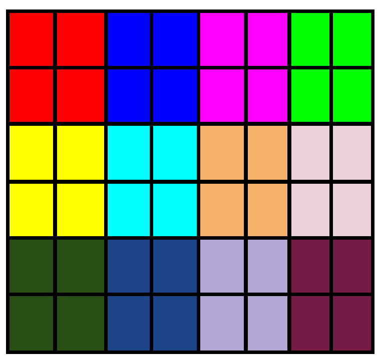

Figure 7: Even-by-even with the bear in one of the shaded 2-by-2 sections

Consider Figure 7. The bear can be any of the differently colored 2-by-2 blocks within the board. Since the game is in a state with a series of 2-by-2 sections that are all complete, using the strategy of Lemma 3.1, the second player is the only person that can remove an entire 2-by-2 section. Since the bear sits on a 2-by-2 block that cannot be hit, we can view the bear as an extension of the wall, but which has the ability to fall. With the block holding the bear acting as a wall, the rest of the game consists of 2-by-2 blocks that are either fully present or previously eliminated. In Lemma 3.1 we covered the scenarios in which sections were missing and showed how this did not affect the second player being the only player that can remove an entire 2-by-2 section. Therefore, since the bear acts as a fully present 2-by-2 block, it does not affect the second player being the only player that can remove an entire

2-by-2 section. Thus, the second player will be the person to eliminate the block that causes the bear to fall, ensuring them the win.

□

6 Future Research

In this section, we provide conjectures on other variations of Don't Break the Ice that may provide avenues for further research. We have worked through game trees and scenarios for some of the smaller cases within these conjectures. However, we do not have any nice ways to prove them at this time. These conjectures are related to the following games:

- Don't Clear the Dam (DCD), Clear the Dam (CD),
- Don't Clear the Bridge (DCB), and Clear the Bridge (CB).

In each of these games, there is no two-by-two block holding up a bear and instead the entire game board is made up of one-by-one blocks of ice. The difference between these two sets of games is that DCD and CD are played on a game board with three walls and DCB and CB are played on a game board only consisting of opposite walls. The misére versions are DCD and DCB. To win in DCD and DCB, you must force your opponent to cause the remaining ice to fall. The normal versions are CD and CB. To win in CD and CB, you must be the player that causes the last piece of ice to fall.

In an m-by-n game, m is the number of rows and n is the number of columns.

Conjecture 6.1. ***Clear the Dam***: *In an m-by-n game, if n is odd the first player can always win, and if n is even the second player can always win.*

Conjecture 6.2. ***Don't Clear the Dam***: *In an m-by-n game, if n is odd the second player can always win, and if n is even the first player can always win.*

Conjecture 6.3. ***Clear the Bridge***: *In an m-by-n game, if n is odd the first player can always win, and if n is even the second player can always win.*

Conjecture 6.4. ***Don't Clear the Bridge***: *In an m-by-n game, if n is odd the second player can always win, and if n is even the first player can always win.*

Bibliography

[1] Alexander Bogomolny. What is a combinatorial game, http://www.cut-the-knot.org/whatis/whatiscombinatorialgame.shtml.

[2] Richard J. Nowakowski Michael H. Albert and David Wolfe. *Lessons in Play: An Introduction to Combinatorial Game Theory.* A K Peters/CRC Press, 2007.