Rose-Hulman Undergraduate Mathematics Journal

Volume 18
Issue 1

Article 11

The Unimodular Determinant Spectrum Problem

Wilson Lough
University of Wisconsin - Madison

Rose-Hulman Undergraduate Mathematics Journal

The Unimodular Determinant Spectrum Problem

Wilson Lough[a]

Volume 18, No. 1, Spring 2017

Sponsored by

Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
mathjournal@rose-hulman.edu
scholar.rose-hulman.edu/rhumj

[a]University of Wisconsin-Madison

Rose-Hulman Undergraduate Mathematics Journal
Volume 18, No. 1, Spring 2017

THE UNIMODULAR DETERMINANT SPECTRUM PROBLEM

Wilson Lough

Abstract. We present results related to the determinant spectrum of matrices with entries restricted to quartic roots of unity. We completely characterize determinant spectra for small orders and present conjectures on the elements and structures of higher-order spectra.

Acknowledgements: This research was supported by the NASA Space Grant at Northern Arizona University.

1 Introduction

The study of determinants of matrices whose entries are restricted to the set $\{1, -1\}$ has a rich and varied history which begins with the 19th century mathematician James Sylvester [1]. The determinant values of these ± 1 matrices have been found by Sylvester and others to posses an interesting structure. The problem of determining the range of the determinant function, when restricted to ± 1 matrices of a given size, is known as the determinant spectrum problem. An obvious generalization of this problem can be obtained by enlarging the set which matrix entries are allowed to be chosen from. One such generalization is explored in this paper.

Some progress has been made classifying the determinant spectrum for matrices with entries restricted to the set $\{1, -1\}$. For all orders up to and including $n = 7$, the $\{1, -1\}$ spectra have been shown to consist of bounded sets of consecutive entries in arithmetic progressions. The determinant spectra for small order matrices appear in Table 1. The left column denotes matrix order; the right column lists the absolute values of all possible determinants. To better illustrate the pattern, Table 2 shows the determinant values scaled by powers of 2.

Table 1: $\{1, -1\}$ spectra for small orders

1	{1}
2	{0,2}
3	{0,4}
4	{0,8,16}
5	{0,16,32,48}
6	{0,32,64,96,128,160}
7	{0,64,128,192,256,320,384,448,512,576}

Table 2: The nature of the nth order spectra is evident when values are scaled by 2^{n-1}.

1	{1}
2	{0,1}
3	{0,1}
4	{0,1,2}
5	{0,1,2,3}
6	{0,1,2,3,4,5}
7	{0,1,2,3,4,5,6,7,8,9}

There was a short-lived conjecture that all $\{1, -1\}$ determinant spectra shared this common structure of consecutive entries in arithmetic progressions. However, it was shown by Metropolis et. al. [2] that gaps appear in the spectra beginning with $n = 8$. Ironically, it is now conjectured that all spectra with orders 8 and larger possess gaps. To date, the $\{1, -1\}$

determinant spectra are completely classified for all orders up to $n = 11$ as well as for $n = 13$, and the elements of several larger spectra have been conjectured. The interested reader may consult the website of Orrick [4] for more information.

In this paper we generalize the classical determinant spectrum problem by expanding the allowed set of matrix entries to include $\pm i$ in addition to ± 1. We find that the resulting spectra, which contains the original ± 1 spectra as a subset, posseses a richer structure and interesting symmetries. We investigate the additional structure of this enlarged spectra, classify the spectra for small orders, and conjecture about larger order spectra.

The rest of this paper is structured as follows. In Section 2, we will formally define determinants and discuss some of their applications. In Section 3, we will present Hadamard's bound on matrix determinants and the role it plays in providing extreme values of matrix determinants, notably those of Hadamard matrices. Finally, in Section 4, we will present new results and conjectures on a generalization of the determinant spectrum problem.

2 Matrix Determinants

Given a square matrix A we can compute a number, denoted $|A|$ or $\det(A)$, called the *determinant* of A. Matrix determinants are used in many areas of pure and applied mathematics, including solving systems of linear equations, finding volumes of parallelepipeds, and computing Jacobians of coordinate transformations. In the special case of a 2-by-2 matrix, the determinant can be computed using the simple formula:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

A *minor* is the determinant of the matrix obtained by deleting a row and column from some larger square matrix. The determinant of an n-by-n matrix can then be defined recursively as follows.

Definition 1. Let M_{ij} denote the minor formed by deleting the ith row and jth column of the n-by-n matrix A. The determinant of A is then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} M_{ij} \tag{2}$$

for any value of i between 1 and n.

Note that any minor which is the determinant of a matrix larger than 2-by-2 can be further decomposed using the above definition. This formula allows us to express the determinant of any square matrix as the sum of 2-by-2 determinants, which we know how to compute.

Example 1. The determinant of a 3-by-3 matrix is a sum of three 2-by-2 determinants.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg). \end{aligned} \tag{3}$$

3 Hadamard's Inequality and Hadamard Matrices

The problem of classifying all possible values taken on by the determinant function applied to matrices of order n whose entries are restricted to the set $\{1, -1\}$ dates back to James Sylvester's investigations in the 19th century. This set of values, now referred to as a determinant *spectrum*, was given an upper bound by Jacques Hadamard in 1893 [1]. His result is actually more general, as it bounds the determinants of matrices of order n with entries in the complex unit disk.

Proposition 1. *(Hadamard, 1893) If M is a matrix of order n with entries in the complex unit disk, then*

$$|\det(M)| \leq n^{n/2}.$$

The bound in Hadamard's inequality is sharp; it is easy to show that Hadamard's upper bound can be achieved, but only by matrices whose entries lie on the boundary of the unit disk in the complex plane $\mathbb{C}$. It then follows that for matrices whose entries are restricted to the real numbers $\mathbb{R}$, Hadamard's upper bound yields an upper bound on Sylvester's determinant spectrum problem, since these matrices have entries restricted to the set $\{1, -1\}$.

A necessary (but not sufficient) condition for the determinant of a $\{1, -1\}$ matrix to meet Hadamard's upper bound is that the matrix have order 1, order 2, or order $4n$ for any positive integer n. Matrices whose determinants achieve this upper bound are known as *Hadamard matrices*. It was first suggested by Paley in 1933 [1] that a Hadamard matrix might exist for every such value $4n$. Before 2005 this open question, known as the *Hadamard Conjecture*, was shown to be true for all orders less than $n = 428$. Following the ingeneous construction of an order 428 Hadamard matrix by Hadi Kharaghani and Behruz Tayfeh-Rezaie in 2005 [5], the conjecture has been verified up to and including $n = 667$.

It is well known that an equivalent definition of a Hadamard matrix of order n is a matrix H with entries in the set $\{1, -1\}$ that satisfies the equation

$$H \cdot H^T = n \cdot I_n,$$

where H^T denotes the transpose of H. It then follows that the columns of a Hadamard matrix are pairwise orthogonal. For additional information about Hadamard matrices and their applications the reader may consult the text by Horadam [1].

Example 2. The 4-by-4 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{4}$$

is an example of a Hadamard matrix. Note that its $\det(A) = 16$ satisfies the upper bound of $4^{4/2} = 16$ given by Hadamard's inequality. The columns of A are also pairwise orthogonal. For example, columns 2 and 3 satisfy

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 0. \tag{5}$$

Of course, the determinants of Hadamard matrices, being maximal, only describe the extreme values of the corresponding determinant spectra. The focus of this paper involves determining entire determinant spectra in a context that is more general than the setting investigated by Sylvester.

4 A Generalized Determinant Spectrum Problem

One generalization of the classic determinant spectrum problem is found by relaxing the restriction on possible matrix entries. Specifically, rather than requiring matrix entries to belong to the set $\{1, -1\}$ (that is, real quadratic roots of unity), we allow matrix entries to take on any value in the set $\{1, -1, i, -i\}$. The elements of this set are known as the complex *quartic roots of unity*, since they are roots of the polynomial $x^4 - 1$, and all lie on the boundary of the unit disk in $\mathbb{C}$.

Definition 2. For a positive integer n, let $\mathcal{M}_n$ denotes the set of all matrices of order n whose entries are quartic roots of unity; that is, $\mathcal{M}_n = \{[a_{ij}], 1 \leq i, j \leq n | a_{ij} \in \{1, -1, i, -i\}\}$.

Definition 3. For a positive integer n, let D_n denotes the determinant spectrum for all order n matrices whose entries are quartic roots of unity; that is, $D_n = \{\det(A) | A \in \mathcal{M}_n\}$.

The Gaussian integers $\mathbb{Z}[i]$ are complex numbers whose real and imaginary parts are integers. It follows immediately from the preceding definitions that for any positive integer n, the corresponding D_n is a subset of the Gaussian integers. In this paper we characterize D_2, D_3, and D_4 and describe several important properties of D_n for any positive integer n. Verifications of the common structures of these generalized spectra are relatively straightforward, and follow using basic facts from linear algebra. Perhaps the most fundamental property of the spectra is the following result.

Lemma 1. *The determinant spectrum D_n is closed under multiplication by quartic roots of unity.*

Proof. Let $A \in \mathcal{M}_n$. We can then always construct another matrix $B \in \mathcal{M}_n$ by multiplying any row or column of A by i^k for some positive integer k – that is, an arbitrary quartic root of unity. Then $\det(B) = i^k \det(A)$, and the result follows. □

As indicated in the previous proof, any element $A \in \mathcal{M}_n$ remains in $\mathcal{M}_n$ when multiplying a row or column of A by a quartic root of unity. Thus, any matrix $A \in \mathcal{M}_n$ can be manipulated so that all leading row and column entries of the resulting matrix are 1. A square matrix whose leading column and row entries are 1's is said to be *normalized.* The following is an immediate consequence of the previous lemma.

Lemma 2. *If $x \in D_n$, then there is a normalized matrix $B \in \mathcal{M}_n$ such that $x = i^k \det(B)$ for some $k \in \{0, 1, 2, 3\}$.*

Proof. If $x \in D_n$ then there is some $A \in \mathcal{M}_n$ whose determinant is x. We can obtain a normalized matrix $B \in \mathcal{M}_n$ by multiplying the rows and columns of A by powers of i. Then $x = i^k \det(B)$ for some $k \in \{0, 1, 2, 3\}$. □

Another important property of D_n is presented in the following lemma.

Lemma 3. *The determinant spectrum D_n is closed under complex conjugation.*

Proof. For a complex number z, let $\bar{z}$ denote its complex conjugate and for a matrix A, let $\overline{A}$ be the matrix obtained by replacing each entry of A with its complex conjugate. If $\det(A) \in D_n$ then $\overline{A} \in \mathcal{M}_n$ since $\overline{i^k}$ is a quartic root of unity for any positive integer k. Thus $\overline{\det(A)} = \det(\overline{A}) \in D_n$. □

Initial attempts at analyzing our generalized determinant spectra using the traditional definition of a determinant were limited. But Chió's determinant formula, an alternative and underappreciated method for computing matrix determinants, proved to be much more useful.

Proposition 2. *(Chió, 1853) Let $A = [a_{ij}]$ be a matrix of order n with $a_{11} \neq 0$. Then*

$$\det A = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}.$$

The key role played by Chió's determinant formula is apparent in the proof of the next result. A more detailed analysis of Chió's determinant formula can be found in the article by Fuller and Logan [3].

Lemma 4. *If $x \in D_4$, the real and imaginary parts of x are congruent modulo 4.*

Proof. Let $x \in D_4$. Then there is a normalized matrix $A \in \mathcal{M}_4$ such that $x = i^k \det(A)$ for some $k \in \{0, 1, 2, 3\}$. By Chió's determinant formula,

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & i^{k_1} & i^{k_2} & i^{k_3} \\ 1 & i^{k_4} & i^{k_5} & i^{k_6} \\ 1 & i^{k_7} & i^{k_8} & i^{k_9} \end{vmatrix} = \begin{vmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{vmatrix},$$

where each $z_n = \begin{vmatrix} 1 & 1 \\ 1 & i^{k_n} \end{vmatrix} = i^{k_n} - 1$. Using the more common determinant formula, we have that

$$\det(A) = z_1z_5z_9 - z_1z_6z_8 - z_2z_4z_9 + z_2z_6z_7 + z_3z_4z_8 - z_3z_5z_7.$$

We will now show that the real and imaginary parts of each product $z_pz_qz_r$ are congruent modulo 4. To begin, note that since $z_n = i^{k_n} - 1$, each product $z_pz_qz_r$ is of the form

$$i^{n_p+n_q+n_r} - i^{n_p+n_q} - i^{n_q+n_r} - i^{n_p+n_r} + i^{n_p} + i^{n_q} + i^{n_r} - 1.$$

Observe that because i has order 4 in $\mathbb{C}$, we may assume without loss of generality that $n_p, n_q, n_r \in \{0, 1, 2, 3\}$. It must be the case that either all three of the exponents n_p, n_q, and n_r are congruent modulo 2 or exactly two are congruent modulo 2. If all three exponents are congruent modulo 2 then they are either all even or all odd. Suppose they are all odd. Then each of the expressions $-i^{n_p+n_q}, -i^{n_q+n_r}, -i^{n_p+n_r}, -1$ is real and each of the expressions $i^{n_p+n_q+n_r}, i^{n_p}, i^{n_q}, i^{n_r}$ is imaginary. We now have four cases to consider.

Case 1: If $n_p = n_q = n_r = 1$, then the real component is $-i^{n_p+n_q} - i^{n_q+n_r} - i^{n_p+n_r} - 1 = 1+1+1-1 = 2$ and the imaginary component is $i^{n_p+n_q+n_r} + i^{n_p} + i^{n_q} + i^{n_r} = -i+i+i+i = 2i$.

Case 2: If $n_p = 1, n_q = n_r = 3$, then the real component is $-i^{n_p+n_q} - i^{n_q+n_r} - i^{n_p+n_r} - 1 = -1+1-1-1 = -2$ and the imaginary component is $i^{n_p+n_q+n_r} + i^{n_p} + i^{n_q} + i^{n_r} = -i+i-i-i = -2i$.

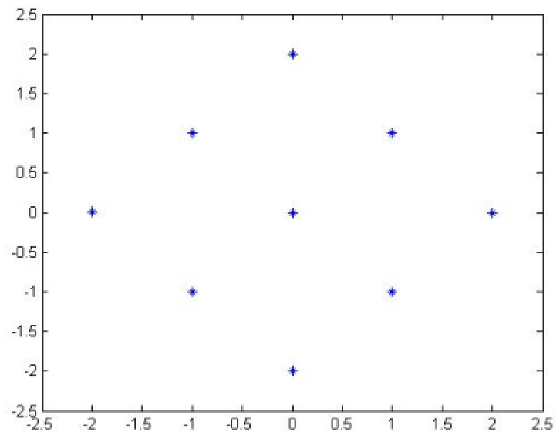
Case 3: If $n_p = n_q = n_r = 3$, then the real component is $-i^{n_p+n_q} - i^{n_q+n_r} - i^{n_p+n_r} - 1 = 1+1+1-1 = 2$ and the imaginary component is $i^{n_p+n_q+n_r} + i^{n_p} + i^{n_q} + i^{n_r} = i-i-i-i = -2i$.

Case 4: If $n_p = n_q = 1, n_r = 3$, then the real component is $-i^{n_p+n_q} - i^{n_q+n_r} - i^{n_p+n_r} - 1 = 1-1-1-1 = -2$ and the imaginary component is $i^{n_p+n_q+n_r} + i^{n_p} + i^{n_q} + i^{n_r} = i+i+i-iki = 2i$.

In all four cases the real and imaginary components of $z_pz_qz_r$ are congruent to 2 modulo 4. Thus the real and imaginary parts of the sum or difference of any pair of products of the form $z_pz_qz_r$ must be congruent to 0 modulo 4. Therefore, since they are both even, the real and imaginary components of the determinant of A, and hence $x = i^k \det(A)$, are congruent modulo 4. The remaining cases concerning the parity of the exponents n_p, n_q, and n_r are similar and left to the reader. □

These results allow us to draw several conclusions about the visual structure of our generalized spectra. They include:

Figure 1: $D_2 = \{0, (1+i)i^k, 2i^k | k \in \{0,1,2,3\}\}$



- Since $D_n \subseteq \mathbb{Z}[i]$ for every positive integer n, each D_n should be a bounded, two-dimensional lattice-like structure.
- Since each D_n is closed under multiplication by quartic roots of unity and complex conjugation, the spectra should be highly symmetric with respect both the real and imaginary axes and with respect to the origin.
- Since the real and imaginary parts of every element in D_4 are congruent modulo 4, the lattice-like structure associated with D_4 should have a fundamental region larger than that of the unit square in the Gaussian integers.

A brute force search using MATLAB yielded complete images of D_n for small values of n. Note that the spectra D_2 and D_3, which appear in Figure 1 and Figure 2, respectively, form complete, highly symmetric, bounded subsets of the lattice of Gaussian integers. Thus, the structure of D_2 and D_3 is, not surprisingly, the two-dimensional complex analogue of the structure of the one-dimensional $\{1, -1\}$ spectra for smaller orders.

The Gaussian integer lattice-like structure formed by the order 4 spectrum contains gaps, and appears in Figure 3. The appearance of gaps in D_4 is somewhat surprising, since as previously noted, gaps do not appear in the one-dimensional $\{1, -1\}$ spectra until the order 8 case.

As seen in Figure 3, D_4 is bounded by the lines $x+y = 16$, $-x+y = 16$, $-x-y = 16$, and $x-y = 16$ in the complex plane. More precisely, D_4 contains all points in the aforementioned restricted region of $\mathbb{C}$ whose real and imaginary parts are pairs of even integers that are congruent modulo 4, with the exception of $i^k(14 \pm 2i)$ for $k \in \{0,1,2,3\}$. These are the 8 points that appear, in pairs, as gaps near the corners of D_4. We summarize these observations in the following result.

Figure 2: $D_3 = \{0, 2i^k, 4i^k, (2 \pm 4i)i^k | k \in \{0, 1, 2, 3\}$

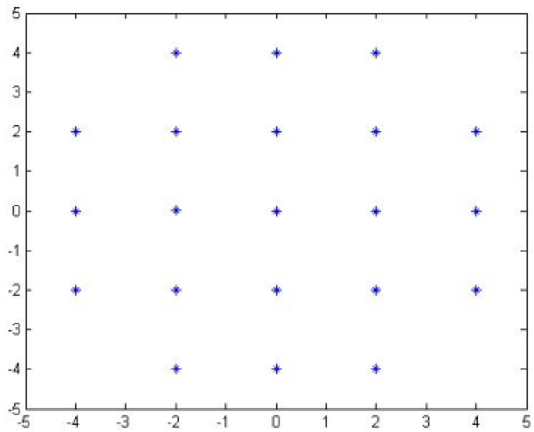
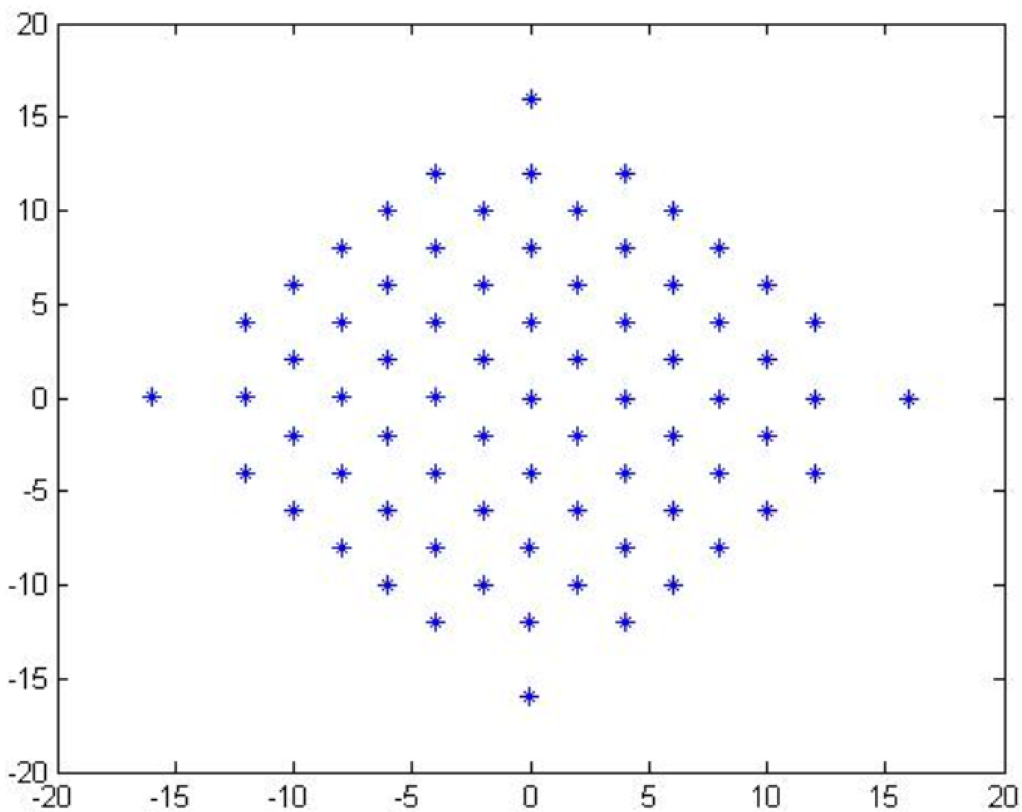


Figure 3: D_4



Theorem 1. *The D_4 spectrum is* $\{0, (2+2i)i^k, 4i^k, (4+4i)i^k, (2\pm 6i)i^k, 8i^k, (6+6i)i^k, (4\pm 8i)i^k, (2\pm 10i)i^k, (8+8i)i^k, (6\pm 10i)i^k, 12i^k, (4\pm 12i)i^k, 16i^k | k \in \{0,1,2,3\}\}$.

The spectra depicted in Figures 1, 2, and 3 show two distinct lattice-like structures. The fundamental domain of D_3 is a square whose sides are parallel to the coordinate axes, whereas the fundamental domain of both D_2 and D_4 is a rhombus. These observations, together with the above results, as well as an extensive search using randomly generated matrices in MATLAB, point toward the following conjecture.

Conjecture 1. *For any positive integer k, $D_{2k-1} \subseteq \{a+bi | a, b \in 2^{k-1}\mathbb{Z}\}$, and $D_{2k} \subseteq \{a+bi | a, b \in 2^{k-1}\mathbb{Z}, a \equiv b \ (mod\ 2^k)\}$.*

That is, the bounded lattice-like structures of these generalized spectra feature fundamental domains, with the fundamental domains associated with the D_{2k-1} spectra being a sequence of squares whose sizes are strictly increasing as k increases, whereas the fundamental domains associated with the D_{2k} spectra comprise a sequence of rhombai whose sizes are likewise strictly increasing.

We also conjecture that for all $k \geq 4$, the lattice-like structures of the corresponding D_k contain gaps.

References

[1] K. J. Horadam, *Hadamard Matrices and Their Applications*, Princeton University Press, 2007.

[2] N. Metropolis, *Spectra of determinant values in (0,1) matrices*, A. O. L. Atkin and B. J. Birch, editors, Computers in Number Theory: Proceedings of the Science Research Atlas Symposium No. 2 held at Oxford, from 18-23 August, 1969, pages 271-276, London, 1971. Academic Press.

[3] L. E. Fuller and J. D. Logan, *On the Evaluation of Determinants by Chio's Method*, The Two-Year College Mathematics Journal, 6, 8-10 (1975).

[4] `http://www.indiana.edu/~maxdet/spectrum.html`, an open source website maintained by Will Orrick (2002).

[5] H. Kharaghani and B. Tayfeh-Rezaie, *A Hadamard matrix of order 428*, J.Combin.Designs, 13, 435-440 (2005).