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### Recommended Citation

Talib, Omar and Mohamed, Souleimane Cheikh Sidi (2017) "Computing the Autocorrelation Function for the Autoregressive Process," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 18 : Iss. 1 , Article 1. Available at: <https://scholar.rose-hulman.edu/rhumj/vol18/iss1/1>

**ROSE-  
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MATHEMATICS  
JOURNAL**

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THE AUTOREGRESSIVE PROCESS**

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# COMPUTING THE AUTOCORRELATION FUNCTION FOR THE AUTOREGRESSIVE PROCESS

Omar Talib      Souleimane Cheikh Sidi Mohamed

**Abstract.** In this document, we explain how complex integration theory can be used to compute the autocorrelation function for the autoregressive process. In particular, we use the deformation invariance theorem, and Cauchy's residue theorem to reduce the problem of computing the autocorrelation function to the problem of computing residues of a particular function. The purpose of this paper is not only to illustrate a method by which one can derive the autocorrelation function of the autoregressive process, but also to demonstrate the applicability of complex analysis in statistical theory through simple examples.

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**Acknowledgements:** The authors would like to thank Dr. Nassor Suleiman Nassor for his encouragement.

## 1 Introduction

A stochastic process is defined as a sequence  $(X_t)_{t \in \mathbb{Z}}$  of random variables. Different values of the variable  $t$  represent different points of time; hence a stochastic process serves a model for time-varying random phenomena. To understand and study a stochastic process we associate with it various functions such as the mean function  $\mu_t = E[X_t]$ , the autocovariance function  $\delta_{t,s} = \text{Cov}(X_t, X_s)$  and the autocorrelation function  $\rho_{t,s} = \text{Corr}(X_t, X_s)$ . Our concern in what follows will be employing complex analytic techniques to compute the autocorrelation function for the autoregressive stochastic process defined below.

A very simple, though important, example of a stochastic process that we shall need later is a sequence  $(e_t)_{t \in \mathbb{N}}$  of independent and identically distributed random variables. Such a stochastic process is called a white noise process. For this process, the mean function  $\mu_t$  is constant and the autocovariance function  $\delta_{t,s}$  is equal to zero when  $t \neq s$  and is equal to the variance of  $e_t$  when  $t = s$ . We may assume that the mean of the white noise process has been subtracted out to produce a white noise process  $\varepsilon_t = e_t - \mu_t$  with zero mean and let  $\sigma_\varepsilon^2$  denote the variance of this process.

A stochastic process that satisfies the two conditions: The mean function  $\mu_t$  is constant for all the values of  $t$ , and the autocovariance function  $\delta$  satisfies  $\delta_{t,t-k} = \delta_{0,k}$  for all  $t$  and  $k$ , is said to be a weakly-stationary process. In other words, a weakly-stationary process is a process for which the autocorrelation and autocovariance functions depend only on the time-lag  $k$  and not on the particular points of time  $t$  and  $t - k$ . Accordingly, for such a process we shall denote  $\delta_{t,t-k}$  and  $\rho_{t,t-k}$  by  $\delta_k$  and  $\rho_k$  respectively, omitting the unnecessary information from the notation. Observe that the white noise process is an example of a weakly-stationary process. Observe further that for a weakly-stationary process  $(X_t)_{t \in \mathbb{Z}}$  if we set  $t' = t - k$  then  $\rho_k = \rho_{t,t-k} = \text{Corr}(X_t, X_{t-k}) = \text{Corr}(X_{t'-(-k)}, X_{t'}) = \rho_{-k}$ . We shall use this property frequently in this paper. Moreover, since weak stationarity is the only form of stationarity that will appear in this paper, we shall omit the word “weak” and simply refer to weak stationarity as stationarity.

In many cases, the current state of a random process depends on the previous states and so, a very natural stochastic process would be one in which this dependence is linear. A  $p$ th-order autoregressive process, abbreviated as  $AR(p)$  process, is a stochastic process in which the current value  $X_t$  is a linear combination of the  $p$  most recent values  $X_{t-1}, X_{t-2}, \dots, X_{t-p}$  plus a stochastic term  $\varepsilon_t$  that includes the portion of variability in  $X_t$  which cannot be explained by the past values. More precisely, an  $AR(p)$  process satisfies

$$X_t = \sum_{i=1}^p \varphi_i X_{t-i} + \varepsilon_t,$$

where  $\varphi_1, \varphi_2, \dots, \varphi_p$  are the model parameters and  $\varepsilon_t$  is a white noise process, independent of the random variables  $X_{t-1}, X_{t-2}, \dots, X_{t-p}$ . We shall not be interested in the stationarity conditions for the  $AR(p)$  process (the conditions under which the process is stationary) in what follows and will work exclusively with stationary autoregressive processes. In other words, throughout this paper,

whenever we are working with an autoregressive process, we shall implicitly assume that it is stationary. For such a process, the mean function is constant and we may assume that the process mean is subtracted out to produce a process with zero mean.

To measure the linear dependence between the random variables  $X_t$  and  $X_{t-k}$  in an autoregressive process, we may use either the autocovariance function  $\delta_k$  or the autocorrelation function  $\rho_k$ . The latter is usually preferred since it is unitless. Our major concern in this paper will accordingly be the computation of the autocorrelation function  $\rho_k$  for the  $AR(p)$  process. In computing this autocorrelation function, we will be using tools from complex analysis that we shall briefly review in the next section. In the third section, we begin our investigation by considering the special cases of the  $AR(1)$  and  $AR(2)$  models. Then, in the fourth section we study the autocorrelation function of the general  $AR(p)$  model. Our work in Section 3 and Section 4 will be based on the assumptions and notations introduced above.

## 2 Preliminary Considerations

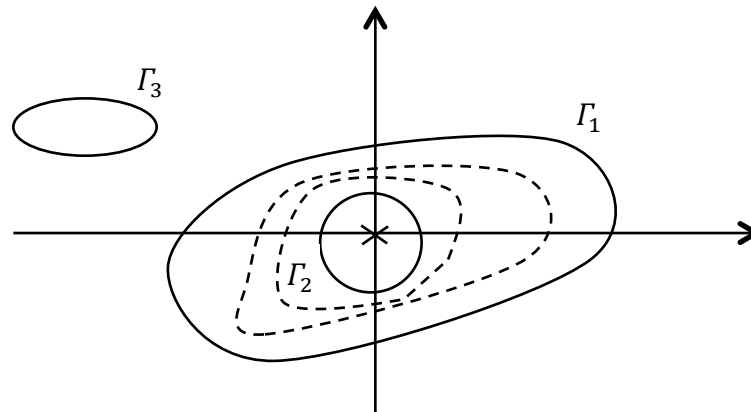
We assume that the reader is familiar with the basic notions of complex analysis such as Cauchy's theorem, Laurent series, computing residues, and Cauchy's residue theorem, and will simply review the tools that will be necessary in our subsequent work. This section will contain a brief review of the deformation invariance theorem, Taylor and Laurent series, and Cauchy's residue theorem. If desired, the reader may skip this section and start reading from Section 3 where our discussion of the main topic begins.

### 2.1 Homotopic Loops

Recall that two oriented, simple, closed contours  $\Gamma_1$  and  $\Gamma_2$  in a domain  $D \subseteq \mathbb{C}$  are said to be homotopic in  $D$  if there exists a continuous function  $\psi: [0,1] \times [0,1] \rightarrow D$  such that  $\psi(t, 0); 0 \leq t \leq 1$  is a parametrization of  $\Gamma_1$ ,  $\psi(t, 1); 0 \leq t \leq 1$  is a parametrization of  $\Gamma_2$ , and for every  $s \in [0,1]$ ,  $\psi(t, s); 0 \leq t \leq 1$  is a parametrization of a loop lying in  $D$ . The function  $\psi$  is called a homotopic deformation from  $\Gamma_1$  to  $\Gamma_2$ . In other words, two simple loops in a domain of the complex plane are homotopic in that domain if one of them can be continuously deformed into the other within the domain (see Figure 2.1.1).

**Example 2.1.1** Let us fix a domain, say the punctured complex plane  $\mathbb{C} - \{z_0\}$ , where  $z_0$  is some complex number, and an analytic function in that domain, say  $f(z) = 1/(z - z_0)$ . If we integrate this function over the positively oriented simple closed contour  $C_r$  which is a circle of radius  $r > 0$  centered at  $z_0$ , then we obtain the following

$$I_r = \oint_{C_r} \frac{dz}{z - z_0} = \int_0^1 \frac{2\pi i r e^{2\pi i t}}{r e^{2\pi i t}} dt = 2\pi i.$$



**Figure 2.1.1** In the domain  $\mathbb{C} - \{0\}$ ,  $\Gamma_1$  and  $\Gamma_2$  are homotopic but  $\Gamma_1$  and  $\Gamma_3$  are not homotopic.

Thus, the integral  $I_r$  is in fact independent of the radius  $r$ . Moreover, it is intuitively clear (and not difficult to prove) that any two contours in the family  $\{C_r\}_{r \in \mathbb{R}^+}$  of contours are homotopic ■

Example 2.1.1 suggests that the integral of an analytic function is invariant under continuous deformations. It turns out that this is true and this result, which we state next, is known as the deformation invariance theorem.

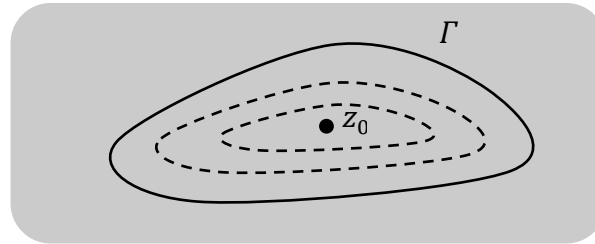
**Theorem 2.1.1 (Deformation Invariance Theorem)** If  $f$  is analytic in a domain  $D$  in which the two loops  $\Gamma_1$  and  $\Gamma_2$  are homotopic, then

$$\oint_{\Gamma_1} f(z) dz = \oint_{\Gamma_2} f(z) dz.$$

For a proof of this theorem, see Conway [1].

As an example to this theorem, consider a function  $f$  analytic inside a simply connected domain  $D$ , and a loop  $\Gamma$  inside  $D$ . Then, by Theorem 2.1.1 above, the integral of  $f$  over the contour  $\Gamma$  is equal to the integral of  $f$  over a point  $z_0$  inside  $D$  since the two are homotopic in  $D$  (see Figure 2.1.2), but the latter integral is zero. Thus, it follows from Theorem 2.1.1 that the integral of a function  $f$  that is analytic in a simply connected domain over a loop inside that domain is zero. This result is sometimes referred to as Cauchy's integral theorem.

The continuous deformation theorem is a very powerful tool in complex integration theory since it allows us to compute integrals over complicated contours by computing the same integrals over much simpler (homotopic) contours. Indeed, this theorem, together with Cauchy's residue theorem (discussed below), will be at the heart of the method described in the next section, which will enable us to compute the autocorrelation function of the autoregressive process.



**Figure 2.1.2** In a simply connected domain a loop  $\Gamma$  is homotopic to a point  $z_0$ .

## 2.2 Taylor and Laurent Series

There are various ways to characterize analytic functions, one of which involves the Cauchy-Riemann Equations. We begin this subsection by describing another characterization of analytic functions which is the local representation of functions by Taylor series. We shall not prove the theorems stated here since the proofs are relatively lengthy and our goal is simply to review these concepts and refer to them later in the paper. The proofs can be found in fifth chapter of the textbook: Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics by Edward B. Saff and Arthur David Snider [3].

**Theorem 2.2.1** A power series  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  that converges to a function  $f(z)$  for all  $z$  in the disk  $|z - z_0| < R$ , converges uniformly in any closed subdisk  $|z - z_0| \leq R' < R$ . Furthermore, the function  $f$  to which the series converges is analytic at every point inside the circle of convergence.

Observe that because of uniform convergence of power series, we can integrate and differentiate them term-by-term. This fact together with the generalized Cauchy integral formula (which does not appear in this document) can be used to prove the next corollary which states that a convergent power series is necessarily the Taylor series of the analytic function to which it converges. Before stating the corollary however, we would like to remind the reader that analytic functions have Taylor series representations at their points of analyticity so that a complex function is analytic at a point if and only if it has a Taylor series representation at that point.

**Corollary 2.2.1** If the series  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  converges to  $f(z)$  in some circular neighborhood of  $z_0$  then

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta, \quad (k = 0, 1, 2, \dots)$$

where  $C$  is a positively oriented circle centered at  $z_0$  and lying inside the circle of convergence of the series.

Next, we discuss Laurent series. Recall first that a point  $z_0$  where  $f$  is not analytic is called a singularity of  $f$ . Furthermore,  $z_0$  is called an isolated singularity of  $f$  if  $f$  is analytic in the domain  $0 < |z - z_0| < R$ , for some positive  $R$ , but not analytic at the point  $z_0$  itself. For example, a rational function has isolated singularities at the roots of its denominator polynomial. Obviously, a function  $f$  with an isolated singularity at  $z_0$  cannot have a Taylor series representation at  $z_0$  (otherwise  $f$  will be analytic at  $z_0$  by the discussion above) but it can be expressed as a sum of two series as stated in the next result.

**Theorem 2.2.2** Let  $f$  be analytic in the domain  $0 < |z - z_0| < R$ . Then  $f$  can be expressed in this domain as the sum of two series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k}$$

both of which converge in that domain. Furthermore, this convergence is uniform in any closed set of the form  $0 < r_1 \leq |z - z_0| \leq r_2 < R$ .

The sum of the two series in Theorem 2.2.2 is often denoted by

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

and is called the Laurent series of  $f$  at  $z_0$ .

If  $f$  has an isolated singularity at  $z_0$  and if the Laurent series of  $f$  takes the form

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + a_0 + a_1(z - z_0) + \cdots,$$

where  $a_{-n} \neq 0$  then  $f$  is said to have a pole of order  $n$  at  $z_0$ . It is not difficult to show that a function  $f$  has a pole of order  $n$  at  $z_0$  if and only if in some neighborhood of  $z_0$ , that excludes  $z_0$ , we can express  $f$  as

$$f(z) = \frac{g(z)}{(z - z_0)^n},$$

where the function  $g$  is analytic at  $z_0$  and  $g(z_0) \neq 0$ . In particular, the singularities of rational functions are poles.



### 2.3 Cauchy's Residue Theorem

Let  $z_0$  be an isolated singularity of the function  $f$  that is analytic in  $D - \{z_0\}$ , for some domain  $D$  containing  $z_0$ , and let

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

be the Laurent series of  $f$  at  $z_0$ . If  $C$  is a positively oriented loop in  $D$  that does not contain  $z_0$  in its interior then by the deformation invariance theorem (Theorem 2.1.1)

$$\oint_C f(z) dz = 0.$$

However, if  $C$  contains the isolated singularity  $z_0$  in its interior then it cannot be continuously deformed to a point in  $D$  so we cannot conclude that the contour integral of  $f$  over such a contour is zero. To compute this integral, we use Theorem 2.2.2 and the fact that a uniformly convergent series can be integrated term-by-term:

$$\begin{aligned} \oint_C f(z) dz &= \oint_C \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k dz \\ &= \sum_{k=-\infty}^{\infty} a_k \oint_C (z - z_0)^k dz = 2\pi i a_{-1}. \end{aligned}$$

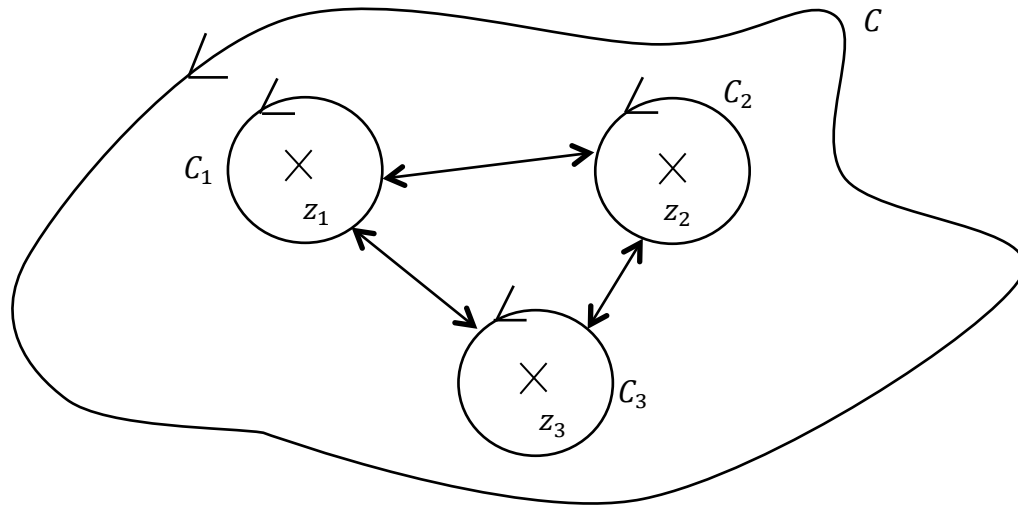
The last equality follows from the deformation invariance theorem and a trivial computation which shows that

$$\oint_{\Gamma} (z - z_0)^k dz = \begin{cases} 2\pi i; & \text{for } k = -1 \\ 0; & \text{for } k \neq -1, \end{cases}$$

where  $\Gamma$  is a positively oriented loop containing  $z_0$  in its interior. The case  $k = -1$  was considered earlier in Example 2.1.1.

The computation above indicates the importance of the coefficient  $a_{-1}$  in evaluating an integral of  $f$ . This coefficient is called the residue of  $f$  at  $z_0$  and is denoted by  $\text{Res}[f(z); z_0]$ .

If the simple closed positively oriented contour  $C$  contains in its interior a finite number of isolated singularities of  $f$ , say  $z_1, z_2, \dots, z_n$ , then by Theorem 2.1.1, the integral of  $f$  over  $C$  can be written as a sum of  $n$  integrals of  $f$  each being over a circular loop  $C_i$  centered at  $z_i$  and



**Figure 2.3.1** The contour  $C$  is homotopic to the contour containing the circles  $C_1, C_2, C_3$  and the lines connecting them.

containing no singularity other than  $z_i$  (see Figure 2.3.1). Combining this observation with the discussion in the previous paragraph we get the following result known as Cauchy's residue theorem.

**Theorem 2.3.1 (Cauchy's Residue Theorem)** Let  $f$  be analytic in a domain  $D$  except at the isolated singularities  $z_1, z_2, \dots, z_n \in D$ . If the simple closed positively oriented contour  $C$  contains in its interior these singularities of  $f$  and is itself contained in  $D$  then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z); z_k].$$

To put Cauchy's residue theorem to use, it remains to discuss the topic of computing the residues at the isolated singularities of a given function  $f$ . One obvious method is to find the Laurent series of  $f$  at each one of the isolated singularities and then read the coefficient  $a_{-1}$  from the series. However, this method is not always the most efficient one, especially when the isolated singularity is a pole of  $f$  which, in fact, is the only case in which we shall be interested. Let us suppose that  $f$  has a pole of order 2 at  $z_0$  then the Laurent series of  $f$  at  $z_0$  takes the form

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots; \quad a_{-2} \neq 0.$$

A simple computation shows that

$$\text{Res}[f(z); z_0] = a_{-1} = \lim_{z \rightarrow z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

A similar argument shows that if  $f$  has a pole of order  $n$  at  $z_0$ , then

$$\text{Res}[f(z); z_0] = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)].$$

### 3 The AR(1) and AR(2) Processes

We begin this section by considering the simple case of the AR(1) process and derive its autocorrelation function using elementary methods. Then we use the various complex analytic tools described in Section 2 of this document to derive the autocorrelation function for the AR(2) process. Although the same techniques that will be used here to derive the autocorrelation function for the AR(2) process can be applied to the general case of the AR( $p$ ) process, we delay the discussion of the general case to the next section and focus now on the second-order autoregressive process. This is because the deformation of contours arguments are easily illustrated with figures in this special case. We shall work with the notation provided in the introduction and under the several assumptions stated there. In particular, we assume that the processes are stationary with zero mean.

#### 3.1 The AR(1) Process

To begin with, let us consider the case of a first-order autoregressive model

$$X_t = \varphi_1 X_{t-1} + \varepsilon_t. \quad (3.1)$$

Taking variance of both sides of this equation gives  $\delta_0 = \varphi_1^2 \delta_0 + \sigma_\varepsilon^2$  so that

$$\delta_0 = \sigma_\varepsilon^2 / (1 - \varphi_1^2). \quad (3.2)$$

For this expression of the variance to be valid (the variance should be nonnegative), we must restrict the possible values of  $\varphi_1$  so that  $|\varphi_1| < 1$ . Next, to compute the autocorrelation function we multiply both sides of Equation 3.1 by  $X_{t-k}$ , take expectations, use the fact that  $E[X_t] = 0$  and that  $\varepsilon_t$  and  $X_{t-k}$  are independent to obtain  $\delta_k = \varphi_1 \delta_{k-1}$ . Combining this with Equation 3.2 we get  $\delta_k = \varphi_1^k [\sigma_\varepsilon^2 / (1 - \varphi_1^2)]$  or

$$\rho_k = \frac{\delta_k}{\delta_0} = \varphi_1^k; \quad k \in \{0, 1, 2, \dots\}. \quad (3.3)$$

Equation 3.3 is an explicit formula for the autocorrelation function of the  $AR(1)$  model. We can conclude from Equation 3.3 and the fact that  $|\varphi_1| < 1$ , that the autocorrelation function  $\rho_k$  approaches zero (exponentially) as the time-lag  $k$  increases without bound. This is just a precise mathematical way of describing the intuitive idea that the (linear) relation between two outcomes of the variable  $X$  becomes weaker as the time interval between them becomes wider.

### 3.2 The $AR(2)$ Process

Next, let us consider the second-order autoregressive process

$$X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \varepsilon_t. \quad (3.4)$$

Taking variance of both sides of Equation 3.4 gives the equation  $\delta_0 = \varphi_1^2 \delta_0 + \varphi_2^2 \delta_0 + \sigma_\varepsilon^2$  which upon solving for  $\delta_0$  yields

$$\delta_0 = \frac{\sigma_\varepsilon^2}{1 - \varphi_1^2 - \varphi_2^2}. \quad (3.5)$$

Multiplying both sides of Equation 3.4 by  $X_{t-k}$  and then taking expectation of both sides gives  $\delta_k = \varphi_1 \delta_{k-1} + \varphi_2 \delta_{k-2}$ . Dividing through by  $\delta_0$  yields the recursive formula

$$\rho_{k+2} = \varphi_1 \rho_{k+1} + \varphi_2 \rho_k; \quad k \geq 0, \quad (3.6)$$

with which the initial conditions  $\rho_0 = 1$  and  $\rho_1 = \varphi_1/(1 - \varphi_2)$  are associated. The first initial condition is true since the correlation between a random variable and itself is always 1 and the second follows from the Equation 3.6 by setting  $k = -1$  and using the fact that  $\rho_k = \rho_{-k}$ .

Although Equation 3.6 together with the initial conditions can be used to compute the autocorrelation function  $\rho_k$ , for a given value of  $k$ , recursively, an explicit formula is desirable in certain situations. To find an explicit expression for  $\rho_k$  we define the (generating) function

$$f(z) = \sum_{k=0}^{\infty} \rho_k z^k, \quad (3.7)$$

which by Theorem 2.2.1 is analytic inside its radius of convergence (see Section 5 below). Our goal is to find an explicit expression of the coefficients  $\rho_k$  appearing in Equation 3.7 and to achieve this goal we will use the various tools described in Section 2. First it follows from Corollary 2.2.1 that

$$\rho_k = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z^{k+1}} dz, \quad (3.8)$$

where the integral is taken over a sufficiently small positively oriented circular contour  $C_R$  (we specify below exactly how small must the radius of this circular contour be) centered at the origin. Equation 3.8 can be useful if a closed form of the function  $f$  is available, and such a form is what we shall seek next.

If we multiply both sides of Equation 3.6 by  $z^k$  and then sum over  $k \in \{0, 1, 2, \dots\}$  we get

$$\sum_{k=0}^{\infty} \rho_{k+2} z^k = \varphi_1 \sum_{k=0}^{\infty} \rho_{k+1} z^k + \varphi_2 \sum_{k=0}^{\infty} \rho_k z^k.$$

If we now multiply and divide the left hand side and the first term in the right hand side of this equation by  $z^2$  and  $z$  respectively, we get

$$\frac{1}{z^2} \sum_{k=0}^{\infty} \rho_{k+2} z^{k+2} = \frac{\varphi_1}{z} \sum_{k=0}^{\infty} \rho_{k+1} z^{k+1} + \varphi_2 \sum_{k=0}^{\infty} \rho_k z^k,$$

or

$$\frac{1}{z^2} [f(z) - \rho_0 - \rho_1 z] = \frac{\varphi_1}{z} [f(z) - \rho_0] + \varphi_2 f(z).$$

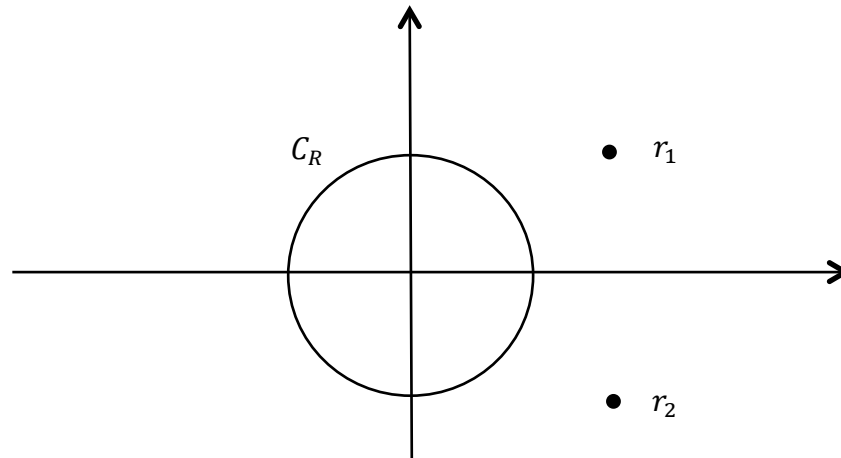
Solving this equation for  $f(z)$  and substituting  $\rho_0 = 1$  gives us the desired closed form

$$f(z) = \frac{(\varphi_1 - \rho_1)z - 1}{\varphi_2 \left( z^2 + \frac{\varphi_1}{\varphi_2} z - \frac{1}{\varphi_2} \right)}. \quad (3.9)$$

It follows from Corollary 2.2.1 that  $\rho_k = f^{(k)}(0)/k!$  and now that we have an expression for the function  $f$ , we might compute the values of  $\rho_k$  by differentiating the expression in Equation 3.9. However, computing higher derivatives of this expression is a very tedious task, and accordingly, we shall avoid this method. Instead, we shall use both Cauchy's residue theorem (Theorem 2.3.1) and the deformation invariance theorem (Theorem 2.1.1) to compute  $\rho_k$ . Let us first rewrite Equation 3.9 after factoring the polynomial in the denominator:

$$f(z) = \frac{(\varphi_1 - \rho_1)z - 1}{\varphi_2(z - r_1)(z - r_2)}, \quad (3.9^*)$$

where  $r_1, r_2 = (-\varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2})/2\varphi_2$ . Observe from Equation 3.9 that both  $r_1$  and  $r_2$  are different from zero. Also  $\varphi_2 \neq 0$ , otherwise this case will reduce to the  $AR(1)$  process discussed earlier. These two observations imply that the rational function  $f$  given by Equation 3.9 is well defined, and analytic at  $z = 0$ .



**Figure 3.2.1** The radius  $R$  of the circle  $C_R$  is less than the minimum of the distances from the two roots  $r_1$  and  $r_2$  to the origin.

From Equation 3.8 we have

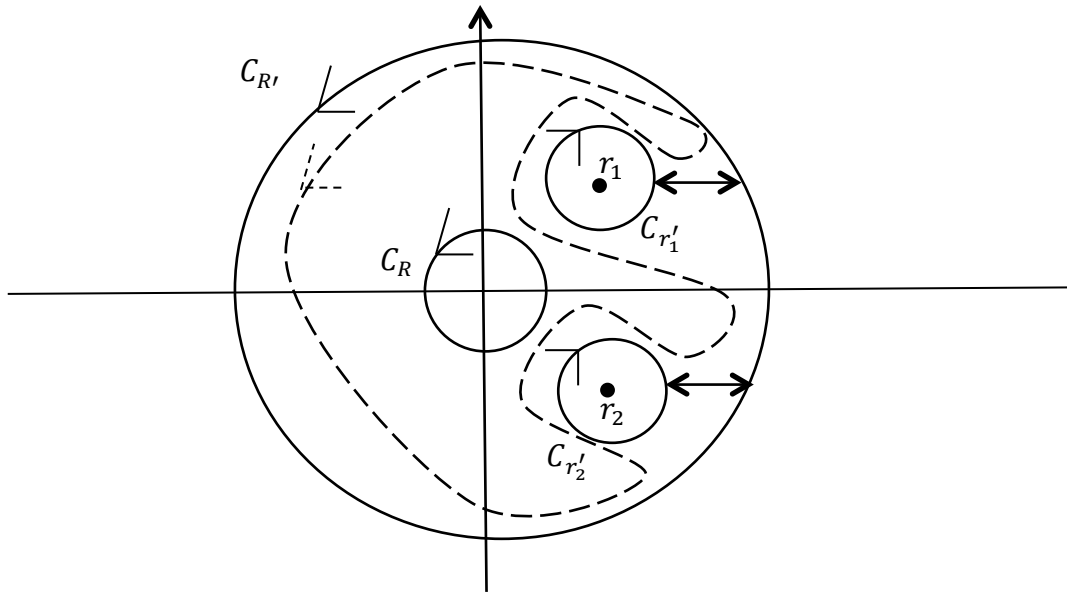
$$\rho_k = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z^{k+1}} dz = \frac{1}{2\pi i} \oint_{C_R} \frac{(\varphi_1 - \rho_1)z - 1}{\varphi_2 z^{k+1}(z - r_1)(z - r_2)} dz, \quad (3.10)$$

where  $C_R$  is a circular contour centered at the origin with a radius  $R$  less than  $\min\{|r_1|, |r_2|\}$  (see Figure 3.2.1).

Cauchy's residue theorem alone will not be very useful in evaluating the integral in Equation 3.10 since the integrand has a pole of order  $k + 1$  at the origin. However, using the deformation invariance theorem we can conclude that

$$\oint_{C_R} \frac{f(z)}{z^{k+1}} dz = \oint_{C_{R'}} \frac{f(z)}{z^{k+1}} dz + \oint_{C_{r'_1}} \frac{f(z)}{z^{k+1}} dz + \oint_{C_{r'_2}} \frac{f(z)}{z^{k+1}} dz, \quad (3.11)$$

where  $C_{R'}$ ,  $C_{r'_1}$ , and  $C_{r'_2}$  are circular contours of radii  $R'$ ,  $r'_1$  and  $r'_2$  and centers  $z = 0$ ,  $z = r_1$  and  $z = r_2$  respectively, as shown in Figure 3.2.2. (Observe the orientations carefully.)



**Figure 3.2.2** The integral over the contour  $C_R$  is equal to the sum of integrals over the contours  $C_{R'}$ ,  $C_{r_1}$  and  $C_{r_2}$  by the deformation invariance theorem.

However, there is no contribution from the integral over the contour  $C_{R'}$ . Indeed,

$$\begin{aligned} \left| \oint_{C_{R'}} \frac{f(z)}{z^{k+1}} dz \right| &\leq \oint_{C_{R'}} \left| \frac{f(z)}{z^{k+1}} \right| |dz| \\ &= \int_0^1 \frac{|f(z)|}{R'^{k+1}} (2\pi R') dt \\ &\leq \int_0^1 \frac{|\varphi_1 - \rho_1| R' + 1}{|\varphi_2| R'^{k+1} (R' - |r_1|)(R' - |r_2|)} (2\pi R') dt \rightarrow 0, \end{aligned}$$

as  $R' \rightarrow \infty$ , and by the deformation invariance theorem

$$\oint_{C_{R'}} \frac{f(z)}{z^{k+1}} dz = \lim_{R' \rightarrow \infty} \oint_{C_{R'}} \frac{f(z)}{z^{k+1}} dz = 0.$$

Thus, we may rewrite Equation 3.11 as

$$\begin{aligned} \oint_{C_R} \frac{f(z)}{z^{k+1}} dz &= \oint_{C_{r_1}} \frac{f(z)}{z^{k+1}} dz + \oint_{C_{r_2}} \frac{f(z)}{z^{k+1}} dz \\ &= -2\pi i \sum_j \operatorname{Res} \left[ \frac{f(z)}{z^{k+1}}; r_j \right], \quad (3.12) \end{aligned}$$

where the last equality follows from Cauchy's residue theorem. Combining Equation 3.10, Equation 3.11 and Equation 3.12 gives us

$$\rho_k = - \sum_{j=1}^m \operatorname{Res} \left[ \frac{f(z)}{z^{k+1}}; r_j \right], \quad (3.13)$$

where  $m$  is the number of distinct complex roots of the polynomial  $h(z) = z^2 + (\varphi_1/\varphi_2)z - 1/\varphi_2$ .

Since the computation of residues will depend on the order of the nonzero pole(s) of the function  $f(z)/z^{k+1}$ , we will have two different formulas of  $\rho_k$  for the  $AR(2)$  process depending on the value of  $m$  which is either 1 or 2. Before we consider the two possible cases, we would like to emphasize that Equation 3.13 is a general expression for  $\rho_k$  and the summation index  $j$  will run over the values 1 and 2 when the polynomial  $h$  has two distinct roots, and will take on a single value,  $j = 1$ , when  $h$  has a single complex root of multiplicity 2. This will be particularly useful when we consider the general  $AR(p)$  process in the next section, where many different cases regarding the orders of the poles will be possible. Now we consider the two possible cases for the  $AR(2)$  process.

#### *Case I ( $r_1 \neq r_2$ )*

If the complex roots of the polynomial  $z^2 + (\varphi_1/\varphi_2)z - 1/\varphi_2$  are distinct, the value of  $m$  will be 2 and the rational function  $f(z)/z^{k+1}$  will have a simple pole at  $z = r_1$  and another simple pole at  $z = r_2$ . By Equation 3.13

$$\begin{aligned} \rho_k &= - \sum_{j=1}^2 \operatorname{Res} \left[ \frac{f(z)}{z^{k+1}}; r_j \right] \\ &= - \left[ \lim_{z \rightarrow r_1} \frac{(z - r_1)f(z)}{z^{k+1}} + \lim_{z \rightarrow r_2} \frac{(z - r_2)f(z)}{z^{k+1}} \right] \end{aligned}$$



$$= \frac{1 + (\rho_1 - \varphi_1)r_1}{\varphi_2 r_1^{k+1}(r_1 - r_2)} + \frac{1 + (\rho_1 - \varphi_1)r_2}{\varphi_2 r_2^{k+1}(r_2 - r_1)}. \quad (3.14)$$

Substituting the expressions of  $\rho_1$ ,  $r_1$  and  $r_2$  in terms of the parameters  $\varphi_1$  and  $\varphi_2$  in Equation 3.14 gives us the desired explicit formula for the autocorrelation function  $\rho_k$  in terms of the parameters  $\varphi_1$  and  $\varphi_2$ .

*Case II ( $r_1 = r_2 = r$ )*

If the polynomial  $z^2 + (\varphi_1/\varphi_2)z - 1/\varphi_2$  has a single complex root  $r$  of multiplicity 2 then the rational function  $f(z)/z^{k+1}$  will have a pole of order 2 at  $z = r$  and by Equation 3.13

$$\begin{aligned} \rho_k &= - \sum_{j=1}^1 \text{Res} \left[ \frac{f(z)}{z^{k+1}}; r_j \right] \\ &= - \left[ \lim_{z \rightarrow r} \frac{d}{dz} \frac{(z-r)^2 f(z)}{z^{k+1}} \right] \\ &= \frac{k[(\varphi_1 - \rho_1)r - 1] - 1}{\varphi_2 r^{k+2}}. \end{aligned} \quad (3.15)$$

Again, substituting the expressions of  $\rho_1$  and  $r$  in terms of the parameters  $\varphi_1$  and  $\varphi_2$  in Equation 3.15 gives us the desired explicit formula for the autocorrelation function  $\rho_k$  in terms of the parameters  $\varphi_1$  and  $\varphi_2$ .

Now that explicit expressions for the autocorrelation function  $\rho_k$  for the two cases discussed above are available, proofs of them can be given using mathematical induction. We leave this task to the reader and consider the general case of the  $AR(p)$  process next.

## 4 The $AR(p)$ Process

Our work in this section will be parallel to that in the previous section. The same ideas and techniques that were applied there in the special case of the  $AR(2)$  process can be applied here in the general case to verify the statements that we shall not verify.

The  $p$ th-order autoregressive process takes the form

$$X_t = \sum_{i=1}^p \varphi_i X_{t-i} + \varepsilon_t. \quad (4.1)$$

Taking the variance of both side of Equation 4.1 gives  $\delta_0 = \delta_0 \sum_{i=1}^p \varphi_i^2 + \sigma_\varepsilon^2$ , which upon solving for  $\delta_0$  gives

$$\delta_0 = \frac{\sigma_\varepsilon^2}{1 - \sum_{i=1}^p \varphi_i^2}. \quad (4.2)$$

Multiplying both sides of Equation 4.1 by  $X_{t-k}$  and then taking expectation of both sides gives  $\delta_k = \sum_{i=1}^p \varphi_i \delta_{k-i}$ . Dividing through by  $\delta_0$  yields the recursive formula

$$\rho_{k+p} = \sum_{i=1}^p \varphi_i \rho_{k+p-i}; k \geq 0. \quad (4.3)$$

As we did earlier in the previous section, the next step is to define the (generating) function

$$f(z) = \sum_{k=0}^{\infty} \rho_k z^k. \quad (4.4)$$

An argument similar to the one given in the previous section (the details are left to the reader) yields a closed form of  $f$ :

$$\begin{aligned} f(z) &= \frac{S(p-1) - [\varphi_1 S(p-2)z + \varphi_2 S(p-3)z^2 + \cdots + \varphi_{p-1} S(0)z^{p-1}]}{1 - [\varphi_1 z + \varphi_2 z^2 + \cdots + \varphi_p z^p]} \\ &= \frac{S(p-1) - [\varphi_1 S(p-2)z + \varphi_2 S(p-3)z^2 + \cdots + \varphi_{p-1} S(0)z^{p-1}]}{-\varphi_p [(z-r_1)(z-r_2)\cdots(z-r_p)]}, \end{aligned} \quad (4.5)$$

where  $S(j) = \sum_{k=0}^j \rho_k z^k$ ;  $j \geq 0$ , and  $r_1, r_2, \dots, r_p$  are the complex roots of the polynomial

$$h(z) = z^p + (\varphi_{p-1}/\varphi_p)z^{p-1} + \cdots + (\varphi_1/\varphi_p)z - 1/\varphi_p. \quad (4.6)$$

Notice that  $z = 0$  is not a root of the polynomial  $h$  and that  $\varphi_p \neq 0$  otherwise this process will reduce to an  $AR(p-1)$  process. Thus, the function  $f$  given by Equation 4.5 is well defined, and analytic at  $z = 0$ .

Next, we follow the same arguments that led to Equation 3.13 in the previous section (the details are omitted) to obtain

$$\rho_k = - \sum_{j=1}^m \text{Res} \left[ \frac{f(z)}{z^{k+1}}; r_j \right], \quad (4.7)$$

where  $f$  is the function given by Equation 4.5, and  $m$  is the number of distinct roots of the polynomial  $h$  defined by Equation 4.6

Notice that the expression of  $f$  in Equation 4.5 includes the sums  $S(j) = \sum_{k=0}^j \rho_k z^k$ ;  $0 \leq j \leq p-1$ . Accordingly, the expression of  $\rho_k$  given by Equation 4.7 above depends on the initial conditions  $\rho_k$ ;  $0 \leq k \leq p-1$  of the recurrence relation given by Equation 4.3. To remove this dependence and have an explicit expression of  $\rho_k$ , we must express these initial values in terms of the model parameters  $\varphi_1, \varphi_2, \dots, \varphi_p$ . This is easily done using the facts that  $\rho_0 = 1$  and  $\rho_k = \rho_{-k}$  once the parameters of the model are specified. We illustrate this procedure in Example 4.1 below.

**Example 4.1** Let us consider the AR(3) process with  $\varphi_1 = 3$ ,  $\varphi_2 = -4$  and  $\varphi_3 = 12$ :

$$X_t = 3X_{t-1} - 4X_{t-2} + 12X_{t-3} + \varepsilon_t$$

First, we note that the roots of the polynomial  $h(z) = z^3 - (1/3)z^2 + (1/4)z - 1/12$  are  $r_1 = i/2$ ,  $r_2 = -i/2$  and  $r_3 = 1/3$ . From Equation 4.7 above we have

$$\rho_k = \sum_{j=1}^3 \text{Res} \left[ \frac{f(z)}{z^{k+1}}; r_i \right],$$

where

$$\begin{aligned} f(z) &= \frac{3S(1)z - S(2) - 4S(0)z^2}{12(z - r_1)(z - r_2)(z - r_3)} \\ &= \frac{3(\rho_0 + \rho_1)z - (\rho_0 + \rho_1 z + \rho_2 z^2) - 4z^2}{12(z - 1/3)(z - i/2)(z + i/2)}. \end{aligned}$$

So we need to express  $\rho_1$  and  $\rho_2$  in terms of  $\varphi_1 = 3$ ,  $\varphi_2 = -4$  and  $\varphi_3 = 12$  to get an explicit formula for  $\rho_k$ . From Equation 4.3 we have the relation  $\rho_{k+3} - 3\rho_{k+2} + 4\rho_{k+1} - 12\rho_k = 0$ . Substituting  $k = -1$  and  $k = -2$  in this relation and using the facts that  $\rho_0 = 1$  and  $\rho_k = \rho_{-k}$  we get the equations  $5\rho_1 - 12\rho_2 = 3$  and  $-15\rho_1 + \rho_2 = -4$ . Solving these two equations gives  $\rho_1 = 9/35$  and  $\rho_2 = -1/7$  ■

As we remarked earlier in the previous section, the formula for  $\rho_k$  will depend on the number of distinct roots of the polynomial  $h$  defined by Equation 4.6 because the roots of this polynomial are poles of the function  $f(z)/z^{k+1}$  appearing in Equation 4.7. However, Equation 4.7 is a general expression of the function  $\rho_k$  that takes into consideration the number of distinct roots of  $h$ . In other words, Equation 4.7 reduces the computation of the autocorrelation function to the computation of the residues of the function  $f(z)/z^{k+1}$  at the roots of  $h$ . For instance, if we want to find an expression for  $\rho_k$  when the polynomial  $h$  has  $p$  distinct complex roots (the function  $f(z)/z^{k+1}$  has simple poles at  $r_1, r_2, \dots, r_p$ ) then we can use Equation 4.7 to get

$$\rho_k = - \sum_{j=1}^p \lim_{z \rightarrow r_j} \frac{(z - r_j) f(z)}{z^{k+1}},$$

where  $f$  is given by Equation 4.5.

## 5 Conclusion and Remarks

Given a zero-mean stationary autoregressive process of order  $p$ , we have shown that the autocorrelation function  $\rho_k$  for this process is given by

$$\rho_k = - \sum_{j=1}^m \operatorname{Res} \left[ \frac{f(z)}{z^{k+1}}; r_j \right],$$

where  $f(z)$  is given by Equation 4.5,  $r_i$  is a root of the polynomial  $h$  defined in Equation 4.6 and  $m$  is the number of distinct roots of this polynomial. Thereby, reducing the computation of the autocorrelation function to a computation of residues. In the process of doing so, we have used the generating function

$$f(z) = \sum_{k=0}^{\infty} \rho_k z^k,$$

the radius of convergence of which we have not yet discussed. Such discussion is necessary for if this generating function converges only when  $z = 0$  then the function  $f(z)$  is nowhere analytic. We claim however, that this series converges uniformly in the open unit disk centered at the origin of the complex plane. Indeed, the autocorrelation function  $\rho_k$  is, in magnitude, less than or equal to 1. Thus,  $|\rho_k z^k| \leq |z|^k$  for all  $z$ , and the series  $\sum_{k=0}^{\infty} |z|^k$  converges in the open unit disk centered at the origin. Our claim now follows from the Weierstrass  $M$ -test.

Finally, we would like to remark that the method described in this paper for computing the autocorrelation function is most efficient when the roots of the polynomial defined in Equation 4.6 are distinct. When this polynomial has a root of high multiplicity, the computation of residues in Equation 4.6 becomes tedious.

## References

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