[Rose-Hulman Undergraduate Mathematics Journal](https://scholar.rose-hulman.edu/rhumj)

Volume 17
Issue 2

Article 10

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Rose-Hulman Undergraduate Mathematics Journal

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Volume 17, No. 2, Fall 2016

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Volume 17, No. 2, Fall 2016

THE CONFORMABLE RATIO DERIVATIVE

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Abstract. This paper proposes a new definition for a conformable derivative. The strengths of the new derivative arise in its simplicity and similarity to fractional derivatives. An inverse derivative (integral) exists showing similar properties to fractional integrals. The derivative is scalable, and exhibits particular product and chain rules. When looked at as a function with a parameter, the ratio derivative $K_\alpha[f]$ of a function f converges pointwise to f as $\alpha \to 0$, and to the ordinary derivative as $\alpha \to 1$. The conformable derivative is nonlinear in nature, but a related operator behaves linearly within a power series and Fourier series. Furthermore, the related operator behaves completely fractionally when acting within an exponential-based Fourier series.

Acknowledgements: Thank you to Dr. Darin J. Ulness for his helpful insight and mentoring while researching and writing this paper. Thank you also to Dr. Douglas R. Anderson for his support with the ideas of this paper. We are extremely grateful for the referee of this manuscript, who provided a thorough analysis as well as a multitude of important corrections, including a great simplification of the proof of Proposition 2. Thanks as well to Professor Tom Langley, this journal's editor, for his extreme care to detail and kindness throughout the finalization of this manuscript.

1 Introduction

The concept of derivatives of non-integer order, commonly known as fractional derivatives, first appeared in a letter between L'Hopital and Leibniz in which the question of a half-order derivative was posed [2]. Since then, many formulations of fractional derivatives have appeared, but one would expect a single definition to emerge out of the many. One might "guess" at what a perfect definition would look like, possibly expecting the following to be true:

1. $\frac{d^\alpha}{dx^\alpha}x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}$ for $n \geq 0$ and $\alpha \leq n+1$,
2. $\frac{d^\alpha}{dx^\alpha}e^{\lambda x} = \lambda^\alpha e^{\lambda x}$, which, assuming the derivative is linear, implies
3. $\frac{d^\alpha}{dx^\alpha}\sin(\lambda x) = \lambda^\alpha \sin(\lambda x + \frac{\pi}{2}\alpha)$, and
4. $\frac{d^\alpha}{dx^\alpha}\cos(\lambda x) = \lambda^\alpha \cos(\lambda x + \frac{\pi}{2}\alpha)$.

This is a result of noticing the patterns of traditional derivatives, and interpolating their results. Thus far, no proposed definition satisfies all four of the above. In the words of Richard Herrmann, "Up to now there is no ultimate definition of a fractional derivative" [1, p.19]. Even so, a commonly used definition for the fractional derivative is the Riemann-Liouville definition, which is a generalization of Cauchy's formula for repeated integration: $\frac{1}{\Gamma(\alpha)}\int_a^x f(\tau)(x-\tau)^{\alpha-1}d\tau$ with a as an integration limit. This, however, is by definition a fractional *integral*. To make the fractional integral into a derivative, a full derivative of the fractional integral is taken. This definition also introduces surprising results, such as the fractional derivative of a constant not being constant [1, p.15-21].

The Grünwald-Letnikov derivative holds properties that work well with the exponential function. It arises from a binomial generalization of repeated limit-based derivatives [1, p.22]. The Grünwald-Letnikov derivative obeys property 2 above and is also very useful in that it can take complex fractional values [1, p.23].

While they do not satisfy all of the aforementioned four properties, the Riemann-Liouville and Grunwald-Letnikov derivatives indeed satisfy the four properties in the following definition, which we will take as the definition of a fractional derivative, as defined by Ortigueira and Machado [4, p.2-3].

Defintion 1. Let $\alpha \in [0,1]$. An operator D^α is a fractional differential operator if it satisfies the following four properties:

1. Linearity: $D^\alpha(af+bg) = aD^\alpha(f)+bD^\alpha(g)$ for all $a,b \in \mathbb{C}$ and $f,g \in \mathrm{Dom}(D^\alpha)$, where $\mathrm{Dom}(D^\alpha)$ is the domain of the operator D^α
2. $D^0[f] = f$ for all functions f
3. $D^1[f] = f'$ for all $f \in \mathrm{Dom}(D^1)$
4. The Index Law: $D^\beta D^\alpha[f] = D^{\beta+\alpha}[f]$ for all $f \in \mathrm{Dom}(D^\beta \circ D^\alpha) \cap \mathrm{Dom}(D^{\beta+\alpha})$.

Because of the difficulty inherent in defining derivatives that satisfy all four of these properties, a new class of fractional-power derivatives have recently surfaced that only satisfy the second and third properties. These derivatives are known as "conformable" derivatives, defined by Anderson and Ulness [2, p.2] as follows.

Defintion 2. Let $\alpha \in [0,1]$. A differential operator D^α is *conformable* if and only if D^0 is the identity operator and D^1 is the classical differential operator, that is, D^α satisfies conditions 2 and 3 of Definition 1.

In this paper we introduce a new conformable derivative that is not linear and does not satisfy the index law. We define this derivative in Section 2 and continue in Section 3 by observing its many properties and peculiarities. Section 4 demonstrates a linearization procedure of our new derivative, and the paper is concluded in Section 5.

2 Definition of the Conformable Ratio Derivative

In this section, we define the conformable ratio derivative and its inverse operator, on a small set of real functions, and a larger set of complex functions.

Defintion 3. For $f(x) \geq 0$, differentiable and $f'(x) \geq 0$, the conformable ratio derivative (represented by the K_α operator) is defined by

$$K_\alpha[f(x)] = \lim_{\epsilon \to 0} f(x)^{1-\alpha}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right)^\alpha, \alpha \in [0,1].$$

Note that

1. $K_0[f(x)] = \lim_{\epsilon \to 0} f(x)^{1-0}\left(\frac{f(x+\epsilon)-f(x)}{\epsilon}\right)^0 = \lim_{\epsilon \to 0} f(x) = f(x)$, and

2. $K_1[f(x)] = \lim_{\epsilon \to 0} f(x)^{1-1}\left(\frac{f(x+\epsilon)-f(x)}{\epsilon}\right)^1 = \lim_{\epsilon \to 0}\left(\frac{f(x+\epsilon)-f(x)}{\epsilon}\right) = f'(x)$.

Therefore this definition holds as a conformable derivative.

However, it may quickly be seen that by limit laws and the definition of the classical derivative this becomes simply

$$K_\alpha[f] = f^{1-\alpha}(f')^\alpha = f\left(\frac{f'}{f}\right)^\alpha, \alpha \in [0,1]. \tag{1}$$

To clarify from the earlier stipulations, if either $f < 0$ or $f' < 0$, then either $f^{1-\alpha}$ or $(f')^\alpha$ will be complex, eliminating the operator's closure on real functions. This alternative formulation prompts the moniker "ratio" derivative, as the operator takes a powered ratio of the function with its first derivative. By the chain rule, one may also obtain the definition

$$K_\alpha[f] = \alpha^\alpha\left(\left(f^{\frac{1}{\alpha}}\right)'\right)^\alpha, \alpha \in (0,1], \tag{2}$$

which allows for the construction of a second limit-based definition:

$$K_\alpha[f] = \lim_{\epsilon \to 0} \left(\alpha \frac{f^{\frac{1}{\alpha}}(x+\epsilon) - f^{\frac{1}{\alpha}}(x)}{\epsilon} \right)^\alpha, \alpha \in (0,1].$$

The definition from equation (2) allows for the definition of an inverse operator K_α^{-1} such that

$$K_\alpha^{-1}[f] = \alpha^{-\alpha} \left(\int_0^x f^{\frac{1}{\alpha}}(x)dx \right)^\alpha, \alpha \in (0,1]. \tag{3}$$

One may verify that

$$K_\alpha[K_\alpha^{-1}[f]] = K_\alpha\left[\alpha^{-\alpha}\left(\int_0^x f^{\frac{1}{\alpha}}dx\right)^\alpha\right] = \alpha^\alpha \alpha^{-\alpha}\left(\frac{d}{dx}\int_0^x f^{\frac{1}{\alpha}}dx\right)^\alpha = \alpha^0 (f^{\frac{1}{\alpha}})^\alpha = f,$$

and

$$K_\alpha^{-1}[K_\alpha[f]] = K_\alpha^{-1}\left[\alpha^\alpha\left((f^{\frac{1}{\alpha}})'\right)^\alpha\right] = \alpha^{-\alpha}\alpha^\alpha\left(\int_0^x (f^{\frac{1}{\alpha}})'dx\right)^\alpha = \alpha^0 (f^{\frac{1}{\alpha}})^\alpha = f.$$

Note that equations (1), (2), and (3) may be generalized to analytic complex functions, while keeping real values of α as follows. For f analytic in domain $D \in \mathbb{C}\backslash\mathbb{R}_0^-$ (where $\mathbb{R}_0^-$ are the nonpositive real numbers),

$$K_\alpha[f(z)] = f^{1-\alpha}(z)(f'(z))^\alpha = f(z)\left(\frac{f'(z)}{f(z)}\right)^\alpha, \alpha \in [0,1],$$

$$K_\alpha[f(z)] = \alpha^\alpha\left(\frac{d}{dz}f^{\frac{1}{\alpha}}(z)\right)^\alpha, \alpha \in (0,1],$$

$$K_\alpha^{-1}[f(z)] = \alpha^{-\alpha}\left(\int_0^z f^{\frac{1}{\alpha}}(z)dz\right)^\alpha, \alpha \in (0,1],$$

where we use the principal branch for the complex power and log functions.

From this point onward, when introducing a new property, the function f will be a complex function analytic in domain D, with the principal branch used for the complex power and log functions, unless otherwise explicitly stated (that is, a property will be introduced with the "*real*" adjective).

3 Properties of the Ratio Derivative

In this section, we observe the conformable ratio derivative's nonlinearity, scalability, a property regarding powers of the operator, and a special case where the index law is upheld. We continue in this section by observing the properties of the product and chain rules, the continuity of the operator as a function, and finally its relation to other fractional derivatives.

One property of the ratio derivative is that it is nonlinear in nature. That is, given f, g analytic in D, and $\lambda, \mu \in \mathbb{C}$, in general

$$K_\alpha[\lambda f + \mu g] \neq \lambda K_\alpha[f] + \mu K_\alpha[g], \lambda, \mu \in \mathbb{C}.$$

However, the ratio derivative is scalable.

Proposition 1. *For $\lambda \in \mathbb{C}$, $K_\alpha[\lambda f] = \lambda K_\alpha[f]$.*

Proof.

$$K_\alpha[\lambda f] = (\lambda f)^{1-\alpha}(\lambda f')^\alpha = \lambda^{1-\alpha}\lambda^\alpha f^{1-\alpha}(f')^\alpha = \lambda f^{1-\alpha}(f')^\alpha = \lambda K_\alpha[f].$$

□

Remark: This allows for a slight enlargement of the family of *real* functions allowed when taking the derivative. Thus if *either* $f(x) \geq 0$, differentiable and non-decreasing, *or* $f(x) \leq 0$, differentiable and non-increasing, then the derivative exists and $K_\alpha[-f] = -K_\alpha[f]$.

The derivative also "favors" functions in the form $f = g^\alpha$. This manifests in taking multiple derivatives of functions in this form.

Proposition 2. *Let g be analytic on D, and let $\alpha \in \mathbb{C}$. For all $n \in \mathbb{N}$,*

$$(K_\alpha)^n[g^\alpha] = \alpha^{n\alpha}\big(g^{(n)}\big)^\alpha$$

where $g^{(n)}$ denotes the usual n^{th} derivative of g.

Proof.

We proceed by induction. If $n = 1$, by definition,

$$(K_\alpha)^1[g^\alpha] = \alpha^\alpha(g')^\alpha.$$

For the $n = 2$ case,

$$(K_\alpha)^2[g^\alpha] = K_\alpha\big[K_\alpha[g^\alpha]\big] = K_\alpha[\alpha^\alpha(g')^\alpha] = \alpha^{2\alpha}(g'')^\alpha.$$

Suppose the $n = m$ case holds. Then for $n = m + 1$
$(K_\alpha)^{m+1}[g^\alpha] = K_\alpha[(K_\alpha)^m[g]] = K_\alpha[\alpha^{m\alpha}\big(g^{(m)}\big)^\alpha] = \alpha^{(m+1)\alpha}\big(g^{(m+1)}\big)^\alpha$.

□

An application of this proposition with $g = f^{\frac{1}{\alpha}}$ yields

$$(K_\alpha)^n[f] = \alpha^{n\alpha}\Big((f^{\frac{1}{\alpha}})^{(n)}\Big)^\alpha,$$

where $f^{(n)}$ denotes the usual n^{th} derivative of f.

Note that this property holds for all integer powers of n, where $(K_\alpha)^0$ is the identity operator, and negative values of n represent positive powers of the inverse operator:

$$(K_\alpha)^0[f] = \alpha^0\Big(\big(f^{\frac{1}{\alpha}}\big)^{(0)}\Big)^\alpha = \big(f^{\frac{1}{\alpha}}\big)^\alpha = f,$$

and

$$(K_\alpha)^{-1}[f] = \alpha^{-\alpha}\Big(\big(f^{\frac{1}{\alpha}}\big)^{(-1)}\Big)^\alpha = \alpha^{-\alpha}\bigg(\int_0^z f^{\frac{1}{\alpha}}(z)dz\bigg)^\alpha = K_\alpha^{-1}[f].$$

Finally, we observe that the ratio derivative obeys the index law when applied to $e^{\lambda z}$:

$$K_\alpha K_\beta[e^{\lambda z}] = K_\beta K_\alpha[e^{\lambda z}] = K_{\alpha+\beta}[e^{\lambda z}] = \lambda^{\alpha+\beta}e^{\lambda z}.$$

3.1 Product and Chain Rules

Because the ratio derivative is nonlinear, the derivative's "product" and "chain" rules are also nonlinear. Applying the derivative to a product of functions one obtains

$$K_\alpha[f\cdot g] = (f\cdot g)^{1-\alpha}(fg' + f'g)^\alpha.$$

When applied to *real* functions, take note that the product of the functions must be non-negative, differentiable, and non-decreasing, or non-positive, differentiable, and non-increasing. It is very interesting to note that with above result one sees the product rule "appear" and "disappear" as $\alpha \to 0$ and $\alpha \to 1$ respectively.

The chain rule is similar in nature. Applying the derivative to a composition of functions yields

$$K_\alpha[f\circ g] = \big(f(g)\big)^{1-\alpha}\big(f'(g)g'\big)^\alpha = \big(f(g)\big)^{1-\alpha}\big(f'(g)\big)^\alpha\cdot(g')^\alpha,$$

where again we see the emergence of the ordinary chain rule as $\alpha \to 1$.

3.2 Continuity of the Operator as a Function of Alpha

A notable quality of the ratio derivative is that if we fix f analytic on D, $z_0 \in D$, and regard $K_\alpha[f(z_0)]$ as a function of α, then the function is continuous. In other words, the value of $K_\alpha[f(z_0)]$ varies continuously with α for every fixed f and $z_0 \in D$. (Note well, this statement may be made similarly with a real, differentiable function and respective real domain.)

Proposition 3. *Let $D \subset \mathbb{C}\backslash\mathbb{R}_0^-$ be a domain, and let f be analytic on D. Then for all $\epsilon > 0$, for all $z \in D$, for all $\alpha \in [0,1]$, there exists $\delta > 0$ such that* $\Big|K_{\alpha+\delta}[f(z)] - K_\alpha[f(z)]\Big| < \epsilon$.

Proof.

Let f be analytic in D. Let $\epsilon > 0$, and let $z_0 \in D$.

If either $f(z_0) = 0$ or $f'(z_0) = 0$, then

$$\Big|K_{\alpha+\delta}[f(z_0)] - K_\alpha[f(z_0)]\Big| = |0 - 0| = 0 < \epsilon.$$

Otherwise, let $f(z_0) = a$ and $f'(z_0) = b$.

Then

$$\Big|K_{\alpha+\delta}[f(z_0)] - K_\alpha[f(z_0)]\Big| = |a|^{1-\alpha}|b|^\alpha \left|\left(\frac{b}{a}\right)^\delta - 1\right|$$

$$\leq |a|^{1-\alpha}|b|^\alpha \left|\left|\frac{b}{a}\right|^\delta \Big(\cos\big(\delta \mathrm{Arg}(b/a)\big) + i\sin\big(\delta \mathrm{Arg}(b/a)\big)\Big) - 1\right|$$

$$\leq |a|^{1-\alpha}|b|^\alpha \left|\left|\frac{b}{a}\right|^\delta \big(1 + \mathcal{O}(\delta)\big) - 1\right| = \mathcal{O}(\delta),$$

for $\delta \ll 1$, and consequently $\Big|K_{\alpha+\delta}[f(z_0)] - K_\alpha[f(z_0)]\Big| \to 0$ as $\delta \to 0$.

□

3.3 Relation to Fractional Derivatives

In some scenarios, the conformable ratio derivative behaves similarly to fractional derivatives. One may compare its effects on elementary functions with those of other fractional derivatives.

One *real* function, $f(x) = \lambda x$, shows an interesting result:

$$K_\alpha[\lambda x] = \lambda x^{1-\alpha} \cdot 1^\alpha = \lambda x^{1-\alpha}, \alpha \in [0, 1].$$

The Riemann-Liouville fractional derivative (with zero as the integration limit) applied gives

$$D_0^\alpha[\lambda x] = \lambda \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-\tau)^{-\alpha}\tau d\tau = \frac{\lambda}{(1-\alpha)\Gamma(1-\alpha)}x^{1-\alpha}, \alpha \in [0, 1].$$

The two derivatives give the same functional form for $f(x) = \lambda x$, but differ by a scalar multiple.

Another relation to fractional derivatives was previously mentioned in the earlier statement that $K_\alpha[e^{\lambda z}] = \lambda^\alpha e^{\lambda z}$. This is the same result that nearly all fractional derivatives, including the GrÃ¼nwald-Letnikov derivative, produce when computed for the exponential [2, p.16].

4 Linearization of Proposed Derivative

In this section we propose a linear version of the conformable ratio derivative. This expands its uses, especially in differential equations, although the results are nearly always limited to series notation.

The nonlinearity of the ratio derivative, along with the fact that it does not follow the index rule, sets it apart from conventional fractional derivatives. This limits its application in some instances, namely differential equations. However, it was noted that the operator behaves similarly to fractional derivatives for the functions $f(z) = \lambda z, \lambda \in \mathbb{C}$ and $f(z) = e^{\lambda z}, \lambda \in \mathbb{C}$.

Therefore, by creating a power series built from the operator applied to monomials, it can exhibit linear properties, even though $K_\alpha[\sum_{n=-\infty}^{\infty} c_n z^n] \neq \sum_{n=-\infty}^{\infty} c_n K_\alpha[z^n]$. Defining a related operator to pass through a summation yields $\overline{K}_\alpha$ which behaves on a Laurent series as follows:

$$\overline{K}_\alpha[f(z)] = \overline{K}_\alpha\left[\sum_{n=-\infty}^{\infty} c_n z^n\right] = \sum_{n=-\infty}^{\infty} c_n \cdot K_\alpha[z^n] = \sum_{n=-\infty}^{\infty} c_n n^\alpha z^{n-\alpha},$$

for all $\alpha \in [0,1]$ where $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$.

Similarly this process may be applied for the inverse derivative within the power series:

$$\overline{K}_\alpha^{-1}[f(z)] = \sum_{n=-\infty}^{\infty} c_n \cdot K_\alpha^{-1}[z^n] = \sum_{n=-\infty}^{\infty} c_n (n+1)^{-\alpha} z^{n+\alpha}.$$

For all $\alpha \in [0,1]$.

An even better application of the operator's fractional derivative character occurs when applying it within an exponential-based Fourier series. By again using $\overline{K}_\alpha$ we see that

$$\overline{K}_\alpha[f(z)] = \sum_{n=-\infty}^{\infty} c_n \cdot K_\alpha[e^{in\pi z}] = \sum_{n=-\infty}^{\infty} c_n (in\pi)^\alpha e^{in\pi z} = \sum_{n=-\infty}^{\infty} c_n (n\pi)^\alpha e^{i\pi(nz+\frac{\alpha}{2})},$$

for all $\alpha \in \mathbb{R}$ where $f(z) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi z}$.

In this case the proposed derivative is both linear and follows the index law. What is important to note is that the *single* result applies for all real values of α, including zero (the identity) and the negative reals (indicating the fractional integral exists in the exact same form as the fractional derivative). For all functions that may be represented by a Fourier series, this definition of a fractional derivative is very easy to utilize, giving it potential for applications in many areas of physics.

5 Conclusion

A definition for a new conformable derivative, called the conformable ratio derivative, was proposed. This definition has the advantage of being local in character and can behave as a continuous function with a parameter. The proposed definition's properties were also discussed. The product and chain rule applied to this definition provide interesting results on the emergence of these properties in integer-valued derivatives. A small number of conformable differential equations may be solved by conventional means with this definition as well. The definition's relation to fractional derivatives was also explored and two linear series-definitions were given. The second, a fractional definition for Fourier series, holds a single result for all real values of alpha, and thus is a very applicable definition. Though the operator may exist in the realm of nonlinearity, this continuously conformable derivative is a very powerful tool for understanding calculus at a deeper level.

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