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Berry, John; Dannenberg, Matthew; Liang, Jason; and Zeng, Yingyi (2016) "Symmetries of Cairo-Prismatic Tilings," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 17 : Iss. 2 , Article 3.
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SYMMETRIES OF CAIRO-PRISMATIC TILINGS

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VOLUME 17, No. 2, FALL 2016

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ROSE-HULMAN UNDERGRADUATE MATHEMATICS JOURNAL

VOLUME 17, No. 2, FALL 2016

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Abstract. We study and catalog isoperimetric, planar tilings by unit-area Cairo and Prismatic pentagons. In particular, in counterpoint to the five wallpaper symmetry groups known to occur in Cairo-Prismatic tilings, we show that the five with order three rotational symmetry cannot occur.

Acknowledgements: This paper is the work of the Williams College NSF “SMALL” 2015 Geometry Group. We thank our advisor Professor Morgan for his support. We would also like to thank the National Science Foundation, Williams College and the MAA for supporting the “SMALL” REU and our travel to MathFest 2015.

1 Introduction

In 2012 Chung et al. [C] proved that perimeter-minimizing, edge-to-edge tilings of the plane by unit-area, convex pentagons are given by any combination of Cairo and Prismatic tiles. Both Cairo and Prismatic pentagons contain two right angles, three angles of $2\pi/3$, and equal perimeter. However, the lengths of the individual edges differ between the two shapes (see Figure 1).

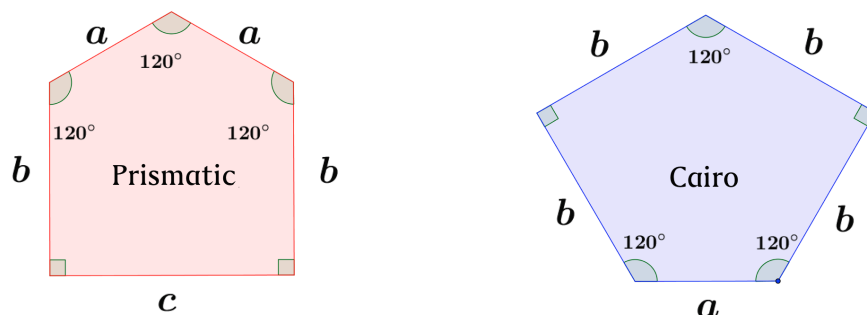


Figure 1: When the pentagons are in the orientation above, the “top” of a Prismatic pentagon, the figure on the left, is formed by two edges of length $a = (2/3)\sqrt{6 - 3\sqrt{3}} \approx 0.5977$, the edges that constitute the left and right sides are of length $b = (3 + \sqrt{3})\sqrt{2 - \sqrt{3}}/3 \approx 0.8165$ and the base has length $c = 2\sqrt{2 - \sqrt{3}} \approx 1.0353$. The Cairo pentagon, right, consists of four edges of length b and a lone edge of length a .

Chung et al. also showed that infinitely many tilings can be formed from these two pentagons. See Figure 2 for many examples.

In this paper we are particularly interested in the wallpaper groups of Cairo-Prismatic tilings. Wallpaper groups are symmetry groups used to classify two-dimensional patterns that contain translational symmetry in two distinct directions. Tilings with wallpaper groups consist of a base pattern that is repeated over and over again in two directions so that it fills the entire space. A familiar example is real-world wrapping paper. Because a roll of wrapping paper must be divided amongst many presents, it cannot portray only a single, unified portrait. Instead it must consist of a chosen base pattern, like a snowman for example, whose image is repeated until it fills the entire roll.

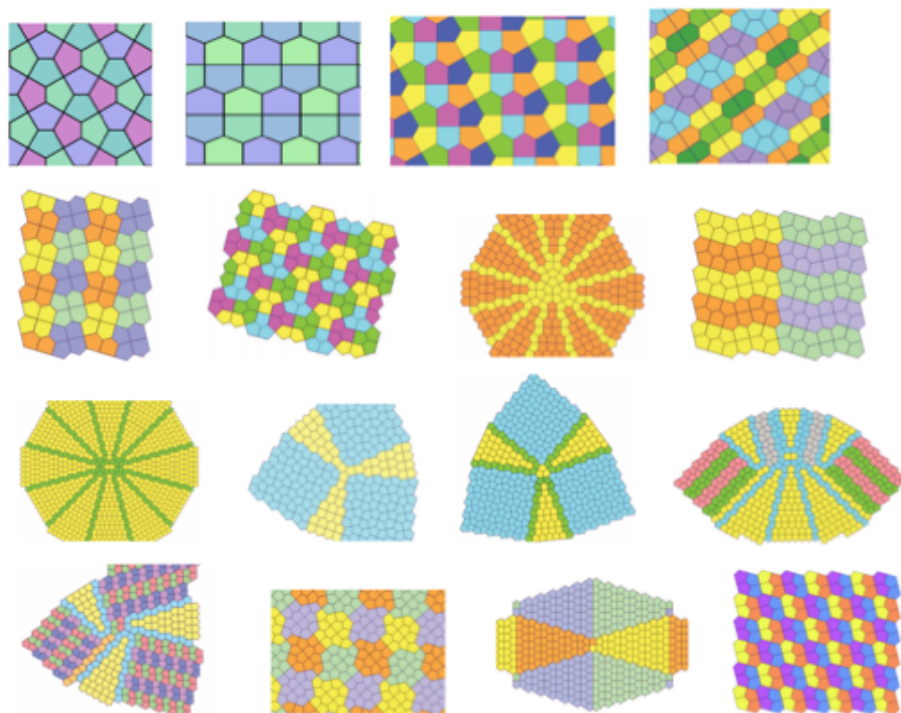


Figure 2: Above are examples of Cairo-Prismatic tilings included in the work of Chung et al. The first and second images are tilings using solely Cairo and Prismatic pentagons, respectively [C, Figures 1-5, 9-12, and 16-22].

There are seventeen different wallpaper groups and they arise from the additional types of symmetries that a tiling might contain. Other than the required translational symmetry, wallpaper groups can also exhibit rotational symmetry of order 1, 2, 3, 4 or 6 or contain axes of reflections and glide reflections. See Figure 3 for a table of the seventeen wallpaper groups and their symmetries.

The 17 Wallpaper Groups				
	lattice	rotational symmetry	reflection axes	glide reflection axes
p1	oblique	none		
p2	oblique	2		
pm	rectangular	none	✓	
pg	rectangular	none		✓
cm	rhombic	none	✓	✓
pmm	rectangular	2	✓	
pmg	rectangular	2	✓	✓
pgg	rectangular	2		✓
cmm	rhombic	2	✓	✓
p4	square	4		
p4m	square	4	✓	
p4g	square	4	✓	✓
p3	hexagon	3		
p31m	hexagon	3	✓	
p3m1	hexagon	3	✓	✓
p6	hexagon	6		
p6m	hexagon	6	✓	✓

Figure 3: The 17 wallpaper groups and their respective symmetries.

We will now analyze a few concrete examples of wallpaper groups in order to better familiarize the reader with this concept.

Figure 4 illustrates the wallpaper group p1. Imagine starting at one of the cats, the base figure, and then translating the entire figure up, down, or diagonally until you reach an area that held a cat originally. At the end of this process the figure would look indistinguishable from how it began. This translational equivalence in two directions is what makes this image a wallpaper group. The cat wallpaper is p1 because it contains only this translational symmetry, that is, you cannot reflect or rotate it and get back the same image.



Figure 4: An example of a design with wallpaper group p1.

Figure 5 illustrates the wallpaper group p4. The tiling can be translated up and down or left and right and maintain equivalence, and thus has the required two-axis translational symmetry required

of a wallpaper group. Notice also that one could rotate the figure 90° , 180° , 270° , and, trivially, 360° around the center of the squares without changing the image. The tiling therefore displays order-4 rotational symmetry. Notice how there is not just one center of rotational symmetry but, in fact, there is one at the center of each square (this must be the case in order to have translational symmetry). Because the tiling has the required translational symmetry, order four rotational symmetry, and no reflection or glide reflection symmetries, it is an example of the wallpaper group $p4$.



Figure 5: An example of a design with wallpaper group $p4$.

As stated above, the different wallpaper groups are distinguished by different reflectional and glide reflectional symmetries, in addition to translational and rotational symmetry. While rotations are centered around each base figure of the tiling, reflections and glide reflections occur along a parallel reflection axis. The reader is no doubt familiar with the concept of reflections. However, glide reflections are slightly more obscure. A glide reflection is a reflection followed by a translation.

Figure 6 provides an example of a glide reflection outside the context of tilings. The triangle in the bottom right is the product of a glide reflection of the triangle in the top left, with the bottom left triangle serving as the intermediate step. In order for a tiling to contain glide reflectional symmetry the tiling must remain unchanged after reflecting and then translating the entire figure along the appropriate axes.

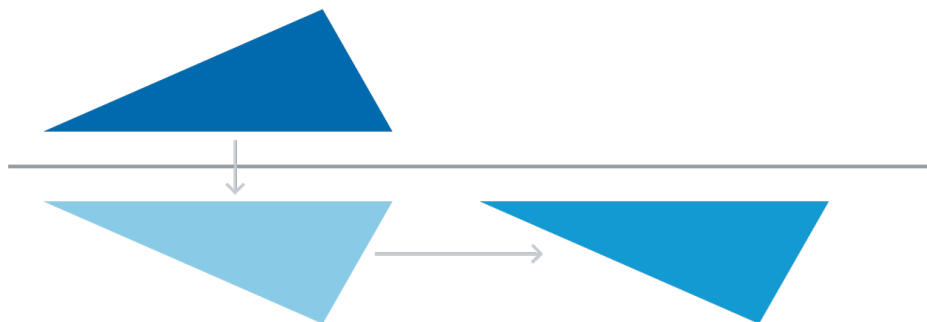


Figure 6: An illustration of a glide reflection.

Shifting our focus back to Cairo-Prismatic tilings, the symmetry group of the tiling in the upper left corner of Figure 2 is a wallpaper group because it contains two linearly independent translations. The tiling in the third row second column has order 3 rotational symmetry but does not contain translational symmetries. Therefore, it is not a wallpaper group. Finally, the tiling in the third row fourth column has no nontrivial symmetries.

Chung et al. found Cairo-Prismatic tilings with symmetries of four different wallpaper groups ($p1$, $p2$, $p4g$, and cmm). Maggie Miller then went on to find four additional Cairo-Prismatic tilings including the Double Pillbox (upper left corner of Figure 7), which adds $p4$ to the known wallpaper group symmetries.

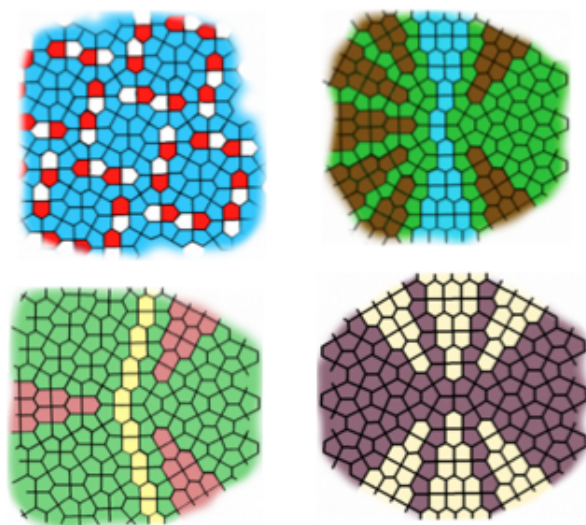


Figure 7: Maggie Miller found four additional Cairo-Prismatic tilings [M2, Figures 2, 3, 4, and 5].

Since the work of Miller eight additional Cairo-Prismatic tilings have been discovered, though no additional wallpaper group symmetries. The first three were discovered by Samantha Petti, by

Victor Luo, and by Lilliana Morris and Byron Perpetua- students in Colin Adams's Tiling class at Williams College (Figure 8).

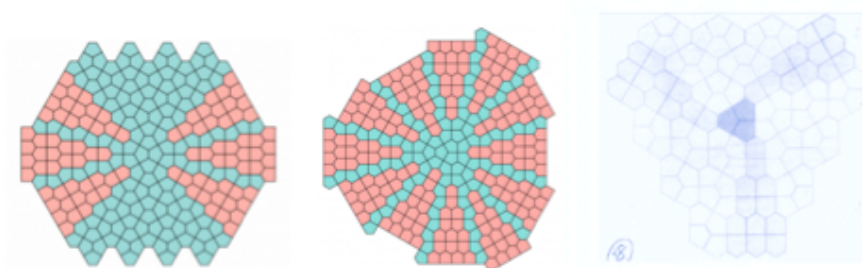


Figure 8: Cairo-Prismatic tilings from a Tiling course at Williams College [M2, Figures 7, 8, and 13].

On January 24, 2015, The National Museum of Mathematics exhibited a working set of Cairo and Prismatic tiles. Visitors, Christian Green and the pair Silvano Bernabel and Daniel Tilkin, discovered two new Cairo-Prismatic tilings (Figure 9).



Figure 9: Christian Green's tiling is on the left while the Cairo-Prismatic tiling discovered by Silvano Bernabel and Daniel Tilkin is shown on the right [M2, Figures 9 and 10].

The latest three were found at The National Museum of Mathematics staff session (Figure 10).

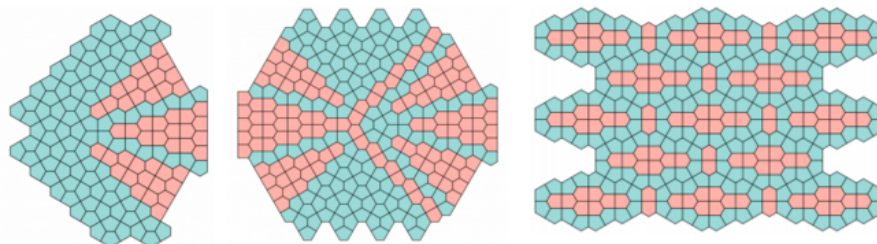


Figure 10: These three latest Cairo-Prismatic tiling discoveries come from Alex Gelman, Ester Schewel and Gemma Gearhart, and David Anderson and Heather Gordon, respectively [M2, Figure 23].

We will begin in Section 2 of this paper with a survey of the wallpaper groups previously found in Cairo-Prismatic tilings. Section 3 will then begin with a proof that a Cairo-Prismatic tiling cannot have a center of order six rotational symmetry. This eliminates the wallpaper groups $p6$ and $p6m$ from consideration as symmetry groups of Cairo-Prismatic tilings. The proof relies on the fact that the interior angles of Cairo and Prismatic pentagons are too large. We then move on to consider wallpaper groups with order three rotational symmetry- $p3$, $p31m$ and $p3m1$. Here, we examine the behavior of Prismatic chains (maximal, connected sets of parallel Prismatic pentagons) and prove that Cairo-Prismatic tilings cannot exhibit these wallpaper groups either. We first show that around any center of order three rotational symmetry there must exist three Prismatic chains. Then, by discovering contradictions if we assume the chains to be either finitely or infinitely long, we find that Cairo-Prismatic tilings cannot contain symmetries of $p3$, $p31m$ or $p3m1$.

2 Tilings with Wallpaper Groups $p1$, $p2$, $p4$, $p4g$ and cmm

Chung et al. [C] proved the existence of Cairo-Prismatic tilings with symmetries of wallpaper groups $p1$, $p2$, $p4g$ and cmm (Figures 11-14). Shortly thereafter Maggie Miller discovered a Cairo-Prismatic tiling with the symmetries of the wallpaper group $p4$ (Figure 15). These are the only five wallpaper groups discovered in Cairo-Prismatic tilings.

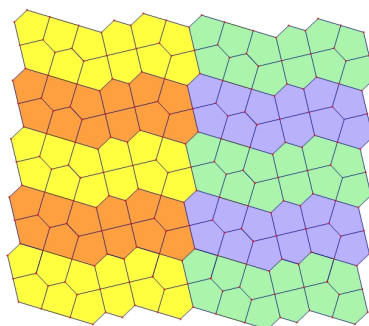


Figure 11: An example of a Cairo-Prismatic tiling with symmetries of wallpaper group $p1$, the most simple of wallpaper groups. The tiling is colored in a manner that highlights the two independent directions of translational symmetry [C, Figure 6].

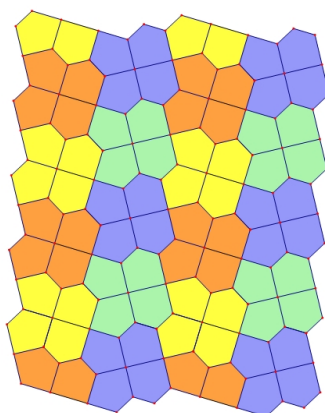


Figure 12: Cairo-Prismatic tiling with symmetries of wallpaper group $p2$. It has translational symmetry in two directions (the minimum requirement for a wallpaper group) as well as centers of order two rotational symmetry [C, Figure 7].

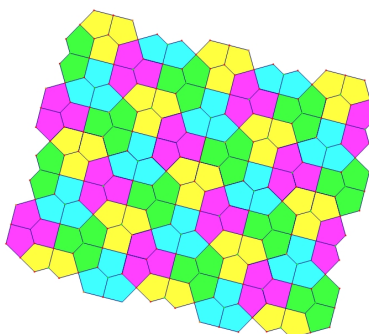


Figure 13: Cairo-Prismatic tiling with symmetries of wallpaper group $p4g$. The tiling has translational symmetry, centers of order two and four rotational symmetry and both reflection and glide reflection axes [C, Figure 8].

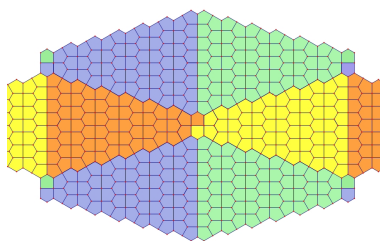


Figure 14: Cairo-Prismatic tiling with symmetries of wallpaper group cmm . The tiling has translational symmetry, centers of order two rotational symmetry, and both reflection and glide reflection axes [C, Figure 10].

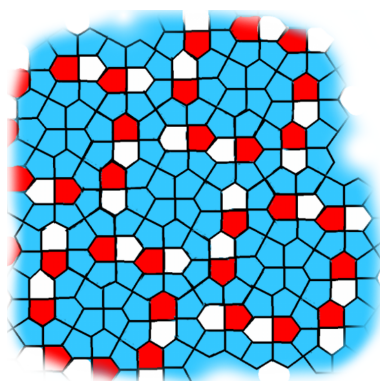


Figure 15: Cairo-Prismatic tiling with symmetries of wallpaper group $p4$. It has translational symmetry and centers of order two and four rotational symmetry [M2, Figure 2].

3 No Tilings with Wallpaper Groups $p6$, $p6m$, $p3$, $p31m$ and $p3m1$

This section contains the theorems proving that a Cairo-Prismatic tiling cannot contain the symmetries of wallpaper groups $p6$, $p6m$, $p3$, $p31m$ or $p3m1$. The impossibility of $p6$ and $p6m$ follow directly from the fact that Cairo-Prismatic tilings cannot contain centers of order six rotational symmetry (Theorem 3.1). However, a Cairo-Prismatic tiling can indeed contain a center of order three rotational symmetry (see Figure 16).

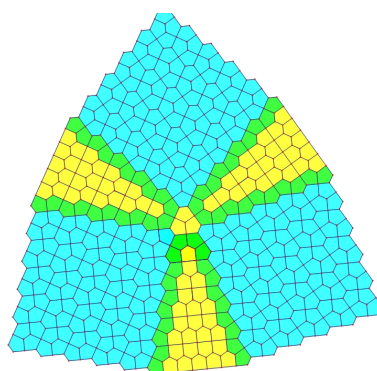


Figure 16: The aptly named Windmill Cairo-Prismatic tiling contains a single center of order three rotational symmetry. However, because it is not translationally symmetric it does not contain the symmetries of a wallpaper group [C, Figure 11].

In order to exclude order three rotations in Cairo-Prismatic wallpaper groups (Theorem 3.13) we will need to study the large-scale behavior of Prismatic pentagons in a tiling. But first we consider order six rotations.

Theorem 3.1. *No Cairo-Prismatic tiling exists with symmetries of the wallpaper groups $p6$ or $p6m$.*

Proof. Suppose that such a tiling exists and let p be a center of order six rotation. If a center of order six rotational symmetry lies in the interior of a tile then the tile must have $6n$ sides for some $n \in \mathbb{N}$. Because pentagons have only five sides, p is not an interior point. If p lies on an edge but is not a vertex point then there exists a neighborhood around p containing only that edge and the interior of the two pentagons meeting along this edge. This area does not have order six rotational symmetry and thus p must be a vertex point. The number of angles about a vertex of order six rotational symmetry must be a multiple of six. This contradicts the fact that the smallest angles in a Cairo or Prismatic pentagon is $\pi/2$. Therefore, a Cairo-Prismatic tiling cannot contain a center of order six rotational symmetry and, because wallpaper groups $p6$ and $p6m$ both contain these centers, there are no Cairo-Prismatic tilings with symmetries of wallpaper groups $p6$ or $p6m$. \square

To rule out order three rotations in a Cairo-Prismatic wallpaper group, we now focus on the Prismatic tiles.

While examining Cairo-Prismatic tilings we noticed that every time a Prismatic pentagon appears in a tiling it is part of a larger, connected set of parallel Prismatic tiles. We call these structures Prismatic chains. We also noticed that every Prismatic chain is either infinitely long or finitely long with additional perpendicular Prismatic chains forming on each side. In every example of the finite case the branching process seems to be infinite. For an example of infinite Prismatic chains refer to the tilings in the third row, second and third columns of Figure 2. Both contain three infinitely long Prismatic chains. For an example of the infinite branching of the finite case, see Figure 17 below.

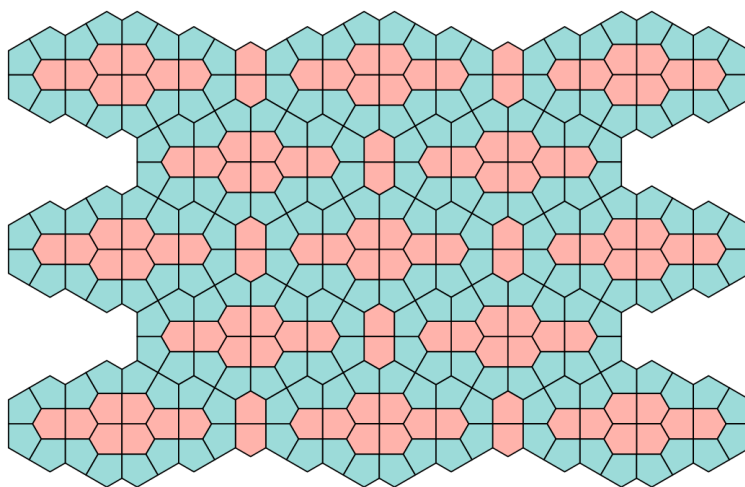


Figure 17: The Cairo-Prismatic tiling Space Pills contains only finite Prismatic chains, seen here in red, and thus exhibits an infinite branching process [M2, Figure 24].

After experimenting with the different possible centers of order 3 rotational symmetry in a Cairo-Prismatic tiling, we determined that sprouting from any such possible center must be a Prismatic pentagon, and consequently a Prismatic chain, and its two 120° rotated copies. The proof that a Cairo-Prismatic wallpaper group cannot have order three rotational symmetry thus comes down to showing that these Prismatic chains can be neither infinite nor finite, thereby contradicting their existence.

The following definitions serve to formalize the discussion above by providing concrete definitions for our objects.

Definition 3.2. Cairo-Prismatic tilings are edge-to-edge tilings of the plane consisting of Cairo and Prismatic pentagons. A *Prismatic chain* is a maximal, connected set of parallel Prismatic tiles. A *row* of Prismatic pentagons is a maximal, connected subset of a Prismatic chain consisting only of Prismatics connected along edges of length b (See Figure 1). The *sides* of a row are the edges of length b not adjacent to Prismatic pentagons in the row. The *outer edges* of a row are edges of length a adjacent to the sides of the row. Due to the uniqueness of the side of length c , each row in a Prismatic chain is attached to another row of equal width along edges of length c . We call these two rows, *row pairs*. If an outer edge is not attached to a Prismatic pentagon in the Prismatic chain we will say the outer edge is *isolated*. The *width* of a Prismatic chain is a local measurement equal to the number of Prismatic pentagons in a row. The *length* of a Prismatic chain is equal to the number of row pairs it contains.

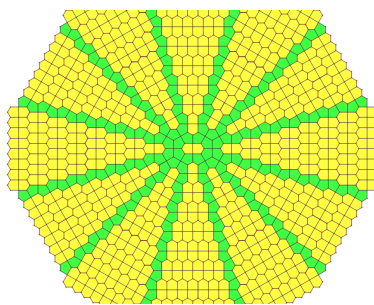


Figure 18: Plaza has twelve infinitely long and wide Prismatic chains with a single finite chain in the center of length one containing two rows of width one (all of which are pictured here in yellow) [C, Figure 13].

The first few propositions below work towards a climax, Proposition 3.8, which shows that any finite Prismatic chain has perpendicular Prismatic chains branching off of each of its sides.

Definition 3.3. A *Prismatic gap* is a 120° angle formed by two edges of length a .

Proposition 3.4. A *Prismatic gap* in a Cairo-Prismatic tiling can only be filled by a single Prismatic pentagon along its edges of length a .

Proof. Cairo and Prismatic pentagons only contain 90° and 120° interior angles. Between the Cairo and Prismatic there is only one interior angle of 120° created by two edges of length a (the top of the Prismatic in Figure 1). A Prismatic gap is thus necessarily filled by this angle. \square

Proposition 3.5. *Adjacent Prismatic row pairs differ in width by at most one.*

Proof. There are $2n$ edges of length a both at the top and bottom of a row pair- two from each of the n Prismatic pentagons in each row. Of these $2n$ edges all but the two outer edges are adjacent to two other edges of length a in the row. These $2n - 2$ edges meet at 120° angles (see Figure 19).

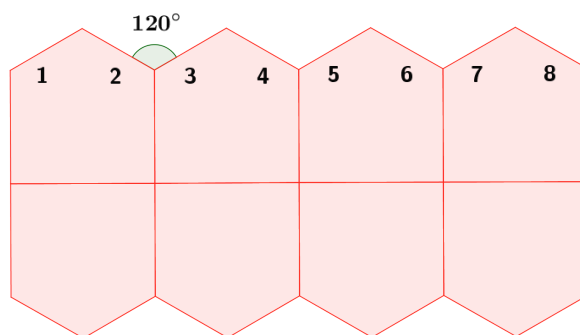


Figure 19: The top of a row pair of width four consists of eight edges of length a and necessarily forms three Prismatic gaps both above and below it. Only the outer edges, edges 1 and 8 and the corresponding edges on the bottom, do not necessarily contribute to Prismatic gaps.

Thus, there are $n - 1$ Prismatic gaps both above and below and the lower bound has been achieved. If a row pair had width n and one of its adjacent row pairs had width $n + 2$ or greater this would contradict the lower bound. Thus, $n + 1$ is an upper bound on the width of a row pair adjacent to a row pair of width n . \square

Proposition 3.6. *Assume a Prismatic row pair has isolated outer edges on a side. Then there must exist two Cairo pentagons connected along the side of the row pair, whose edges of length a form a Prismatic gap perpendicular to the original chain.*

Proof. Consider a Prismatic row pair with isolated outer edges on its right side (see Figure 20).

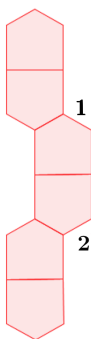


Figure 20: A row pair with isolated outer edges, here denoted 1 and 2, on its right side and its adjacent row pairs.

There are three edges of length a between the Cairo and Prismatic pentagons and so three possible combinations along edges of length a . However, one is not edge to edge (see Figure 21).

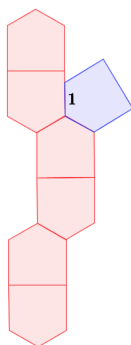


Figure 21: This move is not possible because it places an edge of length a , here labeled 1, next to an edge of length b .

Another possible move is adding a Prismatic parallel to the chain but this contradicts the fact that the outer edge is isolated (see Figure 22).

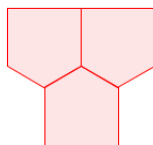


Figure 22: If a parallel Prismatic pentagon was attached to the outer edge the edge would not be isolated.

Therefore, there is only one possible move along these edges- adding a Cairo along its lone edge of length a . Making only necessary moves this gives us Figure 23.

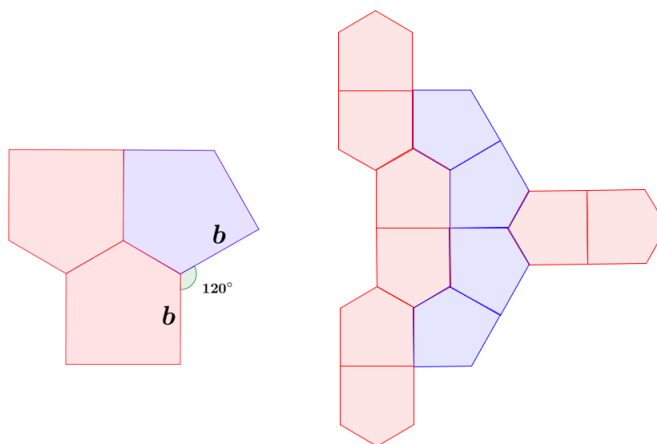


Figure 23: Adding Cairo pentagons to the outer edges creates a 120° exterior angle adjacent to the row pair. This gap is formed by two edges of length b and so can only be filled by a Cairo pentagon along its top (see Figure 1). Thus, two Cairo pentagons must be placed in this specific orientation adjacent to the row pair. These two "interior" Cairo pentagons create a Prismatic gap perpendicular to the original chain.

We must consider the case where the row pair containing isolated outer edges is at the end of the chain. Here, the move shown in Figure 6 is edge-to-edge. However, just as when adding a Cairo pentagon, this move creates a 120° exterior angle adjacent to two edges of length b (see Figure 24).

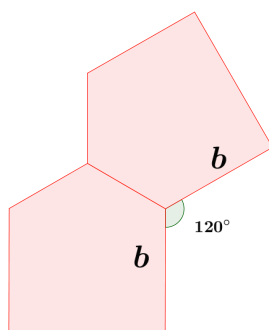


Figure 24: Attaching a Prismatic in this orientation creates the same 120° exterior angle and two edges of length b as in Figure 23.

Thus, this move necessarily forms the same Prismatic gaps seen in Figure 23 (see Figure 25).

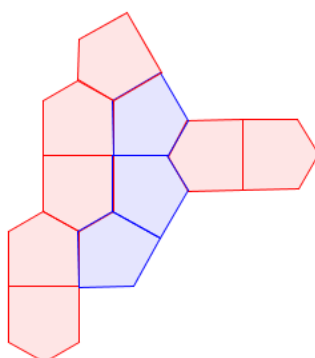


Figure 25: By adding a Prismatic pentagon to the top isolated outer edge and a Cairo pentagon to the bottom edge this figure emphasizes how both moves result in the same Prismatic gap.

The figures depict only the right side of a row pair but the conclusions carry over to the left side as well by symmetry. \square

Proposition 3.7. *A finite, non-trivial Prismatic chain contains a Prismatic row pair such that the two outer edges on its right side are isolated and a row pair with the same property on its left side.*

Proof. Consider a row pair of width one that forms one of the ends of the Prismatic chain. Let this denote the top of the chain. Because the chain is finite and non-trivial, it necessarily contains a row pair such that the adjacent row pair lying further from the top has smaller width. Consider the row pair with this property lying closest to the top of the chain and denote it P . If P has width n then by Proposition 3.5 the row pair below it has width $n - 1$. If the row pair above P has width $n - 1$ then all four outer edges of P are isolated and so the proposition is satisfied (see Figure 26).

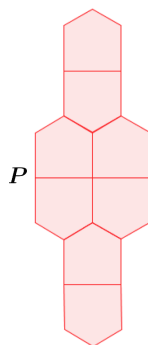


Figure 26: If a row pair of width n is adjacent to two row pairs of width $n - 1$ then the outer edges of the row pair of width n are isolated. Above is the case where $n = 2$.

Now, the row pair above P cannot have width $n + 1$ because this would contradict the fact that P is the closest row pair to the top such that the adjacent row pair below it has smaller width. Thus, the only remaining case is where the row pair above P has width n . The placement of $n - 1$ of these adjacent Prismatic pentagons are determined because they must fill the $n - 1$ inner Prismatic gaps. But, the last Prismatic can be placed along either outer edge (see Figure 27).

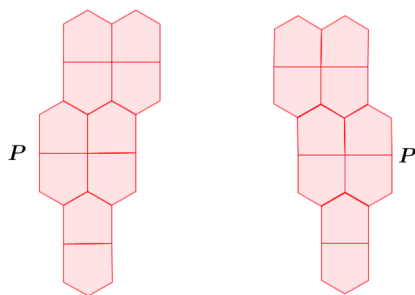


Figure 27: There are two possible orientations for the row pair of width n above P corresponding to when the undetermined Prismatic pentagon is on the left or right side.

Without loss of generality consider the case where the row pair above P is on the right and denote this row pair Q . Once again, because of the choice of P , only row pairs of width n or $n - 1$ lie above Q . If the row pair above Q is on the left the proposition is satisfied (See Figure 28).

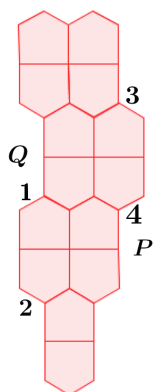


Figure 28: The above shows the case where the row pair above Q is on the left. We see that P satisfies the proposition for the left side and Q for the right.

If instead a row pair of width $n - 1$ is added above Q the proposition is once again satisfied for both sides (see Figure 29).

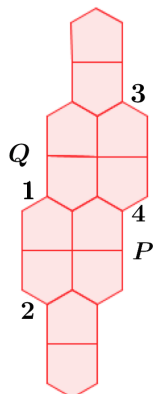


Figure 29: The above shows the case where a row pair of width $n - 1$ is added above Q . P satisfies the proposition for the left side and Q for the right.

The only case left to consider is when the row pair above Q is on the right. Because the Prismatic chain in question is finite there can not exist an infinite number of row pairs on the right above P . Suppose some finite number of row pairs where the undetermined Prismatic is placed on the right exist above P and that the next row pair is of width $n - 1$. The proposition is then satisfied- P will always satisfy the proposition for the left side and the last row pair on the right will satisfy the proposition for the right side. If instead the next row pair is on the left then once again P satisfies the proposition for the left side and the last row pair on the right satisfies the right side. □

Proposition 3.8. *On both sides of a finite Prismatic chain there exists two Cairo pentagons, connected along the side of the row pair, whose edges of length a form a Prismatic gap perpendicular*

to the original chain.

Proof. By Proposition 3.7 a finite Prismatic chain contains a row pair with isolated outer edges on the right side and a row pair with isolated outer edges on the left side. By Proposition 3.6 these two areas must be filled by two Cairo pentagons whose edges of length a in turn form a Prismatic gap perpendicular to the original chain. \square

Thus, we know that in a complete edge-to-edge Cairo-Prismatic tiling on both sides of a finite Prismatic chain there necessarily exist additional Prismatic chains perpendicular to the original. If these perpendicular Prismatic chains are also finite then they too have Prismatic chains branching from both sides. Because these chains must be perpendicular to the chains that were perpendicular to the original chain, they are parallel to the original chain. If every Prismatic chain is finite then there exists an infinite branching process consisting of orthogonal Prismatic chains.

Definition 3.9. A *Prismatic family* is a maximal, connected set of Prismatic chains and the Cairo pentagons that form Prismatic gaps filled by the chains in the family. We will often consider an arbitrary Prismatic chain to serve as the zero-order chain. The first-order Prismatic chains then fill the Prismatic gaps adjacent and perpendicular to the zero-order chain, second-order chains fill the first-order Prismatic gaps and so on.

Given a Prismatic family, a *Prismatic path* is a connected subset of the family consisting of a single Prismatic chain in the family from each order and the Cairo tiles connecting these chains.

We now prove that a Cairo-Prismatic tiling with symmetries $p3$, $p31m$ or $p3m1$ must contain three Prismatic chains at 120° angles.

Proposition 3.10. *Around a center of order three rotational symmetry in a Cairo-Prismatic tiling there necessarily exist a Prismatic pentagon and its 120° and 240° rotated copies.*

Proof. At a center of order three rotational symmetry, three congruent tiles meet with interior angles of 120° and equal edge lengths. The three tiles are either Cairo or Prismatic. If the three tiles are Prismatic, the proposition holds trivially (see Figure 30). If the three tiles are Cairo then there are three options for moves along the edges of length a of these Cairo. Two of these moves are along Prismatic pentagons (Figures 31 and 32). If instead the outside edges are adjacent to Cairo tiles, adding a few necessary tiles shows us that again there must be three Prismatic pentagons at 120° angles (see Figure 32).

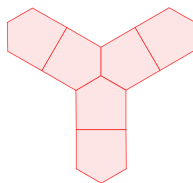


Figure 30: Three Prismatic pentagons forming the center of order three rotational symmetry.

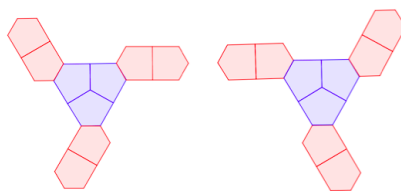


Figure 31: When three Cairo pentagons form the center of order three rotational symmetry, two of the three possible moves along their edges of length a involve adding Prismatic pentagons.

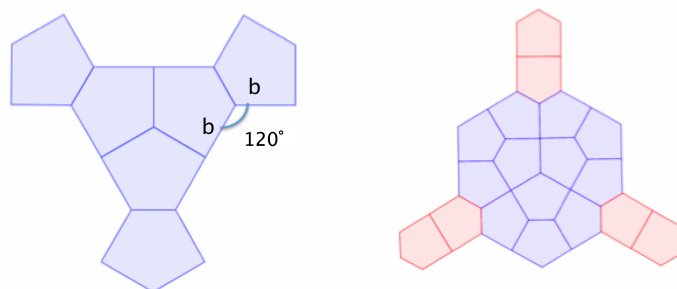


Figure 32: Above, the Cairo pentagons that form the center of order three rotational symmetry attach to three additional Cairo pentagons along edges of length a . As in the proof of Proposition 3.6 this creates 120° angles formed by edges of length b . The only way to tile these gaps are with Cairo pentagons along their tops (Figure 1) and so this necessarily leads to the scenario on the right.

□

Note that, because the three Prismatic pentagons in Proposition 3.10 are neither perpendicular nor parallel, they constitute three different Prismatic chains and generate three separate Prismatic families. The following proposition will be a major tool for our final proof in which we show the impossibility of these three Prismatic families.

Proposition 3.11. *Distinct Prismatic chains cannot intersect at a Prismatic tile. Prismatic paths from different Prismatic families cannot intersect at a Prismatic tile.*

Proof. Assume two Prismatic chains, C and C' , contain the same Prismatic tile. Then, $C \cup C'$ is connected. In addition, the tile contained in $C \cap C'$ must be parallel to both chains and so C and C' must be parallel. So, $C \cup C'$ is a maximal, connected set of parallel Prismatic pentagons containing C and C' . However, this contradicts the existence of C and C' .

Now, assume a Prismatic path of family F intersects another Prismatic path of family F' at a Prismatic tile. By Definition 3.9 every Prismatic pentagon in a Prismatic family is contained in a Prismatic chain and so there exists Prismatic chains from each family, D and D' , each of which contain the tile. By the above paragraph D and D' must be equal. Thus, $F \cap F'$ is nonempty and, because by Definition 3.9 both F and F' are maximal and connected, F and F' must be equal. □

Proposition 3.12. *A Cairo-Prismatic tiling with symmetries $p3$, $p31m$ or $p3m1$ cannot contain an infinitely long Prismatic chain.*

Proof. Assume a Cairo-Prismatic tiling with symmetries $p3$, $p31m$ or $p3m1$ did contain an infinitely long Prismatic chain C . Consider this chain's orientation with respect to a center of order three rotational symmetry. By this symmetry two more infinitely long Prismatic chains exist, one of which is the original chain rotated 120° clockwise with respect to the center of rotation and another that is rotated 120° counterclockwise. By translational symmetry we have three such infinite chains around every center of rotational symmetry (which by Figure 3 lie in a hexagonal grid).

Consider either rotated copy of C and denote it C' . By translation of C' , there is a similar chain C'' which intersects C at a tile. This contradicts Proposition 3.11. Therefore, Cairo-Prismatic tilings with symmetries $p3$, $p31m$ and $p3m1$ cannot contain an infinitely long Prismatic chain. \square

Theorem 3.13. *A Cairo-Prismatic tiling cannot attain the symmetries of wallpaper groups $p3$, $p3m1$ or $p31m$.*

Proof. Suppose that there is a tiling T with wallpaper symmetry group $p3$, $p31m$ or $p3m1$ and hence with centers of order three rotational symmetries. By Proposition 5.11 around every center of rotational symmetry T contains three Prismatic pentagons and thus three Prismatic chains separated by 120 rotations. Consider one such chain C and its 120-degree rotation C' as in Figure 30. By Proposition 5.12, T cannot contain infinitely long Prismatic chains. Thus, C and C' must be finite. By Proposition 5.8, because both C and C' are finite, there exist Prismatic chains perpendicular to C and C' branching off on each side of the original chains, and so on. Consider branching of C left, left, right, left, and continuing to alternate right, left, forming a Prismatic path P . Similarly consider the branchings of C' right, right, left, right, and continuing to alternate left, right, forming a Prismatic path P' . The paths P and P' must either pass through the center of rotational symmetry or intersect. There are three possible centers of order three rotational symmetry, as in Figures 27, 28 and 29. P and P' clearly cannot pass through Figure 29 because it is too densely packed with Cairo pentagons. In the other two cases, the space between the initial Prismatic pentagons of C and C' is at most two tiles wide and so P and P' cannot pass through the centers represented in Figures 27 and 28 without intersecting. Therefore, P and P' must intersect, contradicting Proposition 5.10. Therefore, a Cairo-Prismatic tiling cannot exhibit symmetries $p3$, $p31m$ or $p3m1$.

