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# Isoperimetric Regions on a Weighted 2-Dimensional Lattice

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# ISOPERIMETRIC REGIONS IN A WEIGHTED 2-DIMENSIONAL LATTICE

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*To my mother*

Abstract. We investigate isoperimetric regions in the $1^{st}$ quadrant of the $2-$dimensional lattice, where each point is weighted by the sum of its coordinates. We analyze the isoperimetric properties of six types of regions located in the first quadrant of the Cartesian plane: squares, rectangles, quarter circles, diamonds, crosses and triangles. To compute volume and perimeter of each region we use summation and integration methods which give comparable but not identical results. Among our candidates the diamond has the least perimeter for given volume.

## 1. Introduction

The object of interest in this paper is to investigate isoperimetric regions on the 2-dimensional lattice $(\mathbb{Z}_+ \cup \{0\}) \times (\mathbb{Z}_+ \cup \{0\})$. We use the following definitions of weighted perimeter and volume in the plane to formulate the isoperimetric problem:

**Definition 1.** *For a set $S$ of ordered pairs of whole numbers we define:*

(1) $Vol(S)$ = *sum of all coordinates of points of $S$.*
(2) $Per(S)$ = *sum of all coordinates of points of $S$ at least one of whose four neighbors is not in $S$.*

Geometrically, $S$ is a region on the Cartesian plane whose points are either completely surrounded by four other points in $S$, or are adjacent to some point(s) that is not in $S$. A very simple case is the $3 \times 3$ square depicted in Figure 0.1. According to our definition, we find volume of this region by adding up $x$ and $y$ coordinates of all of its nine points, which gives us volume 18. When finding perimeter it is easy to see that only the middle point $A$ would not contribute since all of its neighbors are in $S$ (axis points always count toward perimeter). Thus, the perimeter is just the sum of the coordinates of the eight outer points of the square, giving perimeter of 16.

The problem we address in this paper is the following:

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*Key words and phrases.* Isoperimetric, Diamonds, Quarter Circles.

The author would like to thank his adviser for his support and patience throughout the process of writing this paper which was his undergraduate thesis in mathematics [S]. He would also like to thank professor Stewart Johnson whose question during a colloquium talk on Isoperimetric Sequences has led the author to futher research of the topic. The author also thanks his fellow Williams student Anna Soybel '11 who presented the early version of the problem at the undergraduate math research conference at Brown University. Finally, the author thanks his second reader Lori Perdersen and fellow Williams student Edward Newkirk '09 for their useful comments on his previous versions of the thesis.

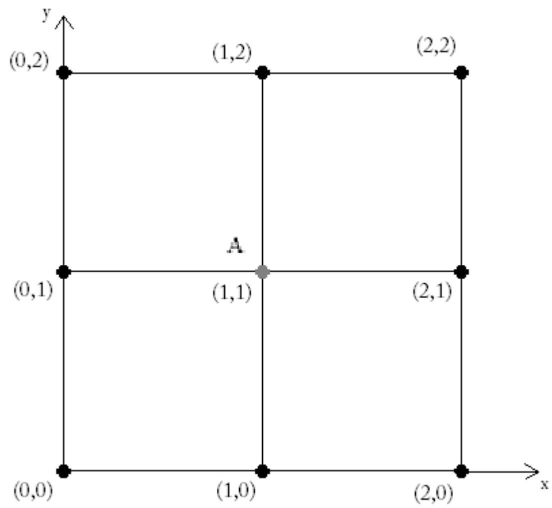


FIGURE 1. A $3 \times 3$ square

**Problem 1.** *For given $Vol(S) = n$, find $S$ to minimize perimeter.*

To illustrate the problem, consider again $n = 18$ — the volume of the $3 \times 3$ square $A$ of Figure 0.1. The perimeter of this region is 16. Is square the perimeter-minimizing shape for $n = 18$? It turns out that it is not. Soybel pointed out that a cross (Figure 0.2), defined as $\{(2,0),(1,1),(2,1),(3,1),(2,2)\}$ with an additional outer point $(0,3)$, has less perimeter. In this case $S$ still has volume $n = 18$ but $Per(S) = 15 < 16$. This example illustrates that for any given volume there might be a variety of different shapes of that volume such as: lines, squares, rectangles, circles, diamonds etc. Our main interest in this paper is to find a solution to the isoperimetric problem on the 2−dimensional lattice for any such given volume $n$.

## Results

Figure 0.3 shows our best candidates for solutions to the isoperimetric Problem 2 in $(\mathbb{Z}_+ \cup \{0\}) \times (\mathbb{Z}_+ \cup \{0\})$ with weight $x + y$. We conjecture that diamond is best. In particular, for given volume $n$ least perimeter regions grow like $n^{2/3}$, just as for the classical isoperimetric problem in $\mathbb{R}^3$.

## Previous Research

The isoperimetric problem on the 2−dimensional lattice is itself an extension of an isoperimetric sequences problem developed by Miller *et al.* (2009) [Mi] . In their paper authors showed that least weighted perimeter on the 1−dimensional lattice in $\mathbb{N}$ is asymtotically related to the classical isoperimetric problem on the plane.

The isoperimetric problem on the unweighted 2−dimensional lattice was solved in 1977 by Wang *et al.* [WW]. The authors showed that diamonds are isoperimetric. Moreover, gradually adding another layer yields isoperimetric shapes for all

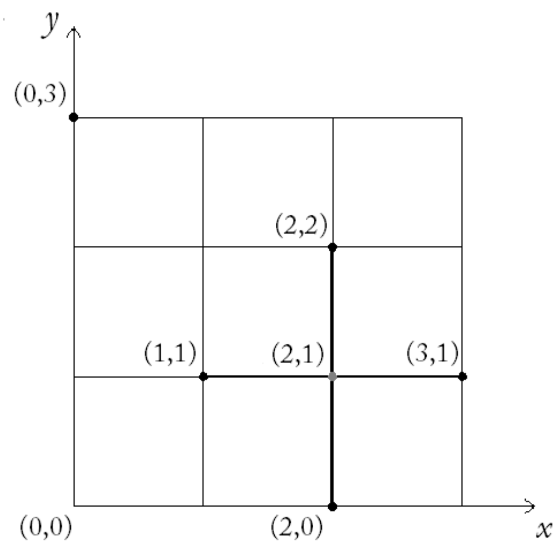


FIGURE 2. Soybel's counterexample

prescribed volumes. With free boundary along the axes, triangles are isoperimetric. Wang also generalizes all of these results to $n$ dimensions.

## Paper Structure

In Section 2 we consider four types of rectangles. For each rectangle we compute weighted volume and perimeter (according to Definition 1) by summation and approximate it by integration. These results agree asymptotically.

In Section 3 we consider four additional shapes that we expect to provide better results than any square: a quarter circle centered at the origin, a diamond in the first quadrant with two of its vertices touching the $x$ and $y$ axes, the cross counterexample provided by Soybel, and a right isosceles triangle. Of those, the diamond provides the least perimeter for prescribed volume.

## Open Questions

The main problem of finding and proving perimeter minimizers remains open. It would also be interesting to consider other densities such as $r$ or $r^p$ (see [DDNT]), $e^{-x^2-y^2}$ (see [Mo], Chapter 18), or perhaps just $x^p$ or $e^{x^2}$; and other dimensions.

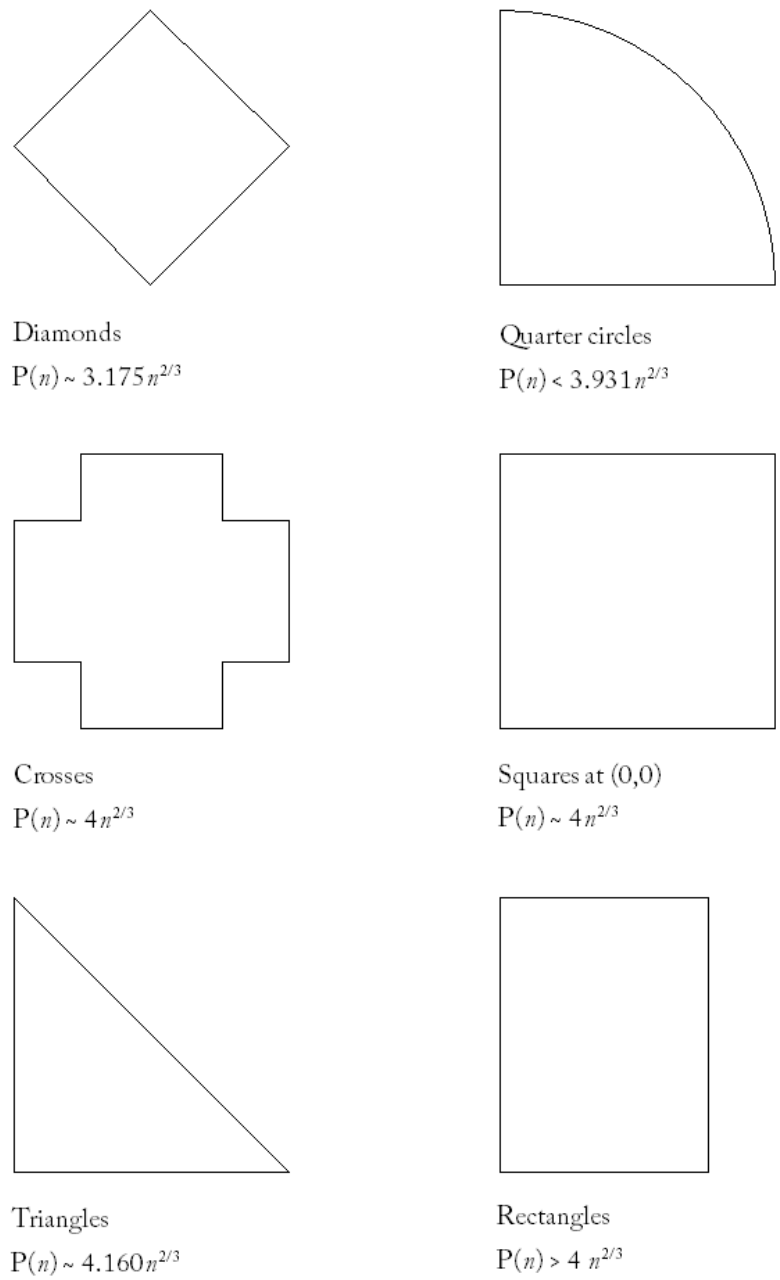


FIGURE 3. The diamond is our best candidate to minimize perimeter $P$ for prescribed volume $n$.

## 2. Rectangles

Rectangles are the simplest of all shapes that we are considering in this paper. It is easy to compute volume and perimeter of squares and rectangles by using simple computation formulas. However, notice that whenever we consider $m \times m$ squares we are really thinking of squares of side length $m-1$. For example, a $2 \times 2$ region is a unit square of side length *one* and *four* lattice points. Keeping this distinction in mind will be important in avoiding double counting mistakes when using the summation method.

### Squares

**Case 1.** $C = (m+1) \times (m+1)$ *square at the left-hand corner of the first quadrant.*

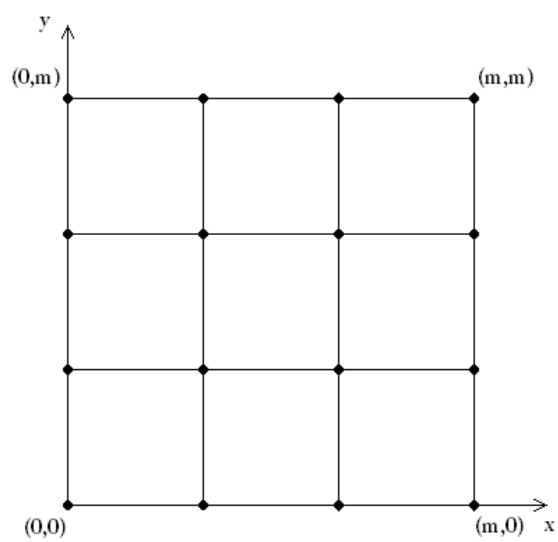


Figure 4. $(m+1) \times (m+1)$ square $C$ at the origin

To obtain the volume of a $m \times m$ square whose two sides coincide with $x$ and $y$ axis, by definition, we need to add up the coordinates of all points of that region. Similarly, to find the perimeter we need to add up the coordinates of its four sides. We can do that either by computing directly using the summation formula (1.1):

$$\sum_{i=0}^{m} i = 0 + 1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{2} \tag{2.1}$$

or by approximating it with an integral. We do both and then compare our results.

2.1. **Summation Method.** To compute volume of $C$ we first find the sum of all $x$ coordinates. Notice that this sum is the same for $y$ coordinates because the square is symmetrical in respect to $x$ and $y$ axes. Therefore, total volume of the region will be just twice the sum of $x$ coordinates:

$$V_C = 2\sum x = 2(m+1)\sum_{i=0}^{m} i = 2(m+1)\frac{m(m+1)}{2},$$

$$V_C = m(m+1)^2. \tag{2.2}$$

For perimeter, notice that because we are considering a square that is placed on the lower left corner of the $1^{st}$ quadrant, points on the $x$ and $y$ axes contribute less to the perimeter than points on the right side and on the top. Points on the $x$ and $y$ axes contribute $m(m+1)/2$ each, while points on the other two sides each contribute $m(m+1)+m(m+1)/2$. In order to avoid double counting the corner points, we also have to subtract $4m$ from the total. Thus,

$$P_C = 2\left(\frac{m(m+1)}{2} + m(m+1) + \frac{m(m+1)}{2}\right) - 4m$$

$$P_C = 4m^2. \tag{2.3}$$

2.2. **Integration Method.** We can also approximate volume and perimeter of the square by integrating with respect to $x$ and $y$. Each point's contribution to total volume or perimeter can be expressed in terms of the sum of its coordinates. That is, each such region has density function $V(x,y) = x+y$. Using this density we can approximate the volume of $C$ using a double integral:

$$V_C \approx \int_0^m \int_0^m (x+y)dxdy,$$

$$V_C \approx m^3. \tag{2.4}$$

For perimeter we integrate along the four sides of the square using the integral:

$$P_C \approx \int_0^m xdx + \int_0^m ydy + \int_0^m (m+y)dy + \int_0^m (m+x)dx,$$

$$P_C \approx 4m^2. \tag{2.5}$$

These results are in agreement with the earlier summation calculations.

Now, since both volume and perimeter are functions of $m$ we can easily derive $P(V)$. Let $V_C = n$, then we find:

$$m = n^{1/3} \Rightarrow P(n) = 4n^{2/3}. \tag{2.6}$$

**Case 2.** $Q = (m+1)\times(m+1)$ *— a general case of a square.*

Now, consider a square whose sides do not coincide with either of the axes. Let $(a,b)$ be the lowest left-most corner of the region as depicted in Figure 1.2. Correspondingly, other corners will have coordinates $(a, b+m)$, $(a+m, b)$ and $(a+m, b+m)$.

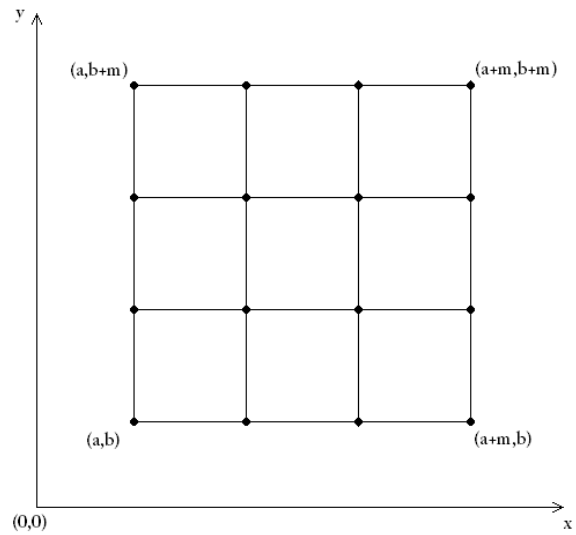


FIGURE 5. $(m+1)\times(m+1)$ square $Q$

2.3. **Summation Method.** Again, we can compute $P_Q$, $V_Q$ and $P(n)$ by using a simple summation method, which gives us comparable results to (1.4) and (1.5). Observe that placing a square at point $(a,b)$ adds $(m+1)^2(a+b)$ to the total volume because each point contributes $a+b$ extra volume and there are $(m+1)^2$ such points. Similarly, perimeter increases by $4m(a+b)$. Thus, we get:

$$V_Q = (m+a+b)(m+1)^2, \tag{2.7}$$

$$P_Q = 4m(m+a+b). \tag{2.8}$$

2.4. **Integration Method.** Similarly to Case 3 we can also integrate along the $x$ and $y$ axes to obtain the perimeter and volume of $Q$. We then get the following result:

$$V_Q \approx \int_0^m \int_0^m (a+b+x+y)\,dxdy,$$

$$V_Q \approx m^2(m+a+b). \tag{2.9}$$

$$P_Q \approx \int_0^m (2a+2b+m+2x)dx + \int_0^m (2a+2b+m+2y)dy,$$

$$P_Q \approx 4m(m+a+b). \tag{2.10}$$

These results are again in agreement with the earlier summation calculations.

Now, we can express $P_Q$ in terms of $V_Q = n$. Let $c = a+b$, where $a,b \in \mathbb{N}-0$, and also let $s = c/m$ such that $s \in Q$. Then we can rewrite:

$$n = m^3(s+1) \Rightarrow m = \sqrt[3]{\frac{n}{s+1}}, \tag{2.11}$$

$$P_Q = 4m^2(s+1) \Rightarrow P(n) = 4n^{2/3}(s+1)^{1/3}. \tag{2.12}$$

Observe that $P(n)$ is minimized when $dP/dn = 0$, that is $s = c/m = 0$ for $c = a + b$. Because we know that $a$ and $b$ are whole numbers, hence, it must be true that $a = b = 0$. That shows that the perimeter of a square at the origin is smaller for any given large volume than the perimeter of a square that does not have one of its corners at point $(0, 0)$.

## Rectangles

Rectangles are a slightly more complicated version of squares because we will not be able to use the symmetry argument when summing up the $x$ and $y$ coordinates. However, the main principles remain the same and we will be able to compare results found by summation and integral approximation.

**Case 3.** $R = (k+1) \times (m+1)$ *rectangle on the lower left-hand corner of the first quadrant.*

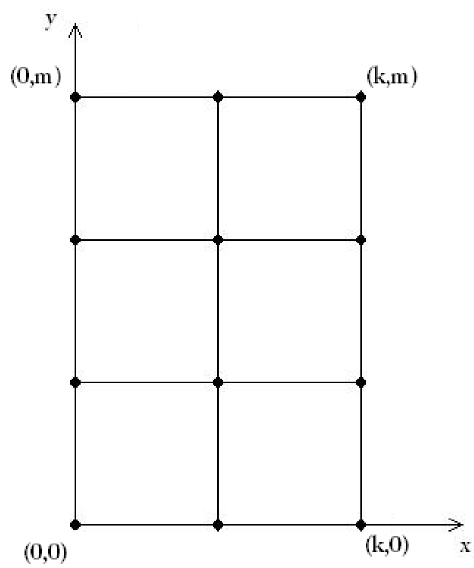


FIGURE 6. $(k+1) \times (m+1)$ rectangle $R$ at the origin

2.5. **Summation Method.** We can again compute $P_R$, $V_R$ and $P(n)$ by the simple summation formula (1.1). For volume, we first add the $x$ coordinates and then similarly add the $y$ coordinates. That is,

$$\sum x = (m+1) \sum_{i=0}^{k} i = (m+1) \frac{k(k+1)}{2},$$

$$\sum y = (k+1) \sum_{i=0}^{m} i = (k+1) \frac{m(m+1)}{2},$$

$$V_R = \frac{1}{2}(k+1)(m+1)(k+m). \tag{2.13}$$

Similarly to Case 3, for perimeter we just need to sum up the coordinates of the four sides of the rectangle. For simplicity we first will only add $x$ coordinates. Thus,

$$\sum x = k(m+1) + 2\sum_{i=0}^{k} i = k(m+1) + k(k+1) - 2k.$$

Using the same method we can compute the sum of $y$ coordinates of all four sides of the rectangles. Therefore,

$$\sum y = m(k+1) + 2\sum_{i=0}^{m} i = m(k+1) + m(m+1) - 2m.$$

Notice that in each summation we subtract $2k$ and $2m$ respectively in order to avoid double counting the corner points. Now, the total perimeter is just the sum of individual components which gives us:

$$P_R = (m+k)^2. \tag{2.14}$$

2.6. **Integration Method.** For rectangles writing down an integral is similar to the previously used density function; however, instead of integrating twice over length $m$, this time we have to integrate over lengths $m$ and $k$. That is,

$$V_R \approx \int_0^m \int_0^k (x+y)dxdy,$$

$$V_R \approx \frac{1}{2}km(k+m). \tag{2.15}$$

$$P_R \approx \int_0^k xdx + \int_0^m ydy + \int_0^k (m+y)dy + \int_0^m (k+x)dx,$$

$$P_R \approx (k+m)^2. \tag{2.16}$$

These results are in agreement with the earlier summation calculations.

Now, observe that for given $(k+m)$, $k=m$ is best if we want to maximize product $mk$. Thus, that proves that the square at the left-hand corner of the $1^{st}$ quadrant is a better perimeter minimizer than the rectangle at the origin.

**Case 4.** *General case of $L = (k+1) \times (m+1)$ rectangle.*

2.7. **Summation Method.** Placing a rectangle at point $(a, b)$ just adds $(m+1)(k+1)(a+b)$ to total volume since each point now contributes $a+b$ extra volume, and there are total of $(m+1)(k+1)$ points. Similarly, the perimeter will increase by $2(m+k)(a+b)$ as there are $2(m+k)$ points on the boundary. Therefore,

$$V_L = \frac{1}{2}(k+1)(m+1)(k+m+a+b), \tag{2.17}$$

$$P_L = 2(m+k)(m+k+a+b). \tag{2.18}$$

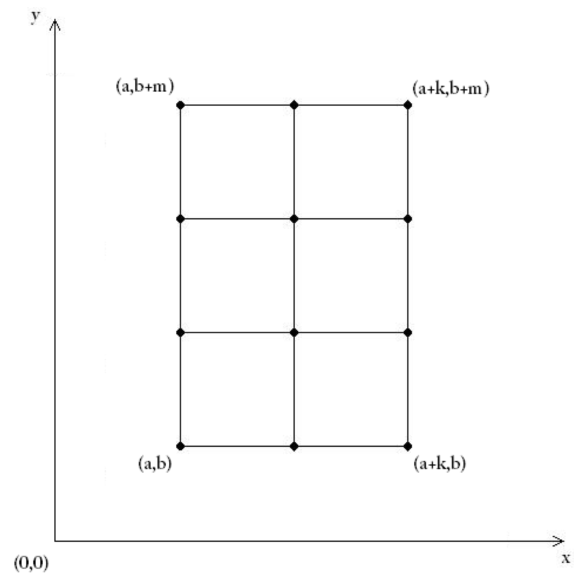


FIGURE 7. $(k+1) \times (m+1)$ rectangle $L$

2.8. **Integration Method.** As usual we also obtain integration approximations for the volume and perimeter of $L$:

$$V_L \approx \int_0^m \int_0^k (a+b+x+y)\, dxdy,$$

$$V_L \approx mk(a+b) + \frac{1}{2}mk(k+m). \tag{2.19}$$

$$P_L \approx \int_0^k (a+b+x)dx + \int_0^m (a+b+y)dy + \int_0^k (a+b+m+y)dy + \int_0^m (a+b+k+x)dx,$$

$$P_L \approx 2(k+m)(m+k+a+b). \tag{2.20}$$

Again, these results are in agreement with the earlier summation calculations.

We see again that for given $(k+m)$ the product $mk$ is maximized when $m=k$; and perimeter is minimized when $a=b=0$. This shows that the rectangle $R$ in the left-hand corner of the $1^{st}$ quadrant is better than the general case of the rectangle $L$ and, therefore, proves that that a square $C$ at the origin has least perimeter for given volume among the four examples considered so far.

## 3. Diamonds, Crosses, Quarter Circles and Triangles

For non-rectangles we, of course, will have many more candidates for the best perimeter minimizer. As illustrated by Soybel's cross counter-example there is a variety of possibilities for obtaining lesser perimeter for given volume by constructing irregular shapes. Indeed, it is not entirely clear or obvious why a regular shape, such as a square, would minimize the perimeter. Therefore, in this chapter we investigate four more cases of regions (a diamond, a cross, a quarter circle and a triangle) which expand our understanding of perimeter minimizing shapes in space.

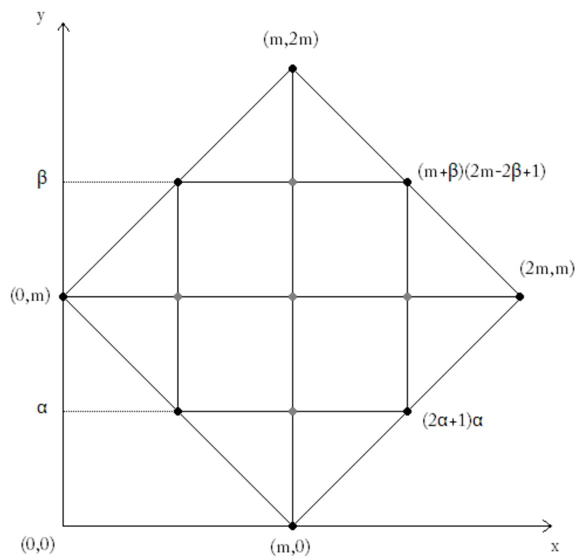


FIGURE 8. Diamond $D$ in the $1^{st}$ quadrant

## DIAMONDS

**Case 5.** *Diamond $D$ in the $1^{st}$ Quadrant.*

Consider a diamond in the $1^{st}$ Quadrant so that two of its vertices touch $x$ and $y$ axes at points $(m, 0)$ and $(0, m)$ respectively. As before, we can compute and approximate volume and perimeter of this region by using summation and integration methods.

3.1. **Summation Method.** First, we start off by summing up $y$ coordinates of all points on the lower triangle of the diamond. Thus, volume of the vertex $(m, 0)$ of this triangle is 0. Now, as we move along the $y$ axis volume of the center point (i.e., the one which is in line with the vertex) increases by 1 and two additional points are added to the left and to the right of the same volume. We can express the volume of some row $\alpha$ on this triangle with the formula:

$$V_\alpha = (2\alpha + 1)\alpha = 2\alpha^2 + \alpha.$$

Now, it is easy to find volume of the whole lower triangle by simply summing up all such rows. That is,

$$V_{Lower} = 2\sum_{\alpha=0}^{m} \alpha^2 + \sum_{\alpha=0}^{m} \alpha = \frac{m(m+1)(2m+1)}{3} + \frac{m(m+1)}{2},$$

$$V_{Lower} = \frac{5}{6}m + \frac{3}{2}m^2 + \frac{2}{3}m^3.$$

For the upper triangle we use similar approach; however, notice that there are a few important differences: (1) we start at $\beta = 1$ because the first row is already included in our lower triangle calculation, (2) the number of points is descending with each row but volume per point is increasing. We can write this down algebraically for some $\beta$ on the upper triangle with the formula:

$$V_\beta = (m+\beta)(2m-2\beta+1) = 2m^2 - 2\beta^2 + \beta + m.$$

Again, we easily find volume of the upper triangle by summing up all such rows, which gives us:

$$V_{Upper} = \sum_{\beta=1}^{m} 2m^2 - \sum_{\beta=1}^{m} 2\beta^2 + \sum_{\beta=1}^{m} \beta + \sum_{\beta=1}^{m} m,$$

$$V_{Upper} = 2m^3 - \frac{m(m+1)(2m+1)}{3} + \frac{m(m+1)}{2} - m^2,$$

$$V_{upper} = \frac{1}{6}m + \frac{1}{2}m^2 + \frac{4}{3}m^3.$$

Thus, the total volume of all $y$ coordinates is:

$$V_y = V_{Upper} + V_{Lower},$$

$$V_y = m + 2m^2 + 2m^3 = m(2m^2 + 2m + 1).$$

Our work is almost done here, because the results for $x$ coordinates are symmetrical to computations long the $y$ axis. Therefore,

$$V_x = m(2m^2 + 2m + 1),$$

$$V_D = V_x + V_y = 2m(2m^2 + 2m + 1). \tag{3.1}$$

Next step is to find perimeter of the diamond. We can do that easily by adding up the $x$ coordinates of all four of its sides (and subtracting $4m$ to avoid double counting), which gives us:

$$P_x = 2m(m+1) + 2m(m+1) - 4m = 4m^2.$$

By symmetry,

$$P_y = 4m^2,$$

$$P_D = P_x + P_y = 8m^2. \tag{3.2}$$

Finally, we can compare the diamond with other already discussed shapes. That is:

$$V_D \backsim 4m^3 \Rightarrow m = \sqrt[3]{\frac{n}{4}},$$

$$P(n) = 8\left(\frac{1}{4}\right)^{2/3} n^{2/3}. \tag{3.3}$$

We find that $P(n) \approx 3.1748n^{\frac{2}{3}}$, which is better than a square $C$ in the corner and, as we will see later, also than a quarter circle $W$ at the origin.

3.2. **Integration Method.** Similarly to our previous approach we can find perimeter and volume of the diamond using integration. The important part here is to notice that for volume we use density $x+y$, but for perimeter we use density $\frac{x+y}{\sqrt{2}}$ because there are only $\frac{1}{\sqrt{2}}$ points per unit distance along the 45° line. Let's find the perimeter first:

$$P_x = 2\int_0^{2m} \frac{x}{\sqrt{2}} ds.$$

Substitute $ds = \sqrt{2}dx$ and using symmetry find:

$$P_x \approx 4m^2,$$

$$P_y \approx 4m^2,$$

$$P_d \approx 8m^2. \tag{3.4}$$

Then we can also find volume of $D$ :

$$V_D \approx \int_0^m \int_{m-y}^{m+y} (x+y)dxdy + \int_m^{2m} \int_{-m+y}^{3m-y} (x+y)dxdy,$$

$$V_D \approx 4m^3. \tag{3.5}$$

These estimates agree asymptotically with the summation formulas (2.5-6).

## Crosses

**Case 6.** $C_r-$ *Cross.*

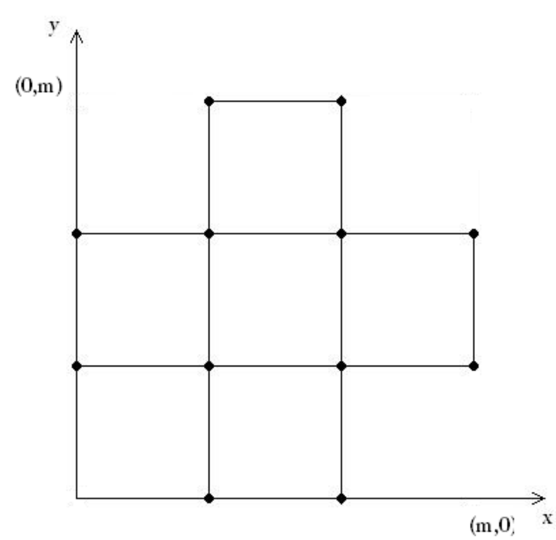


FIGURE 9. Large cross counter example

As I have already briefly discussed, a very interesting counterexample to the proposal that a square in the corner is a perimeter minimizer was provided by Soybel: the cross of Figure 0.2 has smaller perimeter (15) than the square of the same volume (18), which has perimeter 16. This idea can probably be used to show that no square can be perimeter minimizing. To do that simply remove the four corners of the square, reducing perimeter and volume by the same amounts and replace the volume with points with at least one not contributing to the perimeter.

Incidentally the cross formed by taking a square in the corner and removing the four corners has the same asymptotics as the square:

$$V_{C_r} = m(m+1)^2 - 4m, \tag{3.6}$$

$$P_{C_r} = 4m^2 - 4m, \tag{3.7}$$

$$P(n) \backsim 4n^{2/3}. \tag{3.8}$$

Therefore, we see that improvement asymptotically is too small to make any difference between a cross and a square when large volumes are considered. At the same time it is not hard to see that large cross will always be slightly better than a square.

### Quarter Circles

**Case 7.** *Quarter circle $W$ of radius $m$ at the origin.*

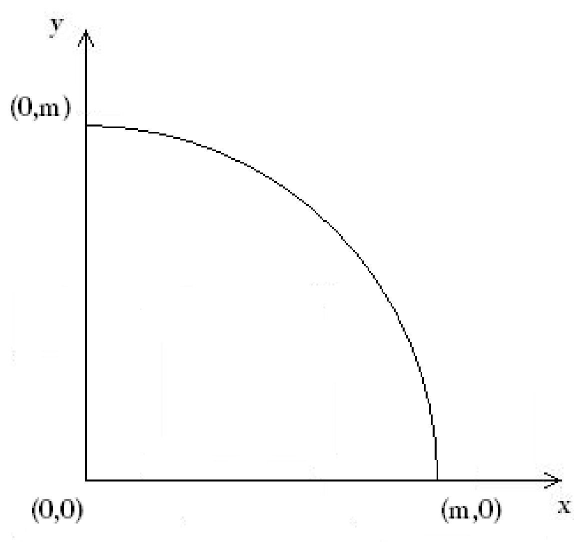


Figure 10. Quarter circle $W$ of radius $m$

By a quarter circle in the lattice we mean the region closest to the $y$ coordinates. We found the perimeter and volume to be too difficult to compute by summation, so we approximated the volume by integration, and attempted to approximate the perimeter by integration. As for a diamond, along the perimeter the lattice points

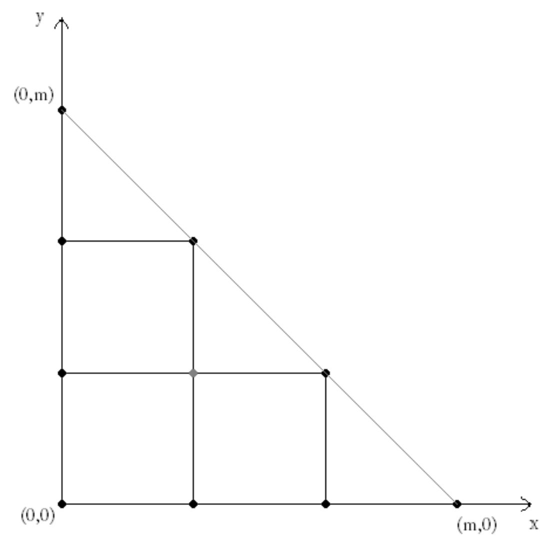


FIGURE 11. Right isosceles triangle $T$ at the origin

occur less frequently than every unit distance depending on the direction. All we can conclude is that the perimeter is less than the integral of $x+y$. Thus, we get:

$$V_W \approx \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=m} (x+y) r dr d\theta \approx \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=m} r(r\cos\theta + r\sin\theta) dr d\theta$$

$$V_W \approx \frac{2}{3} m^3. \tag{3.9}$$

Similarly, for the perimeter of $W$:

$$P_W \lessapprox \int_0^m x dx + \int_0^m y dy + \int_0^{\pi/2} (x+y) m d\theta$$

$$P_W \lessapprox 3m^2. \tag{3.10}$$

Finally, by expressing the minimizing perimeter of $W$ in terms of its volume $V_W = n$ we obtain:

$$m = \sqrt[3]{\frac{3n}{2}} \Rightarrow P = 3n^{2/3} \left(\frac{3}{2}\right)^{2/3} \tag{3.11}$$

This approximates to $P(n) \lessapprox 3.9311 n^{\frac{2}{3}}$, and thus it is better than square $C$ at the corner (Case 3) but probably worse than a diamond $D$ (Case 7).

## TRIANGLES

**Case 8.** *Right isosceles triangle $T$ at the origin.*

Having found that a diamond is probably better than a quarter circle it is natural to ask whether a right isosceles triangle would also perform better. The triangle $T$ provides an interesting case because it gets "cheap" points on the $x$ and $y$ axes (like a quarter circle) as well as high volume points along the efficient $45°$ diagonal (like a diamond). Thus the triangle has properties of all previously considered shapes that made them good perimeter minimizers.

3.3. **Summation Method.** Similarly to our diamond $D$ computations we can find volume and perimeter of triangle $T$ by defining a new formula for the sum of its $y$ coordinates:

$$V_\alpha = \alpha(m+1-\alpha) = \alpha m + \alpha - \alpha^2,$$

where $\alpha$ is any row on the triangle. As before we can find the sum of all $y$ coordinates by simply adding up all such rows:

$$V_y = m\sum_{\alpha=0}^{m}\alpha + \sum_{\alpha=0}^{m}\alpha - \sum_{\alpha=0}^{m}\alpha^2 = \frac{1}{3}m + \frac{1}{2}m^2 + \frac{1}{6}m^3.$$

By symmetry,

$$V_x = \frac{1}{3}m + \frac{1}{2}m^2 + \frac{1}{6}m^3.$$

$$V_T = \frac{1}{3}m(2+3m+m^2). \tag{3.12}$$

Now, we also find the perimeter by adding up the coordinates of points along the diagonal and the $x$ and $y$ axes:

$$P_T = 4\sum_{i=0}^{m} i - 2m = 2m^2. \tag{3.13}$$

3.4. **Integration Method.** Approximating volume and perimeter of $T$ by integration we find:

$$V_T \approx \int_0^m \int_0^{m-y} (x+y)dxdy$$

$$V_T \approx \frac{1}{3}m^3. \tag{3.14}$$

As for a diamond we use density $\frac{x+y}{\sqrt{2}}$ because there are only $\frac{1}{\sqrt{2}}$ points per unit distance along the $45°$ diagonal. We also substitute $ds = \sqrt{2}dx$,

$$P_T \approx \int_0^m \frac{x+y}{\sqrt{2}}ds + 2\int_0^m xdx \approx \int_0^m (x+(m-x))dx + m^2,$$

$$P_T \approx 2m^2. \tag{3.15}$$

These approximations agree asymptotically with the summations. Moreover,

$$m = \sqrt[3]{3n} \Rightarrow P = 2\,(3)^{2/3}\,n^{2/3} \tag{3.16}$$

We find that $P(n) \approx 4.1602n^{\frac{2}{3}}$, which is surprisingly even worse than square $C$ at the origin.

## References

[DDNT] Dahlberg, J., Dubbs, A., Newkirk, E., Tran, H., "Isoperimetric regions in the plane with density $r^p$", NY J. Math., to appear.

[Mi] Miller, S., Morgan, F., Newkirk, E., Pedersen, L., Seferis, D., "Isoperimetric sequences", preprint (2009). See http://blogs.williams.edu/Morgan/2008/11/21/isoperimetric-sequences/.

[Mo] Morgan, F., "Geometric Measure Theory: a Beginner's Guide", fourth edition, Academic Press, Burlington, 2009.

[S] Seferis, D. "Isoperimetric regions in a weighted 2-dimensional lattice", senior thesis, Williams College, 2009.

[WW] Wang, D., Wang, P., "Discrete isoperimetric problems", SIAM J. Appl. Math. 32 (1977), no. 4, 860-870.

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