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Variants of the Game of Nim that have Inequalities as Conditions

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Variants of the Game of Nim that have Inequalities as Conditions.

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Abstract

In this article the authors are going to present several combinatorial games that are variants of the game of Nim. They are very different from the traditional game of Nim, since the coordinates of positions of the game satisfy inequalities. These games have very interesting mathematical structures. For example, the lists of P-positions of some of these variants are subsets of the list of P-positions of the traditional game of Nim. The authors are sure that they were the first people who treated variants of the game of Nim conditioned by inequalities. Some of these games will produce beautiful 3D graphics (indeed, you will see the Sierpinski gasket when you look from a certain view point). We will also present some new results for the chocolate problem, a problem which was studied in a previous paper (see [1]) and related to Nim. The authors make substantial use of Mathematica in their research of combinatorial games.

1 Introduction and Combinatorial Game

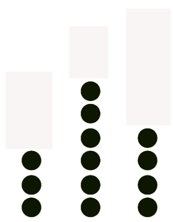
Combinatorial games provide good research topics for undergraduate students or high school students, since they can discover new facts and theorems without the knowledge of graduate students. In combinatorial game theory it is fairly easy to propose a new problem, but we should study problems that are worth studying. Variants of the game of Nim that the authors treat in this article have elegant mathematical structures and beautiful 3D graphs, and the authors are confident they are worth studying. It is often the case that the knowledge of high school mathematics is enough to understand the basics of the theory, but the theory of combinatorial games are very different from traditional mathematics. Therefore the authors provide many examples for readers to understand the methods we are using in this article.

In the combinatorial game Nim, there are two players who take turns alternately. They continue playing until one of the players has no legal moves available. Traditionally the two players of a combinatorial game are called Left(or just L) and Right (R). If the left options and the right options of a position are always the same, then we call the game impartial. In this paper we study impartial games and note that all variables will be non-negative integers.

2 Traditional Nim

Definition 2.1. *We are going to define the game called Nim with 3 piles. There are three piles, and the players alternate by taking all or some of the counters in a single heap. The player who takes the last counter or stack of counters is the winner.*

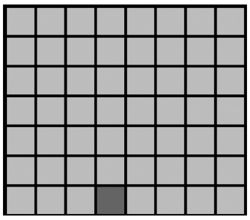
Example 2.1. *In Graph 2.1 we have three piles with 3, 6, 4 counters. We denote the position of Graph 2.1 by* $\{3, 6, 4\}$*. In general we can play the game of Nim with* x, y, z *counters for any non-negative integers* x, y, z*.*

Graph 2.1. 

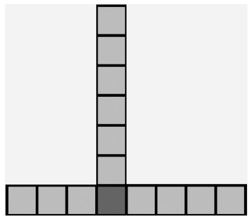
There is another interesting problem called the bitter chocolate problem which is related to Nim.

Definition 2.2. *Start with some pieces of chocolate where the light gray parts are sweet and the dark gray part is very bitter. Each of the two players in turn breaks the chocolate (in a straight line along the grooves) and eats the piece he breaks off. The player to leave his opponent with the single bitter part is the winner.*

Example 2.2. *Here we have two chocolate problems which are essentially the same. These kinds of chocolate problems has been proposed in [3].*



Graph 2.2.



Graph 2.3.

Clearly the game of Nim with the position $\{3, 6, 4\}$ *is mathematically the same as the problem in Graph 2.3, and hence it is mathematically the same as the problem in Graph 2.2. In general we can play chocolate problems with any position* $\{x, y, z\}$ *for non-negative integers* x, y, z.

Definition 2.3. *In the game of Nim and chocolate problems there are two kinds of positions. One kind is a P-position, a previous-player-winning position. The other is an N-position, a Next-player-winning position. In the followings we use the word option to mean "choice of move".*
[1]. *The position* $\{0, 0, 0\}$ *is a P-position.*
[2]. *Every option for a P-position leads to an N-position.*
[3]. *For a N-position there is always at least one option leading to a P-position.*
Note that this definition is recursive.

Remark 2.1. *If you find all the P-positions, then you can find the winning strategy to the game. For example, if you know that particular position is a P-position and you start the game as a previous player, then your opponent is the next player, and any option by your opponent will lead to a N-position. After your opponent moves to a N-position, you can choose a P-position by a proper option. Every time your opponent moves to a N-position, you can move to a P-position, and finally you will reach the final P-position* $\{0, 0, 0\}$ *and you will be the winner.*
Therefore it is very important to know all the P-positions. Note that P-positions and N-positions can be defined for any impartial games.

Next we define the nim-sum which is used to study the game of Nim and its variants.

Definition 2.4. *If x and y are non-negative integers, let $x = \sum_{i=0}^{n} a_i 2^i$ and $y = \sum_{i=0}^{n} b_i 2^i$, where a_i, b_i are 0 or 1 for $i = 1, 2, ...n$.*
We let c_i = 0 or 1, if $a_i + b_i$ is even or odd respectively, and let then deine the nim-sum $x \oplus y = \sum_{i=0}^{n} c_i 2^i$.

We need to demonstrate the calculation of the nim-sum for readers to understand the definition of the nim-sum, and if you get used to this kind of calculation, it will be easy to understand the techniques we use in this article.

Example 2.3. [1]. *We are going to calculate the nim-sum $52 \oplus 21 \oplus 58$.*
[2]. *We are going make the nim-sum equal to 0 by subtracting a natural number from one of three numbers.*

[1]. *Since $52 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$, 52 can be expressed by the list $\{0, 0, 1, 0, 1, 1\}$. Note that the coefficient of 2^5 is the last coordinate of the list. Similarly we can make the same kind of lists $\{0, 1, 1, 0, 1, 0\}$ and $\{0, 1, 0, 1, 1, 1\}$ for 21 and 58 represented graphicall in 2.4.*
Now we are going to calculate the nim-sum $52 \oplus 21 \oplus 58$ by using Definition 2.4.
In the second row of this graph we have two "1"s. Since $1+0+1 = 0$ (mod 2), the value of the nim-sum in this row is 0.
In the third row of this graph we have three "1"s. Since $1+1+1 = 1$ (mod 2), the value of the nim-sum in this row is 1.
In this way we can calculate $0 + 0 + 1 = 1$ (mod 2), $1 + 1 + 0 = 0$ (mod 2), $0 + 1 + 1 = 0$ (mod 2) and $0 + 0 + 0 = 0$ (mod 2), and we get the list $\{0, 0, 0, 1, 1, 0\}$ for the nim-sum.
Therefore the nim-sum $52 \oplus 21 \oplus 58 = 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 24$.

Graph 2.4.

1		52	21	58	*nim-sum = 24*
2	2^5	1	*0*	1	*0*
3	2^4	1	*1*	1	*1*
4	2^3	0	*0*	1	*1*
5	2^2	1	*1*	0	*0*
6	2^1	0	*1*	1	*0*
7	$2^0 = 1$	0	*0*	0	*0*

[2]. *We are going to show that we can make the nim-sum equal to 0 by subtracting a natural number from one of these three numbers.*
We should have even numbers of "1"s in each row to make the nim-sum equal to 0.
For example if we subtract 8 from 52, then we have three numbers $\{44, 21, 58\}$, *then we get Graph 2.5. Since* $1+0+1 = 0+1+1 = 1+0+1 = 1+1+0 = 0+1+1 = 0+0+0 = 0$ $(mod\ 2)$, *the nim-sum* $44 \oplus 21 \oplus 58 = 0$

Graph 2.5.

1		44	21	58	*nim-sum = 0*
2	2^5	1	*0*	1	*0*
3	2^4	0	*1*	1	*0*
4	2^3	1	*0*	1	*0*
5	2^2	1	*1*	0	*0*
6	2^1	0	*1*	1	*0*
7	$2^0 = 1$	0	*0*	0	*0*

Similarly by subtracting 7 from 21 we have three numbers $\{52, 14, 58\}$, *and it is easy to see that* $52 \oplus 14 \oplus 58 = 0$.

Remark 2.2. *It is clear from the definition of nim-sum that* $x \oplus y \oplus z = 0$ *implies that* $z \leq x + y$, $x \leq y + z$ *and* $y \leq z + x$. *For an example please look at Graph 2.5.*

Theorem 2.1. [1]. *The list* $\{\{x,\ y,\ z\},\ x \oplus y \oplus z{=}0\}$ *is the set of P-positions of the game of Definition 2.1.*
[2]. *The list* $\{\{x,\ y,\ z\},\ x \oplus y \oplus z \neq 0\}$ *is the set of N-positions of the game of Definition 2.1.*

Proof. *We are going to omit the proof, since this is a well known result. See [2] in References.*

Corollary 2.1. [1] *If* $x \oplus y \oplus z{=}0$, $u < x$, $v < y$ *and* $w < z$, *then we have* $u \oplus y \oplus z \neq 0$, $x \oplus v \oplus z \neq 0$ *and* $x \oplus y \oplus w \neq 0$.
[2] *If* $x \oplus y \oplus z \neq 0$, *then at least one of the following three conditions is satisfied.*
[2.1]. *There exists* u *such that* $u < x$ *and* $u \oplus y \oplus z = 0$.
[2.2]. *There exists* v *such that* $v < y$ *and* $x \oplus v \oplus z = 0$.
[2.3]. *There exists* w *such that* $w < z$ *and* $x \oplus y \oplus w = 0$.

Proof. [1]. *This is direct from Theorem 2.1 and the definition of P-positions and N-Positions. If $x \oplus y \oplus z = 0$, then by Theorem 2.1 $\{x,\ y,\ z\}$ is a P-position of the game of Definition 2.1. From this position you can move to $\{u,\ y,\ z\}$ with $u < x$ or $\{x,\ v,\ z\}$ with $v < y$ or $\{x,\ y,\ w\}$ with $w < z$. Therefore by* [2] *of Definition 2.3 $\{u,\ y,\ z\}$, $\{x,\ v,\ z\}$ and $\{x,\ y,\ w\}$ are N-positions of the game of Definition 2.1. Therefore we have $u \oplus y \oplus z \neq 0$, $x \oplus v \oplus z \neq 0$ and $x \oplus y \oplus w \neq 0$.*

[2]. *If $x \oplus y \oplus z \neq 0$, then by Theorem 2.1 $\{x,\ y,\ z\}$ is an N-position, and hence by* [3] *of Definition 2.3 there should be an option that leads to a P-position. Therefore at least one of the conditions* [2.1], [2.2] *and* [2.3] *are satisfied.*

3 Other Chocolate Problems which are Variants of the Game of Nim.

In this section we study other chocolate problems, related to the game of Nim, and illustrate their interesting and these variants have a lot of interesting mathematical properties.

Example 3.1. *This time we are gong to use the chocolate in Graph 3.1. This problem has been introduced by our teacher Dr. R.Miyadera and his students in [5].*

Graph 3.1. 

The problem of Graph 2.2 is completely different from the problem of Graph 3.1. In Graph 3.1 you can cut the chocolate in 6 ways, so it is appropriate to represent it with 6 non-negative integers $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. We represent the position in Graph 3.1 with $\{2,1,2,1,2,1\}$.

Note that these 6 coordinates are not independent, i.e., in some cases you cannot subtract a natural number from one coordinate without affecting other coordinates.
It is clear that we have 6 inequalities between these 6 coordinates.

$$x_1 \leq x_2 + x_6, x_2 \leq x_1 + x_3 + 1, x_3 \leq x_2 + x_4,$$
$$x_4 \leq x_3 + x_5 + 1, x_5 \leq x_4 + x_6, x_6 \leq x_5 + x_1 + 1.$$

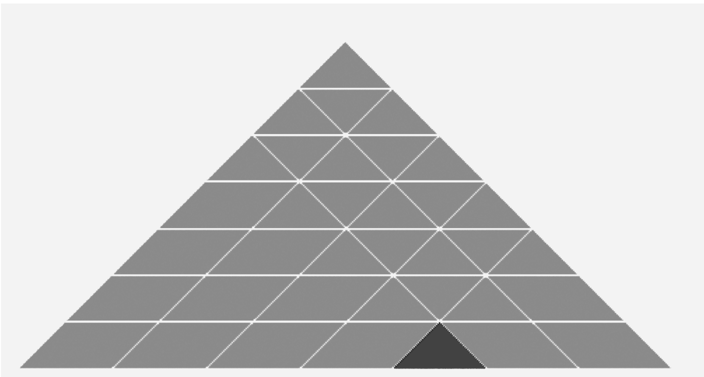
We can study chocolates of any size if we use arbitrary non-negative integers $x_1, x_2, x_3, x_4, x_5, x_6$ that satisfy these 6 inequalities.
We have studied this problem and have presented a strategy to win the game for a limited size of chocolate in [5].

The chocolate problem of Example 3.1 is very difficult to study mathematically, and hence the authors have made an easier version which has a very simple formula for calculating the P-positions.

Example 3.2. *Suppose that you have the chocolate in Graph 3.2. In Graph 3.2 you can cut the chocolate in 3 ways, so it is appropriate to represent it with 3 non-negative integers $\{x, y, z\}$. We represent the position in Graph 3.2 with $\{4, 6, 2\}$. It is clear that we have an inequality*

$$y \leq x + z \tag{3.1}$$

between these 3 coordinates.

Graph 3.2. 

Note that $\{x, y, z\}$ can be a position of this chocolate problem when x, y, z satisfy Inequality (3.1).

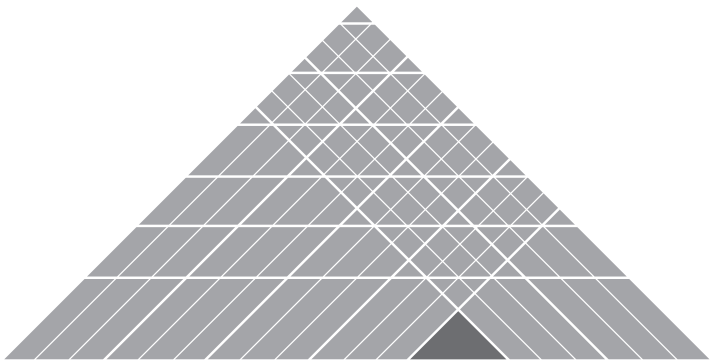
Theorem 3.1. *The chocolate problem of Example 3.2 has the following simple formula to calculate P-positions.*
[1]. *The position $\{x,\ 0,\ z\}$ is a P-position if and only if $x = z$.*
[2]. *For a natural number y the position $\{x+1,\ y,\ z+1\}$ is a P-position if and only if $x \oplus y \oplus z = 0$.*

We are not going to prove Theorem 3.1 in this paper (see [6] for a proof) . In this article we are going to study another chocolate problem.

Example 3.3. *Suppose that you have the chocolate in Graph 3.3. In Graph 3.2 you can cut the chocolate in 3 ways, so it is appropriate to represent it with 3 non-negative integers* $\{x, y, z\}$*. We represent the position in Graph 3.3 with* $\{4, 6, 2\}$*. It is clear that we have an inequality*

$$3y \leq x + z \tag{3.2}$$

between these 3 coordinates. Since y *is a non-negative integer, we have* $y \leq \lfloor \frac{x+z}{3} \rfloor$*, where* $\lfloor \quad \rfloor$ *is the floor function.*



Graph 3.3.

Note that $\{x, y, z\}$ can be a position of this chocolate problem when x, y, z satisfy Inequality (3.2).

Now we are going study the P-positions and N-positions of the game of Example 3.3.
Let $A = \{\{x, y, z\}; x+z \geq 3y \text{ and } x \oplus y \oplus z = 0\}$ and $B = \{\{x, y, z\}; x+z \geq 3y$ and $x \oplus y \oplus z \neq 0\}$.

We are going to prove that A is the list of P-positions and B is the list of N-positions of the game of Example 3.3. We are going to prove this later in Theorem 3.5.

Before we study Theorem 3.2, it is better to see some examples. You can skip this example and read Theorem 3.2 if you like.

Example 3.4. *In this example we are going to use methods that can be generalized for arbitrary non-negative numbers, because this example is for the reader to understand the proofs of theorems that we are going to read later. Therefore we sometimes use a difficult calculation when a very simple calculation can do the job.*
Let $x = \sum_{i=0}^{5} a_i 2^i$, $y = \sum_{i=0}^{5} b_i 2^i$ *and* $z = \sum_{i=0}^{5} c_i 2^i$.
[1]. *Let* $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, 0, 1, 0, 1, 1\}$,
$\{b_0, b_1, b_2, b_3, b_4, b_5\} = \{1, 1, 0, 1, 1, 0\}$ *and* $\{c_0, c_1, c_2, c_3, c_4, c_5\} = \{1, 1, 0, 1, 1, 1\}$, *and hence we have Graph 3.4.*

Graph 3.4.

	$x = 53$	$y = 27$	$z = 59$
2^5	1	*0*	1
2^4	1	*1*	1
2^3	0	*1*	1
2^2	1	*0*	0
2^1	0	*1*	1
$2^0 = 1$	1	*1*	1

Clearly $x + z \geq 3y$ *and* $x \oplus y \oplus z \neq 0$, *and hence* $\{x, y, z\} \in B$.
We are going to show that there is an option that leads to a position in A. *Since there are two "1"s in the row of* 2^5 *and three "1"s in the row of* 2^4, *we can subtract some number from* $y = 27$ *so that there are always even number of "1"s in each row. (Note that 0 is an even number.)*
Let $d_5 = 0, d_4 = 0$ *and* $d_i = a_i + c_i$ (*mod* 2) *for* $i =$ *0,1,2,3. Then we have* $d_3 = 1, d_2 = 1, d_1 = 1$ *and* $d_0 = 0$. *Let* $v = \sum_{i=0}^{5} d_i 2^i$, *then we have* $v = 14 < y$ *and* $x \oplus v \oplus z = 0$ *and* $x + z \geq 3v$. *Therefore* $\{x, v, z\} \in A$ *and we can move from* $\{x, y, z\}$ *to* $\{x, v, z\}$ *by subtracting* $y - v$ *from* y.
As you can see easily, the situation is simple when we can move to a position in A *by reducing* y, *since the inequality* $3y \leq x + z$ *implies that* $3v \leq x + z$ *for* $v < y$.

[2]. *Let* $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{1, 0, 1, 1, 1, 1\}$, $\{b_0, b_1, b_2, b_3, b_4, b_5\} = \{1, 1, 1, 0, 1, 0\}$ *and* $\{c_0, c_1, c_2, c_3, c_4, c_5\} = \{1, 0, 0, 0, 0, 1\}$, *and hence we have Graph 3.5.*

Graph 3.5.

	$x = 61$	$y = 23$	$z = 33$
2^5	1	*0*	1
2^4	1	*1*	0
2^3	1	*0*	0
2^2	1	*1*	0
2^1	0	*1*	0
$2^0 = 1$	1	*1*	1

Clearly $x + z \geq 3y$ *and* $x \oplus y \oplus z \neq 0$, *and hence* $\{x, y, z\} \in B$. *We are going to show that there is an option that leads to a position in* A.
Since there is only one "1" in the row of 2^3, *we have to subtract some number from* x.
Let $d_5 = 1, d_4 = 1, d_3 = 0$ *and* $d_i = b_i + c_i \pmod 2$ *for* i = *0,1,2. Then we have* $d_2 = 1, d_1 = 1$ *and* $d_0 = 0$. *Let* $u = \sum_{i=0}^{5} d_i 2^i = 54$.
Clearly we have $u < x$ *and* $u \oplus y \oplus z = 0$, *and hence we can move to* $\{u, y, z\} \in A$.

In [2] *of this example we move to a position in* A *by reducing* x, *and hence the situation is more difficult than that of* [1] *of this example, since the inequality* $3y \leq x + z$ *does not imply that* $3y \leq u + z$ *for* $u < x$.
Therefore it is meaningful to study the inequality $3y \leq u + z$ *more carefully.*

By the definition of d_i *for* $i = 0, 1, 2, 3, 4$ *we know that*
$\sum_{i=0}^{4} b_i 2^i \oplus \sum_{i=0}^{4} d_i 2^i \oplus \sum_{i=0}^{4} c_i 2^i = 0$, *and hence by Remark 2.2 we have*

$$y = \sum_{i=0}^{4} b_i 2^i \leq \sum_{i=0}^{4} d_i 2^i + \sum_{i=0}^{4} c_i 2^i. \tag{3.3}$$

Clearly we have

$$u + z = 2^5 + \sum_{i=0}^{4} c_i 2^i + 2^5 + \sum_{i=0}^{4} d_i 2^i. \tag{3.4}$$

Since $y \leq 2^5$, *by equations* (3.3) *and* (3.4) *we have*
$3y \leq u + z$.
Therefore the property of nim-sum mentioned in Remark 2.2 plays an important role here.
[3]. *Let* $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{0, 1, 1, 0, 0, 1\}$, $\{b_0, b_1, b_2, b_3, b_4, b_5\} = \{0, 1, 0, 1, 0, 0\}$

and $\{c_0, c_1, c_2, c_3, c_4, c_5\} = \{0, 0, 1, 0, 0, 0\}$, *and hence we have Graph 3.6.*

Graph 3.6.

	$x = 38$	$y = 10$	$z = 4$
2^5	1	*0*	0
2^4	0	*0*	0
2^3	0	*1*	0
2^2	1	*0*	1
2^1	1	*1*	0
$2^0 = 1$	0	*0*	0

Clearly $x + z \geq 3y$ *and* $x \oplus y \oplus z \neq 0$, *and hence* $\{x, y, z\} \in B$.
We are going to show that there is an option that leads to a position in A. *This time the method is a little bit more difficult than the one we used in* [1] *and* [2] *of this example.*
There is a "1" in the row of 2^5, *and hence we have to subtract a natural number from 38, but it is not easy to move to a position that satisfies Inequality* (3.2). *For example if we move to* $\{14, 10, 4\}$, *then we have* $14 \oplus 10 \oplus 4 = 0$, *but* $14 + 4 < 3 \times 10$.
See Graph 3.7. In this graph we have managed to make the nim-sum equal to 0, but the position does not satisfy the inequality.

Graph 3.7.

	$x = 14$	$y = 10$	$z = 4$
2^5	0	*0*	0
2^4	0	*0*	0
2^3	1	*1*	0
2^2	1	*0*	1
2^1	1	*1*	0
$2^0 = 1$	0	*0*	0

Therefore we have to reduce two numbers at the same time. This method is very different from the traditional game of nim in Example 2.1.
First we reduce x, *and by so doing we can reduce* y *by Inequality* (3.2).

Let $d_i = 0$ *for* $i = 3, 4, 5$, $d_2 = 1$ *and* $d_i = 1 - c_i$ *(mod 2) for* $i =$ *0,1. Then we have* $d_1 = 1$ *and* $d_0 = 1$. *Let* $u = \sum_{i=0}^{5} d_i 2^i$.

When we subtract $x-u$ from x, the first coordinate will be u and by Inequality (3.2) *the second coordinate should be* $v = \lfloor \frac{u+z}{3} \rfloor$
$= \lfloor \frac{2^2+\sum_{i=0}^{1} d_i 2^i + 2^2 + \sum_{i=0}^{1} c_i 2^i}{3} \rfloor = \lfloor \frac{2^2+2^2+2^2-1}{3} \rfloor = 2^2-1$. *Therefore y is going to be reduced to* $v = e_1 2^1 + e_0 2^0$ *and* $e_1 = e_0 = 1$.

Then we have Graph 3.8.

Graph 3.8.

	$u=7$	$v=3$	$z=4$
2^5	0	*0*	0
2^4	0	*0*	0
2^3	0	*0*	0
2^2	1	*0*	1
2^1	1	*1*	0
$2^0=1$	1	*1*	0

By Graph 3.8 it is clear that
$u \oplus v \oplus z = 0$, *and hence we have*

$$\sum_{i=0}^{1} e_i 2^i \oplus \sum_{i=0}^{1} d_i 2^i \oplus \sum_{i=0}^{1} c_i 2^i = 0. \tag{3.5}$$

Next we are going to prove that $3v \leq v+z$. *By Equation* (3.5) *and Remark 2.2 we have*

$$v = \sum_{i=0}^{1} d_i 2^i \leq \sum_{i=0}^{1} e_i 2^i + \sum_{i=0}^{1} c_i 2^i. \tag{3.6}$$

By the definition of u and z we have

$$u+z = 2^2 + \sum_{i=0}^{1} d_i 2^i + 2^2 + \sum_{i=0}^{1} c_i 2^i. \tag{3.7}$$

Since $v \leq 2^2$, *by equations* (3.6) *and* (3.7) *we have*
$3v \leq u+z$. *Therefore we can move to* $\{u, v, z\} \in A$.

[4]. *Let* $\{a_0, a_1, a_2, a_3, a_4, a_5\} = \{0, 0, 1, 0, 0, 1\}$, $\{b_0, b_1, b_2, b_3, b_4, b_5\} = \{1, 0, 1, 0, 0, 0\}$ *and* $\{c_0, c_1, c_2, c_3, c_4, c_5\} = \{0, 1, 1, 1, 0, 0\}$, *and hence we have Graph 3.9.*

Graph 3.9.

	$x = 36$	$y = 5$	$z = 14$
2^5	1	*0*	0
2^4	0	*0*	0
2^3	0	*0*	1
2^2	1	*1*	1
2^1	0	*0*	1
$2^0 = 1$	0	*1*	0

Clearly $x + z \geq 3y$ and $x \oplus y \oplus z \neq 0$, and hence $\{x, y, z\} \in B$.
We are going to show that there is an option that leads to a position in A.
Since there is only one "1" in the row of 2^5, we have to reduce x to move to a position in A.
Let $d_i = 0$ for $i = 4, 5$, $d_3 = 1$ and $d_i = b_i + c_i (mod\ 2)$ for $i =$ 0,1,2. Then we have
$d_2 = 0$ *and* $d_1 = d_0 = 1$.
Let $u = \sum_{i=0}^{5} d_i 2^i$.
Then it is clear from the definition of d_i for $i = 0, 1, ...5$ that
$u \oplus y \oplus z = 0$, *and hence we have*

$$\sum_{i=0}^{2} d_i 2^i \oplus \sum_{i=0}^{2} b_i 2^i \oplus \sum_{i=0}^{2} c_i 2^i = 0. \tag{3.8}$$

By Equation (3.8) *and Remark 2.2 we have*

$$y = \sum_{i=0}^{2} b_i 2^i \leq \sum_{i=0}^{2} d_i 2^i + \sum_{i=0}^{2} c_i 2^i. \tag{3.9}$$

By the definition of u and z we have

$$u + z = 2^3 + \sum_{i=0}^{2} d_i 2^i + 2^3 + \sum_{i=0}^{2} c_i 2^i. \tag{3.10}$$

Since $y < 2^3$, by Inequality (3.9) *and Equation* (3.10) *we have*
$3y \leq u + z$.
Therefore we can move to the position $\{u, y, z\} \in A$.

Remark 3.1. *In* [3] *of Example 3.4 we reduced x to u, and in so doing we reduced y to v. Then we got the equation $v = \lfloor \frac{u+z}{3} \rfloor$. This kind of equation is valid when we reduce two numbers at the same time.*

Theorem 3.2. *Let $\{x, y, z\} \in B$, then there is an option that leads a position in A.*

Proof. *Let n, p, q be natural numbers, and we suppose that $n = \lfloor\ log_2 x\rfloor$, $p = \lfloor\ log_2 y\rfloor$ and $q = \lfloor\ log_2 z\rfloor$, where $\lfloor\ \rfloor$ is the floor function.*
Since $3y \leq x + z$, we assume without any loss of generality that $n \geq p, q$.
Let $x = \sum_{i=0}^{n} a_i 2^i$, $y = \sum_{i=0}^{n} b_i 2^i$ and $z = \sum_{i=0}^{n} c_i 2^i$, where a_i, b_i and c_i are 0 or 1 for $i = 1, 2, ...n$.

[1]. *Suppose that $n = q$. Let k be the biggest non-negative integer such that $a_k + b_k + c_k \neq 0 \ (mod\ 2)$. Clearly $k \leq n$.*

[1.1]. *Suppose that $b_k = 1$. Then* [1] *of Example 3.4 is a special case of this situation, and hence we are going to use the method that is similar to the one used in* [1] *of Example 3.4.*
We want to move to a position in A by reducing y.
Let $d_i = b_i$ $(i = k + 1, ..., n)$, $d_k = 0$ and $d_i = a_i + b_i \ (mod\ 2)$ for $i = 0, 1, ..., k - 1$.
Let $v = \sum_{i=0}^{n} d_i 2^i$, then we have $v < y$, $x \oplus v \oplus z = 0$ and $3v < 3y \leq x + z$. Therefore we can move to $\{x, v, z\} \in A$.

[1.2]. *Suppose that $b_k = 0$, then we have $a_k = 1$ or $c_k = 1$. We can assume without any loss of generality that $a_k = 1$ and $c_k = 0$.* [2] *of Example 3.4 is a special case of this situation.*
Let $d_i = a_i$ for $i = k + 1, k + 2, ..., n$, $d_k = 0$ and $d_i = b_i + c_i (mod\ 2)$ for $i = 0, 1, ..., k - 1$.
Let $u = \sum_{i=0}^{n-1} d_i 2^i$.
By the definition of u it is clear that

$$u \oplus y \oplus z = 0, \tag{3.11}$$

and hence we have

$$\sum_{i=0}^{n-1} b_i 2^i \oplus \sum_{i=0}^{n-1} d_i 2^i \oplus \sum_{i=0}^{n-1} c_i 2^i = 0. \tag{3.12}$$

Therefore by Equation (3.12) *and Remark 2.2 we have*

$$y = \sum_{i=0}^{n-1} b_i 2^i \leq \sum_{i=0}^{n-1} d_i 2^i + \sum_{i=0}^{n-1} c_i 2^i. \tag{3.13}$$

By the definition of u, z *we have*

$$u + z = 2^n + \sum_{i=0}^{n-1} c_i 2^i + 2^n + \sum_{i=0}^{n-1} d_i 2^i. \tag{3.14}$$

Since $y \leq 2^n$, *by equations* (3.13) *and* (3.14) *we have*
$3y \leq u + z$. *We have* $u < x$ *and* $u \oplus y \oplus z = 0$, *and hence we can move to* $\{u, y, z\} \in A$.

[2]. *We suppose that* $n > p, q$.

[2.1]. *We suppose that* $n > p \geq q$. [3] *of Example 3.4 is a special case of the situation of* [2.1], *and hence we are going to reduce* x *and in so doing we will reduce* y.
Let $d_i = 0$ *for* $i = q+1, q+2, ..., n$, $d_q = 1$ *and* $d_i = 1 - c_i$ *for* $i = 0, 1, ..., q-1$. *Let* $u = \sum_{i=0}^{n} d_i 2^n$.

When we subtract $x - u$ *from* x, *the first coordinate will be* u *and by Inequality* (3.2) *the second coordinate should be*
$v = \lfloor \frac{u+z}{3} \rfloor = \lfloor \frac{2^q + \sum_{i=0}^{q-1} d_i 2^i + 2^q + \sum_{i=0}^{q-1} c_i 2^i}{3} \rfloor = \lfloor \frac{2^q + 2^q + 2^q - 1}{3} \rfloor = 2^q - 1$.
Therefore y *is going to be reduced to* $v = \sum_{i=0}^{q-1} e_i 2^i$, *where* $d_i = 1$ *for* $i = 0, 1, ..., q-1$.
By the definition of e_i *and* d_i *we have*

$$u \oplus v \oplus z = 0, \tag{3.15}$$

and hence we have

$$\sum_{i=0}^{q-1} e_i 2^i \oplus \sum_{i=0}^{q-1} d_i 2^i \oplus \sum_{i=0}^{q-1} c_i 2^i = 0. \tag{3.16}$$

Therefore by Equation (3.16) *and Remark 2.2 we have*

$$v = \sum_{i=0}^{q-1} e_i 2^i \leq \sum_{i=0}^{q-1} d_i 2^i + \sum_{i=0}^{q-1} c_i 2^i. \tag{3.17}$$

Clearly

$$u + z = 2^q + \sum_{i=0}^{q-1} d_i 2^i + 2^q + \sum_{i=0}^{q-1} c_i 2^i. \tag{3.18}$$

Since $v \leq 2^q$, *by Inequality* (3.17) *and Equation* (3.18) *we have* $3v \leq u + z$. *By Equation* (3.15) *we can move to* $\{u, v, z\} \in A$.

[2.2]. *We suppose that* $p < q$. [4] *of Example 3.4 is a special case of this situation, and hence we are going to reduce* x *without affecting* y.
Let $d_i = 0$ *for* $i = q+1, q+2, ..., n$, $d_q = 1$ *and* $d_i = b_i + c_i$ *(mod* 2) *for* $i = 0, 1, ..., q-1$.
Let $u = \sum_{i=0}^{n} d_i 2^n$.

By the definition of d_i *we have*

$$u \oplus y \oplus z = 0, \tag{3.19}$$

and hence we have

$$\sum_{i=0}^{q-1} d_i 2^i \oplus \sum_{i=0}^{q-1} b_i 2^i \oplus \sum_{i=0}^{q-1} c_i 2^i = 0. \tag{3.20}$$

Therefore by Equation (3.20) *and Remark 2.2 we have*

$$y = \sum_{i=0}^{q-1} b_i 2^i \leq \sum_{i=0}^{q-1} d_i 2^i + \sum_{i=0}^{q-1} c_i 2^i. \tag{3.21}$$

Clearly we have

$$u + z = 2^q + \sum_{i=0}^{q-1} d_i 2^i + 2^q + \sum_{i=0}^{q-1} c_i 2^i. \tag{3.22}$$

Since $y \leq 2^q$, *by equations* (3.21) *and* (3.22) *we have* $3y \leq u + z$. *By Equation* (3.19) *we can move to* $\{u, v, z\} \in A$.

Theorem 3.3. *For any non-negative integers* p, q, r *the following* [1] *and* [2] *are equivalent.*
[1]. $p \oplus q \oplus r{=}0$ *and* $q = \lfloor \frac{p+r}{3} \rfloor$.
[2]. *There exist non-negative integers* n, m *such that* $0 \leq m < 2^n$, $p = 2^n + m$, $q = 2^n - 1$ *and* $r = 2^{n+1} - m - 1$.

Proof. *We are going to prove that* [1] *implies* [2].
We suppose that

$$p \oplus q \oplus r = 0 \tag{3.23}$$

and

$$q = \lfloor \frac{p+r}{3} \rfloor. \tag{3.24}$$

By Equality (3.24) *we have* $Max(p, r) \geq q$,
and hence we can assume without any loss of generality that $p \geq q, r$.
Let $n = \lfloor log_2 p \rfloor$.
Let

$$p = \sum_{i=0}^{n} a_i 2^i, q = \sum_{i=0}^{n} b_i 2^i \text{ and } r = \sum_{i=0}^{n} c_i 2^i, \tag{3.25}$$

, where a_i, b_i *and* c_i *are 0 or 1 for* $i = 1, 2, ...n$.
By (3.23) *and* (3.24) *we have*
$b_n = 0$ *and* $c_n = 1$. *Therefore we have Graph 3.10.*

Graph 3.10.

	p	q	r
2^n	$a_n = 1$	$b_n = 0$	$c_n = 1$
2^{n-1}	a_{n-1}	b_{n-1}	c_{n-1}
2^{n-2}	a_{n-2}	b_{n-2}	c_{n-2}
2^{n-3}	a_{n-3}	b_{n-3}	c_{n-3}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^3	a_3	b_3	c_3
2^2	a_2	b_2	c_2
2^1	a_1	b_1	c_1
$2^0 = 1$	a_0	b_0	c_0

By Equation (3.23) *we have*

$$\sum_{i=0}^{n-1} a_i 2^i \oplus \sum_{i=0}^{n-1} b_i 2^i \oplus \sum_{i=0}^{n-1} c_i 2^i = 0, \tag{3.26}$$

and hence by Remark 2.2 we have

$$q = \sum_{i=0}^{n-1} b_i 2^i \leq \sum_{i=0}^{n-1} d_i 2^i + \sum_{i=0}^{n-1} c_i 2^i. \tag{3.27}$$

Next we are going to prove that

$$q = \sum_{i=0}^{n-1} b_i 2^i = \sum_{i=0}^{n-1} d_i 2^i + \sum_{i=0}^{n-1} c_i 2^i. \tag{3.28}$$

If $q = \sum_{i=0}^{n-1} b_i 2^i < \sum_{i=0}^{n-1} d_i 2^i + \sum_{i=0}^{n-1} c_i 2^i$, *then by the fact that* $q < 2^n$ *and Equation* (3.25) *we have*
$3q \leq p + r - 3$, *and hence* $q \leq \lfloor \frac{p+r}{3} \rfloor - 1$, *which contradicts Equation* (3.24). *Therefore we have Equation* (3.28). *If* $b_i = 0$ *for some* $0 \leq i \leq n-1$, *then* $q \leq 2^n - 2$, *and hence by Equation* (3.25) *we have* $3q \leq p + r - 4$, *which contradicts Equation* (3.24.)
Therefore we have $b_i = 1$ *for* $i = 0, 1, ...n-1$. *Then we have Graph 3.11.*

Graph 3.11.

	p	q	r
2^n	$a_n = 1$	*0*	1
2^{n-1}	a_{n-1}	*1*	$1 - a_{n-1}$
2^{n-2}	a_{n-2}	*1*	$1 - a_{n-2}$
2^{n-3}	a_{n-3}	*1*	$1 - a_{n-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^3	a_3	*1*	$1 - a_3$
2^2	a_2	*1*	$1 - a_2$
2^1	a_1	*1*	$1 - a_1$
$2^0 = 1$	a_0	*1*	$1 - a_0$

Let $m = a_0 + 2a_1 + 2^2 a_2 + 2^3 a_3 + ... + 2^{n-1} a_{n-1}$.
Then by Graph 3.11 we have $p = 2^n + m$, $q = 2^n - 1$ *and* $r = 2^{n+1} - m - 1$
$= (1 - a_0) + 2(1 - a_1) + 2^2(1 - a_2) + 2^3(1 - a_3) + ... + 2^{n-1}(1 - a_{n-1}) + 2^n$.
Next we are going to prove that [2] *implies* [1].
Suppose that there exist non-negative integers m *and* n *such that* $0 \leq m < 2^n$, $p = 2^n + m$, $q = 2^n - 1$ *and* $r = 2^{n+1} - m - 1$.
Then we have Graph 3.11, and hence we have the conditions in [1].

Theorem 3.4. *For any* $\{x, y, z\} \in A$ *any option leads to a position* $\in B$.

Proof. *Let* $\{x, y, z\} \in A$, *then we have*

$$x \oplus y \oplus z = 0 \tag{3.29}$$

and

$$3y \leq x + z. \tag{3.30}$$

By Inequality (3.30) *we can assume without any loss of generality that* $x \geq y, z$.
Let $t = \lfloor log_2 x \rfloor$.
[1]. *By Corollary 2.1 we have* $u \oplus y \oplus z \neq 0$, $x \oplus v \oplus z \neq 0$ *and* $x \oplus y \oplus w \neq 0$ *for non-nagative integers* u, v, w *such that* $u < x, v < y$ *and* $w < z$. *Therefore any option that reduces only one number leads to a position* $\in B$.
[2]. *You can reduce* x *and* y *at the same time by the method we used in* [3] *of Example 3.4.*
Suppose that we moved from $\{x, y, z\}$ *to* $\{p, q, z\}$, *and we are going to prove that* $\{p, q, z\} \in B$ *by contradiction.*
We assume that $\{p, q, z\} \in A$, *then we have*

$$p \oplus q \oplus z = 0. \tag{3.31}$$

Since we reduced two numbers at the same time, and hence by Remark 3.1 we have

$$q = \lfloor \frac{p+z}{3} \rfloor. \tag{3.32}$$

Let $n = \lfloor log_2 p \rfloor$.
By (3.31), (3.32), *Theorem 3.3 and Graph 3.11 we have Graph 3.12 and* $n = \lfloor log_2 z \rfloor$.
Let $p = \sum_{i=0}^{n} a_i 2^i$, $q = \sum_{i=0}^{n} b_i 2^i$ *and* $z = \sum_{i=0}^{n} z_i 2^i$, *where* a_i, b_i *and* z_i *are 0 or 1 for* $i = 1, 2, ...n$.

Graph 3.12.

	p	q	z
2^n	$a_n = 1$	*0*	1
2^{n-1}	a_{n-1}	*1*	$1 - a_{n-1}$
2^{n-2}	a_{n-2}	*1*	$1 - a_{n-2}$
2^{n-3}	a_{n-3}	*1*	$1 - a_{n-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^3	a_3	*1*	$1 - a_3$
2^2	a_2	*1*	$1 - a_2$
2^1	a_1	*1*	$1 - a_1$
$2^0 = 1$	a_0	*1*	$1 - a_0$

Let $x = \sum_{i=0}^{t} x_i 2^i$ *and* $y = \sum_{i=0}^{t} y_i 2^i$, *where* x_i, y_i *are 0 or 1 for* $i = 1, 2, ...t$.
Suppose that $t = \lfloor log_2 x \rfloor > n$, *then* $x_t = 1$. *Therefore by Equation* (3.29) *and Inequality* (3.30) *we have* $y_t = 0$ *and* $z_t = 1$, *which contradicts the fact*

that $n = \lfloor log_2 z \rfloor$.
Therefore we can assume that $t = n$*, and by Equation* (3.29) *and Inequality* (3.30) *we have* $y_n = 0$ *and* $z_n = 1$.
By Graph 3.12 $q = 2^n - 1$*, and by the fact that* $y_n = 0$ *we have* $y \leq q$*, and hence you cannot reduce* y *to* q*. This contradicts to the fact that we reduce* x *and* y *at the same time. Therefore* $\{p, q, r\} \in B$.

Theorem 3.5. *Let* $A = \{\{x, y, z\}; x + z \geq 3y$ *and* $x \oplus y \oplus z = 0\}$ *and* $B = \{\{x, y, z\}; x + z \geq 3y$ *and* $x \oplus y \oplus z \neq 0\}$*. Then* A *is the list of P-positions and* B *is the list of N-positions of the game of Example 3.3.*

Proof. *This is direct from Theorem 3.4 and Theorem 3.2. Note that* $\{0, 0, 0\} \in A$ *and by reducing numbers one of the players will finally reach* $\{0, 0, 0\}$.

4 Beautiful Graphs produced by Games

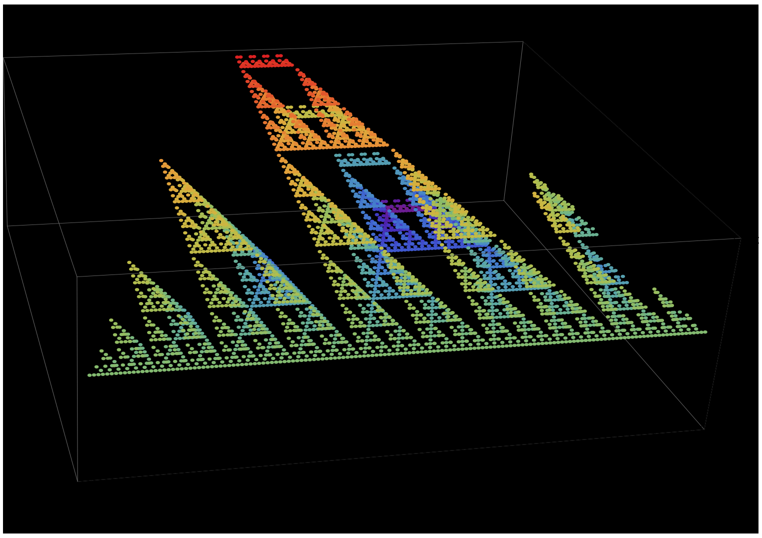
Here we are going to study the 3D graph of the list of P-positions. By Theorem 3.5 we can easily calculate P-positions by computers.

{{0, 0, 0}, {1, 0, 1}, {2, 0, 2}, {3, 0, 3}, {4, 0, 4}, {5, 0, 5}, {6, 0, 6}, {7, 0, 7}, {8, 0, 8}, {9, 0, 9}, {10, 0, 10}, {11, 0, 11}, {12, 0, 12}, {13, 0, 13}, {14, 0, 14}, {15, 0, 15}, {16, 0, 16}, {17, 0, 17}, {18, 0, 18}, {19, 0, 19}, {20, 0, 20}, {21, 0, 21}, {22, 0, 22}, {23, 0, 23}, {24, 0, 24}, {2, 1, 3}, ... {3, 1, 2}, {4, 1, 5}, {5, 1, 4}, {6, 1, 7}, {7, 1, 6}, {8, 1, 9}, {9, 1, 8}, {10, 1, 11}, {11, 1, 10}, {12, 1, 13}, {13, 1, 12}, {14, 1, 15}, {15, 1, 14}, {16, 1, 17}, {17, 1, 16}, {18, 1, 19}, {19, 1, 18}, {20, 1, 21}, {21, 1, 20}, {22, 1, 23}, {23, 1, 22}, ... {4, 2, 6}, {5, 2, 7}, {6, 2, 4}, {7, 2, 5}, {8, 2, 10}, {9, 2, 11}, {10, 2, 8}, {11, 2, 9}, {12, 2, 14}, {13, 2, 15}, {14, 2, 12}, {15, 2, 13}, {16, 2, 18}, {17, 2, 19}, {18, 2, 16}, {19, 2, 17}, {20, 2, 22}, {21, 2, 23}, {22, 2, 20}, {23, 2, 21}, {24, 2, 26}, {25, 2, 27}, ...}

After that we replace {a, b, c} by {a, b, c-a}, then we can get a very interesting strucuture of data. We denote this data by Data. Then
Data ={{0, 0, 0}, {1, 0, 0}, {2, 0, 0}, {3, 0, 0}, {4, 0, 0}, {5, 0, 0}, {6, 0, 0}, {7, 0, 0}, {8, 0, 0}, {9, 0, 0}, {10, 0, 0}, {11, 0, 0}, {12, 0, 0}, {13, 0, 0}, {14, 0, 0}, {15, 0, 0}, {16, 0, 0}, {17, 0, 0}, {18, 0, 0}, {19, 0, 0}, {20, 0, 0}, {21, 0, 0}, {22, 0, 0}, {23, 0, 0}, {24, 0, 0},... {2, 1, 1}, {3, 1, -1}, {4,

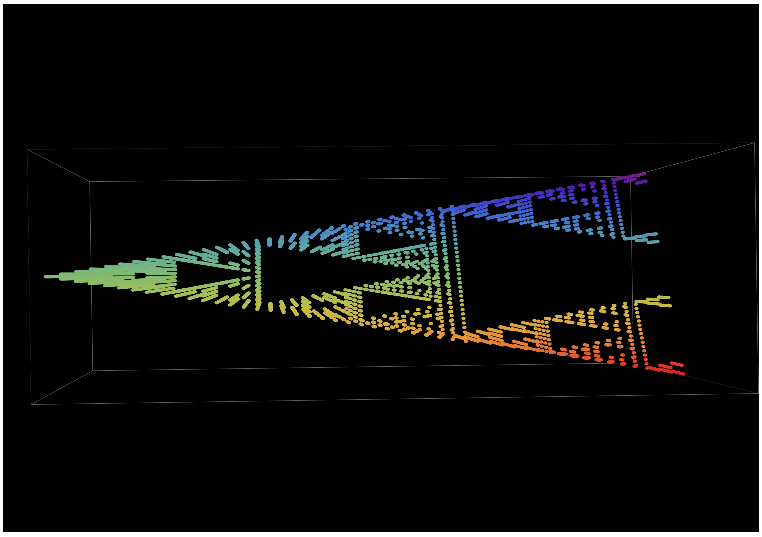
1, 1}, {5, 1, -1}, {6, 1, 1}, {7, 1, -1}, {8, 1, 1}, {9, 1, -1}, {10, 1, 1}, {11, 1, -1}, {12, 1, 1}, {13, 1, -1}, {14, 1, 1}, {15, 1, -1}, {16, 1, 1}, {17, 1, -1}, {18, 1, 1}, {19, 1, -1}, {20, 1, 1}, {21, 1, -1}, {22, 1, 1}, {23, 1, -1},... {4, 2, 2}, {5, 2, 2}, {6, 2, -2}, {7, 2, -2}, {8, 2, 2}, {9, 2, 2}, {10, 2, -2}, {11, 2, -2}, {12, 2, 2}, {13, 2, 2}, {14, 2, -2}, {15, 2, -2}, {16, 2, 2}, {17, 2, 2}, {18, 2, -2}, {19, 2, -2}, {20, 2, 2}, {21, 2, 2}, {22, 2, -2}, {23, 2, -2}, {24, 2, 2}, {25, 2, 2}}

The authors used Mathematica to create 3D graphs from the above data.

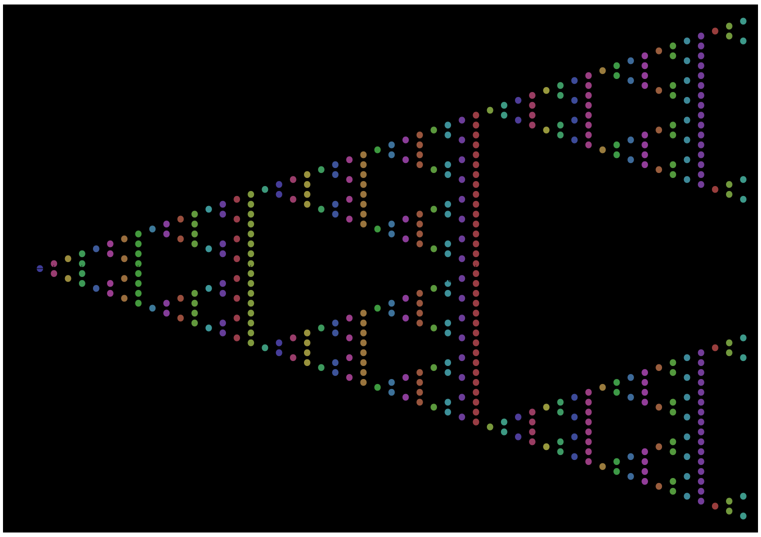


Graph 4.1.

You can rotate the 3D graphics made by Mathematica, and hence the authors have rotated the 3D graph to find any interesting pattern. After some attempts the authors have discovered a pattern that looked like a Sierpinski-like gasket.

Graph 4.2.


Finally the authors discovered Graph 4.3 which is the Sierpinski gasket itself.

Graph 4.3.


Remark 4.1. *The authors have not discovered the reason why we get the Sierpinski gasket in the graph, and they are studying the reason now. You can study the chocolate problem that satisfies the inequality* $ay \leq x + z$ *for any natural number* a*. So far the authors have not proved any results for the case when* a *is an even number.*
However, the case when a *is an even number and even non-integer values of* a *produce interesting graphics.*

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References

[1] T. Yamauchi, M. Naito, R. Miyadera and etc., Combinatorial games, Kokyuroku, The Research Institute of Mathematical Science to appear.

[2] Charles Bouton, Nim, a game with a complete mathematical theory. Ann of math.,Princeton,3. 1901-2.

[3] A.C.Robin, A poisoned chocolate problem, Problem corner, The Mathematical Gazette Volume 73 1989.

[4] Michael H. Albert, Richard J. Nowakowski and David Wolfe, Lessons In Play , A K Peters.

[5] Bitter Chocolate Problem, " Material added 8 Jan 06 (Happy New Year)", MathPuzzle.com.

[6] R.Miyadera, M. Naito and etc.,Combinatorial Games and Beautiful Graphs Produced by them, under review for publication.