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Abstract

We are going to study the Josephus Problem and its variants under various moduli in this article. Let n be a natural number. We put n numbers in a circle, and we are going to remove every second number. Let $J(n)$ be the last number that remains. This is the traditional Josephus Problem. The list $\{J(n)$, n = 1,2,...,20 $\}$ is $\{1, 1, 3, 1, 3, 5, 7, 1, 3, 5, 7, 9, 11, 13, 15, 1, 3, 5, 7, 9\}$. When this sequence is reduced *mod* 4, then we have $\{1, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1\}$.
Next we are going to study a variant of the Josephus Problem in which two numbers are to be eliminated at the same time, and let $J2(n)$ be the last number that remains. If the sequence $\{J2(2n)$, n = 1, 2, ...63 $\}$ is reduced *mod* 2, then we have $\{1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$. The pattern that exists in the sequence is obvious if you look at the sequence carefully.
In this way we get interesting patterns of sequences for the Josephus Problem and its variants under various moduli. The authors were the first people who began to study these problems under various moduli. The authors have discovered many interesting facts, and they are going to present them in this article. They have also studied the graphs that are made from these

problems. They are going to present a program of Java applet. With this applet you can study variants of the Josephus Problems.

1 Introduction.

In this article we are going to study the Josephus Problem and its variants, and we are going to study the sequences and graphs produced by these variants.
The authors hope that this article will be interesting for many people, because this deals with the Josephus Problem under various moduli for the first time in the history of the Josephus Problem.
There are some mathematicians who have studied the variants of the Josephus Problem. See [8]. Our teacher Dr.Miyadera and his students have also studied the Josephus Problem and its variants, and they have talked at the conference [7]. They have published their result in [1]. The authors have used some of their results, and they have studied the Josephus Problem for more than 3 years. The authors have presented our discovery at the Research Institute of Mathematical Science of Kyoto Universty. See [5] and [6]. They have published the resulf of the research in [4]. Their article on the variant of the Josephus Problem is going to be published in [9].
The authors hope that mathematics used in this article is easy enough for freshmen in college to understand, but some of the proofs are a little bit too complicated to read. If you do not like these long proofs, you can skip the proof and read examples. You can find many interesting graphs and sequences in these example. The authors presented a Mathematica program and Java program to calculate variants of the Josephus Problem in this article. In Section 2 the authors studied old problems with a new perspective, and in Section 3,4 they studied new problems.

2 the traditional Josephus problem.

First we are going to study the traditional Josephus Problem. This problem was originated from an ancient story.

Example 2.1. *According to a legend, Josephus was the leader of 40 Jewish rebels trapped by the Romans. His subordinates preferred suicide to surrender, so they decided to form a circle and eliminate every third person until no* *one was left. Josephus wanted to live, so he calculated where to stand and managed to be the last person. He surrendered to the Romans, and he became a famous historian. Where did Josephus stand?*

The number of persons involved in this problem is $40+1=41$*. By the rule of this problem they eliminate* $3rd$*,* $6th$*,* $9th$*,* $12th$*,* $15th$*,* $18th$*, ...*
In this way they eliminate $\{$*3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 1, 5, 10, 14, 19, 23, 28, 32, 37, 41, 7, 13, 20, 26, 34, 40, 8, 17, 29, 38, 11, 25, 2, 22, 4, 35, 16* $\}$*, and the position of the person that remains is 31.*
Therefore Josephus must have been at the position of 31.

We generalize the problem presented in this story, and make a definition of the traditional Josephus Problem. We are going to use numbers instead of persons in the definition.

Definition 2.1. *Let* n *and* r *be natural numbers. We put* n *numbers in a circle. We start with the 1st number removing every* r*th number.*
We denote by $J(n,r)$ *the last number that remains.*

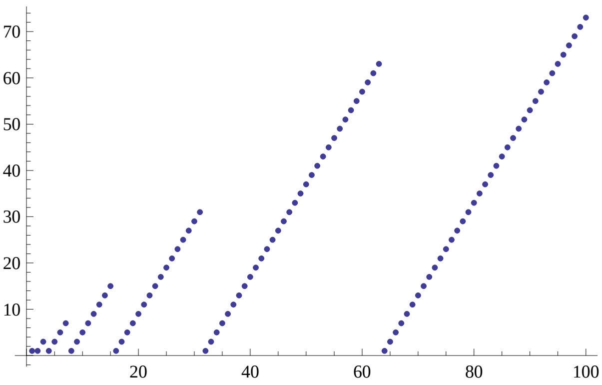
When $r=2$, the Josephus Problem has a very simple formula. We denote $J(n,2)$ by $J(n)$.

Theorem 2.1. $J(2^m+k)=2k+1$ $(m \geq 0$ *and* $0 \leq k < 2^m)$.

proof. *This is a well known formula. See [2].*

By Theorem 2.1 we can calculate $J(n)$ for any natural number n.

Example 2.2. *The graph of the list* $\{$*J(n) , n* $=$ *1, 2, 3, ..., 100*$\}$*. The horizontal coordinate is for the number of numbers (or people in the original Josephus Problem) involved in the game, and the vertical coordinate is for the number that remains when the game is over.*



Graph 2.1.

As you can see, Graph 2.1 is very simple.

Example 2.3. *The list* $\{J(n),\ n=1,2,...64\} = \{$*1, 1, 3, 1, 3, 5, 7, 1, 3, 5, 7, 9, 11, 13, 15, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 1* $\}$.
These numbers are odd numbers, but if this sequence is reduced mod 4*, then we got the following sequence.* $\{$*1, 1, 3, 1* $\}$.
This sequence has a very simple pattern.

We can explain the pattern in Example 2.3 by Theorem 2.2.

Theorem 2.2. *J(1) = 1 and for* $n > 1$

$$J(n) = \begin{cases} 1 \pmod 4, & \textit{if } n \textit{ is even.} \\ 3 \pmod 4, & \textit{if } n \textit{ is odd.} \end{cases}$$

proof. *This is direct from Theorem 2.1.*

Next we are going to study $J(n,3)$. We denote this by $J1(n)$. In this Josephus Problem we remove every third numbers. We need some recursive relations to study this problem.

Theorem 2.3. $J1(n)$ *has the following recursive relations.*
$J1(1) = 1, J1(2) = 2.$

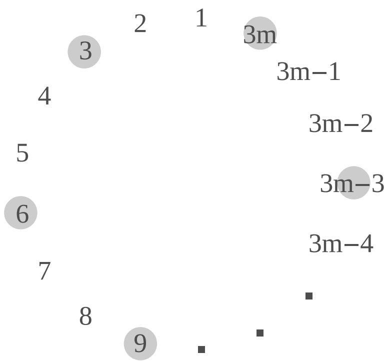
$$J1(3m) = J1(2m) + \lfloor \frac{J2(2m)-1}{2} \rfloor. \tag{1}$$

(2) $$J1(3m+1) = \begin{cases} 3m+1, & (J1(2m+1)=1). \\ J1(2m+1) + \lfloor \frac{J1(2m+1)}{2} \rfloor - 2, & (J1(2m+1)>1). \end{cases}$$

(3) $$J1(3m+2) = J1(2m+1) + \lfloor \frac{J1(2m+1)}{2} \rfloor + 1.$$

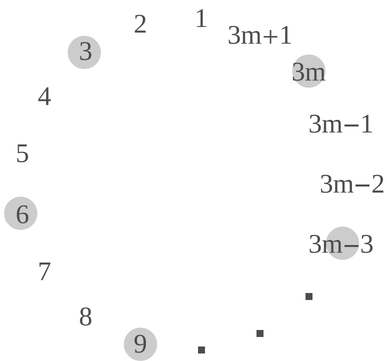
In particular $J1(3m+2) = J1(3m+1) + 3$ *when* $J1(2m+1) > 1$.

proof. *It is clear from the Definition of the Josephus Problem that* $J1(1) = 1$ *and* $J1(2) = 2$.
[1] *First we are going to prove (1). We suppose that there are* $3m$ *numbers. We remove 3, 6, 9, ...3m-3, 3m, then* $2m$ *numbers remain. See Graph 2.2.*



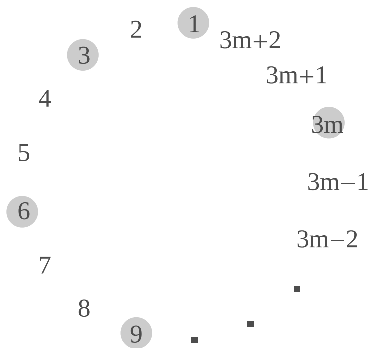
Graph 2.2.

The value of $J1(3m)$ *depends on the value of* $J1(2m)$.
If $J1(2m)$ = *1,2,3,4,5,...2m, then* $J1(3m)$ = *1,2,4,5,7,...3m-1. We can make a recursive relation by comparing the values of* $J1(3m)$ *and* $J1(2m)$ *in this way. Therefore we can prove (1).*
[2] *We are going to prove (2). We suppose that there are* $3m+1$ *numbers. We remove 3, 6, 9, ...3m-3, 3m, then* $2m+1$ *numbers remain. See Graph 2.3.*



Graph 2.3.

The value of $J1(3m+1)$ *depends on the value of* $J1(2m+1)$*.*
The process of eliminating numbers starts with $3m+1$*, and hence if* $J1(2m+1)$ *= 1,2,3,4,5,...2m+1, then* $J1(3m+1)$ *= 3m+1, 1,2,4,5,7,...3m-1. Therefore we can prove (2).*
[3] *We are going to prove (3). We suppose that there are* $3m+2$ *numbers. We remove 3, 6, 9, ...3m-3, 3m,1, then* $2m+1$ *numbers remain. See Graph 2.4.*

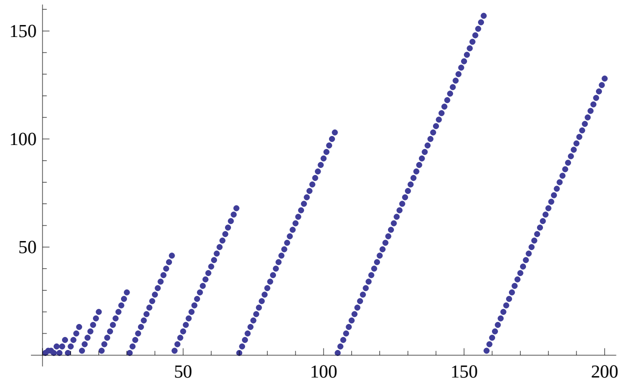


Graph 2.4.

The value of $J1(3m+2)$ *depends on the value of* $J1(2m+1)$*.*
If $J1(2m+1)$ *= 1,2,3,4,5,...2m+1, then* $J1(3m+2)$ *= 2,4,5,7,8...3m+2. Therefore we can prove (3).*

By using recursive relations in Theorem 2.3, we can calculate the value of $J1(n)$.

Example 2.4. *The graph of the list {J1(n) , n =* 1*, 2, 3, ..., 200}. This graph is very similar to Graph 2.1*



Graph 2.5.

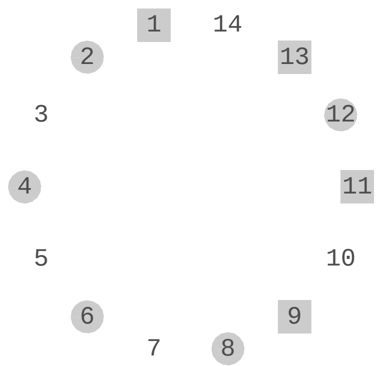
Example 2.5. [1] *The list* $\{J1(n),\ n = 1, 2, ...69\} = \{$*1, 2, 2, 1, 4, 1, 4, 7, 1, 4, 7, 10, 13, 2, 5, 8, 11, 14, 17, 20, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53, 56, 59, 62, 65, 68* $\}$.
[2] *If this sequence is reduced mod* 3*, then we got the following sequence.* $\{$*1, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2* $\}$.
This sequence has a very interesting pattern.

3 A Josephus Problem with two processes of elimination.

In the traditional Josephus Problem they eliminate one by one, in this variant two persons are to be eliminated at the same time.

Let's begin by an example.

Example 3.1. *Suppose that there are 14 persons. The Romans are coming soon, and they do not have enough time. Therefore they are going to kill every second person, but two at the same time. The first process of elimination starts with the 1st person, and the second process elimination starts with the 8th person. Please see Graph 3.1. Numbers enclosed in a circle are to be eliminated by the first process and numbers enclosed in a rectangle are by the second. We suppose that the first process comes first at every stage. Therefore they are going to eliminate 2,9,4,11,6,13,8,1,12,5,3,10,14, and 7 remains.*



Graph 3.1.

We are going to use numbers instead of persons in the definition.

Definition 3.1. *Let m and k be natural numbers. When there are $(2m)$-numbers, the first process of elimination starts with the 1st number, and the kth, $(2k)$th, $(3k)$th number, ... are to be eliminated. The second with the $(m+1)$th number, and the $(m+k)$th, $(m+2k)$th, $(m+3k)$th number, ... are to be eliminated. We suppose that the first process comes first and the second process second at every stage. We denote the number that remains by $J2(2m,k)$.*

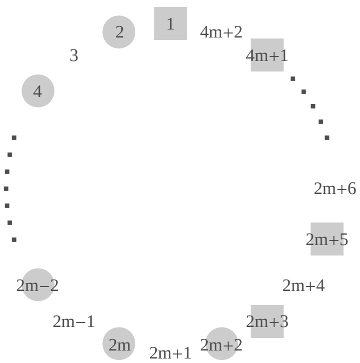
Example 3.1 is the case of $k = 2$. We are going to study $J(n,2)$. We denote by $J2(n,2)$ by $J2(n)$. By Definition 3.1 we study $J(n)$ for an even number n.

Theorem 3.1. *The function $J2(n)$ has the following simple recurrence relations.*

$$(1) \qquad J2(4m+2) = \begin{cases} 2J2(2m) + 2m + 2, & (1 \le J2(2m) \le m). \\ 2J2(2m) - 2m + 1, & (m < J2(2m) \le 2m). \end{cases}$$

$$(2) \qquad J2(4m) = \begin{cases} 2J2(2m) + 2m - 1, & (1 \le J2(2m) \le m). \\ 2J2(2m) - 2m - 1, & (m < J2(2m) \le 2m). \end{cases}$$

proof. *(1) First we are going to prove (1) of this theorem. Suppose that we have $(4m+2)$ numbers. The first process starts with the first number and the second process starts with $(2m+2)$th number. After $(m+1)$ steps for each process we eliminate $(2m+2)$ numbers, and $2m$ numbers remain. See the following Graph 3.3.*

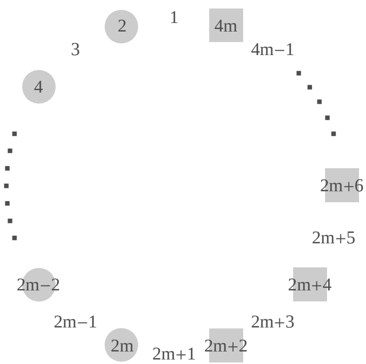


Graph 3.2.

Now we have the process of elimination again with $2m$ *numbers. The first process starts with* $(2m+4)th$ *number and the second process starts with the 3rd number*

If $J2(2m) = k$ *=1, 2, 3, ... , m, then* $J2(4m+2) = 2m+4$, $2m+6$, ... , $(4m+2)$ *respectively. By comparing the values of* $J2(2m)$ *and* $J2(4m+2)$ *we can make a recursive relation, and hence* $J2(4m+2) = 2k+2m+2 = 2J2(2m)+2m+2$. *If* $J2(2m) = k = m+1, m+2, ..., 2m$, *then* $J2(4m+2) = 3, 5, ..., (2m+1)$ *respectively. Therefore* $J2(4m+2) = 2k-2m+1 = 2J2(2m)-2m+1$.

(2) Next we are going to prove (2) of this theorem. Suppose that we have $4m$ *numbers. The first process starts with the first number and the second process starts with* $(2m+1)th$ *number. After* m *steps for each process we eliminate* $2m$ *numbers, and* $2m$ *numbers remain. See the following Graph 3.3.*



Graph 3.3.

Now we have the process of elimination again with $2m$ *numbers. The first process starts with* $(2m+1)th$ *number and the second process starts with the*

1st number. If $J2(2m) = k = 1, 2, 3, ..., m$, then $J2(4m) = 2m + 1, 2m + 3, ..., (4m - 1)$th number respectively. Therefore $J2(4m) = 2k + 2m - 1 = 2J2(2m) + 2m - 1$. If $J2(2m) = k = m + 1, m + 2, ..., 2m$, then $J2(4m) = 1, 3, ..., (2m - 1)$th number respectively. Therefore $J2(4m) = 2k - 2m - 1 = 2J2(2m) - 2m - 1$.

Theorem 3.2. *For any non negative integer h we have the following closed forms for $J2(n)$.*

$$(1) \qquad J2(2(2^{2h} + s)) = 2s + 1 \qquad (0 \leq s < 2^{2h}).$$

$$(2) \qquad J2(2(2^{2h+1} + s)) = 2^{2h+1} + 3s + 1 \qquad (0 \leq s < 2^{2h+1}).$$

proof. *We are going to prove by mathematical induction.*
We suppose that (1) and (2) of this theorem is valid for $h \leq t$, and we are going to prove (1) and (2) for $h = t + 1$.
[1] *First we are going to prove (1) for $h = t+1$ and any odd number $s = 2k+1$ with the condition $k \geq 0$ and $2k + 1 < 2^{2(t+1)}$.*
Since $2k + 1 < 2^{2(t+1)}$, we have $k < 2^{2t+1}$. Therefore by the assumption of mathematical induction for (2) of this theorem we have

$$J2(2(2^{2t+1} + k)) = 2^{2t+1} + 3k + 1. \tag{3.1}$$

By (3.1) we have $J2(2(2^{2t+1} + k)) > 2^{2t+1} + k$, and hence by the second part of (1) of Theorem 3.1

$$J2(2(2^{2(t+1)} + 2k + 1)) = 2J2(2(2^{2t+1} + k)) - 2(2^{2t+1} + k) + 1. \tag{3.2}$$

By (3.2) and (3.1) we have

$$\begin{gathered} J2(2(2^{2(t+1)} + s)) \\ = J2(2(2^{2(t+1)} + 2k + 1)) = 2J2(2(2^{2t+1} + k)) - 2(2^{2t+1} + k) + 1 \\ = 2(2^{2t+1} + 3k + 1) - 2(2^{2t+1} + k) + 1 \\ = 4k + 3 = 2(2k + 1) + 1 = 2s + 1. \end{gathered}$$

Therefore we have proved (1) of this theorem for $h = t+1$ and any odd number s.
[2] *We are going to prove (1) for $h = t + 1$ and any even number $s = 2k$ with the condition that $k \geq 0$ and $2k < 2^{2(t+1)}$.*

Since $k < 2^{2t+1}$*, by using the assumption of mathematical induction for (2) of this theorem we have*

$$J2(2(2^{2t+1}+k)) = 2^{2t+1}+3k+1. \tag{3.3}$$

By (3.3) *we have* $J2(2(2^{2t+1}+k)) > 2^{2t+1}+k$*, and hence by the second part of (2) of Theorem 3.1 we have*

$$J2(2(2^{2(t+1)}+2k)) = 2J2(2(2^{2t+1}+k)) - 2(2^{2t+1}+k) - 1. \tag{3.4}$$

By (3.4) *and* (3.3)

$$\begin{gathered} J2(2(2^{2(t+1)}+s)) \\ = J2(2(2^{2(t+1)}+2k)) = 2J2(2(2^{2t+1}+k)) - 2(2^{2t+1}+k) - 1 \\ = 2(2^{2t+1}+3k+1) - 2(2^{2t+1}+k) - 1 \\ = 4k+1 = 2(2k)+1 = 2s+1. \end{gathered}$$

Therefore we have proved (1) of this theorem for $h = t+1$ *and any even number* $s = 2k$*.*
[3] *We are going to prove (2) for* $h = t+1$ *and any odd number* $s = 2k+1$ *with the condition that* $k \geq 0$ *and* $2k+1 < 2^{2(t+1)+1}$*.*
Since $k < 2^{2(t+1)}$*, by* [1] *and* [2] *of the proof of this theorem we have*

$$J2(2(2^{2(t+1)}+k)) = 2k+1. \tag{3.5}$$

By (3.5) *and the fact that* $k < 2^{2(t+1)}$ *we have* $J2(2(2^{2(t+1)}+k)) = 2k+1 \leq 2^{2(t+1)}+k$*, and hence by the first part of (1) of Theorem 3.1*

$$J2(2(2^{2(t+1)+1}+2k+1)) = 2J2(2(2^{2(t+1)}+k)) + 2(2^{2(t+1)}+k) + 2. \tag{3.6}$$

By (3.6) *and* (3.5)

$$\begin{gathered} J2(2(2^{2(t+1)+1}+s)) \\ = J2(2(2^{2(t+1)+1}+2k+1)) = 2J2(2(2^{2(t+1)}+k)) + 2(2^{2(t+1)}+k) + 2 \\ = 2(2k+1) + 2(2^{2(t+1)}+k) + 2 \\ = 2^{2(t+1)+1}+6k+4 = 2^{2(t+1)+1}+3s+1. \end{gathered}$$

Therefore we have proved (2) of this theorem for h = t+1 and any odd number s.
[4] *We are going to prove (2) for $h = t+1$ and any even number $s = 2k$ with the condition that $k \geq 0$ and $2k < 2^{2(t+1)+1}$.*
Since $k < 2^{2(t+1)}$, by [1] and [2] of the proof of this theorem we have

$$J2(2(2^{2(t+1)} + k)) = 2k + 1. \tag{3.7}$$

By (3.7) and the fact that $k < 2^{2(t+1)}$ we have $J2(2(2^{2(t+1)} + k)) = 2k + 1 \leq 2^{2(t+1)} + k$, and by the first part of (2) of Theorem 3.1

$$J2(2(2^{2(t+1)+1} + 2k)) = 2J2(2(2^{2(t+1)} + k)) + 2(2^{2(t+1)} + k) - 1. \tag{3.8}$$

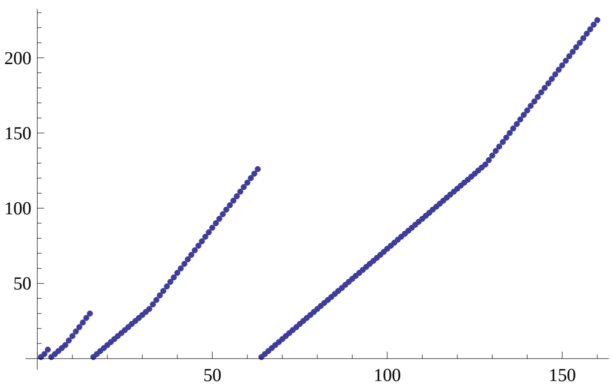
By (3.8) and (3.7)

$$\begin{aligned} & J2(2(2^{2(t+1)+1} + s)) \\ &= J2(2(2^{2(t+1)+1} + 2k)) = 2J2(2(2^{2(t+1)} + k)) + 2(2^{2(t+1)} + k) - 1 \\ &= 2(2k + 1) + 2^{2(t+1)+1} + 2k - 1 \\ &= 2^{2(t+1)+1} + 6k + 1 = 2^{2(t+1)+1} + 3s + 1. \end{aligned}$$

Therefore we have proved (2) of this theorem for h = t+1 and any even number s.

The graph produced by $J2(n)$ is quite interesting.

Example 3.2. *The graph of the list $\{J2(n)$, n = 1, 2, 3, ..., 200$\}$. The horizontal coordinate is for the number of numbers (or people in the original Josephus Problem) involved in the game, and the vertical coordinate is for the number that remains when the game is over. This graph is very similar to Graph 2.1 and Graph 2.5, but there are two kinds of slopes in this graph.*



Graph 3.4.

Example 3.3. *The list* $\{J2(2n),\ n = 1, 2, ...63\}$ = $\{$*1, 3, 6, 1, 3, 5, 7, 9, 12, 15, 18, 21, 24, 27, 30, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90, 93, 96, 99, 102, 105, 108, 111, 114, 117, 120, 123, 126* $\}$*, and if this sequence is reduced mod 2, then we have* $\{$*1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0* $\}$.
If we divide this sequence into subsequences, the pattern of the sequence becomes obvious.
$\{$*1*$\}$
$\{$*1,0*$\}$
$\{$*1, 1, 1, 1*$\}$
$\{$*1, 0, 1, 0, 1, 0, 1, 0*$\}$
$\{$*1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1*$\}$
$\{$*1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0* $\}$
We are going to prove the existence of the pattern mathematically.

Theorem 3.3. *For any non negative integer* h *we have the following formula for* $J2(n)$*.*

$$J2(2(2^{2h} + s)) = 1 \qquad (mod\ 2) \qquad (0 \le s < 2^{2h}). \tag{1}$$

$$J2(2(2^{2h+1}+s)) = \begin{cases} 1\ (mod\ 2), & (0 \le s < 2^{2h+1} \text{ and } s \text{ is even.}) \\ 0\ (mod\ 2), & (0 \le s < 2^{2h+1} \text{ and } s \text{ is odd.}) \end{cases} \tag{2}$$

proof. *This theorem is direct from Theorem 3.2.*

4 Some Interesting Facts about the Josephus Problem in both Direction Under Various Moduli.

We are going to study another variant of the Josephus Problem.

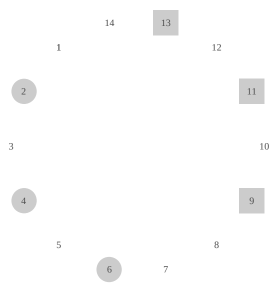
Definition 4.1. *In this variant of the Josephus Problem two numbers are to be eliminated at the same time, but two processes of elimination go for* *different directions. Let n and k be natural numbers such that $k \geq 2$. Suppose that there are n numbers. Then the first process of elimination starts with the 1 st number and the kth, $(2k)$th, $(3k)$th number, ... are to be eliminated. The second process starts with the nth number, and the $(n-k+1)$th, $(n-2k+1)$th, $(n-3k+1)$th number, ... are to be eliminated. We suppose that the first process comes first and the second process second at every stage. We denote the position of the survivor by $JB(n,k)$.*

Here we are going to study an example of Definition 4.1.

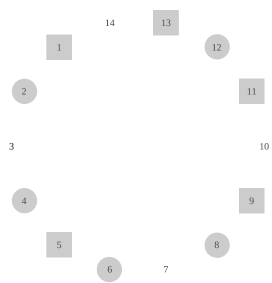
Example 4.1. *Suppose that there are $n = 14$ numbers and $k = 2$. Then the 2nd, 4th, 6th number will be eliminated by the first process. Similarly the 13th, 11th, 9th number will be eliminated by the second process. Then we have Graph 4.1. Here we covered eliminated numbers by the first process and the second process with gray color disks and gray color rectangles respectively.*

Now two directions are going to overlap. The first process will eliminate the 8, 12 and the second process will eliminate 5, 1. See Graph 4.2.

After this the first process will eliminate 3, 14, and the second process will eliminate 10. The number that remains is 7.



Graph 4.1.



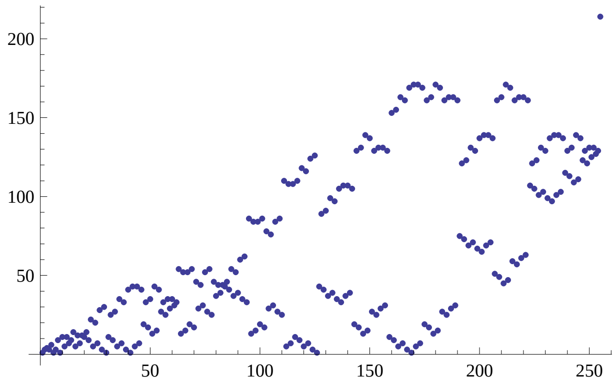
Graph 4.2.

Next we are going to study $B(n,2)$. We denote $JB(n,2)$ by $JB(n)$. The function $JB(n)$ has very interesting properties. It has fractal-like graphs. The sequence $\{JB(n), n = 1, 2, ...\}$ has a remarkable property when divided by 2.

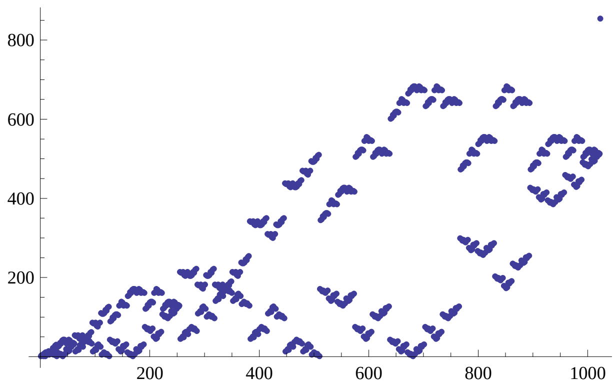
Example 4.2. *Graph 4.3 is the graph of the list $\{JB(n)$, $n = 2$, 3, ..., 256$\}$. The horizontal coordinate is for the number of numbers involved in the game,*

and the vertical coordinate is for the number that remains when the game is over. For example by $JB(256) = 214$ *we have the point* $(256, 214)$ *in the graph.*
Graph 4.4 is the graph of the list $\{JB(n)\ ,\ n = 2,\ 3,\ \ldots,\ 1024\}$.
If we compare these graphs, we can find the the self-similarity.

Graph 4.3.



Graph 4.4.



As to the research of self-similarity by the authors see [9].
There is a very interesting fact about the function $JB(n)$.

Example 4.3. *The list of the sequence* $\{JB(n), n = 1, 2, 3, ..., 63\}$*is*
$\{1, 1, 3, 4, 3, 6, 1, 3, 9, 1, 11, 5, 11, 7, 9, 14, 5, 12, 7, 12, 11, 14, 9, 22, 5, 20, 7, 28, 3, 30, 1, 11, 25, 9, 27, 5, 35, 7, 33, 3, 41, 1, 43, 5, 43, 7, 41, 19, 33, 17, 35, 13, 43, 15, 41, 27, 33, 25, 35, 29, 35, 31\}$.
We denote this sequence $JB(n)$ *modulo 2 by* $JB(\ mod\ 2)$.
Then $JB(\ mod\ 2)$ *is*
$\{1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1,$

$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.

We can find a very beautiful pattern if we divide it into subsequences.
$\{1\}$,
$\{1, 1\}$,
$\{1, 0, 1, 0\}$,
$\{1, 1, 1, 1, 1, 1, 1, 1\}$,
$\{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$,
$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.
We have presented this pattern in [9], and we have proved this mathematically using recursive relations. See [10].

Example 4.4. *Similarly the list of the sequence* $\{JB(n), n = 1, 2, 3, ..., 126\}$*is* *{1, 1, 3, 4, 3, 6, 1, 3, 9, 1, 11, 5, 11, 7, 9, 14, 5, 12, 7, 12, 11, 14, 9, 22, 5, 20, 7, 28, 3, 30, 1, 11, 25, 9, 27, 5, 35, 7, 33, 3, 41, 1, 43, 5, 43, 7, 41, 19, 33, 17, 35, 13, 43, 15, 41, 27, 33, 25, 35, 29, 35, 31, 33, 54, 13, 52, 15, 52, 19, 54, 17, 46, 29, 44, 31, 52, 27, 54, 25, 46, 37, 44, 39, 44, 43, 46, 41, 54, 37, 52, 39, 60, 35, 62, 33, 86, 13, 84, 15, 84, 19, 86, 17, 78, 29, 76, 31, 84, 27, 86, 25, 110, 5, 108, 7, 108, 11, 110, 9, 118, 5, 116, 7, 124, 3, 126 }, and if this sequence is reduced modulo 4 , then we get*
{1, 3, 0, 3, 2, 1, 3, 1, 1, 3, 1, 3, 3, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 3, 1, 1, 3, 1, 3, 3, 1, 3, 1, 1, 3, 1, 3, 3, 1, 3, 1, 1, 3, 1, 3, 3, 1, 3, 1, 1, 3, 1, 3, 3, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1, 2, 1, 0, 3, 0, 3, 2, 1}.
We can find a very beautiful pattern if we divide them into subsequences.
{1, 1, 3 }
{0, 3, 2, 1 }
{3, 1, 1, 3, 1, 3, 3, 1 }
{2, 1, 0, 3, 0, 3, 2, 1 } {2, 1, 0, 3, 0, 3, 2, 1 }
{3, 1, 1, 3, 1, 3, 3, 1 } {3, 1, 1, 3, 1, 3, 3, 1 } {3, 1, 1, 3, 1,3, 3, 1}
{3, 1, 1, 3, 1, 3, 3, 1}
{2, 1, 0, 3, 0, 3, 2, 1 } {2, 1, 0, 3, 0, 3, 2, 1 } {2, 1, 0, 3, 0,3, 2, 1 }
{2, 1, 0, 3, 0, 3, 2, 1} {2, 1, 0,3, 0, 3, 2, 1} {2, 1, 0, 3, 0, 3, 2, 1 }
{2, 1, 0, 3, 0, 3, 2, 1 } {2, 1, 0, 3, 0, 3, 2, 1 }

If we ignore the first 7 terms, then the remaining terms of the sequence show a very simple pattern.

5 Computer Programs for the Josephus Problem and its Variants.

Here we present a Mathematica program and a Java program.

Example 5.1. *This Mathematica function jose*$[m, k]$ *returns the last number that remains in the Josephus Problem in both direction with* m *numbers and each process eliminates every* k *th number.*

```
jose[m_, k_] := Block[{t, p, q, u, v, w}, w = k - 1;
t = Range[m];
p = t;
q = t;
  Do[p = RotateLeft[p, w];
u = First[p];
p = Rest[p];
q = Drop[q, Position[q, u][[1]]];
If[Length[p] == 1, Break[],];
q = RotateRight[q, w];
v = Last[q];
q = Drop[q, -1];
p = Drop[p, Position[p, v][[1]]];
If[Length[q] == 1, Break[],], {n, 1, Ceiling[m/2]}];p[[1]]];
```

Example 5.2. *This is a Java applet to show how we remove numbers in the Josephus Problem in both direction.*

```
import java.applet.*;
import java.awt.*;
import java.awt.event.*;
public class crossjose_x_ver2 extends Applet
  implements ActionListener
{
  double S,C,S2,C2;
  Button b_con=new Button("NEXT");//Here we make a button.
  Button b_uncon=new Button("BACK");//Here we make a button.
  int L;    //the number of numbers (players) we use in the game.
  int R;    //We remove every Rth number.
  int k=0;  //the number of numbers (players) we have removed.
```

```
  private TextField box=new TextField(1);//textfield
  private Label moji=new Label("  PEOPLE  ");//Label
  private TextField box2=new TextField(1);//textfield
  private Label moji2=new Label("  TH  ");//Label
  private Button ok=new Button("OK");//Label

  //Here we make buttons and labels.
  public void init()
  {
    add(b_con);  add(b_uncon);
    b_con.addActionListener(this);
    b_uncon.addActionListener(this);

    resize(750,650);

    add(box);
    add(moji);
    add(box2);
    add(moji2);
    add(ok);
    ok.addActionListener(this);
  }

  //Here we define the function of buttons.
  public void actionPerformed(ActionEvent e)
  {
//We get a number from the text field.
    String t=box.getText();
    String t2=box2.getText();
    //Once the button is pushed,
    String s=e.getActionCommand();
    //move one step forward.
    if(s=="NEXT")
    {
      if(k<L-1) ++k;/*When we finish the game,
      we have no more step forward.*/
      repaint();
    }
```

```
  //go back one step.
  else if(s=="BACK")
  {
    if(1<=k) --k;/
    *When we have not started the game,
    we cannot go back.*/
    repaint();
  }
  else if(s=="OK")
  {
  //We get the number.
    L=Integer.parseInt(t);
    R=Integer.parseInt(t2);
    R=R-1;
    if(L>20)  L=20;  //We restrict the number under 20.
    if(L<=R)  R=1;  /*We cannot jump more than
    the number of the numbers (players).*/
    if(R<1)    R=1;
    k=0;        //To reset the previous moves.
    repaint();
  }
}

public void paint(Graphics g)
{
  for(int i=1;i<=L;i++)
  {
  //We display numbers.
    S=Math.sin(2*Math.PI/L*i)*250;
    C=Math.cos(2*Math.PI/L*i)*250;
    String M=Integer.toString(i);
    g.setFont(new Font("Serif", Font.BOLD, 50));
    g.drawString(M,350+(int)S, 350+(int)C);
  }
  int[] p;
  p=new int[L];//We prepare array.
  /*We put 1 for every number. If we remove it, we put 0.*/
  for(int s=0;s<=L-1;s++)
```

```
{
  p[s]=1;
}
//This is the main problem of Josephus Problem.
int Ac1=0; //This is for the first process to jump.
int Ac2; //This is for the first process to remove.
int Bc1=L-1;//This  is for the second process to jump.
int Bc2;//This is for the second process to remove.

for(int F=1;F<=k;F++)
{
int con=0;
   //We use this variable to jump the given times.
  //First process A jumps to the next number.
  if(F%2==1)
  {
    //jump once.
    for(;Ac1<=L;Ac1++)
    {
      if(con==R)  break;
      if(p[Ac1]==1 && Ac1<=L-2)  //jump once.
      {
        con=con+1;
        continue;
      }
      if(p[Ac1]==1 && Ac1==L-1)
    /*Since the process is at the last number,
    it will jump to the first number.*/
      {
        Ac1=-1;
        con=con+1;
        continue;
      }
      if(p[Ac1]==0 && Ac1==L-1)  Ac1=-1;
    /*Since the last number is removed,
    the process goes to the first number.*/
    }
```

```
        //remove
        for(Ac2=Ac1;Ac2<=L-1;Ac2++)
        {
        //We look for the number to remove.
          if(p[Ac2]==1)
         {
            p[Ac2]=0;//We removed.
            break;
          }
          if(Ac2==L-1)  Ac2=-1;
        /*Since the last number is removed,
        the process goes to the first number.*/
        }

        if(Ac2<=L-2)  Ac1=Ac2+1;
        if(Ac2==L-1)  Ac1=0;
        //Put a disk on the number.
        g.setColor(Color.black);
        S2=Math.sin(2*Math.PI/L*(Ac2+1))*250;
        C2=Math.cos(2*Math.PI/L*(Ac2+1))*250;
        g.fillOval(350+(int)S2-5, 350+(int)C2-38, 60, 60);

        //If it is the last number, we display it in red.
        if(k==L-1)
        {
          //We look for the number that remains.
          int w=0;
          for(;w<=L-1;w++)
          {
            if(p[w]==1)  break;
          }
//display in red.
          S=Math.sin(2*Math.PI/L*(w+1))*250;
          C=Math.cos(2*Math.PI/L*(w+1))*250;
          g.setColor(Color.red);
          String N=Integer.toString(w+1);
          g.drawString(N,350+(int)S, 350+(int)C);
        }
```

```
}
//Next the second process (B) jumps to the next number.
if(F%2==0)
{
  //Jump once.
  for(;Bc1>=0;Bc1--)
  {
    if(con==R)  break;
    if(p[Bc1]==1 && Bc1>=1)  /Jump once.
    {
      con=con+1;
      continue;
    }
    if(p[Bc1]==1 && Bc1==0)
    /*Since we are at the first number,
     we go to the last number.*/
    {
      Bc1=L;
      con=con+1;
      continue;
    }
    if(p[Bc1]==0 && Bc1==0)  Bc1=L;
    /*Since the first number is removed,
     we got to the last*/
  }
  //remove
  for(Bc2=Bc1;Bc2>=0;Bc2--)
  {
  //We look for a number to remove.
    if(p[Bc2]==1)
    {
      p[Bc2]=0;//We removed the number.
      break;
    }
    if(Bc2==0)  Bc2=L;  /*Since the first number is removed,
    we go to the first number.*/
  }
  if(Bc2>=1)  Bc1=Bc2-1;
```

```
        if(Bc2==0)  Bc1=L-1;
       //We put a disk on the number.
        g.setColor(Color.blue);
        S2=-Math.sin(2*Math.PI/L*(L-Bc2-1))*250;
        C2=Math.cos(2*Math.PI/L*(L-Bc2-1))*250;
        g.fillOval(350+(int)S2-5, 350+(int)C2-38, 60, 60);

        //if this is the last number, display it in red.
        if(k==L-1)
        {
        //We look for the number to remove.
          int w=0;
          for(;w<=L-1;w++)
          {
            if(p[w]==1)  break;
          }
          //display in red.
          S=-Math.sin(2*Math.PI/L*(L-w-1))*250;
          C=Math.cos(2*Math.PI/L*(L-w-1))*250;
          g.setColor(Color.red);
          String N=Integer.toString(w+1);
          g.drawString(N,350+(int)S, 350+(int)C);
        }
      }//the end of "if".
    }//the end of "for".
  }//the end of "paint".
}//the end of the program.
```

If you are willing to use Mathematica or Mathematica player (a free software), please download the following demonstrations.
See [11],[12] and [13].
If you want to learn to use Mathematica program to study the Josephus Problem, [3] is a very good book to read.

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