Rose-Hulman Undergraduate Mathematics Journal

Volume 10
Issue 1

Article 9

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The Period and the Distribution of the Fibonacci-like Sequence Under Various Moduli.

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Abstract

We reduce the Fibonacci sequence mod m for a natural number m, and denote it by F (mod m). We are going to introduce the properties of the period and distribution of F (mod m). That is, how frequently each residue is expected to appear within a single period. These are well known themes of the research of the Fibonacci sequence, and many remarkable facts have been discovered.
After that we are going to study the properties of period and distribution of a Fibonacci-like sequence that the authors introduced in [4]. This Fibonacci-like sequence also has many interesting properties, and the authors could prove an interesting theorem in this article. Some of properties are very difficult to prove, and hence we are going to present some predictions and calculations by computers.

1 Introduction.

The Fibonacci sequence has a lot of remarkable properties. One of them is the fact that we get a periodic sequence when we reduce the Fibonacci sequence mod m for a natural number m, and many people have studied the period of the Fibonacci sequences. see [1]
In [2] [3], [4] and [5] the authors have studied Fibonacci-like sequences. It is a proper theme to study the periods of Fibonacci-like sequences, and the formulas in [4] can be very useful tools in the research.
The content of Section 2 is well known fact, but the contents of Section 3 and 4 are works of the authors.

2 The Periods of the Fibonacci Sequence.

Definition 2.1. *We are going to define the Fibonacci sequence.*
We define $F(0) = 0$, $F(1) = F(2) = 1$ *and for natural number* n *such that* $n \geq 2$

$$F(n) = F(n-1) + F(n-2).$$

Note that here we define $F(n)$ for non-negative number n. In this article it is convenient to have $F(0) = 0$.

We denote Fibonacci sequence modulo m by $F(\ mod\ m)$ for a natural number m.

Example 2.1. *We are going to study* F $(mod\ 10\ )$ *for n = 1, 2, 3, ..., 62.* $F(n)$ *= 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,..., and hence* F $(mod\ 10\ )$ *= 1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,7,7,4,1,5,6,1,7,8,5,3,8,1,9,0,9,9,8,7,5,2,7,9,6, 5,1,6,7,3,0,3,3,6,9,5,4,9,3,2,5,7,2,9,1,0,1,1.*
Both of the 61th and 62th terms are 1, and hence next terms should be 2, 3, 5, 8, ... Therefore it is easy to see that it has a period of 60.

Theorem 1. *Let* $a_n = F\ (\ mod\ m)$ *, then there exists a positive integer* s *such that* $(a_s,a_{s+1}) = (1,1)$*. In particular* $F(\ mod\ m)$ *has a period.*

Proof. *Since* $0 \leq a_n < m$*, there are only* m^2 *possible pairs of residues, and hence there must be natural numbers* s *and* t *such that* $s < t$ *and* $(a_s, a_{s+1}) = (a_t, a_{t+1})$*. Therefore* $F(mod\ m)$ *is 1, 1, ...,*a_s*,* a_{s+1}*,...,* a_t*,* a_{t+1}*, ..., where we have* $a_t = a_s$*,* $a_{t+1} = a_{s+1}$*. By the property of* F $(mod\ m)$ *any pair will completely determine a sequence both forward and backward. For example by the fact that* $a_{t-1} + a_t = a_{t+1}$ *and* $a_{s-1} + a_s = a_{s+1}$ $(mod\ m)$ *we have* $a_{t-1} = a_{t+1} - a_t$*,* $a_{s-1} = a_{s+1} - a_s$ $(mod\ m)$ *and* $a_{t-1} = a_{s-1}$*. By going backward in this way we can find the smallest natural number* u *such that* $u < t$ *and* $a_u = 1, a_{u+1} = 1$*. Therefore* F $(mod\ m)$ *must has the period, and i if we denote this period by* $k(m)$*, then* $k(m) = u - 1$*. Clearly* $k(m) < m^2$*.*

Theorem 2. F $(mod\ 2^n)$ *has a period of* $3 \cdot 2^{n-1}$ *for any natural number* n*.*

Proof. *This is a well known property of the Fibonacci sequence. See Lemma 1 and the remark after it in [1]. You can also find this fact in some textbooks on Fibonacci sequences.*

Example 2.2. *We are going to see an example of the formula presented in Theorem 2. In this example we are going to study* F *(* $mod\ 2^5 = 32$*).*
$\{F(s)\ (mod\ 32), s = 1, 2, 3, ...48\}$
$=$ *{1, 1, 2, 3, 5, 8, 13, 21, 2, 23, 25, 16, 9, 25, 2, 27, 29, 24, 21, 13, 2, 15, 17, 0, 17, 17, 2, 19, 21, 8, 29, 5, 2, 7, 9, 16, 25, 9, 2, 11, 13, 24, 5, 29, 2, 31, 1, 0 }.*
$\{F(s)\ (mod\ 32), s = 49, 50, 51, ...96\}$
$=$ *{1, 1, 2, 3, 5, 8, 13, 21, 2, 23, 25, 16, 9, 25, 2, 27, 29, 24, 21, 13, 2, 15, 17, 0, 17, 17, 2, 19, 21, 8, 29, 5, 2, 7, 9, 16, 25, 9, 2, 11, 13, 24, 5, 29, 2, 31, 1, 0 } .*
It is clear that $F(s)\ (mod\ 32)$ *has a period of 48.*

There are many interesting facts about the period of the Fibonacci sequence. For example see Graph4.1 and Graph4.2 in Section 4.

3 The Periods of the Fibonacci Like Sequence.

Now we are going to study periods of Fibonacci-like sequences that the authors studied in [2] [3], [4] and [5].

First we define the Fibonacci-like sequence.

Definition 3.1. *Let* p *be a fixed natural number with* $p \geq 2$*. We define* $B_p(0) = 0$, $B_p(1) = B_p(2) = 1$ *and for natural number* n *such that* $n \geq 2$

$$B_p(n) = B_p(n-1) + \begin{cases} B_p(n-2) + 1, & \text{if } n = 1 \text{ (mod } p). \\ B_p(n-2), & \text{if } n \neq 1 \text{ (mod } p). \end{cases}$$

There are some very interesting relationships between the sequence $B_p(n)$ and $F(n)$.

Theorem 3. *Let* $p = 4q$ *for a natural number* q*. Here we denote* $B_p(n)$ *by* $f(n)$*. Note that by Definition 3.1 and 2.1 we have* $F(0) = f(0) = 0$*.*
Then $f(n)$ *satisfies the following equations for any natural number* t*.*

$$\begin{cases} f(4qt) = \frac{F(2qt)F(2qt+2q)}{F(2q)}, \\ f(4qt+1) = \frac{F(2qt+1)F(2qt+2q)}{F(2q)}, \\ f(4qt+2) = \frac{F(2qt+2)F(2qt+2q)}{F(2q)}, \\ \vdots \\ f(4qt+4q-1) = \frac{F(2qt+4q-1)F(2qt+2q)}{F(2q)}. \end{cases}$$

For a proof see [4].

Theorem 4. *Let m be a natural number such that $m \geq 2$. Here we denote $B_{2^m}(s)$ by $f(s)$.*
Then $f(n)$ satisfies the following equations for any natural number t.

$$\begin{cases} f(2^m t) = \frac{F(2^{m-1}t)F(2^{m-1}t+2^{m-1})}{F(2^{m-1})}, \\ f(2^m t+1) = \frac{F(2^{m-1}t+1)F(2^{m-1}t+2^{m-1})}{F(2^{m-1})}, \\ f(2^m t+2) = \frac{F(2^{m-1}t+2)F(2^{m-1}t+2^{m-1})}{F(2^{m-1})}, \\ \vdots \\ f(2^m t+2^m-1) = \frac{F(2^{m-1}t+2^m-1)F(2^{m-1}t+2^{m-1})}{F(2^{m-1})}. \end{cases}$$

Proof. *Let $4q = 2^m$ for natural number m with $m \geq 2$, then this theorem is direct from Theorem 3 .*

Lemma 3.1. *$F(m)$ is even if and only if m is a multiple of 3.*

Proof. *Since $F(n)$ = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,..., F (mod 2) = 1,1,0,1,1,0,1,1,0, ... It is clear that the period of the F (mod 2) is 3, and $F(m)$ is even if and only if m is a multiple of 3.*

Theorem 5. *Let $a_n = B_p$ (mod m) for natural numbers p and m. Then there exists the smallest natural number s such that $(a_{ps+1}, a_{ps+2}) = (1, 1)$. In particular a_n has a period of the length of $p \times s$. We denote the period by $kp(m)$.*

Proof. *Since $0 \leq a_n < n$, there are only m^2 possible pairs of residues, and hence there must be natural numbers s and t such that $s < t$ and $(a_{ps+1}, a_{ps+2}) = (a_{pt+1}, a_{pt+2})$.*

Therefore B_p (mod m) is 1, 1, ..., a_{ps+1}, a_{ps+2},..., a_{pt+1}, a_{pt+2}, where $a_{pt+1} = a_{ps+1}$ and $a_{pt+2} = a_{ps+2}$. By the property of Fibonacci-like sequence

any pair will completely determine a sequence both forward and backward. For example by the property of Fibonacci-like sequence $a_{pt} + a_{pt+1} = a_{pt+2}$ and $a_{ps} + a_{ps+1} = a_{ps+2}$, and hence we have $a_{pt} = a_{pt+2} - a_{pt+1}$, $a_{ps} = a_{ps+2} - a_{ps+1}$ and $a_{pt} = a_{ps}$.
Similarly by the property of the Fibonacci-like sequence $a_{pt-1} + a_{pt} + 1 = a_{pt+1}$ and $a_{ps-1} + a_{ps} + 1 = a_{ps+1}$, and hence we have $a_{pt-1} = a_{pt+1} - a_{pt} - 1$, $a_{ps-1} = a_{ps+1} - a_{ps} - 1$ and $a_{pt-1} = a_{ps-1}$.
By going backward in this way we can find the smallest natural number u such that $u < t$, $a_{pu+1} = 1$ and $a_{pu+2} = 1$. Therefore B_p (mod m) must has the period. If we denote the period by $kp(m)$, then $kp(m) = u \times p$. Clearly $kp(m) < p \times m^2$.

Remark 3.1. *By Theorem 5 the period of B_p is $p \times s$ when $B_{ps+1} = B_{ps+2} = 1$ (mod m). Then by the property of B_p we have $B_{ps} + B_{ps+1} = B_{ps+2}$ and $B_{ps-1} + B_{ps} + 1 = B_{ps+1}$, and hence $B_{ps-1} = B_{ps} = 0$ (mod m).*

Theorem 6. *For any natural number k*
$F(3 \cdot 2^k) = 2^{k+2}$ (mod 2^{k+4}).

Proof. *For a proof see Lemma 1 of [1]. In this Lemma 1 the author of the article [1] proved it with the condition that $k \geq 2$, but k can be 1, since $F(3 \cdot 2^1) = F(6) = 8 = 2^3$.*

Theorem 7. *Let n, m be natural numbers such that $n > m$ and $m \geq 2$. Here we denote $B_{2^m}(s)$ by $f(s)$.*
Then the period of $f(s)$ (mod 2^n) is $3 \cdot 2^{n-1}$.

Proof. *Let $t = 3 \cdot 2^{n-m-1}$. Then $2^m t = 3 \cdot 2^{n-1}$ and $2^{m-1} t = 3 \cdot 2^{n-2}$, and by the first equation of Theorem 4 we have*

$$f(3 \cdot 2^{n-1}) = \frac{F(3 \cdot 2^{n-2}) F(3 \cdot 2^{n-2} + 2^{m-1})}{F(2^{m-1})}. \tag{3.1}$$

Let $t = 3 \cdot 2^{n-m-1} - 1$, then we have $2^m t + 2^m = 3 \cdot 2^{n-1}$, $2^{m-1} t + 2^m = 3 \cdot 2^{n-2} + 2^{m-1}$ and $2^{m-1} t + 2^{m-1} = 3 \cdot 2^{n-2}$. Therefore by the last equation of Theorem 4 we have

$$f(3 \cdot 2^{n-1} - 1) = \frac{F(3 \cdot 2^{n-2} + 2^{m-1} - 1) F(3 \cdot 2^{n-2})}{F(2^{m-1})}. \tag{3.2}$$

By Theorem 6 we have

$$F(3 \cdot 2^{n-2}) = 2^n \ (mod\ 2^{n+2}). \tag{3.3}$$

By Lemma 3.1 $F(2^{m-1})$ *is an odd number, and hence by* (3.1), (3.2) *and* (3.3) *we know that*

$$f(3 \cdot 2^{n-1}) = f(3 \cdot 2^{n-1} - 1) = 0 \ (mod\ 2^n). \tag{3.4}$$

By (3.4)*, Definition 3.1 and the fact that* $3 \cdot 2^{n-1}$ *is a multiple of* $p = 2^m$ *we have*

$$f(3 \cdot 2^{n-1} + 1) = f(3 \cdot 2^{n-1} - 1) + f(3 \cdot 2^{n-1}) + 1 = 1 \ (mod\ 2^n). \tag{3.5}$$

Similarly we can get

$$f(3 \cdot 2^{n-1} + 2) = f(3 \cdot 2^{n-1}) + f(3 \cdot 2^{n-1} + 1) = 1 \ (mod\ 2^n), \tag{3.6}$$

and by (3.4), (3.5) *and* (3.6) *the period* $kp(2^n)$ *is a factor of* $3 \cdot 2^{n-1}$ *for* $p = 2^m$.
Next we are going to prove that $kp(2^n) = 3 \cdot 2^{n-1}$.
A factor of $3 \cdot 2^{n-1}$ *is* $3 \cdot 2^{n-1}$ *itself or* $3 \cdot 2^k$ *with* $k \leq n-2$ *or* 2^k *with* $k \leq n-1$.
By Theorem 5 the period of f *is a multiple of* 2^m.
Therefore we can suppose that $m \leq k$.
Let $t = 2^{k-m}$*, then by the first equation of Theorem 4*

$$f(2^k) = \frac{F(2^{k-1})F(2^{k-1} + 2^{m-1})}{F(2^{m-1})}, \tag{3.7}$$

and hence

$$f(2^k) > 0 \ (mod\ 2^n), \tag{3.8}$$

since by Lemma 3.1 $F(2^{k-1})$, $F(2^{m-1})$ *and* $F(2^{k-1} + 2^{m-1})$ *are odd numbers.*
By Remark 3.1 $kp(2^n) \neq 2^k$.
Next we are going to prove that $kp(2^n) \neq 3 \cdot 2^k$ *with* $(k \leq n-2)$. *By Remark 3.1 it is sufficient to prove that* $f(3 \cdot 2^k) > 0 \ (mod\ 2^n)$ *for* $m \leq k \leq n-2$.

Let $t = 3 \cdot 2^{k-m-1}$. *Then by the first equation of Theorem 4*

$$f(3 \cdot 2^k) = \frac{F(3 \cdot 2^{k-1})F(3 \cdot 2^{k-1} + 2^{m-1})}{F(2^{m-1})}. \tag{3.9}$$

By Theorem 6 we have

$$F(3 \cdot 2^{k-1}) = 2^{k+1} \ (mod\ 2^{k+3}). \tag{3.10}$$

[1] *If* $n \leq k+3$, *then by the assumption that* $k \leq n-2$ *we have* $k+1 < n \leq k+3$, *and hence by* (3.10)

$$F(3 \cdot 2^{k-1}) = 2^{k+1} > 0 \ (mod\ 2^n). \tag{3.11}$$

By Remark 3.1 $3 \cdot 2^{k-1}$ *is not the period* $kp(2^n)$.
[2] *Next we suppose that* $k+3 < n$. *By* (3.10) *there exists a natural number* K *such*

$$F(3 \cdot 2^{k-1}) = 2^{k+1} + K \cdot 2^{k+3}. \tag{3.12}$$

Then there exists a sequence of natural numbers $a_0, a_1, ...$ *such that* $a_t = 0$ *or 1 and* $K = a_0 + a_1 2^1 + a_2 2^2 + a_3 2^3 + ...$ *Then by* (3.12) *we have* $F(3 \cdot 2^{k-1}) = 2^{k+1} + a_0 2^{k+3} + a_1 2^{k+4} + a_2 2^{k+5} + a_3 2^{k+6} + ...$, *and hence*

$$F(3 \cdot 2^{k-1}) > 0 \ (mod\ 2^n). \tag{3.13}$$

By Remark 3.1 $3 \cdot 2^{k-1}$ *is not the period* $kp(2^n)$.

4 Some Interesting Graphs and Predictions about the periods of Fibonacci and Fibonacci-like sequences.

The periods of Fibonacci sequences and Fibonacci-like sequences is a difficult theme to study. Although it is fairly easy to make many predictions, it is often the case that proving them are very difficult.

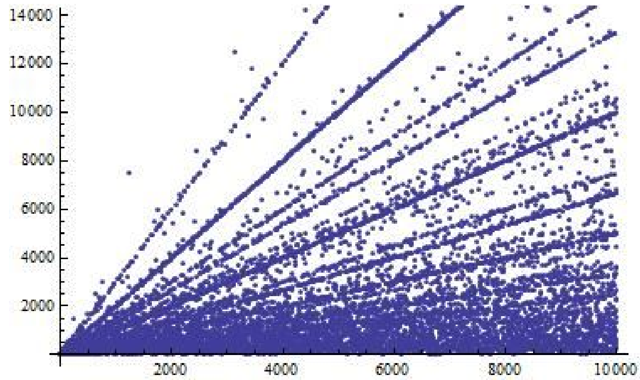
Here we are going to introduce many predictions and graphs that are very interesting, however, the authors think that it will be difficult to prove them generally by mathematical proof.

Throughout this section we use the computer algebra system Mathematica to draw graphs and make predictions. Although Mathematica is a very powerful tool in our research, you can understand the contents of this section without the knowledge of Mathematica.

Many people have studied the periods of the Fibonacci sequence, but we believe that we were the first people who made Graph 4.1 and Graph 4.2.

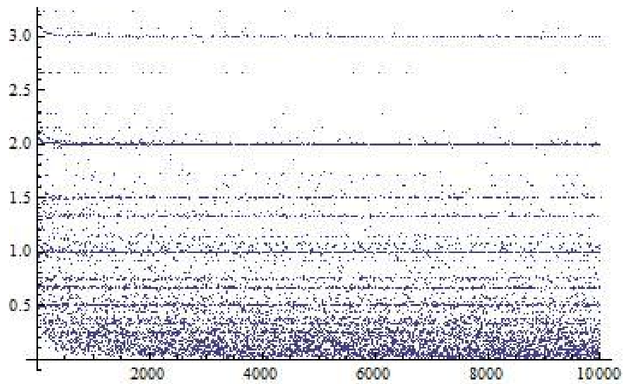
We used the Mathematica function in 5.1 to make Graph 4.1 and 4.2.

Example 4.1. *Let $k(m)$ be the period of F (mod m) and Graph 4.1 shows the list* $\{(m, k(m)),\ m = 1,2,...,10000\ \}$.



Graph 4.1.

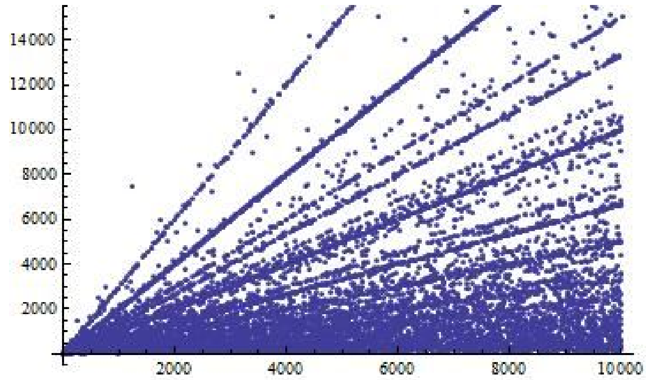
There are many lines in Graph 4.1, and we are going to study the slopes of the lines. We made the list $\{k(m)/m$ for $m = 1, 2, 3, ..., 10000\}$, and got Graph 4.2. From Graph 4.2 we can see that the slopes of the lines in Graph 4.1 are 3, 2, 1.5, 1, 0.5, etc. This fact indicates that for many natural numbers m we have $k(m) = 3m, 2m, 1.5m, m, 0.5m$.



Graph 4.2.

The Graphs 4.1 and 4.2 are very interesting, and hence we are going to make the same kind of graphs for our Fibonacci-like sequence. We used the Mathematica function in 5.2 to make Graph 4.3.

Example 4.2. *Let $k4(m)$ be the period of $B_4(s)$ (mod m) and the following graph shows the list* $\{(m, k4(m)),\ m = 1,2,...,10000\ \}$.



Graph 4.3.

The similarities between of Graph 4.1 and Graph 4.3 are obvious. This implies the existence of many simple relations between $k(x)$ and $k4(x)$.

Next we are going to study the distribution of residues of $F(mod\ m)$. That is, how frequently each residue is expected to appear within a single period. This is a well known topic to study.

Example 4.3. *The period of the sequence $F(mod\ 25)$ is 100, and the list of first 100 terms is { 1, 1, 2, 3, 5, 8, 13, 21, 9, 5, 14, 19, 8, 2, 10, 12, 22, 9, 6, 15, 21, 11, 7, 18, 0, 18, 18, 11, 4, 15, 19, 9, 3, 12, 15, 2, 17, 19, 11, 5, 16, 21, 12, 8, 20, 3, 23, 1, 24, 0, 24, 24, 23, 22, 20, 17, 12, 4, 16, 20, 11, 6, 17, 23, 15, 13, 3, 16, 19, 10, 4, 14, 18, 7, 0, 7, 7, 14, 21, 10, 6, 16, 22, 13, 10, 23, 8, 6, 14, 20, 9, 4, 13, 17, 5, 22, 2, 24, 1, 0 }.*
In this list the number 0 appears 4 times, the number 1 appears 4 times, ..., the number 24 appears 4 times. The following Chart shows how frequently each residue 0,1,...24 is expected to appear within a single period of 100.

residue	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
frequency	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

The distribution of residues of $F(mod\ m)$ is a very interesting topic, so we are going to the same thing for Fibonacci-like sequences.

Example 4.4. *The period of the sequence $B_2(mod\ 25)$ is 60, and the list of first 60 terms is {1, 1, 3, 4, 8, 2, 1, 3, 5, 8, 4, 2, 7, 9, 7, 6, 4, 0, 5, 5, 1, 6, 8, 4, 3, 7, 1, 8, 0, 8, 9, 7, 7, 4, 2, 6, 9, 5, 5, 0, 6, 6, 3, 9, 3, 2, 6, 8, 5, 3, 9, 2, 2, 4, 7, 1, 9, 0, 0, 0}.*
In this list the number 0 appears 6 times, the number 1 appears 6 times, ..., the number 9 appears 6 times. The following Chart shows how frequently

each residue 0,1,...9 is expected to appear within a single period of 60.

residue	0	1	2	3	4	5	6	7	8	9
frequency	6	6	6	6	6	6	6	6	6	6

Example 4.5. *The period of the sequence $B_3(mod\ 5)$ is 60, and the list of first 60 terms is { 1, 1, 2, 4, 1, 0, 2, 2, 4, 2, 1, 3, 0, 3, 3, 2, 0, 2, 3, 0, 3, 4, 2, 1, 4, 0, 4, 0, 4, 4, 4, 3, 2, 1, 3, 4, 3, 2, 0, 3, 3, 1, 0, 1, 1, 3, 4, 2, 2, 4, 1, 1, 2, 3, 1, 4, 0, 0, 0, 0}.*
In this list the number 0 appears 12 times, the number 1 appears 12 times, ..., the number 4 appears 12 times. The following Chart shows how frequently each residue 0,1,2,3,4 is expected to appear 12 times within a single period of 60.

residue	0	1	2	3	4
frequency	12	12	12	12	12

Example 4.4 and Example 4.5 are special cases of the following prediction.

Prediction 4.1. *In the following cases we have the same number of frequency of each number in the distribution of residues.*
(1) B_2 $(mod\ 2 \times 5^s)$ for non negative integer s.
(2) B_p $(mod\ 5^s)$ for non negative integer s and any prime number p with $p \neq 5$.

5 Computer programs the authors used in their research

Here we are going to present programs of Mathematica, Java, C, Python. Readers are encouranged to use these programs to do calculations and discover new formulas and theorems.

Example 5.1. *This is a Mathematica function to calculate the periods of the Fibonacci sequence $(mod\ x)$ for a natural number x.*
$findloopcom = Compile[\{\{x, Integer\}\},$
$Block[\{k = 1, s = x, a = 1, b = 2\},$
$If[x == 1, 1,\ While[!\{a, b\} == \{1, 1\}, \{a, b\} = \{b, Mod[a + b, s]\};$
$k = k + 1]]; k]];$

Let's explain about the algorithm used in the above function. Here we denote by $g[n]$ *the sequence* $F(modm)$*. If* $m = 1$, $k(1) = findloop[1] = 1$*. If* $m > 1$*, then the function* $findloop[m]$ *continues to produce* $g[2] = 1$, $g[3], g[4], \ldots$ *until there appears a natural number* h *such that* $g[h] = 1$ *and* $g[h+1] = 1$*. Then* $k(m) = h - 1$*.*

Example 5.2. *This is a Mathematica function to calculate the periods of Fibonacci-like sequence* $B_p(s)$ *(mod* x*) for a natural number* x*.*
By $fp[x, p]$ *we can calculate the period.*
$fp =$
$Compile[\{\{x, Integer\}, \{p, Integer\}\}$,
$Block[\{k = 0, s = x, a = 1, b = 1, aa = 1, bb = 1, cc = 2\},$
$Do[\{aa, bb, cc\} = \{bb, cc, Mod[cc + bb, s]\}, \{ns, 1, p - 2\}];$
$\{a, b\} = \{Mod[1 + aa * a + bb * b, s], Mod[1 + bb * a + cc * b, s]\};$
$If[x == 1, 1, While[!\{a, b\} == \{1, 1\},$
$\{a, b\} = \{Mod[1 + aa * a + bb * b, s], Mod[1 + bb * a + cc * b, s]\};$
$k = k + 1]]; p(k + 1)]]$

Example 5.3. *This is a C-program to present the distribution of* B_p *(mod* m*). We use* P *and* M *for* p *and* m *in* B_p *(mod* m*).*

```
#include <stdio.h>
#include <stdlib.h>

int main(int argc, char* args[]){
  int P, M;
  int n, cur, pre, tmp, i;
  int *cnt;

  P = atoi(args[1]);
  M = atoi(args[2]);
  n = 3;
  cur = pre = 1;
  cnt = (int *)calloc(M, sizeof(int));

  while(1){
    tmp = cur;
    cur += pre;
    pre = tmp;
```

```
    if(n % P == 1) cur += 1;
    cur %= M;
    cnt[cur]++;
    if(cur == 1 && pre == 1 && (n - 1) % P == 1) break;
    n++;
  }
```

Example 5.4. *This is a Java-program to present the distribution of B_p (mod m).*
We use P and M for p and m in B_p (mod m).

```
class Fibonacci{
  public static void main(String[] args){
    int P, M;
    int n, cur, pre, tmp;
    int[] cnt;

    P = Integer.parseInt(args[0]);
    M = Integer.parseInt(args[1]);
    n = 3;
    cur = pre = 1;
    cnt = new int[M];

    while(true){
      tmp = cur;
      cur += pre;
      pre = tmp;
      if(n % P == 1) cur++;
      cur %= M;
      cnt[cur]++;
      if(cur == 1 && pre == 1 && (n - 1) % P == 1)
        break;
      n++;
    }

    System.out.println("Loop:" + (n - 2));
    System.out.print("[");
    for(int i=0; i<M-1; i++) System.out.print(cnt[i] + ", ");
    System.out.println(cnt[M-1] + "]");
```

```
  }
}
```

Example 5.5. *This is a Python program to present the distribution of B_p (mod m).*
We use P and M for p and m in B_p (mod m).

```
import sys

def fp(P, M):
  n = 3
  pre = 1
  cur = 1
  cnt = [0] * M

  while True:
    pre, cur = cur, cur + pre
    if n % P == 1: cur += 1
    cur %= M
    cnt[cur] += 1
    if cur == 1 and pre == 1 and (n - 1) % P == 1:
      break
    n += 1

  print "Loop:%d"%(n - 2)
  print cnt

if __name__ == "__main__":
  P = int(sys.argv[1])
  M = int(sys.argv[2])
  fp(P, M)
```

6 An Interesting Prediction.

We are going to study graphs of distributions of residues. Here we let $m = 5$. In Graph 6.1 we let p = 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18, 19,20,21, and in Graph 6.2 we let p = 22,23,24,25,26,27,28,29,30,31, 32,33,34,35,36,37,38,39,40,41.

residue	0	1	2	3	4
frequency	4	4	4	4	4

residue	0	1	2	3	4
frequency	12	12	12	12	12

residue	0	1	2	3	4
frequency	7	4	1	3	5

residue	0	1	2	3	4
frequency	8	3	2	4	3

residue	0	1	2	3	4
frequency	12	12	12	12	12

residue	0	1	2	3	4
frequency	28	28	28	28	28

residue	0	1	2	3	4
frequency	15	8	6	9	2

residue	0	1	2	3	4
frequency	36	36	36	36	36

residue	0	1	2	3	4
frequency	12	3	1	3	1

residue	0	1	2	3	4
frequency	44	44	44	44	44

residue	0	1	2	3	4
frequency	21	14	7	10	8

residue	0	1	2	3	4
frequency	52	52	52	52	52

residue	0	1	2	3	4
frequency	28	28	28	28	28

residue	0	1	2	3	4
frequency	24	9	8	10	9

residue	0	1	2	3	4
frequency	29	11	13	15	12

residue	0	1	2	3	4
frequency	68	68	68	68	68

residue	0	1	2	3	4
frequency	36	36	36	36	36

residue	0	1	2	3	4
frequency	76	76	76	76	76

residue	0	1	2	3	4
frequency	36	16	16	16	16

residue	0	1	2	3	4
frequency	84	84	84	84	84

Graph 6.1.

residue	0	1	2	3	4
frequency	4	4	4	4	4

residue	0	1	2	3	4
frequency	12	12	12	12	12

residue	0	1	2	3	4
frequency	7	4	1	3	5

residue	0	1	2	3	4
frequency	8	3	2	4	3

residue	0	1	2	3	4
frequency	12	12	12	12	12

residue	0	1	2	3	4
frequency	28	28	28	28	28

residue	0	1	2	3	4
frequency	15	8	6	9	2

residue	0	1	2	3	4
frequency	36	36	36	36	36

residue	0	1	2	3	4
frequency	12	3	1	3	1

residue	0	1	2	3	4
frequency	44	44	44	44	44

residue	0	1	2	3	4
frequency	21	14	7	10	8

residue	0	1	2	3	4
frequency	52	52	52	52	52

residue	0	1	2	3	4
frequency	28	28	28	28	28

residue	0	1	2	3	4
frequency	24	9	8	10	9

residue	0	1	2	3	4
frequency	29	11	13	15	12

residue	0	1	2	3	4
frequency	68	68	68	68	68

residue	0	1	2	3	4
frequency	36	36	36	36	36

residue	0	1	2	3	4
frequency	76	76	76	76	76

residue	0	1	2	3	4
frequency	36	16	16	16	16

residue	0	1	2	3	4
frequency	84	84	84	84	84

Graph 6.2.

Example 6.1. *From Graph 6.1 we know that for p = 2,3,6,7,9,11,13,14,17,18,19,21 we have the same number of frequency of each number in the distribution of residues. From Graph 6.2 we know that for p = 22,23,26,27,29, 31,33,34,37,38,39,41 we have the same situation. If we compare these two sequences, we can discover many interesting things.*
For example, if we add 20 to each number in the first sequence, we get all the numbers in the second sequence.
We are going to calculate $\frac{kp(5)}{5p}$. *For example, if we let p = 2, then the destribution is Graph 6.3.*

Graph 6.3.

residue	0	1	2	3	4
frequency	4	4	4	4	4

From this graph we know that the period is $5 \times 4 = 20$. *Note that the length of the period is equal to the number of all the numbers that appear in one period. If we divide 20 by* $5p = 10$, *then we get 2. In this way we can calculate* $\frac{kp(5)}{5p}$ *for various values of p.*

If we calculate $\frac{kp(5)}{5p}$ *for p = 2,3,6,7,9,11,13,14,17,18,19,21, then we get 2, 4, 2, 4, 4, 4, 4, 2, 4, 2, 4, 4.*
If we calculate $\frac{kp(5)}{5p}$ *for p = 22,23,26,27,29, 31,33,34,37,38,39,41, then we get 2, 4, 2, 4, 4, 4, 4, 2, 4, 2, 4, 4.*
It seems that the sequence { $\frac{kp(5)}{5p}$ *, p = 1,2,3,..., } has a period of the length of 20. This is very interesting prediction.*

If you use the computer programs in this article, you may be able to discover many new facts about the Fibonacci-like sequences.

acknowledgments.
We would like to thank Takuya Takimoto for his contribution in making the graph of the periods of the Fibonacci sequences. We would like to thank Mr. Harrison Gray, Mr.Daisuke Minematsu and Dr. Ryohei Miyadera for helping us to prepare this article.

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