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Extreme Rays of AND-Measures in Circuit Complexity

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Abstract

This paper is motivated by the problem of proving lower bounds on the formula size of boolean functions, which leads to lower bounds on circuit depth. We know that formula size is bounded from below by all formal complexity measures. Thus, we study formula size by investigating "AND-measures", which are weakened forms of formal complexity measures. The collection of all AND-measures is a pointed polyhedral cone; we study the extreme rays of this cone in order to better understand AND-measures. From the extreme rays, we attempt to discover useful properties of AND-measures that may help in proving new lower bounds on formula size and circuit depth. This paper focuses on describing some of the properties of AND-measures, especially those that are extreme rays. Furthermore, it describes some algorithms for finding the extreme rays.

1 Introduction

We give a brief introduction to circuit complexity. A *boolean circuit* is a directed acyclic graph in which every node is labeled as an input node, an AND gate, an OR gate, or a NOT gate. Input nodes have in-degree 0, AND gates and OR gates have in-degree 2, and NOT gates have in-degree 1. The nodes can have arbitrary out-degree, but exactly one node has out-degree 0 and is referred to as the output node. Each input node is labeled by a boolean variable or its negation, with repeats allowed. Given a value in $\{0, 1\}$ for each boolean variable, the circuit computes an output value by propagating values along the edges and computing the output for each logic gate until the output node is assigned an output value. Every boolean function $f : \{0, 1\}^n \to \{0, 1\}$ on n variables has a circuit containing n or fewer input nodes that computes it. Figure 1 gives an example of a boolean circuit that computes XOR (exclusive OR).

The *size* of a circuit refers to the number of logic gates it contains, while the *depth* of a circuit refers to the length of the longest path from the output node to an input node. The *circuit size complexity* of a boolean function f, denoted $S(f)$, is the minimal size of any circuit that computes f. The *circuit depth complexity* of f, denoted $D(f)$, is defined similarly, using depth instead of size. (See [Weg87] for a more detailed introduction to circuit complexity.)

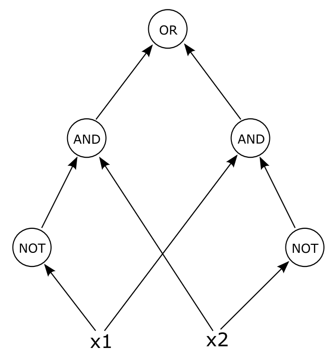


Figure 1: A boolean circuit that computes XOR.

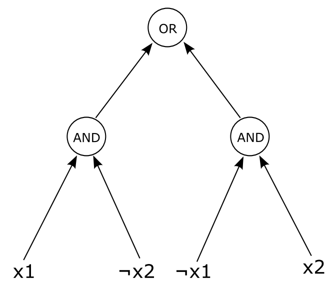


Figure 2: A boolean formula that computes XOR.

In this paper, we focus on a special type of boolean circuit called a *boolean formula*, where the out-degree of every node is at most 1. These circuits correspond directly to the usual expressions that we also call boolean formulas, such as $x_1 \wedge (x_2 \vee \neg x_3)$. Figure 2 gives an example of a boolean formula that computes XOR. Any (boolean) formula that computes a boolean function f can be converted to a formula without NOT gates that also computes f. This can be done by pushing all NOT gates to the input nodes by applying De Morgan's Law repeatedly. The resulting circuit is a binary tree that has fewer or the same number of logic gates.

For convenience, we define the *formula size complexity* of f, denoted $L(f)$, to be the minimal number of *leaves* in any binary tree formula that computes f (as opposed to the number of logic gates; the number of leaves in a binary tree formula is only 1 more than the number of logic gates.) It can be shown that any circuit that computes f can be converted to a formula that computes f and has the same depth. Thus, the "formula depth" and circuit depth complexity of a boolean function are the same.

The size complexity of a sequence $(f_n) = (f_1, f_2, \ldots)$ of boolean functions is the function $F\colon \mathbb{Z}^+ \to \mathbb{Z}_{\geq 0}$ where $F(n)$ is the circuit size complexity of f_n.

The depth and formula size complexity of (f_n) are defined similarly. One of the main challenges in circuit complexity is to prove lower bounds on circuit size and circuit depth for a sequence (f_n) of boolean functions. This paper is mainly concerned with formula size; however, a lower bound on formula size immediately leads to a lower bound on circuit depth, since it has been shown that $D(f_n) \in \Theta(\log L(f_n))$ (See [Spi71]).

Let B_n denote the set of all boolean functions on n variables. A formal complexity measure is a function $\mu\colon B_n \to \mathbb{R}_{\geq 0}$ such that

1. $\mu(f) = \mu(\neg f)$ for all $f \in B_n$,
2. $\mu(f \wedge g) \leq \mu(f) + \mu(g)$ for all $f, g \in B_n$, and
3. $\mu(x_i) \leq 1$ for $i = 1, \ldots, n$.

It can be shown that formula size is a formal complexity measure. In fact, formula size is the largest formal complexity measure, meaning that if μ is any formal complexity measure, then $L(f) \geq \mu(f)$ for all boolean functions f (See [Weg87]). Thus, studying formal complexity measures may help in proving new lower bounds on formula size and circuit depth. In this paper, we study formal complexity measures indirectly by investigating "AND-measures", which are weakened and generalized forms of formal complexity measures. We will see later that the collection of all AND-measures forms a pointed polyhedral cone. We investigate and describe some of the properties of AND-measures and the extreme rays of the cone. We also describe some algorithms for finding the extreme rays.

2 AND-Measures

We first generalize the usual concept of a boolean function on n boolean variables. Let $\mathbb{B} = \{0, 1\}$, and let S be any finite set. A boolean function on S is a function $f\colon S \to \mathbb{B}$. We observe that when $|S| = 2^n$, a boolean function on S corresponds to a boolean function on n variables. Throughout this paper, we will mainly work with boolean functions on a set S as opposed to boolean functions on n variables.

Now, let $\mathbb{B}^S$ denote the collection of all boolean functions on S. On $\mathbb{B}^S$, we have the usual operations: negation ($\neg$), conjunction ($\wedge$), and disjunction ($\vee$). We shall denote the identically 0 and identically 1 boolean functions by $\vec{0}$ and $\vec{1}$, respectively. Other boolean functions are written by listing their values in an arbitrary but consistent order, such as 0110 when $|S| = 4$. We now introduce the notion of an AND-measure.

Definition 1. *Let S be any finite set. An AND-measure on $\mathbb{B}^S$ is a function $\mathcal{F}\colon \mathbb{B}^S \to \mathbb{R}_{\geq 0}$ such that*

$$\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1}) \text{ for all } f, g \in \mathbb{B}^S. \tag{1}$$

Furthermore, $\mathcal{F}$ is said to be centered *if $\mathcal{F}(\vec{1}) = 0$, and said to be* negation-invariant *if $\mathcal{F}(f) = \mathcal{F}(\neg f)$ for all $f \in \mathbb{B}^S$.*

The notion of AND-measures and ways of constructing them have been studied previously (See [Fri06] and [Fri07]). The term $\mathcal{F}(\vec{1})$ in (1) above is normally not included in the definition of a formal complexity measure, but we include it here because it appears naturally when constructing AND-measures using the methods described in [Fri06] and [Fri07]. However, when an AND-measure $\mathcal{F}$ is centered (i.e., $\mathcal{F}(\vec{1}) = 0$), the $\mathcal{F}(\vec{1})$ term disappears anyway. We also note that we can "remove" the $\mathcal{F}(\vec{1})$ term by defining a new AND-measure $\mathcal{G}$ via $\mathcal{G}(f) = \mathcal{F}(f) + \mathcal{F}(\vec{1})$, which satisfies $\mathcal{G}(f \wedge g) \leq \mathcal{G}(f) + \mathcal{G}(g)$ without the $\mathcal{G}(\vec{1})$ term.

Throughout this paper, we shall write AND-measures by listing their values in an arbitrary but consistent order, such as (0,0,1,1,0,0,1,1) when $|S| = 3$. Also, we shall order the elements of $\mathbb{B}^S$ lexicographically (reading from left to right, with $0 < 1$) and write AND-measures according to this order. E.g., for $|S| = 2$, we have $00 < 01 < 10 < 11$, and an AND-measure $\mathcal{F}$ on $\mathbb{B}^S$ would be of the form $(\mathcal{F}(00), \mathcal{F}(01), \mathcal{F}(10), \mathcal{F}(11))$.

3 The Extreme Rays of AND-Measures

Each AND-measure on $\mathbb{B}^S$ corresponds to a vector in $\mathbb{R}^{(2^{|S|})}$. For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ denote the collection of all AND-measures on a set S with $|S| = n$. We shall view $\mathcal{C}_{|S|}$ as a set of vectors in $\mathbb{R}^{(2^{|S|})}$. From the definition of an AND-measure, we see that $\mathcal{C}_{|S|}$ is closed under vector addition and non-negative scalar multiplication. Furthermore, $\mathcal{C}_{|S|}$ is described by a finite number of homogeneous linear inequalities. Thus, $\mathcal{C}_{|S|}$ is a polyhedral cone[1] in $\mathbb{R}^{(2^{|S|})}$. By Minkowski's Theorem for polyhedral cones, this cone can be described by a finite set of generating vectors $\vec{w_1}, \ldots, \vec{w_m}$; i.e., $\mathcal{C}_{|S|} = \{\sum_{i=1}^m c_i w_i \mid c_i \geq 0, i = 1, \ldots, m\}$.

We now introduce some terminology from the theory of polyhedral cones. Let C be a polyhedral cone in $\mathbb{R}^n$. A non-zero vector $\vec{w} \in C$ is said to be an *extreme ray* of C if $\vec{w}$ cannot be written as the sum of two vectors $\vec{u}, \vec{v} \in C$ such that $\vec{u} \neq c\vec{w}$ and $\vec{v} \neq c\vec{w}$ for all $c \in \mathbb{R}_{\geq 0}$. For convenience, when $\vec{w}$ is an extreme ray, we shall call the set $\{c\vec{w} \mid c \in \mathbb{R}_{\geq 0}\}$ an extreme ray also (it should be clear from context which definition we are referring to). A hyperplane H in $\mathbb{R}^n$ is said to be supporting C if C intersects H and is contained in one of the closed half-spaces defined by H. A subset $F \subseteq C$ is a *face* of C if F is $\emptyset, C$ itself, or the intersection of C with a supporting hyperplane. An alternate but equivalent definition of an extreme ray of C is a 1-dimensional[2] face of C that is a half-line.

[1] A subset $C \subseteq \mathbb{R}^n$ is a polyhedral cone if $C = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \leq \vec{0}\}$ for some $m \times n$ matrix A.

[2] The dimension of a face F is the dimension of the span of F, which is the smallest subspace containing F.

We say that two extreme rays are equivalent when they differ by a scalar multiple. Thus, when we refer to the set of extreme rays of a polyhedral cone C, the set only includes a single extreme ray chosen from each equivalence class; we do not include two extreme rays that differ by a scalar multiple. A point $\vec{x} \in C$ is an *extreme point* of C if there does not exist two distinct points $\vec{u}, \vec{v} \in C$ and a $c \in (0,1)$ such that $\vec{x} = c\vec{u} + (1-c)\vec{v}$. C is said to be *pointed* if $\vec{0}$ is an extreme point of C.

Firstly, we note that the polyhedral cone $\mathcal{C}_{|S|}$ of all AND-measures is pointed; this follows immediately from the constraint that the values of an AND-measure are non-negative. It is clear that every set of generating vectors for $\mathcal{C}_{|S|}$ must contain all the extreme rays of $\mathcal{C}_{|S|}$ up to positive scaling. Furthermore, it is known in polyhedral theory that the set of (distinct) extreme rays of a pointed polyhedral cone is a generating set of vectors for the cone. Thus, every pointed polyhedral cone has a unique minimal[3] set of generating vectors (up to positive scaling), namely the set of extreme rays of the cone. Since $\mathcal{C}_{|S|}$ is a pointed polyhedral cone, we can describe $\mathcal{C}_{|S|}$ by its set of extreme rays. We shall study AND-measures by investigating the extreme rays of this cone.

3.1 The Set of Extreme Rays for $\mathcal{C}_2$

We now describe the set of extreme rays of the polyhedral cone of all AND-measures on a set S with $|S| = 2$. For $|S| = 2$, the property

$$\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1}) \text{ for all } f, g \in \mathbb{B}^S$$

reduces to the single inequality:

$$\mathcal{F}(00) \leq \mathcal{F}(01) + \mathcal{F}(10) + \mathcal{F}(11).$$

All other inequalities are trivially satisfied due to the nonnegativity of $\mathcal{F}$'s values. In this simple case, we can see that the set of extreme rays is $\{\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}, \vec{w_6}\}$, where $\vec{w_i}$ has the form $(w_i(00), w_i(01), w_i(10), w_i(11))$ and

$$\begin{array}{lll} \vec{w_1} = (0,1,0,0), & \vec{w_2} = (0,0,1,0), & \vec{w_3} = (0,0,0,1), \\ \vec{w_4} = (1,1,0,0), & \vec{w_5} = (1,0,1,0), & \vec{w_6} = (1,0,0,1). \end{array}$$

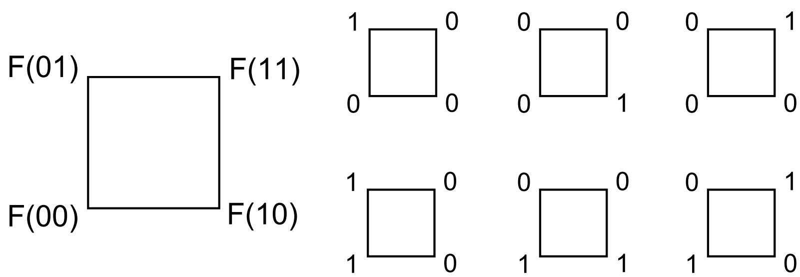
A graphical representation of the set of extreme rays is shown in Figure 3. Each extreme ray is represented by a square with 4 vertices. The 4 vertices and their corresponding values represent the 4 components of the extreme ray, which are the values of the AND-measure for the 4 boolean functions on S. The square on the left indicates which value of the AND-measure a vertex represents.

3.2 A Simple Algorithm for Finding the Extreme Rays

For $|S| = 2$, finding the extreme rays by inspection is easy due to the small number of inequalities involved. For larger cardinalities of S, however, finding

[3] A set $\mathfrak{B}$ of generating vectors for a polyhedral cone C is said to be *minimal* if no subset of $\mathfrak{B}$ is a generating set for C.

Figure 3: The extreme rays of $\mathcal{C}_2$. The cone has 6 extreme rays.



the extreme rays by inspection is impractical. Thus, we now describe a simple brute force algorithm for finding the extreme rays for any cardinality of S. We shall use $|S| = 3$ as an example. In this case, the property

$$\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1}) \text{ for all } f, g \in \mathbb{B}^S$$

reduces to the following set of inequalities:

$$\begin{aligned}
\mathcal{F}(000) &\leq \mathcal{F}(001) + \mathcal{F}(010) + \mathcal{F}(111) \\
\mathcal{F}(000) &\leq \mathcal{F}(010) + \mathcal{F}(100) + \mathcal{F}(111) \\
\mathcal{F}(000) &\leq \mathcal{F}(001) + \mathcal{F}(100) + \mathcal{F}(111) \\
\mathcal{F}(000) &\leq \mathcal{F}(001) + \mathcal{F}(110) + \mathcal{F}(111) \\
\mathcal{F}(000) &\leq \mathcal{F}(010) + \mathcal{F}(101) + \mathcal{F}(111) \\
\mathcal{F}(000) &\leq \mathcal{F}(011) + \mathcal{F}(100) + \mathcal{F}(111) \\
\mathcal{F}(001) &\leq \mathcal{F}(011) + \mathcal{F}(101) + \mathcal{F}(111) \\
\mathcal{F}(010) &\leq \mathcal{F}(011) + \mathcal{F}(110) + \mathcal{F}(111) \\
\mathcal{F}(100) &\leq \mathcal{F}(101) + \mathcal{F}(110) + \mathcal{F}(111)
\end{aligned}$$

We shall also add inequalities to ensure the nonnegativity of $\mathcal{F}$'s values:

$$\begin{array}{cccc}
\mathcal{F}(000) \geq 0 & \mathcal{F}(001) \geq 0 & \mathcal{F}(010) \geq 0 & \mathcal{F}(011) \geq 0 \\
\mathcal{F}(100) \geq 0 & \mathcal{F}(101) \geq 0 & \mathcal{F}(110) \geq 0 & \mathcal{F}(111) \geq 0
\end{array}$$

The 17 inequalities define 17 closed half-spaces in $\mathbb{R}^8$, and the intersection of these half-spaces forms the pointed polyhedral cone. Associated with each half-space is the hyperplane at the boundary of the half-space. These 17 hyperplanes are described by the 17 equations that correspond to the above inequalities. Geometrically, we observe that each extreme ray of the cone lies in the intersection of 7 of these hyperplanes in $\mathbb{R}^8$. However, not every intersection of 7 hyperplanes gives an extreme ray, as the intersection may not be 1-dimensional, or may be a line that only intersects the cone at the origin.

We now see a simple algorithm for finding the extreme rays of the cone. We solve every linear system of 7 equations chosen from the set of 17 equations, and if the solution is a one-dimensional line, we check whether half of the line lies in the cone. If it does, then the half-line in the cone must be an extreme ray. In all other cases, the solution does not give an extreme ray. We note that every linear system must be consistent, since they are all homogeneous. We also note that we can safely ignore linear systems whose solution is of dimension greater than 1; if an extreme ray lies in such a solution space, it will be found by the algorithm, since we are checking all systems of 7 equations. We now generalize and summarize this algorithm:

1. From the property $\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1})$ for all $f, g \in \mathbb{B}^S$, we generate the set of non-trivial inequalities that need to be satisfied by every AND-measure. Also, we add inequalities to ensure the nonnegativity of $\mathcal{F}$'s values.

2. We generate the set $\mathcal{E}$ of equations that correspond to the inequalities.

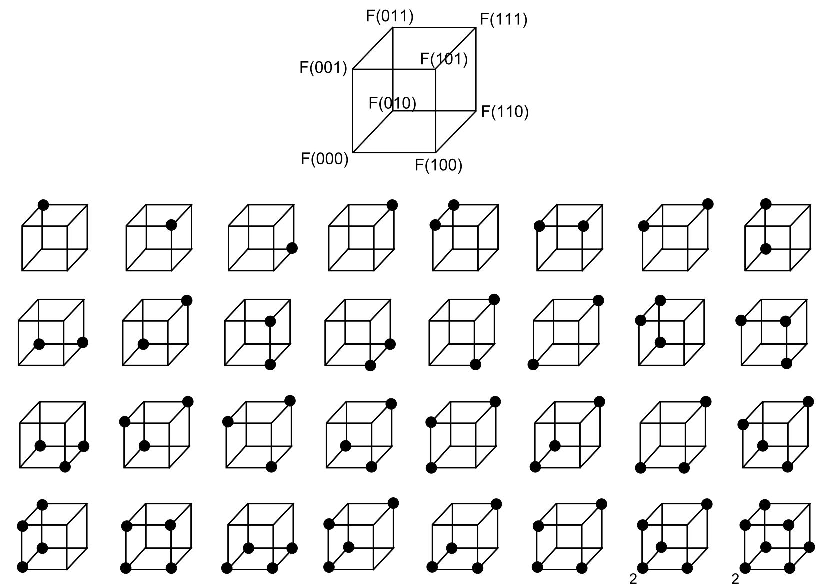
3. For every system of $2^{|S|} - 1$ equations chosen from $\mathcal{E}$, we solve the system. If the solution is a one-dimensional line, we choose a vector on the line that has at least one positive component (since all non-zero vectors in the cone have non-negative components and at least one positive component). If the vector is in the cone (i.e., satisfies all the inequalities) and no scalar multiple of the vector already exists in the set of extreme rays, add the vector to the set of extreme rays.

This algorithm is not meant to be an efficient method for finding the extreme rays. However, it gives a description of the extreme rays that may be helpful when investigating the properties of the extreme rays. More efficient algorithms for finding the extreme rays will be discussed later.

3.3 The Set of Extreme Rays for $\mathcal{C}_3$

Figure 4 displays the set of extreme rays for $\mathcal{C}_3$ (the cone of all AND-measures when $|S| = 3$). Each extreme ray is represented by a cube with 8 vertices. The vertices represent the 8 components of the extreme ray, which are the values of the AND-measure for the 8 boolean functions on S. The cube at the top indicates which value of the AND-measure a vertex represents. The vertices with a dot on them are the only non-zero components, while the others have value 0; the dot represents a 1, except for the last two extreme rays, where the AND-measure value for the $\vec{0}$ boolean function is 2, as indicated.

Figure 4: The extreme rays of $\mathcal{C}_3$. The cone has 32 extreme rays.



4 Properties of AND-Measures

Let $S = \{s_1, s_2, \ldots, s_n\}$. We shall write an arbitrary boolean function f on S by listing its values in the following form: $f(s_1)f(s_2)...f(s_n)$. We now describe some of the properties of AND-measures.

Definition 2. *Let S be any finite set. A boolean function f on S is said to be* conjunction-basic *if $f = \vec{1}$ or $f(s) = 0$ for exactly one $s \in S$.*

AND-Measure Property 1. *Let $S = \{s_1, \ldots, s_n\}$, and for $i = 1, \ldots, n$, let g_i be the conjunction-basic boolean function where $g_i(s_i) = 0$ and $g_i(s) = 1$ if $s \neq s_i$. E.g., for $n = 3$, we have $g_1 = 011, g_2 = 101, g_3 = 110$. For every $h \in \mathbb{B}^S$, let $zeros(h) = \{i \mid h(s_i) = 0\}$. Let $\mathcal{F}$ be any AND-measure. Then, for every $f \in \mathbb{B}^S \setminus \{\vec{1}\}$, we have $\mathcal{F}(f) \leq \sum_{i \in zeros(f)} \mathcal{F}(g_i) + (|zeros(f)| - 1) \cdot \mathcal{F}(\vec{1})$.*

Proof. Let $f \in \mathbb{B}^S \setminus \{\vec{1}\}$. If $|zeros(f)| = 1$, we have $f = g_i$ and $zeros(f) = \{i\}$ for some $i \in \{1, \ldots, n\}$. Thus, $\mathcal{F}(f) = \mathcal{F}(g_i) = \sum_{j \in zeros(f)} \mathcal{F}(g_j)$, and the property holds. Now, suppose that $|zeros(f)| > 1$, and let $j \in zeros(f)$. We observe that f can be written in the form $g_j \wedge h$, where $h(s) = f(s)$ if $s \neq s_j$, and $h(s) = 1$ if $s = s_j$. h satisfies the inductive hypothesis, so we have

$$\begin{aligned}\mathcal{F}(f) &= \mathcal{F}(g_j \wedge h) \\ &\leq \mathcal{F}(g_j) + \mathcal{F}(h) + \mathcal{F}(\vec{1}) \\ &\leq \mathcal{F}(g_j) + \sum_{i \in zeros(h)} \mathcal{F}(g_i) + (|zeros(f)| - 2) \cdot \mathcal{F}(\vec{1}) + \mathcal{F}(\vec{1}) \\ &= \sum_{i \in zeros(f)} \mathcal{F}(g_i) + (|zeros(f)| - 1) \cdot \mathcal{F}(\vec{1}).\end{aligned}$$

□

Terminology 1. *Let S be any finite set, and let $\mathcal{F}$ be any AND-measure on $\mathbb{B}^S$. For any boolean function $f \in \mathbb{B}^S$, we shall refer to $\mathcal{F}(f)$ as the complexity of f (with respect to $\mathcal{F}$).*

Remark 1. *The above property merely states that the complexity of every boolean function except for $\vec{1}$ is bounded in terms of the complexity of the conjunction-basic boolean functions. This comes from the defining property of AND-measures and the fact that every boolean function can be written as a conjunction of conjunction-basic boolean functions. We immediately see that imposing a bound on the complexity of the conjunction-basic boolean functions would give a bound on the complexity of all the other boolean functions. The complexity of $\vec{1}$ is unbounded for general AND-measures, but we can insist that $\mathcal{F}(\vec{1}) = 0$ (in the case of* centered *AND-measures) or at least $\mathcal{F}(\vec{1}) \leq 1$.*

AND-Measure Property 2. *Let $\mathcal{F}$ be any AND-measure on $\mathbb{B}^S$ that is not identically 0. Then, $\mathcal{F}(h) > 0$ for some conjunction-basic boolean function h.*

Proof. Suppose that $\mathcal{F}(h) = 0$ for every conjunction-basic boolean function h. Then, from the previous property, we have $\mathcal{F}(f) \leq 0$ for every $f \in \mathbb{B}^S \setminus \{\vec{1}\}$. This implies that $\mathcal{F}$ is identically 0, contrary to assumption.

□

Terminology 2. *Let $\mathcal{F}$ be any AND-measure on $\mathbb{B}^S$. We say that the maximal complexity of a boolean function $f_m \in \mathbb{B}^S$ (with respect to $\mathcal{F}$) is x if there does not exist an AND-measure $\mathcal{G}$ such that $\mathcal{G}(f_m) > x$ and $\mathcal{G}(h) = \mathcal{F}(h)$ for all $h \neq f_m$. We define the term "minimal complexity" in a similar manner.*

AND-Measure Property 3. *Recall that we order boolean functions lexicographically; e.g., for $|S| = 2$, we have $00 < 01 < 10 < 11$. For any AND-measure $\mathcal{F}$, the maximal complexity of a boolean function f is precisely determined by the the complexity of the boolean functions greater than f. E.g., for $|S| = 3$, the maximal complexity of the boolean function 100 is precisely determined by the complexity of 101, 110, and 111, since the only inequality giving an upper bound on 100 is $\mathcal{F}(100) \leq \mathcal{F}(101) + \mathcal{F}(110) + \mathcal{F}(111)$.*

Proof. We note that the maximal complexity of a boolean function h is the largest value of $\mathcal{F}(h)$ (with the other values of $\mathcal{F}$ fixed) such that the property $\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1})$ for all $f, g \in \mathbb{B}^S$ holds. We immediately observe that the only inequalities that give an upper bound on $\mathcal{F}(h)$ are the ones where $\mathcal{F}(h)$ appears on the left hand side of the inequality. For such inequalities, the right hand side only involves boolean functions whose conjunction is h, and since the conjunction operation always returns a lower boolean function (for non-trivial inequalities), we see that the property holds.

□

AND-Measure Property 4. *For any AND-measure $\mathcal{F}$, the minimal complexity of a boolean function f is precisely determined by the complexity of the boolean functions less than f.*

Proof. This property can be easily proven in a manner similar to the proof for maximal complexity. Both properties can be easily seen by observing the nature of the inequalities given by the property $\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1})$ for all $f, g \in \mathbb{B}^S$.

□

AND-Measure Property 5. *Let $\mathcal{F}$ be any AND-measure, and let f_l be the least boolean function with non-zero complexity. Then, for every $\mathcal{G} \colon \mathbb{B}^S \to \mathbb{R}_{\geq 0}$ such that $\mathcal{G}(f_l) < \mathcal{F}(f_l)$ and $\mathcal{G}(h) = \mathcal{F}(h)$ for all $h \neq f_l$, $\mathcal{G}$ is an AND-measure.*

Proof. From the previous two properties, we know that the maximal and minimal complexities of a boolean function f are precisely determined by the complexity of the boolean functions greater than f and the boolean functions less than f, respectively. We note that $\mathcal{G}(h) = \mathcal{F}(h)$ for all $h > f_l$, so the maximal complexity of f_l with respect to $\mathcal{G}$ is the same as that with respect to $\mathcal{F}$. Since $\mathcal{G}(f_l) < \mathcal{F}(f_l)$, we know that $\mathcal{G}(f_l)$ is less than its maximum. Since $\mathcal{G}(h) = \mathcal{F}(h) = 0$ for all $h < f_l$, the minimal complexity of f_l is 0. Thus, $\mathcal{G}(f_l)$ is within its valid range and thus does not violate any inequalities. The complexity of the other boolean functions with respect to $\mathcal{G}$ are also in their valid ranges. Thus, $\mathcal{G}$ is an AND-measure.

□

4.1 Properties of Extreme Rays

We now describe some of the properties of the extreme rays of $\mathcal{C}_{|S|}$ (the cone of all AND-measures).

Extreme Ray Property 1. *Let $\mathfrak{B}_{|S|}$ be the set of extreme rays for $\mathcal{C}_{|S|}$. Then, for every extreme ray $\mathcal{B} \in \mathfrak{B}_{|S|}$, there exists an $f \in \mathbb{B}^S$ such that $\mathcal{B}(f) = 0$.*

Proof. Recall that all extreme rays lie in the intersection of $2^{|S|} - 1$ hyperplanes in $\mathbb{R}^{(2^{|S|})}$. The hyperplanes that we have available are the coordinate planes and the planes defined by the equations derived from the property

$$\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1}) \text{ for all } f, g \in \mathbb{B}^S. \tag{2}$$

Let $\mathcal{B} \in \mathfrak{B}_{|S|}$, and suppose that $\mathcal{B}$ has no zero component. Then, $\mathcal{B}$ does not lie on any of the coordinate planes, so it must lie on a line formed by the intersection of $2^{|S|} - 1$ hyperplanes defined by $2^{|S|} - 1$ equations derived from (2). Now, observe that the vector $\vec{v} = (1, 1, \ldots, 1, -1)$ lies on the line, since it satisfies all the equations derived from (2). Since the line is one-dimensional, all other vectors on the line are scalar multiples of $\vec{v}$. Since $\vec{v}$ contains both negative and positive components, we see that the line only intersects the cone at the origin. Since $\mathcal{B}$ is in the cone and on the line, $\mathcal{B}$ must be the zero vector, which is not an extreme ray of the cone.

□

Remark 2. *The above property states that all the extreme rays lie on the coordinate planes.*

Extreme Ray Property 2. *Let $\vec{w} \in \mathcal{C}_{|S|}$ be any non-zero AND-measure. Then, $\vec{w}$ is an extreme ray if and only if for every AND-measure $\vec{a} \in \mathcal{C}_{|S|}$ that is not a scalar multiple of $\vec{w}$, we have $\vec{w} - \vec{a} \notin \mathcal{C}_{|S|}$.*

Proof. Suppose that $\vec{w}$ is an extreme ray. Let $\vec{a} \in \mathcal{C}_{|S|}$ such that $\vec{a}$ is not a scalar multiple of $\vec{w}$. Now, suppose that $\vec{w} - \vec{a} \in \mathcal{C}_{|S|}$. Then, $\vec{w}$ is the sum of two vectors in $\mathcal{C}_{|S|}$, namely $\vec{a}$ and $\vec{w} - \vec{a}$. Also, $\vec{w} - \vec{a} \neq c\vec{w}$ for all $c \in \mathbb{R}_{\geq 0}$, since if $\vec{w} - \vec{a} = c\vec{w}$ for some $c \in \mathbb{R}_{\geq 0}$, we would have $\vec{a} = (1 - c)\vec{w}$, which is a contradiction. Thus, $\vec{a}$ and $\vec{w} - \vec{a}$ show that $\vec{w}$ is not an extreme ray, contrary to assumption. Hence, we must have $\vec{w} - \vec{a} \notin \mathcal{C}_{|S|}$, as required.

Now, suppose that $\vec{w}$ is not an extreme ray. Then, $\vec{w}$ can be written as the sum of two vectors $\vec{u}, \vec{v} \in \mathcal{C}_{|S|}$ such that $\vec{u} \neq c\vec{w}$ and $\vec{v} \neq c\vec{w}$ for all $c \in \mathbb{R}_{\geq 0}$. Then, $\vec{u}$ is in $\mathcal{C}_{|S|}$ and is not a scalar multiple of $\vec{w}$, but $\vec{w} - \vec{u} = \vec{v} \in \mathcal{C}_{|S|}$, as required. □

Remark 3. *This property gives a characterization of extreme rays that may be helpful in testing whether an AND-measure is an extreme ray or not. For example, suppose that we have a subset $\mathfrak{B}'$ of the set of extreme rays for $\mathcal{C}_{|S|}$, and we are trying to find the remaining extreme rays. Now, suppose that $\vec{v}$ is an AND-measure that is not a scalar multiple of any of the existing extreme rays in $\mathfrak{B}'$. The above property gives us a method for testing whether $\vec{v}$ could possibly be an extreme ray. If $\vec{v}$ is an extreme ray, then by the above property, we have $\alpha\vec{v} - \vec{w} \notin \mathcal{C}_{|S|}$ for every $\vec{w} \in \mathfrak{B}'$ and $\alpha \in \mathbb{R}^+$. If this condition fails to hold, we can be certain that $\vec{v}$ is not an extreme ray. If this condition does hold, then we are still uncertain, since $\mathfrak{B}'$ is only a subset of the extreme rays for $\mathcal{C}_{|S|}$. (α above should be chosen sufficiently large so that $\alpha\vec{v} - \vec{w}$ has non-negative values, if possible.)*

Extreme Ray Property 3. *Let $\mathcal{G}$ be any AND-measure on $\mathbb{B}^S$ such that exactly one conjunction-basic boolean function has non-zero complexity, and all the other boolean functions have 0 complexity. Then, $\mathcal{G}$ is an extreme ray of $\mathcal{C}_{|S|}$.*

Proof. We observe that $\mathcal{G}$ has exactly one non-zero component, and thus, lies in the intersection of $2^{|S|} - 1$ coordinate planes. These coordinate planes intersect in a line, and since $\mathcal{G}$ lies on the line and is in the cone, half of the line must be in the cone and must be an extreme ray. Thus, $\mathcal{G}$ is an extreme ray of $\mathcal{C}_{|S|}$. □

Remark 4. *The above property tells us that the set of extreme rays for $\mathcal{C}_{|S|}$ must contain the AND-measures described, up to a positive scaling.*

Extreme Ray Property 4. *Let $\mathcal{F}$ be any extreme ray of $\mathcal{C}_{|S|}$ such that $\mathcal{F}(h) > 0$ for some non-conjunction-basic boolean function h. Then, all conjunction-basic boolean functions have minimal complexity with respect to $\mathcal{F}$.*

Proof. Suppose that some conjunction-basic boolean function g does not have minimal complexity. Then, there exists an AND-measure $\mathcal{G}$ such that $\mathcal{G}(g) < \mathcal{F}(g)$ and $\mathcal{G}(f) = \mathcal{F}(f)$ for all $f \neq g$. Since $\mathcal{F}(h) > 0$ for some non-conjunction-basic boolean function h, we can be sure that $\mathcal{G}$ is not a scalar multiple of $\mathcal{F}$. We now note that $\mathcal{F} - \mathcal{G}$ is 0 for all boolean functions except for g, which is a conjunction-basic boolean function, and $(\mathcal{F} - \mathcal{G})(g) > 0$. We recognize that $\mathcal{F} - \mathcal{G}$ is an AND-measure, so by Extreme Ray Property 2, $\mathcal{F}$ is not an extreme ray, contrary to assumption.

□

Remark 5. *The above property gives us another way of quickly recognizing that some AND-measure cannot possibly be an extreme ray. If an AND-measure violates the above property, then we can be certain that it is not an extreme ray.*

Extreme Ray Property 5. *Recall that we order boolean functions lexicographically. E.g., for $|S| = 2$, we have $00 < 01 < 10 < 11$. Now, let $\mathfrak{B}_{|S|}$ be the set of extreme rays for $\mathcal{C}_{|S|}$, and let $\mathcal{F}$ be any extreme ray in $\mathfrak{B}_{|S|}$ whose least boolean function f_l with non-zero complexity is not conjunction-basic. Then, f_l has maximal complexity.*

Proof. Without loss of generality, suppose that $\mathfrak{B}_{|S|} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n\}$ and that $\mathcal{F} = \mathcal{F}_1$. Now, suppose that f_l does not have maximal complexity with respect to $\mathcal{F}_1$. Then, there exists an AND-measure $\mathcal{G}$ such that $\mathcal{G}(f_l) > \mathcal{F}_1(f_l)$ and $\mathcal{G}(h) = \mathcal{F}_1(h)$ for all $h \neq f_l$. Since $\mathcal{G} \in \mathcal{C}_{|S|}$, we have $\mathcal{G} = c_1\mathcal{F}_1 + c_2\mathcal{F}_2 + \cdots + c_n\mathcal{F}_n$ for some $c_1, c_2, \ldots, c_n \in \mathbb{R}_{\geq 0}$, not all 0, with $c_1 \leq 1$. Now, let $\mathcal{H}$ be the AND-measure where $\mathcal{H}(f_l) = 0$ and $\mathcal{H}(h) = \mathcal{F}_1(h)$ for all $h \neq f_l$. Since $\mathcal{H}(f_l) = 0$ while $\mathcal{F}_1(f_l) \neq 0$, we can write $\mathcal{H}$ in the form $d_2\mathcal{F}_2 + \cdots + d_n\mathcal{F}_n$, where $d_2, \ldots, d_n \in \mathbb{R}_{\geq 0}$. Since $\mathcal{F}_1, \mathcal{G}$, and $\mathcal{H}$ all have the same values except for the complexity of f_l, and since $0 = \mathcal{H}(f_l) < \mathcal{F}_1(f_l) < \mathcal{G}(f_l)$, we can write $\mathcal{F}_1$ in the form $\mathcal{F}_1 = \alpha\mathcal{G} + \beta\mathcal{H} = (\alpha)(c_1\mathcal{F}_1 + c_2\mathcal{F}_2 + \cdots + c_n\mathcal{F}_n) + (\beta)(d_2\mathcal{F}_2 + \cdots + d_n\mathcal{F}_n)$, where $\alpha, \beta \in (0, 1)$. Thus, we have $\mathcal{F}_1 = \frac{c_2\alpha + d_2\beta}{1 - c_1\alpha}\mathcal{F}_2 + \cdots + \frac{c_n\alpha + d_n\beta}{1 - c_1\alpha}\mathcal{F}_n$. This implies that F_1 is not an extreme ray, contrary to assumption.

□

5 Extreme Rays for the Cone of Centered and Negation-Invariant AND-Measures

Recall that an AND-measure $\mathcal{F}$ is said to be centered if $\mathcal{F}(\vec{1}) = 0$, and said to be negation-invariant if $\mathcal{F}(f) = \mathcal{F}(\neg f)$ for all $f \in \mathbb{B}^S$. Let $\mathcal{C}(C)_n$ denote the collection of all centered AND-measures on a set S with $|S| = n$. Similarly, let $\mathcal{C}(N)_n$ denote the collection of all negation-invariant AND-measures, and let $\mathcal{C}(C,N)_n$ denote the collection of all AND-measures that are both centered and negation-invariant. We shall view $\mathcal{C}(C)_n$, $\mathcal{C}(N)_n$, and $\mathcal{C}(C,N)_n$ as sets of vectors in $\mathbb{R}^{(2^n)}$. Similar to the cone of all AND-measures, $\mathcal{C}(C)_n$, $\mathcal{C}(N)_n$, and $\mathcal{C}(C,N)_n$ are also pointed polyhedral cones in $\mathbb{R}^{(2^n)}$. We now investigate the problem of finding the extreme rays of these cones.

We observe that $\mathcal{C}(C)_n$ is simply the intersection of $\mathcal{C}_n$ (the collection of all AND-measures) with the coordinate plane $\mathcal{F}(\vec{1}) = 0$. Since there are no AND-measures $\mathcal{F} \in \mathcal{C}_n$ such that $\mathcal{F}(\vec{1}) < 0$, the set of extreme rays for $\mathcal{C}_n$ must contain a subset that generates $\mathcal{C}(C)_n$, namely all the extreme rays $\mathcal{G}$ for which $\mathcal{G}(\vec{1}) = 0$. Thus, if we have the extreme rays of $\mathcal{C}_n$, the extreme rays of $\mathcal{C}(C)_n$ can be easily found.

We observe that $\mathcal{C}(N)_n$ is the intersection of $\mathcal{C}_n$ with $2^{(n-1)}$ hyperplanes defined by the equations obtained from the property $\mathcal{F}(f) = \mathcal{F}(\neg f)$ for all $f \in \mathbb{B}^S$. In the set of extreme rays for $\mathcal{C}_n$, there may be extreme rays on both sides of each hyperplane, so we cannot easily obtain the set of extreme rays for $\mathcal{C}(N)_n$ from the set of extreme rays for $\mathcal{C}_n$. However, if we did have the set of extreme rays for $\mathcal{C}(N)_n$, we could easily obtain the set of extreme rays for $\mathcal{C}(C,N)_n$ by taking all extreme rays $\mathcal{G}$ for which $\mathcal{G}(\vec{1}) = 0$. Thus, we see that adding the "centered" property does not make the problem of finding the extreme rays harder. We now revise our previous simple algorithm so that it can find the extreme rays of $\mathcal{C}(C)_n$, $\mathcal{C}(N)_n$, and $\mathcal{C}(C,N)_n$ also.

5.1 Revised Algorithm to Find $\mathcal{C}(C)_n, \mathcal{C}(N)_n$, and $\mathcal{C}(C,N)_n$

We shall use $|S| = 3$ as an example to describe the revised algorithm. Recall that our previous algorithm involves solving every linear system of 7 equations chosen from the set of 17 equations derived from the constraints

$$\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1}) \text{ for all } f, g \in \mathbb{B}^S$$

and

$$\mathcal{F}(f) \geq 0 \text{ for all } f \in \mathbb{B}^S.$$

If we are finding the extreme rays of $\mathcal{C}(C)_3$ or $\mathcal{C}(C,N)_3$ (a cone where all the AND-measures are centered), then every extreme ray $\mathcal{B}$ will have to satisfy $\mathcal{B}(111) = 0$. Thus, we can remove the equation $\mathcal{F}(111) = 0$ from our set of equations to choose from and just insist that every linear system that we solve must contain the equation $\mathcal{F}(111) = 0$.

If we are finding the extreme rays of $\mathcal{C}(N)_3$ or $\mathcal{C}(C,N)_3$ (a cone where all the AND-measures are negation-invariant), then every extreme ray $\mathcal{B}$ will have to satisfy the negation-invariant property: $\mathcal{B}(f) = \mathcal{B}(\neg f)$ for all $f \in \mathbb{B}^S$. Thus, we insist that every linear system that we solve must contain the equations $\mathcal{F}(000) = \mathcal{F}(111), \mathcal{F}(001) = \mathcal{F}(110), \mathcal{F}(010) = \mathcal{F}(101)$, and $\mathcal{F}(011) = \mathcal{F}(100)$. Furthermore, we can choose exactly one coordinate plane equation from each $\{\mathcal{F}(f) = 0, \mathcal{F}(\neg f) = 0\}$ pair and remove it from our set of equations to choose from, since the equations $\mathcal{F}(f) = 0$ and $\mathcal{F}(\neg f) = 0$ are the same when the equation $\mathcal{F}(f) = \mathcal{F}(\neg f)$ is in every linear system.

Depending on what characteristics the AND-measures in the cone have (e.g. centered, or negation-invariant, or both), instead of choosing 7 equations from our set of equations to choose from, we now choose 6, 3, or 2 equations. Our goal is to choose enough equations so that every linear system has exactly 7 equations altogether. These 7 equations represent hyperplanes in $\mathbb{R}^8$, and solving the system corresponds to finding their intersection. As before, if their intersection is a one-dimensional line and half of the line lies in the cone, we can be sure that the half-line is an extreme ray. Thus, the same procedure as before gives us the set of extreme rays for the cone. We now generalize and summarize this revised algorithm:

1. From the constraints $\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1})$ and $\mathcal{F}(f) \geq 0$ for all $f, g \in \mathbb{B}^S$, we generate the set of non-trivial inequalities that need to be satisfied by every AND-measure.

2. We then generate the set $\mathcal{E}$ of equations that correspond to the inequalities. If the cone only contains centered AND-measures, we omit the equation $\mathcal{F}(\vec{1}) = 0$. If the cone only contains negation-invariant AND-measures, we omit one coordinate-plane equation chosen from every pair $\{\mathcal{F}(f) = 0, \mathcal{F}(\neg f) = 0\}$, and if the cone also has the "centered" property, we omit both $\mathcal{F}(\vec{1}) = 0$ and $\mathcal{F}(\vec{0}) = 0$. We can simplify the equations if we desire to do so.

3. If the cone only contains centered AND-measures, we will add the equation $\mathcal{F}(\vec{1}) = 0$ to every linear system that we form. If the cone only contains negation-invariant AND-measures, we will add the $2^{|S|-1}$ equations derived from the property $\mathcal{F}(f) = \mathcal{F}(\neg f)$ for all $f \in \mathbb{B}^S$ to every linear system that we form.

4. We form a new linear system containing any default equations mentioned above. We now choose enough equations from $\mathcal{E}$ so that our linear system contains exactly $2^{|S|} - 1$ equations, and we solve the system. If the solution is a one-dimensional line, we choose a vector on the line that has at least one positive component. If the vector is in the cone (i.e., satisfies all the inequalities) and no scalar multiple of the vector already exists in the set of extreme rays, add the vector to the set of extreme rays.

5. We repeat the previous step for all possible combinations of equations chosen from $\mathcal{E}$.

6 Symmetries in the Extreme Rays

There are certain symmetries in the set of extreme rays for the cone of AND-measures (with any combination of properties, such as centered and negation-invariant). We will discuss some of these symmetries here.

Let $S = \{s_1, \ldots, s_n\}$, and let $\mathfrak{B}$ be the set of extreme rays. By a *permutation* of S, we mean a bijective function from S to S; also, we shall denote the group of all permutations of S by $Sym(S)$.

Recall that we write boolean functions on S in the form $f(s_1)...f(s_n)$. For every permutation $\sigma \in Sym(S)$, let $\pi_\sigma : \mathbb{B}^S \to \mathbb{B}^S$ be defined by $\pi_\sigma(f(s_1)...f(s_n)) = f(\sigma(s_1))...f(\sigma(s_n))$. For any $f \in \mathbb{B}^S$, $\pi_\sigma(f)$ is simply the boolean function obtained by "permuting" the values of f according to σ. Each π_σ is a permutation of $\mathbb{B}^S$, and it can be easily verified that the set $G = \{\pi_\sigma \mid \sigma \in Sym(S)\}$ is a subgroup of $Sym(\mathbb{B}^S)$.

Firstly, we note that for every $f, g \in \mathbb{B}^S$ and every $\sigma \in Sym(S)$, we have $\pi_\sigma(f \wedge g) = \pi_\sigma(f) \wedge \pi_\sigma(g)$. This can be easily seen by the fact that π_σ simply permutes the values of a boolean function and that conjunction ($\wedge$) of boolean functions is performed component-wise. We now describe the structure in the set of inequalities derived from the property $\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1})$ of AND-measures. If σ is any permutation of S and $\mathcal{F}(f_o \wedge g_o) \leq \mathcal{F}(f_o) + \mathcal{F}(g_o) + \mathcal{F}(\vec{1})$ is any inequality in our set of inequalities, then $\mathcal{F}(\pi_\sigma(f_o \wedge g_o)) \leq \mathcal{F}(\pi_\sigma(f_o)) + \mathcal{F}(\pi_\sigma(g_o)) + \mathcal{F}(\pi_\sigma(\vec{1}))$ is also an inequality in our set. This follows immediately from the fact that $\pi_\sigma(f_o \wedge g_o) = \pi_\sigma(f_o) \wedge \pi_\sigma(g_o)$ and $\pi_\sigma(\vec{1}) = \vec{1}$.

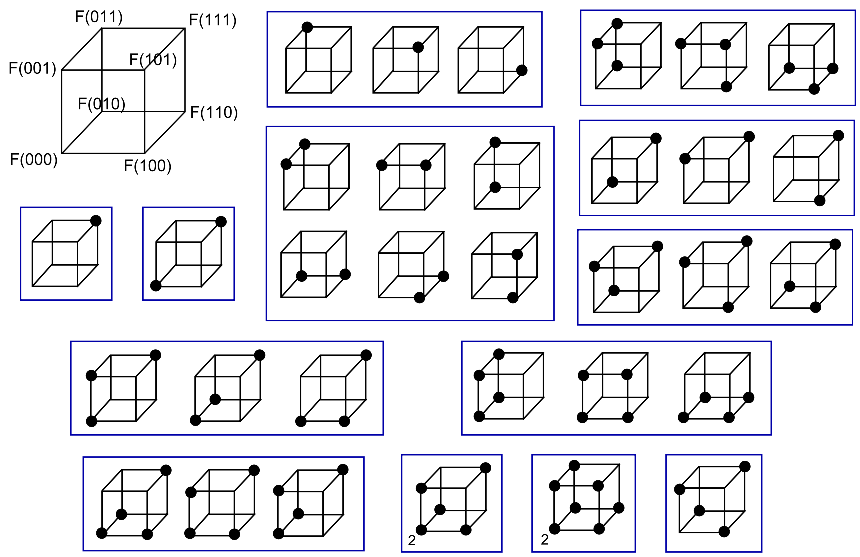
Let A be the set of linear constraints that define the cone. Each constraint in A is in one of the following forms:

1. $\mathcal{F}(f \wedge g) \leq \mathcal{F}(f) + \mathcal{F}(g) + \mathcal{F}(\vec{1})$
2. $\mathcal{F}(f) \geq 0$
3. $\mathcal{F}(\vec{1}) = 0$
4. $\mathcal{F}(f) = \mathcal{F}(\neg f)$

From our discussion above, we know that if *con* is a constraint of the form (1) and σ is any permutation of S, we can apply π_σ to all the boolean functions in *con* and the resulting constraint is still in A. It is easy to see that this also applies to constraints of the form (2), (3) and (4). Due to this symmetry in the constraints, it is not hard to see that if σ is any permutation of S and $\mathcal{F} = (\mathcal{F}(f_1), \ldots, \mathcal{F}(f_{2^n}))$ is any AND-measure, then $(\mathcal{F}(\pi_\sigma(f_1)), \ldots, \mathcal{F}(\pi_\sigma(f_{2^n})))$ is also an AND-measure. This also applies to extreme rays, so the set of extreme rays for the cone also has these symmetries.

Now, we define a group action $\circ$ of $Sym(S)$ on $\mathfrak{B}$ (the set of extreme rays) by $\sigma \circ (\mathcal{B}(f_1), \ldots, \mathcal{B}(f_{2^n})) = (\mathcal{B}(\pi_\sigma(f_1)), \ldots, \mathcal{B}(\pi_\sigma(f_{2^n})))$. The set of orbits of $\mathfrak{B}$ forms a partition of $\mathfrak{B}$. We see that if we have a subset $\mathfrak{B}'$ of $\mathfrak{B}$ that contains (at least) one extreme ray from each orbit, we can determine the full set of extreme rays by applying the group action on the extreme rays in $\mathfrak{B}'$. Thus,

Figure 5: The orbits of the extreme rays for $\mathcal{C}_3$. There are 13 orbits.



it suffices to only find one extreme ray in each orbit. This symmetry can be exploited to find the extreme rays more efficiently. Figure 5 displays the orbits of the extreme rays for $\mathcal{C}_3$ (the cone of all AND-measures when $|S| = 3$).

7 The Number of Extreme Rays

The number of extreme rays for $\mathcal{C}_3$ and $\mathcal{C}_4$ is 32 and 3201, respectively. The number of extreme rays seems to grow extremely quickly as $|S|$ increases. Tables 1 and 2 display the number of extreme rays and orbits when $|S| = 3$ and $|S| = 4$, respectively, with the various combinations of settings for the "centered" and "negation-invariant" properties.

From the tables, we see that the cones with the negation-invariant property has significantly fewer extreme rays than the cones without the negation-invariant property. It is not totally clear why this is the case. Perhaps, the number of extreme rays can be further reduced by adding additional constraints of a certain form. Also, by adding additional constraints of a certain form, we can focus on studying AND-measures that have certain properties.

8 Complexity of Finding the Extreme Rays

The problem of finding the set of extreme rays for a pointed polyhedral cone has been studied previously (e.g., see [MRTT53], [FP96], [Fuk04], and [MD73]).

Centered	Negation-Invariant	Number of Extreme Rays	Number of Orbits
T	T	3	1
T	F	16	5
F	T	7	3
F	F	32	13

Table 1: The number of extreme rays and orbits when $|S| = 3$.

Centered	Negation-Invariant	Number of Extreme Rays	Number of Orbits
T	T	49	10
T	F	971	94
F	T	145	26
F	F	3201	290

Table 2: The number of extreme rays and orbits when $|S| = 4$.

For the cones in this paper, the problem of finding the extreme rays is said to be *degenerate*. This means that there exists a point in the cone $\mathcal{C}_{|S|}$ that satisfies more than $2^{|S|}$ inequalities (describing the cone) with equality. Finding the extreme rays when the problem is degenerate is difficult; there is no known algorithm that runs in polynomial time in both the size of the input and the size of the output. Furthermore, for the cones of AND-measures, both the number of constraints and the dimension of the cone grows very quickly as $|S|$ increases. Thus, it appears that finding the extreme rays of the cone for larger sizes of S is currently infeasible.

One algorithm for finding the set of extreme rays for a polyhedral cone is the *Double Description Method* (See [MRTT53, FP96]). The algorithm first finds the extreme rays of the polyhedral cone defined by a small subset of the constraints (e.g., the empty set with no constraints). Then, it incrementally adds the remaining constraints in some order, determining the set of extreme rays for the new cone (with the added constraint) at each step. After all the constraints have been added, the algorithm outputs the set of extreme rays. Since the extreme rays of the cone of AND-measures have certain structure to them, we may be able to make this algorithm run faster by exploiting the structure.

Stronger results and more significant properties of the extreme rays (and of AND-measures in general) are still being investigated.

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