

A Reverse Sierpinski Number Problem

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A REVERSE SIERPIŃSKI NUMBER PROBLEM

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ABSTRACT. A generalized Sierpiński number base b is an integer $k > 1$ for which $\gcd(k+1, b-1) = 1$, k is not a rational power of b , and $k \cdot b^n + 1$ is composite for all $n > 0$. Given an integer $k > 0$, we will seek a base b for which k is a generalized Sierpiński number base b . We will show that this is not possible if k is a Mersenne number. We will give an algorithm which will work for all other k provided that there exists a composite in the sequence $\{(k^{2^m} + 1) / \gcd(k+1, 2)\}_{m=0}^{\infty}$.

1. INTRODUCTION

A Sierpiński number $k > 0$ is an odd number such that $k \cdot 2^n + 1$ is composite for all integers $n > 0$. Waclaw Sierpiński, in 1960, proved that there are infinitely many such numbers [12] but found no exact values. (This is a dual of a problem of Euler that Erdős solved in 1950 [6].) In 1962 John Selfridge discovered what may be the smallest Sierpiński number, 78,557. He showed that each term in the sequence $78557 \cdot 2^n + 1$ is divisible by one of the primes in the covering set $\{3, 5, 7, 13, 19, 37, 73\}$. After 45 years of computing there remains only 6 possible numbers less than 78,557 that must be eliminated to prove this is true [9].

In our previous paper [1], we extended the definition of Sierpiński numbers to include other bases (as below) and found Sierpiński numbers for each of the bases $2 \leq b \leq 100$. We proved that for 33 of the bases, these were the least possible Sierpiński numbers.

Definition 1.1. A **Sierpiński number base** b is an integer $k > 1$ for which $\gcd(k+1, b-1) = 1$, k is not a rational power of b , and $k \cdot b^n + 1$ is composite for all $n > 0$.

The gcd condition is to avoid having a single prime which divides every term of the sequence; these are called trivial covers (or 1-covers). The rational power condition avoids numbers of the form $b^{2^n} + 1$, the generalized Fermat numbers. Some researchers do not exclude these and instead exclude those k which allow polynomial factorization [8]. Others have generalized the notion of Sierpiński numbers by altering the conditions on k without changing the base b [11, 2, 3, 4, 5, 7].

Key words and phrases. Sierpiński number, covering set, generalized Fermat number, Mersenne number.

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Definition 1.2. A **cover** for the sequence $k \cdot b^n + 1$ ($n > 0$) is a finite set of primes $S = \{p_1, p_2, \dots, p_m\}$ for which each element of the sequence is divisible by a prime in S . S is called a **N-cover** if N is the least positive integer for which each prime in S divides $k \cdot b^n + 1$ if and only if it divides $k \cdot b^{n+N} + 1$. We will call this integer N the **period of the cover** S .

In our previous paper we showed that every integer base $b > 1$ admitted a Sierpiński number k . In this paper we will reverse the problem and ask for each integer $k > 1$ is there a base b for which k is a Sierpiński number.

Using a program to find Sierpiński numbers for small bases [1] we found that every positive integer less than 3000 is a Sierpiński number (with a cover with period dividing 5040) for some base less than 10,000,000 except for $k = 2, 3, 5, 7, 15, 31, 63, 65, 127, 255, 511, 1023$, and 2047.

A Mersenne is a number of the form $2^m - 1$ where (e.g., $2^8 - 1$). We will show that when k is a Mersenne number, $k \cdot b^n + 1$ can not have a non-trivial cover. Most Sierpiński numbers arise through covers, though it is also possible for them to arise by algebraic factorization or a combination of the two [1]. But it is very unlikely that such factorizations arise when k is a Mersenne [8, 10]. Once we eliminate the Mersenne numbers from the list above we are left with 2, 5, and 65.

It was conjectured that there is no base which makes 2 a Sierpiński number [13]. We will find bases for which 2, 5, and 65 are Sierpiński numbers, and then characterize those k which cannot be Sierpiński numbers.

2. PRELIMINARY THEOREMS

Theorem 2.1. *Suppose $k \cdot b^n + 1$ has a cover S with period N . Then each prime in S divides $(-k)^N - 1$ and $b^N - 1$.*

Proof. Let S be such a cover and let $p \in S$. Then p divides $k \cdot b^n + 1$ for some n with $1 \leq n \leq N$. Since N is our period and $p \in S$ we know p divides $k \cdot b^n + 1$, $k \cdot b^{n+N} + 1$, and their difference $k \cdot b^n(b^N - 1)$, but does not divide $k \cdot b^n$. So $b^N \equiv 1 \pmod{p}$. Now $k \cdot b^n + 1 \equiv 0 \pmod{p}$ so $k \equiv -b^{-n} \pmod{p}$ and $(-k)^N \equiv (b^{-n})^N \equiv 1 \pmod{p}$. \square

Notice 2 cannot be a member of a non-trivial cover because if 2 divides $k \cdot b^n + 1$ and 2 divides $b^N - 1$, then 2 divides $\gcd(b - 1, k + 1)$.

Theorem 2.2. *If k is a Mersenne number and $n \geq 0$, then $k \cdot b^n + 1$ cannot have a non-trivial cover.*

Proof. We will prove the contrapositive. Let S be a non-trivial cover of $k \cdot b^n + 1$ with period N . Then there is some odd prime $p \in S$ which divides $k \cdot b^n + 1$. By the previous theorem, $b^N \equiv 1 \pmod{p}$, so p divides $k + 1$. This means $k + 1$ has an odd prime divisor. Therefore, $k + 1$ is not a power of 2. \square

These theorem suggests the following approach to finding a base b which makes k a Sierpiński number. First, select a period N for the cover. Then

factor $(-k)^N - 1$. Using each of its prime divisors p , we would find a specific b value such that p divides $b^N - 1$, but not $b - 1$, and p divides $k \cdot b^n + 1$ for some $n < N$. If this is possible, then a list could be formed showing the specific b values and n values satisfying the requirements for each prime. After repeating this for each prime, determine whether there is a set of bases and primes such that the entire set of N terms would be covered. If so, we then solve the problem by using the Chinese Remainder Theorem.

We used this method to solve for b when $k = 12$. We have $(-12)^6 - 1 = 7 \cdot 11 \cdot 13 \cdot 19 \cdot 157$. Using the primes 7, 13, and 19 coupled respectively with base residues 4, 3, and 7 for b , the Chinese Remainder Theorem gave the result $b = 900$. The primes in the cover $\{7, 13, 19\}$ divide the sequence of terms $12 \cdot 900^n + 1$ with period 3 in the following pattern.

$$\underbrace{7, 19, 13}, \underbrace{7, 19, 13}, \dots$$

However, with $k = 2$ we ran into difficulties. The approach appeared to fail for all periods $N \leq 60$. For example, when we set our period at 60 (a smooth number which is likely to yield results), we cover either $2 \cdot b^{15} + 1$ or $2 \cdot b^{45} + 1$ but not both. When we set the period at 48, this process would fail at one of these specific n values: $\{3, 9, 17, 15, 21, 27, 33, 39, 41, 45\}$. The key problem is that $k \cdot b^n + 1 \equiv 0 \pmod{p}$ may be impossible to solve for a certain k , n , and p ; so we will give an alternative approach in the next section.

3. PERIOD LENGTH A POWER OF TWO

We need a few more results before we present our alternate method.

Theorem 3.1. *If $n > m \geq 0$, then $\gcd(k^{2^m} + 1, k^{2^n} + 1) = \gcd(k + 1, 2)$.*

Proof. Note $k^{2^m} + 1$ is one of the terms on the left of

$$(k - 1)(k + 1)(k^2 + 1) \cdot \dots \cdot (k^{2^{n-1}} + 1) = k^{2^n} - 1 = (k^{2^n} + 1) - 2,$$

so $\gcd(k^{2^m} + 1, k^{2^n} + 1)$ divides 2. If k is even then the greatest common divisor is one; otherwise it is 2. \square

Theorem 3.2. *If $n > 0$ and $k > 1$, then $k^{2^n} + 1$ has an odd prime factor.*

Proof. If k is even, then $k^{2^n} + 1$ is an odd number greater than 1, so we are done. If instead k is odd, then $\gcd(k^{2^n} + 1, 4) = 2$ and $(k^{2^n} + 1)/2$ is an odd number greater than or equal to 5, so again it has an odd prime factor. \square

Theorem 3.3. *Let p be an odd prime which divides $k^{2^m} + 1$ for some fixed integer $m > 0$. For all odd integers n it is possible to solve $k \cdot b^n + 1 \equiv 0 \pmod{p}$ for a solution $b = (-k)^j$. This solution satisfies $k \cdot b^M + 1 \equiv 0 \pmod{p}$ if and only if $\text{ord}_p(b)$ divides $M - n$.*

Proof. Let $m > 0$. From what we are given $(-k)^{2^m} \equiv -1 \pmod{p}$, hence $(-k)^{2^{m+1}} \equiv 1 \pmod{p}$. So we know $-k$ has order 2^{m+1} modulo p . To solve $k \cdot b^n + 1 \equiv 0 \pmod{p}$ for b , let $b = (-k)^j$ for some $j \geq 0$ and solve

$(-k)^{j \cdot n + 1} \equiv 1 \pmod{p}$. It is sufficient to solve $j \cdot n + 1 \equiv 0 \pmod{\text{ord}_p(-k)}$ which is possible because n and the order of $-k \pmod{p}$ are relatively prime. Note that j must be odd, so the solution b has order 2^{m+1} also. \square

We are now able to present our new approach as Algorithm 1.

Algorithm 1: Find b so that k is a Sierpiński number base b

```

1  input  $k > 1$ 
2  if  $k$  is a Mersenne number then
3    | return “[probably] not possible”
4  else
5    |  $m \leftarrow -1$ 
6    | repeat
7    |   |  $m \leftarrow m + 1$ 
8    |   |  $p_m \leftarrow$  the least odd prime which divides  $k^{2^m} + 1$ 
9    |   |  $n \leftarrow 2^m + 1$ 
10   |   | solve  $k \cdot b_m^n + 1 \equiv 0 \pmod{p_m}$  for  $b_m$ 
11   |   | until  $m > 0$  and there is another odd prime which divides
12   |   |  $k^{2^m} + 1$ 
13   |   | call this second prime  $p_{m+1}$ 
14   |   | define  $b_{m+1}$  by  $k \cdot b_{m+1} + 1 \equiv 0 \pmod{p_{m+1}}$ 
15   |   | find  $b$  so that  $b \equiv b_i \pmod{p_i}$  for  $0 \leq i \leq m + 1$ , and
16   |   |  $\text{gcd}(k + 1, b - 1) = 1$ 
17   |   | return base  $b$ , cover  $S = \{p_0, p_1, \dots, p_{m+1}\}$ , period  $2^{m+1}$ 
18   | endif

```

Theorem 3.4. *Suppose $k > 1$ is not a Mersenne number. If there is a term in the sequence*

$$(3.1) \quad \{(k^{2^m} + 1) / \text{gcd}(k + 1, 2)\} \quad (m > 0)$$

divisible by at least two distinct primes, then Algorithm 1 will find a base b for which k is a Sierpiński number.

Before we present the proof, we will illustrate the use of this algorithm with $k = 2$. After the first time through the loop (steps 6 through 11), $m = 0$, $p_0 = 3$, $n = 2$, and $b_0 \equiv 2 \pmod{3}$. So after we solve for $b \equiv b_0$, 3 will divide the second term of the sequence $k \cdot b^v + 1$ for $v = \{1, 2, \dots\}$ and every other term because b has period 2 modulo 3. This prime begins our cover and at this stage the pattern with which the primes in the cover divides the terms $k \cdot b^v + 1$ for $v = \{1, 2, \dots\}$ and looks like the following:

$$\underbrace{\quad, 3, \quad, 3, \dots}$$

That is, 3 divides $2 \cdot b^v + 1$ for $v = 2, 4, 6, \dots$

After the second time through the loop, $m = 1$, $p_1 = 5$, $n = 3$, and $b_1 \equiv 3 \pmod{5}$. So 5 divides the third term and then every fourth because $2^{1+1} = 4$ is the period of $b_1 \pmod{p_1}$ by Theorem 3.3. So our divisibility pattern now looks like:

$$\underbrace{--, 3, 5, 3}, \underbrace{--, 3, 5, 3}, \dots$$

After the third time through $p_2 = 17$ and b_2 has order 8, so we now have a pattern with period 2^3

$$\underbrace{--, 3, 5, 3, 17, 3, 5, 3}, \underbrace{--, 3, 5, 3, 17, 3, 5, 3}, \dots$$

The primes for iterations four and five are 257 and 65537 see (Table 1), leaving a pattern with period 2^5 .

Notice that each time through the loop the period doubles; however, since we have added just one prime to each stop we are still left with one hole. If we were to ever get a case in which there were two odd primes dividing $k^{2^m} + 1$ then we can fill in that hole! In our example, it is the sixth iteration in which this first occurs (when $2^{2^5} + 1 = 641 \cdot 6700417$), so we double the period again but we can also now fill in the last hole. Finally, we solve for b using the Chinese Remainder Theorem. Thus, when $k = 2$, we find $b = 16979062410086072498$ and the cover $\{3, 5, 17, 257, 641, 65537, 6700417\}$ that repeats with period 64.

Similarly, with $k = 5$, we find $b = 140324348$ and the cover $\{3, 13, 17, 313, 11489\}$ with a period of 16. Finally, for $k = 65$, $b = 19030688904264$ with cover $\{3, 17, 113, 2113, 8925313\}$ with a period of 16.

Proof of Theorem 3.4. The algorithm starts with an input $k > 1$. If k is a Mersenne number, the algorithm throws it out because of Theorem 2.2. Next, we start the loop with $m = 0$, and find the least odd prime p_0 that divides $k + 1$ (we know this exists because k is not a Mersenne). When $m = 0$, it is also possible to solve $k \cdot b_0^2 + 1 \equiv 0 \pmod{p}$, because $k \equiv -1 \pmod{p}$. So if we let $b_0 \equiv -1 \pmod{p}$, then $k \cdot b_0^2 + 1 \equiv 0 \pmod{p}$. Note that if $b \equiv b_0 \pmod{p_0}$ then p_0 divides every other term in the sequence $k \cdot b^n + 1$ because b_0 has order 2 mod p_0 .

During the next loop $m = 1$ and we know that $k^{2^1} + 1$ has an odd prime factor (for all $m > 0$) from Theorem 3.2. Thus, we solve $k \cdot b_1^n + 1 \equiv 0 \pmod{p_1}$ for base b_1 , which is possible (for all $m > 0$) by Theorem 3.3. By Theorem 3.2, we know our solution has order $2^{m+1} = 4$ so we have a pattern

$$\underbrace{--, p_0, p_1, p_0}, \underbrace{--, p_0, p_1, p_0}, \dots$$

We repeat this process (steps 6 – 11) until there are two odd primes which divide $k^{2^m} + 1$. For the first prime p_m , the algorithm solves (as usual) $k \cdot b_m^n + 1 \equiv 0 \pmod{p_m}$. With the second prime, p_{m+1} , the algorithm solves $k \cdot b_{m+1} + 1 \equiv 0 \pmod{p_{m+1}}$ for base b_{m+1} to fill in the last hole in our pattern. So we have a cover $\{p_1, \dots, p_{m+1}\}$ which divide the terms $k \cdot b^n + 1$

with a pattern determined by Theorem 3.3 to be

$$\underbrace{p_{m+1}, p_0, p_1, p_0, p_2, p_0, \dots, p_m, \dots, p_0, p_1, p_0}.$$

We then use the Chinese Remainder Theorem (step 13) to solve for b . However, we must make sure the cover is not trivial so if $\gcd(k+1, b-1) \neq 1$, we can add $p_1 p_2 \cdots p_{m+1}$ to b to get b' . Now $\gcd(k+1, b'-1) = 1$ by Theorem 3.1. We are then left with a base b that makes k a Sierpiński Number with a cover S and a period equal to 2^{m+1} . \square

TABLE 1. The Algorithm Table

m	$k^{2^m} + 1$	p_m	n	$b_m \pmod{p}$
0	3	3	2	2
1	5	5	3	3
2	17	17	5	9
3	257	257	9	129
4	65537	65537	17	32769
5	641 · 6700417	641	33	321
		6700417	1	3350208

4. CONCLUSION

It is highly unlikely that there exists an integer k for which the sequence of generalized Fermat numbers of equation 3.1 are all prime. So given a k , we have shown how to find bases b for which $k \cdot b^n + 1$ has non-trivial cover, for all k for which these exist! However, this does not mean our choice of b is the least. Computations have shown that our solution for $k = 5$ is the smallest with period dividing 5040, but we do not know if the others are the smallest.

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