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Thomas Cuchta
Marshall University Kathryn A. Lokken, cuchta@marshall.edu

Kathryn A. Lokken
University of Wisconsin, Madison

William Young
Purdue University

ZERO-DIVISOR GRAPHS OF LOCALIZATIONS AND MODULAR RINGS

THOMAS CUCHTA, KATHRYN A. LOKKEN, WILLIAM YOUNG

Abstract. In this paper, we examine the algebraic properties of localizations of commutative rings and how localizations affect the zero-divisor graph's structure of modular rings. We also classify the zero-divisor graphs of modular rings with respect to both the diameter and girth of their resultant zero-divisor graphs.

1. Introduction

In [4], Beck introduced the concept of the graph of a ring. Beck's main purpose was to examine the colorings of commutative rings. The work was continued by D.D. Anderson and Naseer in [1]. In these two papers all ring elements were included in the graph. However, this paper only includes the non-zero zero-divisors as vertices in the graph, just as D.F. Anderson and Livingston introduced in [2]. The diameter and girth of these zero-divisor graphs, among other things, were examined by D.F. Anderson and Livingston in [2], by Mulay in [9], and by DeMeyer and Schneider in [6]. In [2] it was shown that all zero-divisor graphs of commutative rings must be connected with diameter less than or equal to three and girth three, four, or infinity.

Throughout, R will denote a commutative ring with unity. The *powerset* of R will be denoted $\mathcal{P}(R)$. In this paper we will only consider proper ideals of R. A *prime ideal* P of a commutative ring R is an ideal of R such that if $ab \in P$, then $a \in P$ or $b \in P$. A *zero-divisor* is an element $z \in R$ such that $zr = 0$ for some nonzero $r \in R$. The set of zero-divisors of R will be denoted by $Z(R)$, and $Z(R)^* = Z(R)\backslash\{0\}$.

Let R be a commutative ring with unity. Let S be a multiplicatively closed subset of R. Define a binary relation $\sim$ on $R \times S$ by $(r,s) \sim (r',s')$ if and only if there exists $s^* \in R\backslash S$ such that $s^*(rs' - r's) = 0$. This relation is an equivalence relation. Define $\frac{r}{s} = (r,s)$. The localization of R at S is the set $R_S = \{[\frac{r}{s}] \mid r \in R$ and $s \in S\}$ together with two binary operations $+, \cdot : R_S \times R_S \to R_S$ defined by $[\frac{r}{s}] + [\frac{r'}{s'}] = [\frac{rs'+r's}{ss'}]$ and $[\frac{r}{s}] \cdot [\frac{r'}{s'}] = [\frac{rr'}{ss'}]$. By the usual convention, $\cdot$ is replaced by juxtaposition, that is, $x \cdot y = xy$. Note that R_S is a commutative ring with additive identity $[\frac{0}{1}]$ and unity $[\frac{1}{1}]$. Notice that the complement of a prime ideal is always multiplicatively closed. By convention, a localization $R_{R\backslash P}$, where P is a prime ideal, is denoted R_P.

The *zero-divisor graph* of R is denoted $\Gamma(R) = \{V, E\}$, where the vertices $V = \{z \mid z \in Z(R)^*\}$ and the edges $E = \{(z,z') \mid zz' = 0$ and $z,z' \in V\}$ (edges are sometimes denoted as (a,b) where a and b are vertices). A *path* in a graph is a sequence of vertices, $a_0, a_1, ..., a_n$ such that each adjacent pair, a_i, a_{i+1}, is a

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nonrepeated valid edge in the edge set; (a_n, a_0) may or may not be an edge. The *distance* between two vertices in a graph is a path with the least number of edges. The *diameter* of a graph G, denoted $\mathrm{diam}(G)$, is the largest distance between any two vertices. A *cycle* is a path $a_0, a_1, ..., a_n$ such that (a_n, a_0) is an edge. The *girth* of a graph G, denoted $\mathrm{g}(G)$, is the length of a cycle with the least number of edges. A graph has girth ∞ if it contains no cycles. A graph is *connected* if there exists a path between all vertices of the graph.

A reference for localizations can be found in [8], and a reference for graph theory can be found in [5]. We compare the zero-divisor graphs of R and R_P, the localization of R around P. We consider the localizations of modular rings, and show they are isomorphic to a particular $\mathbb{Z}_m$. We also completely classify the zero-divisor graphs of modular rings by diameter and girth.

2. $\Gamma(R)$ AND $\Gamma(R_P)$

From the definition of $\sim$, an element $[\frac{r}{s}]$ of R_S is in the equivalence class of $[\frac{0}{1}]$ if and only if there exists an $s^* \in S$ such that $s^*r = 0$. Clearly, if $[\frac{r}{s}] = [\frac{0}{1}]$, then for any $s' \in S$ $[\frac{r}{s'}] = [\frac{0}{1}]$.

Lemma 1. *If $[\frac{r}{s}] \in Z(R_P)$, then for any $\bar{s} \in R\backslash P$, $[\frac{r}{\bar{s}}] \in Z(R_P)$.*

Proof. Assume $[\frac{r}{s}] \in Z(R_P)$. Then there exists $[\frac{r'}{s'}] \neq [\frac{0}{1}]$ such that $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$. So there exists $s^* \in R\backslash P$ such that $s^*rr' = 0$. Let $[\frac{r}{\bar{s}}] \in R_P$. Then since $s^*rr' = 0$, $[\frac{r}{\bar{s}}][\frac{r'}{s'}] = [\frac{0}{1}]$. Thus, $[\frac{r}{\bar{s}}] \in Z(R_P)$. □

Define the numerator function $n : R_S \to \mathcal{P}(R)$ by $n([\frac{r}{s}]) = \{r' \in R \mid [\frac{r}{s}] = [\frac{r'}{s'}]\}$.

Lemma 2. *If $[\frac{r}{s}] \in Z(R_P)$, then for every $\hat{r} \in n([\frac{r}{s}])$, $\hat{r} \in P \cap Z(R)$.*

Proof. Suppose $[\frac{r}{s}] \in Z(R_P)$. Let $[\frac{r'}{s'}] \neq [\frac{0}{1}]$ such that $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$. By definition, there exists $s^* \in R\backslash P$ such that $s^*rr' = 0$, so $r \in Z(R)$. We know for every $\hat{r} \in n([\frac{r}{s}])$, $[\frac{r}{s}] = [\frac{\hat{r}}{\hat{s}}]$ for some $\hat{s}$. So, suppose $\hat{r} \notin P$. Then we know that $s^*\hat{r} \notin P$, since P is a prime ideal. If $\bar{s} = s^*\hat{r}$, then $\bar{s}r' = 0$, which implies that $[\frac{r'}{s'}] = [\frac{0}{1}]$, a contradiction. Thus $\hat{r} \in P$. □

Definition 3. *The total quotient ring of R, denoted $T(R)$, is the localization R_S where $S = R\backslash Z(R)$.*

The following lemmas present some relations between elements in $T(R)$ and elements in R_P.

Lemma 4. *Assume $Z(R) \subseteq P$. Then the following are equivalent.*
i) $[\frac{r}{1}][\frac{r'}{1}] = [\frac{0}{1}]$ in $T(R)$.
ii) $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in $T(R)$ for all $s, s' \in R\backslash Z(R)$.
iii) $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in R_P for all $s, s' \in R\backslash P$.
iv) $rr' = 0$.

Proof. $(i \Rightarrow ii)$ Assume $[\frac{r}{1}][\frac{r'}{1}] = [\frac{0}{1}]$ in $T(R)$. Then there exists $s^* \in R\backslash Z(R)$ such that $s^*rr' = 0$. But $s^* \notin Z(R)$, so $rr' = 0$ which implies $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in $T(R)$ for any $s, s' \in R\backslash Z(R)$.

$(ii \Rightarrow iii)$ Assume $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in $T(R)$ for all $s, s' \in R\backslash Z(R)$. Then there exists $s^* \in R\backslash Z(R)$ such that $s^*rr' = 0$. Since $s^* \notin Z(R)$, we know $rr' = 0$. Therefore, $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in R_P for all $s, s' \in R\backslash P$.

$(iii \Rightarrow iv)$ Assume $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in R_P. Then there exists $s^* \in R\backslash P$ such that $s^*rr' = 0$. We know $s^* \notin Z(R)$ since $Z(R) \subseteq P$, so $rr' = 0$.

$(iv \Rightarrow i)$ Clear. □

The following lemma shows that a pair of equivalence classes that are equal in R_P are also equal in $T(R)$, under a particular condition.

Lemma 5. *Assume $Z(R) \subseteq P$. Then $[\frac{r}{s}] = [\frac{r'}{s'}]$ in R_P if and only if $[\frac{r}{s}] = [\frac{r'}{s'}]$ in $T(R)$.*

Proof. $(\Rightarrow)$ Assume $[\frac{r}{s}] = [\frac{r'}{s'}]$ in R_P. Then there exists $s^* \in R\backslash P$ such that $s^*(rs' - r's) = 0$. Since $s^* \notin Z(R)$, we know $rs' - r's = 0$. Hence, $[\frac{r}{s}] = [\frac{r'}{s'}]$ in $T(R)$.

$(\Leftarrow)$ Assume $[\frac{r}{s}] = [\frac{r'}{s'}]$ in $T(R)$. Then there exists $s^* \in R\backslash Z(R)$ such that $s^*(rs' - r's) = 0$. Since $s^* \notin Z(R)$, we know $rs' - r's = 0$. Thus, $[\frac{r}{s}] = [\frac{r'}{s'}]$ in R_P. □

The next lemma establishes an expected result between zero-divisors of R and their respective canonical fractions.

Lemma 6. *Assume $Z(R) \subseteq P$. Let $r, r' \in Z(R)^*$. Then, $r = r'$ if and only if $[\frac{r}{1}] = [\frac{r'}{1}]$ in R_P.*

Proof. $(\Leftarrow)$ Assume $[\frac{r}{1}] = [\frac{r'}{1}]$ in R_P. Then, there exists $s^* \in R\backslash P$ such that $s^*(r - r') = 0$. Since $s^* \notin Z(R)$, we know $r - r' = 0$, which implies $r = r'$.

$(\Rightarrow)$ Trivial. □

It would be convenient to be able to create a homomorphism between R and $T(R)$, and the next lemma establishes that under certain circumstances, the obvious candidate for a homomorphism will suffice.

Lemma 7. *If $R = Z(R) \cup U(R)$, then for every $[\frac{r}{s}] \in T(R)$, there exists $r' \in R$ such that $[\frac{r}{s}] = [\frac{r'}{1}]$.*

Proof. Consider $r' = s^{-1}r$. Then, for all $s^* \in R\backslash Z(R)$, we have $s^*(r - sr') = s^*(r - ss^{-1}r) = 0$, which implies $[\frac{r}{s}] = [\frac{r'}{1}]$. □

A generalized version of the following theorem was presented and proved as Theorem 2.2 in [3]. However, we will now present a different proof below.

Theorem 8. *If R is a commutative ring with identity such that $R = Z(R) \cup U(R)$, then $R \cong T(R)$.*

Proof. Consider the function $\phi : R \to T(R)$ defined by $\phi(r) = [\frac{r}{1}]$. Consider $[\frac{r}{1}], [\frac{r'}{1}] \in T(R)$ such that $[\frac{r}{1}] = [\frac{r'}{1}]$. Then, there exists $s^* \in R\backslash Z(R)$ such that $s^*(r - r') = 0$. But $s^* \notin Z(R)$, so $r - r' = 0$ implies $r = r'$, thus ϕ is injective. Consider any $[\frac{r}{s}] \in T(R)$. By Lemma 7, we know that $[\frac{r}{s}] = [\frac{r'}{1}]$ for some $r' \in R$. Thus, $\phi(r') = [\frac{r'}{1}] = [\frac{r}{s}]$, so ϕ is surjective. Let $r, r' \in R$. Then, $\phi(r+r') = [\frac{r+r'}{1}] = [\frac{r}{1}] + [\frac{r'}{1}] = \phi(r) + \phi(r')$ and $\phi(rr') = [\frac{rr'}{1}] = [\frac{r}{1}][\frac{r'}{1}] = \phi(r)\phi(r')$, so ϕ is an operation preserving function. Thus, $R \cong T(R)$. □

Note that since $R \cong T(R)$, it follows trivially that $\Gamma(R) \cong \Gamma(T(R))$ under the above conditions.

3. Localizations of $\mathbb{Z}_n$

Consider $\mathbb{Z}_n$, where $n{=}{p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$ and each p_i is prime. Since $\mathbb{Z}_n$ is a principal ideal ring, every ideal is generated by a single element. Notice that prime ideals in $\mathbb{Z}_n$ will be generated by prime factors of n. Thus, we will only be concerned only with localizations of $\mathbb{Z}_n$ around ideals of the form (p_i), where $1 \leq i \leq k$.

Lemma 9. *If* $[\frac{q{p_i}^{e_i}+r}{1}] \in \mathbb{Z}_{n(p_i)}$*, then* $[\frac{q{p_i}^{e_i}+r}{1}] = [\frac{r}{1}]$.

Proof. All elements in $\mathbb{Z}_n$ can be written as $q{p_i}^{e_i} + r$ for some $0 \leq r < {p_i}^{e_i}$ by the division algorithm. Thus $[\frac{q{p_i}^{e_i}+r}{1}] = [\frac{r}{1}]$, since if $s^* \in \mathbb{Z}_n \backslash (p_i)$ such that $s^* = {p_1}^{e_1}...{p_{i-1}}^{e_{i-1}}{p_{i+1}}^{e_{i+1}}...{p_k}^{e_k}$, then $s^* q{p_i}^{e_i} = 0$ in $\mathbb{Z}_n$. $\square$

The following lemma is in a similar spirit to Lemma 7. It establishes that we can use the canonical mapping between $\mathbb{Z}_n$ and prime ideal localizations of $\mathbb{Z}_n$ as the basis of an isomorphism.

Lemma 10. *If* $[\frac{r}{s}] \in \mathbb{Z}_{n(p_i)}$*, then* $[\frac{r}{s}] = [\frac{r'}{1}]$ *in* $\mathbb{Z}_{n(p_i)}$ *for some* $r' \in \mathbb{Z}_n$*.*

Proof. Let $[\frac{r}{s}] \in \mathbb{Z}_{n(p_i)}$. Since $s \in \mathbb{Z}_n \backslash (p_i)$, $\gcd(p_i^{e_i}, s) = 1$, and thus $({p_i}^{e_i}) \cong \mathbb{Z}_s$. So, $r \equiv m{p_i}^{e_i} (\text{mod } s)$ for some m. Therefore, there exists $r' = m'{p_i}^{e_i}$ for some $r', m' \in \mathbb{Z}_n$ such that $sr' \equiv r - mp_i^{e_i} (\text{mod } s)$. Hence $m{p_i}^{e_i} \equiv r - sr' (\text{mod } s)$. Then, there exists $\bar{s} \in \mathbb{Z}_n \backslash (p_i)$, namely $\bar{s} = {p_1}^{e_1}...{p_{i-1}}^{e_{i-1}}{p_{i+1}}^{e_{i+1}}...{p_k}^{e_k}$, such that $\bar{s}(r - sr') = \bar{s}(mp_i^{e_i} - sm'p_i^{e_i}) = 0$ in $\mathbb{Z}_n$ which, by definition, implies $[\frac{r}{s}] = [\frac{r'}{1}]$. $\square$

In general, it is a difficult task to determine a ring that is isomorphic to a given localization. The following theorem yields a very simple isomorphism for localizations around prime ideals of modular rings.

Theorem 11. *Let* $n{=}{p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$*. Then* $\mathbb{Z}_{n(p_i)} \cong \mathbb{Z}_{{p_i}^{e_i}}$*.*

Proof. Consider $\phi : \mathbb{Z}_{{p_i}^{e_i}} \to \mathbb{Z}_{n(p_i)}$ defined by $\phi(r) = [\frac{r}{1}]$. Assume $\phi(r) = \phi(\bar{r})$. Then $[\frac{r}{1}] = [\frac{\bar{r}}{1}]$. By definition, there exists an $s^* \in \mathbb{Z}_n \backslash (p_i)$ such that $s^*(r - \bar{r}) = 0$. Thus, $p_i^{e_i}$ must divide $r - \bar{r}$, which is impossible unless $r = \bar{r}$. So, ϕ is injective.

Let $[\frac{r}{s}] \in \mathbb{Z}_{n(p_i)}$. Then by the previous lemma, we know $[\frac{r}{s}] = [\frac{r'}{1}]$ for some $r' \in \mathbb{Z}_n$. So, $r' \equiv m \pmod{n}$. Then, since $n = p_1^{e_1} \dots p_k^{e_k}$, $r' \equiv m \pmod{{p_i}^{e_i}}$. Then, $\phi(r') = \phi(m) = [\frac{m}{1}]$, and since $r' \equiv m \pmod{n}$, this implies $[\frac{m}{1}] = [\frac{r'}{1}] = [\frac{r}{s}]$, and hence ϕ is surjective.

Take $r, \bar{r} \in \mathbb{Z}_{{p_i}^{e_i}}$. Then, $\phi(r + \bar{r}) = [\frac{r+\bar{r}}{1}] = [\frac{r}{1}] + [\frac{\bar{r}}{1}] = \phi(r) + \phi(\bar{r})$. Similarly, $\phi(r\bar{r}) = [\frac{r\bar{r}}{1}] = [\frac{r}{1}][\frac{\bar{r}}{1}] = \phi(r)\phi(\bar{r})$. $\square$

Corollary 12. *Let* $n = {p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$*. Then* $\Gamma(\mathbb{Z}_{n(p_i)}) \cong \Gamma(\mathbb{Z}_{{p_i}^{e_i}})$*.*

The isomorphism established above inspires a question about whether localizations around non-prime ideals have other nice isomorphisms. It also hints at the possibility of an unexplored generalization of the theorem to localizations of direct product decompositions, since a modular ring can be decomposed into a direct product of powers of primes.

4. Classification of $\Gamma(\mathbb{Z}_n)$

Theorem 13. *The following table holds true:*

Factorization of n	*Diameter*	*Girth*
p; p is prime	-	-
2^2	*0*	∞
3^2	*1*	∞
p^2; p is prime and $p > 3$	*1*	*3*
2^3, or $2p$; p odd prime	*2*	∞
pq; p,q, distinct odd primes	*2*	*4*
p^m; p is prime, $m > 2$, and $p^m \neq 8$	*2*	*3*
$4p$; p is an odd prime	*3*	*4*
pqk; p, q distinct primes, $k \in \mathbb{Z}^+$ and pqk does not meet any criteria listed above	*3*	*3*

Proof. Let $n = p$ where p is prime, then $\Gamma(\mathbb{Z}_n) = \emptyset$, since $\mathbb{Z}_p$ is a field.

Let $n = 2^2$, then $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 0$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = \infty$ by observation.

Let $n = 3^2$, then $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 1$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = \infty$ by observation.

Let $n = p^2$ where p is prime and $p > 3$. Then all zero-divisors of $\mathbb{Z}_{p^2}$ are multiples of p. Consider the zero-divisors m_1p, m_2p, and m_3p. Then, there is a 3-cycle m_1p — m_2p — m_3p — m_1p and it is clear that all zero-divisors are attached to each other, so $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 1$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = 3$.

Let $n = 2^3$, then $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = \infty$ by observation.

Let $n = 2p$ where p is an odd prime. Thus, $\mathbb{Z}_n \cong \mathbb{Z}_2 \times \mathbb{Z}_p$. By Theorem 2.5 in [2], we know that $\mathbb{Z}_n$ is a star graph. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = \infty$.

Let $n = pq$ where p and q are distinct odd primes. Then clearly, $\Gamma(\mathbb{Z}_n)$ is complete bipartite since $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are fields, so $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = 4$.

Let $n = p^m$ where p is prime, $m > 2$, and $p^m \neq 8$. Then, since multiples of p are zero-divisors, consider m_1p and m_2p, any two arbitrary multiples of p. There will always be a 2-path m_1p — p^{m-1} — m_2p and a 3-cycle m_1p^{m-1} — p — m_2p^{m-1} — m_1p^{m-1}. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = 3$.

Let $n = 4p$. Then $Z(\mathbb{Z}_n) = \{p^\ell, 2p^\ell, 3p^\ell, 2\cdot 2, 2\cdot 3, \ldots, 2\cdot(p-1), 2\cdot(p+1), \ldots, 2\cdot(n-1)\}$ for all $\ell \in \mathbb{N}$. Consider 2, $2p^{\ell_1}$, $p^{\ell_2}m_1$, and $2m_2$ where $m_1 \in \{1,3\}$, m_2 is an even element, and $\ell_i \in \mathbb{N}$. There is a shortest 3-path, namely $p^{\ell_1}m_1$ — $2m_2$ — $2p^{\ell_2}$ — 2, so $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 3$. Now consider $2m_3$ where m_3 is a distinct even element such that $m_3 \neq m_2$. There is a 4-cycle $p^{\ell_1}m_1$ — $2m_2$ — $2p^{\ell_2}$ — $2m_3$ — $p^{\ell_1}m_1$. Since the multiples of p must connect to some multiple of 2, and all multiples of 2 must connect to a multiple of p, any cycle in $\Gamma(\mathbb{Z}_n)$ must have an even number of edges. So, since we have a exhibited a 4-cycle and the girth cannot be 3, $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = 4$.

Let $n = pqk$; p, q distinct primes, $k \in \mathbb{Z}^+$ and pqk does not meet any criteria listed above. Then, p — qk — pk — q is a shortest 3-path, and qk — pq — pk is a 3-cycle. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 3$ and $\operatorname{g}(\Gamma(\mathbb{Z}_n)) = 3$. □

5. Conclusions and Acknowledgments

Localizations provide a valuable way to extend rings in general commutative ring theory. Looking at the implications to zero-divisor graphs may provide a deeper understanding of the structure of zero-divisor graphs and, in turn, of the zero-divisors themselves. The classification of modular rings is a step that we hope will lead to a more complete classification of zero-divisor graphs, relying less on number theoric devices and more on broad algebraic properties.

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