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Natural Families of Triangles I: Parametrizing Triangle Space

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Natural Families of Triangle I: Parametrizing Triangle Space

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Abstract

We group triangles into families based on three parameters: the distance between the circumcenter O and the centroid G, the circumradius, and the measure of angle $\angle GOA$ where A is one vertex. Using these parameters, we present triangle space, a subset of $\mathbb{R}^3$ in which every triangle is represented by exactly one point.

1 Introduction

Modern geometry software, e.g., *Cabri, Cinderella*, or *Geometer's Sketchpad*, enables the user to interact dynamically with geometric constructions. Given, for example, a triangle and some construction or constructions dependent on it, say, one or more triangle centers, it is possible to vary the construction continuously and observe the evolution of the dependent constructions. It is natural to conceive of the resulting images as illustrating motion taking place in the space of all triangles and, then, even more natural to wonder how one might parametrize the motion. In order to do so, one must first parametrize "triangle space," the collection of all triangles. Many, seemingly natural, parametrizations present themselves. In this paper we describe one such parametrization that seems especially felicitous; a triangle is described by its circumradius, r, the distance between its circumcenter and centroid, g, and the minimal angle formed by the centroid, circumcenter, and one vertex, θ. This choice, based as it is on the Euler line, seems more intrinsic and potentially interesting than say, side-lengths. Similarity is especially nice in these coordinates; each flat slice $r = c$ through triangle space contains precisely one representative of each similarity class. The choice is also well adapted to dynamic geometry software.

This paper describes this parametrization and the shape of the resulting triangle space. We also introduce a metric that puts a natural topology on the space. Fixing two coordinates and varying the third organizes the space into natural fibers (or families). We explore some of the uses of this parametrization in "Natural Families of Triangles II: A Locus of Symmedian Points" [Carr2].

2 Natural Families of Triangles

Let us begin by returning to the ideas of grouping triangles into natural families. Undoubtedly, there are many ways to associate triangles with each other. However, in an effort to ensure that our triangles share certain intrinsic features, let us base that grouping off of one special line of a triangle—its Euler line.

*We would like to thank our advisor Stephen Kennedy of the Carleton College Mathematics Department for all of his help and support.

Definition 1. The Euler line of a triangle is the line through its centroid and circumcenter.

The Euler line of a given triangle contains a great number of the triangle's special points. For example, the orthocenter, the nine-point center, and the DeLongchamps point all lie along it. Another perhaps surprising feature is that the ratio of distances between certain points on the Euler line remains constant. Three such points are the circumcenter, the centroid, and the orthocenter. If we call the circumcenter O, the centroid G, and the orthocenter H, then, $OH = 3OG$ [Kimberling].

We will use the above properties of the Euler line and to create our families of triangles. Specifically, we will construct a triangle given its circumcenter, its centroid, and one vertex. These three pieces of geometric information will be the basis of our groupings. Before we begin, however, we must state a proposition we will use in the construction. We have not included a proof of it here, but the fact is well-known.

Proposition 1. *Given $\triangle ABC$ and its centroid G, let M_a, M_b, and M_c be the midpoints of segments $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Then, $AG = 2GM_a$, $BG = 2GM_b$, and $CG = 2GM_c$.*

2.1 Triangle Construction

Let us now construct a triangle given its circumcenter, its centroid, and one vertex.

Let points O and G be given, and let l be the line connecting them. Choose a point A not on l and create circle c centered at O with radius OA. O will be the circumcenter of our triangle and A one of its vertices. Note that the other two vertices of our triangle will also lie on circle c. Let M and N be the intersections of l with c. Now, choose a point $G \neq O$ on l such that G lies between M and N. G will be the centroid of our triangle. Construct point H on $\overrightarrow{OG}$ such that $3OG = OH$. By the observation above, H is the orthocenter of the triangle. Now, construct lines $\overline{AG}$ and $\overline{AH}$. Since H is the orthocenter of our triangle, $\overline{AH}$ will be the altitude from vertex A and thus perpendicular to side $\overline{BC}$. By Proposition 1, the midpoint M_a of $\overline{BC}$ will lie on $\overrightarrow{AG}$ such that $AM_a = \frac{3}{2}AG$. Now, construct the line m through M_a such that $m \perp \overline{AH}$. By construction, m must be side $\overline{BC}$ of $\triangle ABC$. Thus, B and C will be the intersections of m with circle c. Therefore, we have constructed $\triangle ABC$ from a circumcenter, a centroid, and one vertex.

With further exploration into this construction, we find that when G lies on or past the point one-third of the way from O to N, then certain angles $\angle GOA$ will not produce a triangle. This arises from the fact that in this case, the midpoint of the side opposite vertex A will lie on or outside the disk enclosed by the given circumcircle. When this occurs, the construction of a triangle is impossible. Keep this in mind, because we will return to this idea later in our discussion of triangle space.

Returning to our construction, we see that the three pieces of geometric information above correspond to the distance between the circumcenter and the centroid (which we will call g), the distance between the circumcenter and a vertex, (which we will call r), and the angle $\angle GOA$ (which we will call θ). As we can see in the construction, by fixing any two of these parameters and keeping the third constant, we create a family of triangles.

2.2 Special Cases of the Construction

At this point, it is natural to ask where special triangles such as isosceles and right triangles occur in the construction. Perhaps not so surprisingly, they always fall at specific locations. For example, isosceles triangles occur when one vertex of the triangle falls on the Euler line. Even more interestingly, all of the right triangles occur when $r = 3g$. Before we can show this, we need the next proposition which we will not prove here. The result follows from a proof by contradiction.

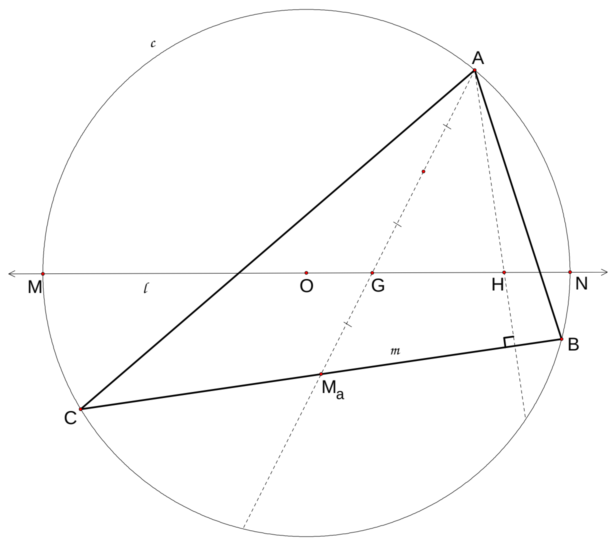

Figure 1: The elements of the triangle construction.

Proposition 2. *Let $\triangle ABC$ with orthocenter H be given. If H lies on the circumcircle of $\triangle ABC$, then H is coincident with one vertex of $\triangle ABC$.*

With Proposition 2, we can quite easily prove the following corollary:

Corollary 1. *Let $\triangle ABC$ with orthocenter H be given. Then, H is coincident with vertex B if and only if $m\angle ABC = \frac{\pi}{2}$.*

Proof. ($\Rightarrow$) Assume that H is coincident with vertex B. Since the altitude from vertex A to $\overline{BC}$ passes through H and intersects $\overline{BC}$ at a right angle, we see that $m\angle ABC = \frac{\pi}{2}$.

($\Leftarrow$) Assume that $m\angle ABC = \frac{\pi}{2}$. Then, $\overline{BC}$ is the altitude from vertex C to $\overline{AB}$, and $\overline{AB}$ is the altitude from vertex A to $\overline{BC}$. Since the altitudes of a triangle coincide at H, it follows that B and H are coincident. □

Now, we can finally show that all right triangles (up to scaling) occur in a single $\theta-$ family.

Proposition 3. *Triangle $\triangle ABC$ is a right triangle if and only if $g = \frac{r}{3}$.*

Proof. ($\Leftarrow$) If $g = \frac{r}{3}$, then $OH = r$. That is, H lies on the circumcircle of $\triangle ABC$. Therefore, by Proposition 2 and Corollary 1, $\triangle ABC$ is a right triangle.

($\Rightarrow$) If $\triangle ABC$ is a right triangle, then by Corollary 1, H is coincident with one vertex. Without loss of generality, let that vertex be vertex B. Thus, H is on the circumcircle. Consequently, $OH = r$. Since G is one-third of the way from O to H, it follows that $g = \frac{r}{3}$. □

Now that we have seen some special cases of our construction, let us now turn to finding the bounds of our parameters.

3 Triangle Space

As mentioned in the introduction, one of our goals is to figure out what triangle space looks like. One could argue that this can be done by considering triangles in the traditional sense: defining a triangle by the lengths of its sides. In either case, you are given three numbers, and those three numbers correspond to a point in $\mathbb{R}^3$. This point corresponds to the triangle defined by the three numbers. The problem with considering triangles in the traditional sense is that every triangle is corresponds to six different points (one for each permutation of the three sides). It is not a trivial problem to define the space in such a way that every triangle is represented only once. We will show that every triangle (by our definition) is represented by exactly one ordered triple; note that not every point in $\mathbb{R}^3$ represents a triangle. Now we shall set out on the quest of demonstrating exactly which values of θ, g, and r give us legitimate triangles.

To help us reach our goal, let us first examine similar triangles.

3.1 Similar Triangles

Theorem 1. *Two triangles $\triangle ABC$ defined by (θ, g, r) and $\triangle A'B'C'$ defined by (θ', g', r') are similar if and only if $\theta = \theta'$, $g = kg'$, and $r = kr'$ for $k > 0$.*

Proof. ($\Leftarrow$) Let $\triangle ABC$ be similar to $\triangle A'B'C'$.

Let G and G' be the centroids of and O and O' the circumcenters of $\triangle ABC$ and $\triangle A'B'C'$, respectively. Also, let M_a and $M_a{}'$ be the midpoints of segments $\overline{BC}$ and $\overline{B'C'}$, respectively. Let M_b and $M_b{}'$ be the midpoints of $\overline{AC}$ and $\overline{A'C'}$, respectively. Since the two triangles are similar, it follows that $\frac{AB}{A'B'} = \frac{BM_a}{B'M_a{}'} = k$. Therefore, $\triangle ABM_a \sim \triangle A'B'M_a{}'$. So, $\frac{AM_a}{A'M_a{}'} = k$. Now, G and G' must lie on $\overline{AM_a}$ and $\overline{A'M_a{}'}$,respectively. Moreover, $GM_a = \frac{1}{3}AM_a$ and $G'M_a{}' = \frac{1}{3}A'M_a{}'$. So, $\frac{GM_a}{G'M_a{}'} = \frac{AM_a}{A'M_a{}'}$. Also, $\frac{BC}{B'C'} = \frac{M_bC}{M_b{}'C'} = k$. So, $\triangle BCM_b \sim \triangle B'C'M_b{}'$. So, $\frac{BM_b}{B'M_b{}'} = k$. But, $GM_b = \frac{1}{3}BM_b$ and $G'M_b{}' = \frac{1}{3}B'M_b{}'$. Thus, $\frac{GM_b}{G'M_b{}'} = k$. This implies that $\triangle BM_bM_a \sim \triangle G'M_b{}'M_a{}'$ by SSS similarity. Thus, $\angle GM_bM_a \cong \angle G'M_b{}'M_a{}'$. Now,

$$m\angle O'M_b{}'G' + m\angle G'M_b{}'M_a{}' + m\angle M_a{}'M_b{}'C' = \pi$$
$$m\angle GM_bM_a = m\angle G'M_b{}'M_a{}'$$
$$m\angle M_aM_bC = m\angle BM_bC - m\angle GM_bM_a = m\angle B'M_b{}'C' - m\angle G'M_b{}'M_a{}' = m\angle M_a{}'M_b{}'C'$$
$$m\angle OM_bG = m\angle O'M_b{}'G'$$

Thus, by SAS similarity, $\triangle OM_bG \cong \triangle O'M_b{}'G'$. This implies that $\frac{OG}{O'G'} = \frac{GM_b}{G'M_b{}'} = k$. So, $OG = kO'G'$. But, $O'G' = g'$ and $OG = g$. Thus, $g = kg'$.

Now, we know that $\frac{M_bC}{M_b{}'C'} = \frac{M_aC}{M_a{}'C'} = k$. So, $\triangle M_bM_aC \sim \triangle M_b{}'C'M_a{}'$ by SAS similarity. Therefore, $\angle M_b{}'M_a{}'C' \cong \angle M_bM_aC$, $\angle M_aM_bC \cong \angle M_a{}'M_b{}'C'$, and $\frac{MM_a}{M'M_a{}'}$. Thus, by subtraction, $\angle OM_aM \cong \angle O'M_a{}'M_b{}'$ and $\angle OM_bM_a \cong \angle O'M_b{}'M_a{}'$. This implies that $\triangle OM_bM_a \sim \triangle O'M_b{}'M_a{}'$. So, $\frac{OM_b}{O'M_b{}'} = \frac{M_bM_a}{M_b{}'M_a{}'} = k$. Thus, $\triangle OM_bC \sim \triangle O'M_b{}'C'$. Therefore, $\frac{OC}{O'C'} = k$. But, $OC = r$ and $O'C' = r'$. So we know $r = kr'$.

Now, construct lines $\overline{OG}$, $\overline{O'G'}$, and segments $\overline{OC}$ and $\overline{O'C'}$. We have shown that the segments connecting the centroids to the vertices are proportional by k, and $OG = kO'G'$. Thus, $\triangle OGC \sim \triangle O'G'C'$ by SSS similarity. This implies that $\angle GOC \cong \angle G'O'C'$. So, since $\angle GOC = \theta$ and $\angle G'O'C' = \theta'$, $\theta = \theta'$.

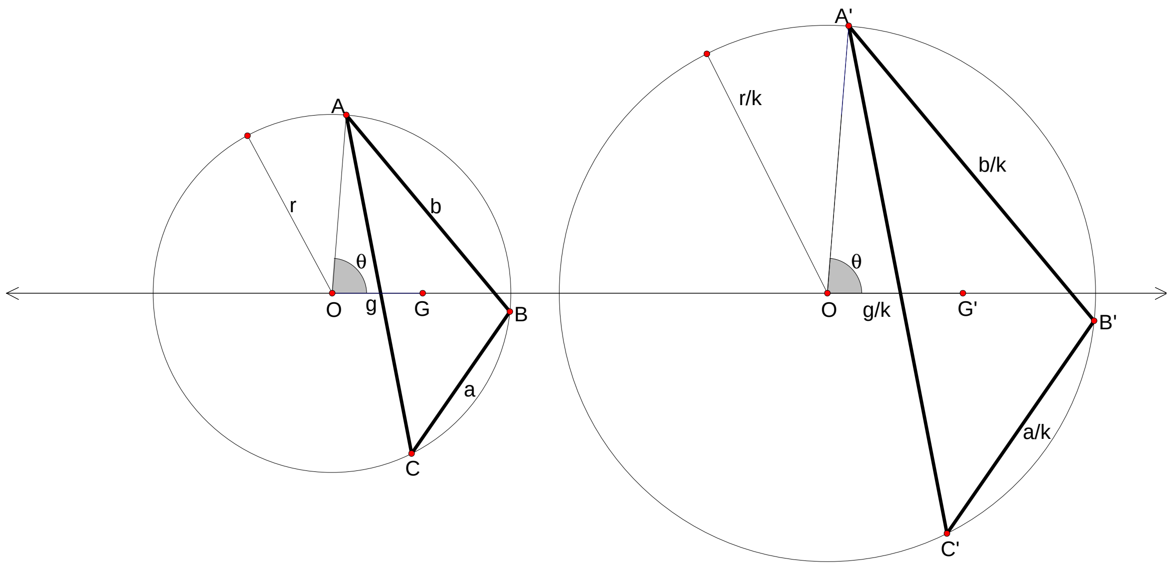

Figure 2: Angle θ is the same in both triangles.

($\Rightarrow$) Let $\triangle ABC$ defined by (θ, g, r) and $\triangle A'B'C'$ defined by (θ', g', r') be two triangles with the property that $r = kr'$, $g = kg'$, and $\theta = \theta'$.

First, let M_a be the midpoint of $\overline{CB}$. Now, we can see that $\triangle AOG \sim \triangle A'O'G'$ by SAS similarity. Thus, $\frac{AG}{A'G'} = k$. So,

$$\frac{AM_a}{A'M_a'} = \frac{\frac{3}{2}AG}{\frac{3}{2}A'G'} = \frac{AG}{A'G'} = k$$

Then, $\frac{GM_a}{G'M_a'} = k$ and $GM_a = kG'M_a'$. Also, $\angle AGO \cong \angle A'G'O'$ and thus $\angle OGM_a \cong \angle O'G'M_a'$. Therefore, $\triangle OGM_a \sim \triangle O'G'M_a'$ by SAS similarity. So, $\frac{OM_a}{O'M_a'} = k$ and $OM_a = kO'M_a'$. Also, $\angle OM_aG \cong \angle O'M_a'G'$. Because O and O' lie on the perpendicular bisectors to sides $\overline{BC}$ and $\overline{B'C'}$ respectively, we know,

$$\begin{aligned} CM_a^2 &= (kr)^2 - OM_a^2 \\ (C'M_a')^2 &= r^2 - (O'M_a')^2 \\ CM_a^2 &= (kr)^2 - (kO'M_a')^2 \\ CM_a^2 &= k^2(r^2 - (O'M_a')^2) \\ CM_a^2 &= k^2(C'M_a')^2 \\ CM_a &= kC'M_a' \end{aligned}$$

Also, $BC = 2CM_a = 2kC'M_a' = kB'C'$. Thus, $\triangle ACM_a \sim \triangle A'C'M_a{}'$ by SAS similarity. Thus, $\angle ACM_a \cong \angle A'C'M_a{}'$ and $\triangle ABC \sim \triangle A'B'C'$ by SAS similarity.

□

3.2 Bounding the Parameters

Let us now begin looking at the bounds on θ, g, and r.

Proposition 4. *If (θ, g, r) represents a triangle, then $r > 0$.*

Proof. The only acceptable values of r are $r > 0$ because $r = 0$ would mean that the circumcircle is only a point, which does not allow for any triangles. Moreover, $r < 0$ is absurd. Also, if r is bounded above, then we restrict ourselves to triangles with side lengths less than $2r$, so r can be any positive number. □

That being said, Theorem 1 implies that r is just a scaling variable. Therefore we can get a clear picture of what triangle space looks like by examining cross-sections of $\mathbb{R}^3$ parallel to the θg-plane.

Proposition 5. *If* (θ, g, r) *represents a triangle, then* $0 \le g < r\dfrac{\cos[\frac{1}{2}(\theta + \sin^{-1}(\frac{1}{2}\sin\theta))]}{\cos[\frac{1}{2}(\theta - \sin^{-1}(\frac{1}{2}\sin\theta))]}$

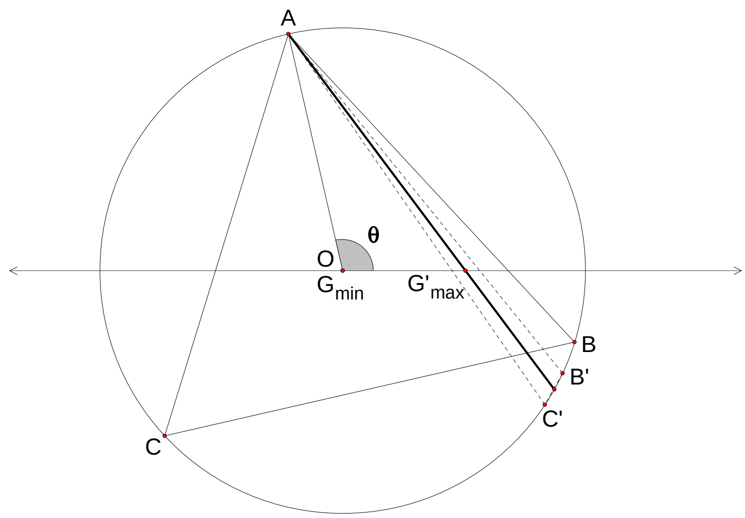

Figure 3: Note that vertices B' and C' are coincident.

Proof. As motivation for this proof, imagine being given a triangle inscribed in a circle. Take the centroid G of the triangle, which lies on the Euler line, and start moving it away from the circumcenter toward the closest edge of the circle. Since G is the intersection of the lines connecting the vertices of the triangle with the midpoints of the opposite sides, it follows that as G approaches the edge of the circle, two vertices of the triangle (say B and C) begin to come closer together. This is because G is approaching one side of the triangle (say $\overline{AB}$), and as it gets closer and closer to that side, the line connecting vertex A and the midpoint of $\overline{BC}$, M_a moves closer and closer to the edge of the circumcircle. Thus, when G reaches side $\overline{AB}$, $\overline{BC}$ and consequently $\triangle ABC$ disappears. It seems that this point is the limit for OG or g. Therefore, we first identify a way to calculate this limit.

Let M_a again be the midpoint of the side opposite vertex A. We want to find the value of g so that M_a lies on the the circumcircle. Now, consider $\triangle OAM_a$. Let A be the vertex of $\triangle ABC$ inscribed in the circle and M_a be the point on ray $\overrightarrow{AG}$ such that $AG = 2GM_a$. Let $y = GM_a$ and $g = OG$.

Then, since $\overline{OM_a}$ and $\overline{OA}$ are both radii of the circle, it follows that $\triangle OAM$ is isosceles. Thus, $\angle OAM_a \cong \angle OM_aA$. Let $\alpha = \angle M_aOG$ and $\angle GOA = \theta$. Now, by the Law of Sines, $\frac{\sin\theta}{2y} = \frac{\sin\angle OAM_a}{g}$ in $\triangle OAG$. Also, $\frac{\sin\alpha}{y} = \frac{\sin OM_aA}{g} = \frac{\sin\angle OAM_a}{g}$ in $\triangle M_aOG$. Therefore, $\frac{\sin\theta}{2y} = \frac{\sin\alpha}{y}$, which implies that $\sin\alpha = \frac{\sin\theta}{2}$. This in turn implies that $\alpha = \sin^{-1}\frac{1}{2}\sin\theta$. Now, $\frac{\pi-(\theta+\alpha)}{2} = \beta$. If we define $\gamma = \angle OGA$, then $\gamma = \pi - \theta - \beta = \pi - \theta - \frac{\pi}{2} + \frac{\theta}{2} + \frac{\alpha}{2} = \frac{\pi}{2} - \frac{\theta}{2} + \frac{\alpha}{2}$. Finally, again by the Law of Sines, $\dfrac{\sin\frac{\pi}{2} - \frac{\theta}{2} - \frac{\alpha}{2}}{x} = \dfrac{\sin\frac{\pi}{2} - \frac{\theta}{2} + \frac{\alpha}{2}}{r}$. $\therefore g = r\dfrac{\sin\frac{\pi}{2} - \frac{\theta}{2} - \frac{\alpha}{2}}{\sin\frac{\pi}{2} - \frac{\theta}{2} + \frac{\alpha}{2}}$. Replacing α with $\sin^{-1}\frac{1}{2}\sin\theta$ and realizing that $\sin(\frac{\pi}{2}-\phi)=\cos\phi$ provides the desired result,

$$g < r\frac{\cos[\frac{1}{2}(\theta+\sin^{-1}(\frac{1}{2}\sin\theta))]}{\cos[\frac{1}{2}(\theta-\sin^{-1}(\frac{1}{2}\sin\theta))]} .$$

Now that we have g bounded above, we consider the lower bound. Clearly we need to include $g=0$ because we need to include equilateral triangles. We have always looked at our triangles with G on the right side of O. If we look at the triangles we get with G on the left side of O we realize they are the same as the ones with G on the right side. They can be mapped to each other by reflecting across the line perpendicular to the Euler line at O. Thus,

$$0 \le g < r\frac{\cos[\frac{1}{2}(\theta+\sin^{-1}(\frac{1}{2}\sin\theta))]}{\cos[\frac{1}{2}(\theta-\sin^{-1}(\frac{1}{2}\sin\theta))]} .$$

□

The bounds for θ are a lot harder to sort out because they come in cases which depend on g For the following propositions, we will denote the θ value for an isosceles triangle with a vertex and G lying on the Euler line on the same side of O as θ_R. Also, we will denote the θ value for an isosceles triangle with a vertex and G lying on the Euler line on the opposite sides of O as θ_L.

Proposition 6. *If* $g<\frac{r}{3}$, *then* $\cos^{-1}(\frac{3g+r}{2r}) \le \theta \le \cos^{-1}(\frac{3g-r}{2r})$.

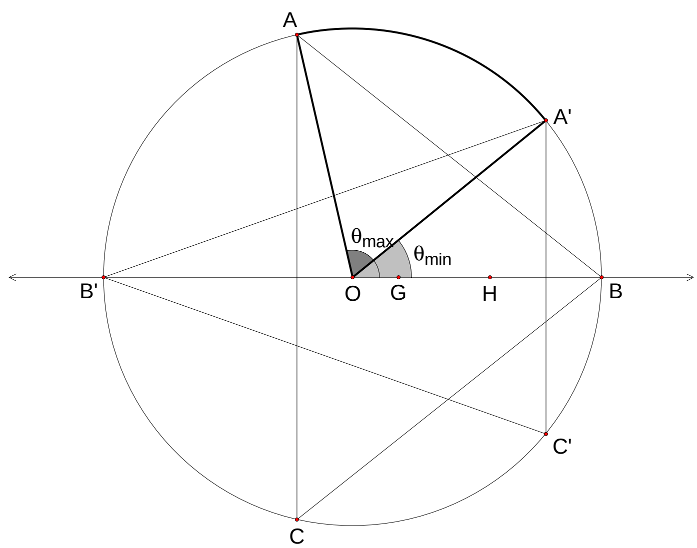

Figure 4: The bounds are created by an isosceles triangle on either end.

Proof. Since $g<\frac{r}{3}$, every triangle with a vertex on the Euler line is an isosceles triangle. Consider the isosceles triangle with vertex B on the Euler line on the same side of O that G is on. So, $\theta_R = m\angle GOA$. Let M_b be the midpoint of $\overline{AC}$. Since this is an isosceles triangle, M_b lies on the Euler line and $m\angle AM_bO=\frac{\pi}{2}$. Also, $r-g=BG=2(GM_b)$. Therefore, $OM=\frac{r-3g}{2r}$. We also know $m\angle AOM=\pi-\theta_R$. So $\cos(\pi-\theta_R)=-\cos\theta_R=\frac{r-3g}{2r}$. Therefore,

$$\theta_R=\cos^{-1}(\frac{3g-r}{2r}) .$$

A consequence of this isosceles triangle case is that $\angle GOA \cong \angle AOB$. Also, $\angle GOA \cong \angle GOC$, and both measure θ, so we can think of $\angle GOA$ as θ. By the law of cosines, we know that

$$\angle AOB = \cos^{-1}(\frac{2r^2-(AB)^2}{2r^2}) .$$

In the appendix of "A New Way to Think About Triangles" **??**, we prove that

$$AB = \sqrt{3r^2 - 3rg\cos\theta - 3rg\sin\theta\frac{\sqrt{3r^2 - 9g^2 + 6rg\cos\theta}}{\sqrt{r^2 + 9g^2 - 6rg\cos\theta}}}.$$

When $\theta = \cos^{-1}(\frac{3g-r}{2r})$, (after some simplification) we see

$$\frac{\partial(\theta - m\angle AOB)}{\partial\theta} = 1 - \frac{3g}{r}.$$

So, since $g < \frac{r}{3}$,

$$\frac{\partial(\theta - m\angle AOB)}{\partial\theta} > 0.$$

This means that as we increase $m\angle GOA$, the difference $m\angle GOA - m\angle BOA$ increases. So,

$$m\angle GOA - m\angle BOA > 0.$$

Therefore, A and B must be on the same side of the Euler line. This means that θ is made by $\angle GOC$, and

$$\angle GOC < \cos^{-1}(\frac{3g-r}{2r}),$$

otherwise C and B would be on the same side of the Euler line (and B cannot be on both sides of the Euler line). This proves that θ_R is the upper bound for θ when $g < \frac{r}{3}$.

Likewise, the lower bound for θ is found in the other isosceles triangle case. That is, when we have an isosceles triangle with a vertex (without loss of generality, C) on the opposite side of O from G. Let θ_L be made by $\angle GOA$. Again, let M_c be the midpoint of $\overline{AB}$. Since this is an isosceles triangle, M_c lies on the Euler line and $m\angle AM_cO = \frac{\pi}{2}$. Also, $g + r = CG = 2(GM_c)$. Therefore, $OM_c = \frac{3g+r}{2r}$. We also know $m\angle AOG = \theta_L$. So,

$$\theta_L = \cos^{-1}(\frac{3g+r}{2r}) .$$

The consequence of this isosceles triangle case is that $\angle GOA + \angle AOC = \pi$. Also, $\angle GOA \cong \angle GOB$, and both measure θ, so we can think of $\angle GOA$ as θ. By the law of cosines, we know that

$$\angle AOC = \cos^{-1}(\frac{2r^2 - (AC)^2}{2r^2}.$$

In the appendix of the aforementioned paper **??**, we also prove that

$$AC = \sqrt{3r^2 - 3rg\cos\theta + 3rg\sin\theta\frac{\sqrt{3r^2 - 9g^2 + 6rg\cos\theta}}{\sqrt{r^2 + 9g^2 - 6rg\cos\theta}}}.$$

When $\theta = \cos^{-1}(\frac{3g+r}{2r})$, (after some simplification) we see

$$-\frac{\partial(\theta + m\angle AOC - \pi)}{\partial\theta} = -(1 + \frac{3g}{r}) .$$

So,

$$-\frac{\partial(\theta + m\angle AOC - \pi)}{\partial\theta} < 0 .$$

This means that as we decrease $m\angle GOA$, the $m\angle GOA + m\angle BOA - \pi$ decreases. So,

$$m\angle GOA + m\angle COA < \pi.$$

Therefore, A and C must be on the same side of the Euler line. This means that θ is made by $\angle GOB$, and

$$\angle GOB > \cos^{-1}(\frac{3g+r}{2r}),$$

otherwise C and B would be on the same side of the Euler line (and C cannot be on both sides of the Euler line). This proves that θ_L is the lower bound for θ when $g < \frac{r}{3}$.

Combining the upper and lower bounds we get,

$$\cos^{-1}(\frac{3g+r}{2r}) \leq \theta \leq \cos^{-1}(\frac{3g-r}{2r}) .$$

□

Note that if we consider the $g = 0$ case, we have defined $\theta = \frac{\pi}{2}$ which falls between the bounds.

Proposition 7. *If* $g = \frac{r}{3}$, *then* $0 < \theta \leq \frac{\pi}{2}$.

Proof. To find the upper bound for θ when $g = \frac{r}{3}$ we again look at the isosceles triangle with vertex B on the same side of O as G. We know that in this case, O and M_b are the same point, and that $m\angle AOG = \frac{\pi}{2}$. We also know that A, O, and C are collinear, so if we increase $\angle AOG$ we necessarily decrease $\angle GOC$, and the two angles must sum to π. The right triangle case is the reason we had to add the requirement that θ be the smaller of the two angles if we have two "one-vertex sides." In this case, the largest value that the smaller angle can take is $\frac{\pi}{2}$. When we go to look at the isosceles triangle with vertex C on the opposite side of O from G, we realize that we cannot make this triangle because two vertices would have to fall on on the Euler line, which is impossible. So if we keep B at the right angle, we can move vertex C arbitrarily close to the Euler line on the opposite side of the circle from B. This means that we can make $m\angle GOC$ arbitrarily close to π, so $m\angle GOA$ can be made arbitrarily close to 0. So when $g = \frac{r}{3}$,

$$0 < \theta \leq \frac{\pi}{2}.$$

□

Proposition 8. *If* $g > \frac{r}{3}$, *then* $\cos^{-1}(\frac{3g-r}{2r}) \leq \theta \leq \cos^{-1}(\frac{3g^2-r^2}{2rg})$.

Proof. Again we consider the isosceles triangle with vertex B on the Euler line on the same side of O that G is on. So,

$$m\angle GOA = \theta_R = \cos^{-1}(\frac{3g-r}{2r}) .$$

As before we have $\angle GOA \cong \angle AOB$. Also, $\angle GOA \cong \angle GOC$, and both measure θ, so we can think of $\angle GOA$ as θ. Following the same procedure for the isosceles triangle with $\theta = \theta_R$ (from the $g < \frac{r}{3}$ case), we see,

$$\frac{\partial(\theta - m\angle AOB)}{\partial\theta} = 1 - \frac{3g}{r}.$$

But this time, since $g > \frac{r}{3}$,

$$\frac{\partial(\theta - m\angle AOB)}{\partial\theta} < 0.$$

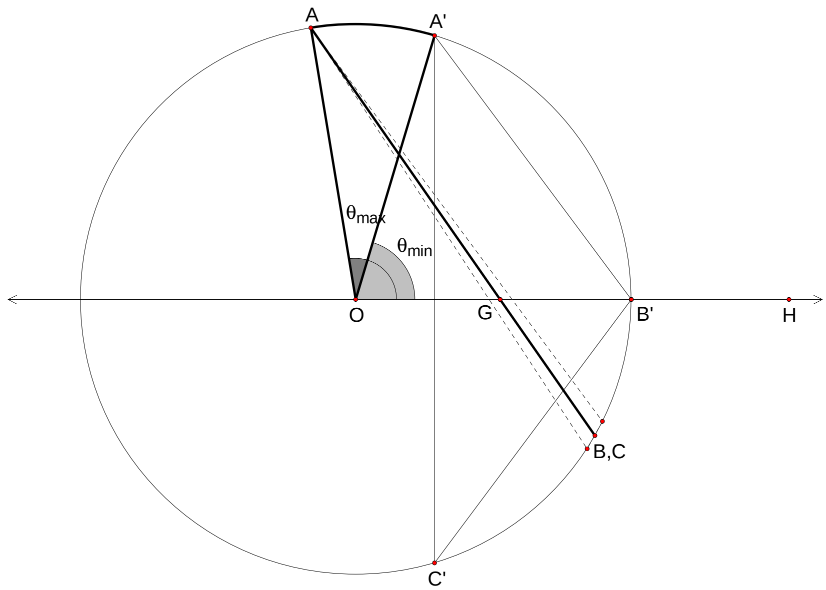

Figure 5: The bounds are created by an isosceles triangle and a line.

This means that as we increase $m\angle GOA$, the difference $m\angle GOA - m\angle BOA$ decreases! So,

$$m\angle GOA - m\angle BOA < 0.$$

Therefore, A and B must be on opposite sides of the Euler line. This proves that θ_R is the lower bound for θ when $g > \frac{r}{3}$.

When $g > \frac{r}{3}$ there is no other isosceles triangle to look at. This is because $BM_b = \frac{3}{2}(r+g) > \frac{3}{2}(\frac{4r}{3}) = 2r$, so the midpoint of the side opposite B and therefore the side itself is not contained in the circumcircle. Even though this means that our usual means of finding the other bound is not possible, it gives us another possibility. Now we set out to find where the triangle disappears. To do this, we find the angle that forces M_a to land on the circumcircle (also forcing A, M_A, and G to be collinear. Therefore, we are interested in the case where $OA = r = OM_a$. This means $\triangle AOM_a$ is an isosceles triangle, and $\angle OAM_a \cong \angle AM_aO$. If we define $l = \overline{AG}$, then $AM_a = \frac{3l}{2}$. By the law of cosines, $r^2 = r^2 + \frac{9l^2}{4} - 3rl\cos\angle AM_aO$. After some simple algebra we discover,

$$2r(\cos\angle AMO) = \frac{3l}{2}\ .$$

By the Law of Cosines in $\triangle AOG$ we know $g^2 = r^2 + l^2 - 2rl\cos\angle OAM_a$. Substituting and solving for l^2, we get $l^2 = 2(r^2 - g^2)$. We again use the law of cosines in $\triangle AOG$ and get $l^2 = r^2 + g^2 - 2rg\cos\theta$. Substituting for l^2 and solving for θ we get

$$\theta = \cos^{-1}(\frac{3g^2 - r^2}{2rg})\ .$$

Thus,

$$\cos^{-1}(\frac{3g - r}{2r}) \le \theta \le \cos^{-1}(\frac{3g^2 - r^2}{2rg})\ .$$

□

A cross-section of the space can be found in Figure 6. As you can see, we have a built in flipping effect that happens when g crosses the $\frac{r}{3}$ line. This stems directly from our definition of θ as the

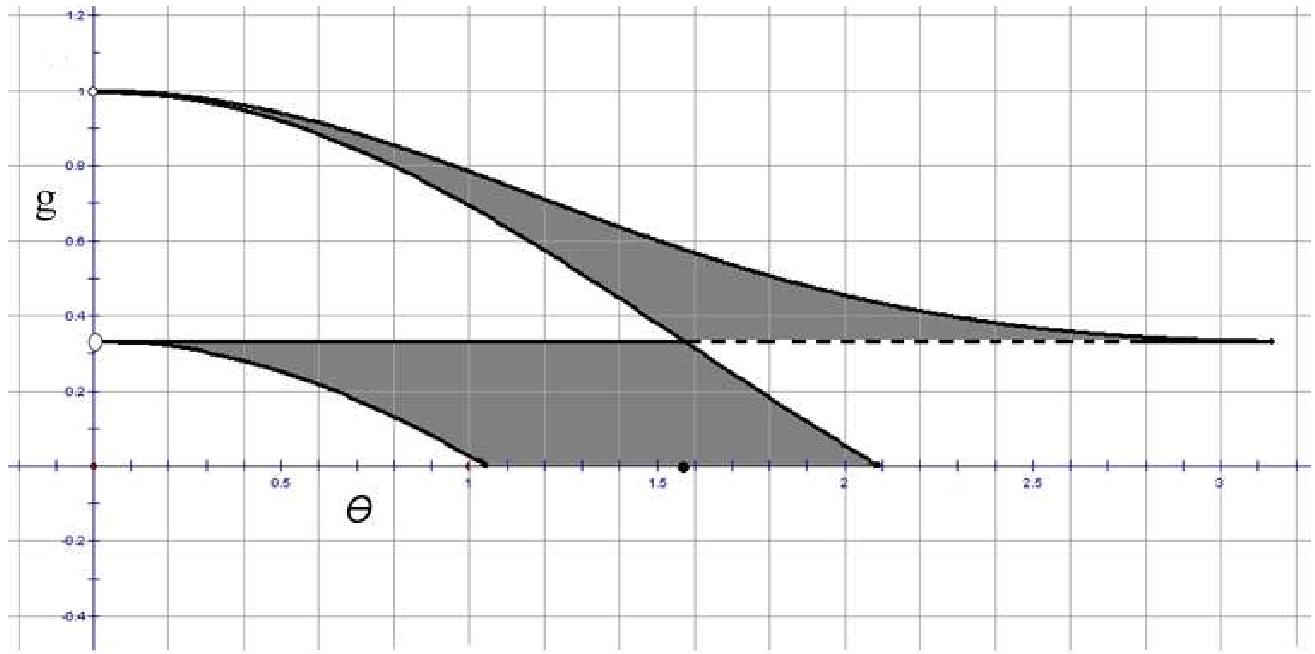

Figure 6: A cross-section of triangle space when $r = 1$.

angle formed to the one-vertex side. It is a problem we can live with and will deal with shortly, but we do so with the belief that our definition of θ is the most natural way to eliminate overcounting. First, we must quickly return to the situation where we fix θ and allow g to vary. If we do this and allow g to cross the $\frac{r}{3}$ threshold, it is not immediately clear what happens. Upon reflection of the $g = \frac{r}{3}$ case, we realize that $(\theta, g, r) \cong (\pi - \theta, g, r)$.

3.3 Metric

Now that we know where all of our triangles are and what triangle space looks like, it is natural to define a metric to determine how close two triangles are to being congruent. We would like the metric to have two additional properties:

i) Account for the fact that all triangles with $g = 0$ and the same r are congruent, and

ii) Account for the flipping effect when g crosses $\frac{r}{3}$.

First, we will account for the flipping by defining a new function,

$$\bar{\theta} = \begin{cases} \theta_i & g_i \leq \frac{r_i}{3} \\ \pi - \theta_i & g_i > \frac{r_i}{3} \end{cases} .$$

Let s and t be triangles such that $s = (\theta_1, g_1, r_1)$ and $t = (\theta_2, g_2, r_2)$. Define

$$D(s,t) = \sqrt{(g_1\bar{\theta_1} - g_2\bar{\theta_2})^2 + (g_1 - g_2)^2 + (r_1 - r_2)^2} .$$

Attaching the g 's onto the $\bar{\theta}$ terms accounts for how alike triangles with small g values are. It also leads to two nice propositions dealing with similar triangles, but first we will show a result about triangles who have the same r and g values.

Definition 2. A θ-family of triangles is a family of triangles in which θ varies while g and r remain constant.

Proposition 9. *If triangles s and t are in the same θ-family, with $s = (\theta_1, g, r)$ and $t = (\theta_2, g, r)$, then $D(s,t) = g|\bar{\theta}_1 - \bar{\theta}_2|$.*

Proof.

$$D(s,t) = \sqrt{(g\bar{\theta}_1 - g\bar{\theta}_2)^2 + (g-g)^2 + (r-r)^2} = \sqrt{g^2(\bar{\theta}_1 - \bar{\theta}_2)^2} = g|\bar{\theta}_1 - \bar{\theta}_2|\ .$$

□

Proposition 10. *Let s and t be triangles such that $s = (\theta, g, r)$ and $t = (\theta, kg, kr)$. Then*

$$D(s,t) = |k-1|\sqrt{(g\bar{\theta})^2 + g^2 + r^2}$$

Proof. Begin with the definition of

$$D(s,t) = \sqrt{(kg\bar{\theta} - g\bar{\theta})^2 + (kg-g)^2 + (kr-r)^2}.$$

Then factor a $(k-1)^2$ from each term, and bring it outside the square root as a $|k-1|$. □

Proposition 11. *Given triangles $p = (\theta_1, g, r)$, $p' = (\theta_1, kg, kr)$, $q = (\theta_2, g, r)$, $q' = (\theta_2, kg, kr)$, then*

$$D(p', q') = kD(p,q).$$

Proof.

$$D(p', q') = \sqrt{(kg\bar{\theta}_1 - kg\bar{\theta}_2)^2 + (kg-kg)^2 + (kr-kr)^2} = kg|\bar{\theta}_1 - \bar{\theta}_2| = kD(p,q).$$

□

4 Conclusion

It is important to note that we did not actually show you a picture of the space, but rather a cross-section of the space parallel to the θg-plane with $r = 1$. If you were to take cross-sections for bigger r values, the picture would stretch vertically (clearly, if r is bigger, then $\frac{r}{3}$ is bigger). The picture would not stretch horizontally, because $0 < \theta < 2\pi$, for all values of g and r.

Using our construction and its parameters, we were able to derive a way to think about and picture triangle space. By defining a metric on our space, we were able to come to an even better understanding of what is actually happening. Even so, we feel that we have barely scraped the surface of what our construction can actually do. It would be fascinating to see what how much more we could learn about triangle space.

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