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# Course Function Value Theorems

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**COARSE FUNCTION VALUE THEOREMS**

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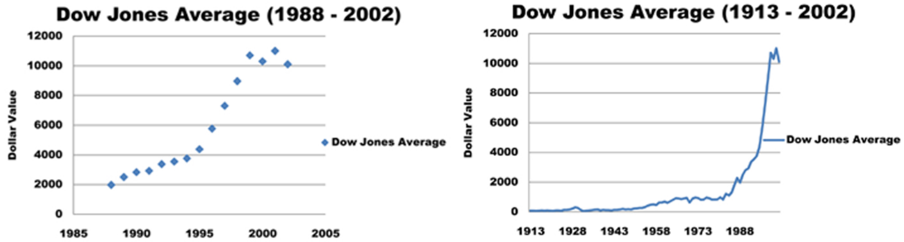
Abstract. Coarse functions are functions whose graphs appear to be continuous at a distance, but in fact may not be continuous. In this paper we explore examples and properties of coarse functions. We then generalize several basic theorems of continuous functions which apply to coarse functions.

## 1. Introduction

In this paper we will explore several properties of coarse functions. In particular, we will establish the conditions necessary to generalize several basic theorems of calculus for use with such functions. Unlike continuous functions, where attention is placed on the small scale structure defined by a given metric, coarse functions focus instead on the large scale structure of a function. This large scale focus is the defining aspect of coarse geometry, where the coarse equivalence of two metric spaces is established by their similarity when viewed from a great distance [1].

In the stock market, prices are updated at discrete intervals, namely every second. Such sampling gives us a discrete set of data rather than some continuous functions. However, because we are more concerned with the long-term price trends of such stocks, whether that be hourly, daily, weekly or monthly, we need not concern ourselves with the small gaps in our data set. Though there exists discontinuity between the stock prices on our second to second intervals, we can "take a step back" and find coarse continuity as we look at the prices on larger and larger intervals. This gives us the information we need, as we can use the principles of coarse geometry to analyze and interpret such data. The graphs in Figure 1 represent Dow Jones Industrial Average closings taken from two time periods, one short term and the other long term. The short term data on the left appears rough and unrelated, while the long term data on the right appears smooth and can be approximated by a continuous function over $\mathbb{R}^2$.

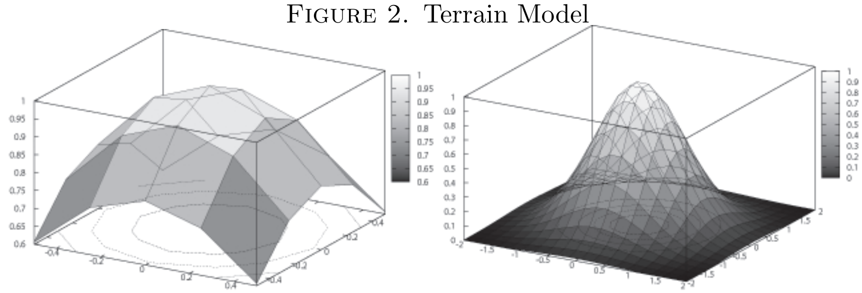
Figure 1. Stock Prices



In engineering, these are the cases where functions are defined over a discrete domain but viewed in large scale. The material properties of a specific model's microstructure, for example, are defined only at certain points on the given structure, a "grain space" [2]. The application of coarse geometric principles to this discrete set of data allows us to make useful observations and conclusions about the model's behavior and properties, as the data can be treated more like a continuous function.

The appearance of perfect, continuous functions in real world situations is rare. More common is the appearance of discrete data sets. This data contains much useful information. However, making sense of that information proves a more difficult task. Take, for example, the terrain elevation samplings shown in Figure 2. The vertices of the piecewise linear surface on the left represent a small scale sampling from the terrain data. Note the relative jaggedness of the graph. The image on the right, however, is a much larger sampling from the same terrain data. Though still discrete, the graph appears smoother and significantly less jagged.

Figure 2. Terrain Model



The purpose of this paper is to create a broader mathematical basis for interpreting such discrete sets of data. The figures on the following pages are indicative of the types of data sets for which such analysis would prove useful.

Coarse functions and geometry, in the context of geometric group theory, have been extensively studied by John Roe of Penn State University. Please refer to [3] for more information on the subject. This paper relies on several fundamental ideas behind coarse-geometry, primarily the characterization of functions as large-scale Lipschitz, bornologous and weak-bornologous as found in [3]. By establishing the equivalence of these conditions under certain conditions, we created a framework for adapting fundamental theorems for continuous functions, as found in most calculus textbooks, such as Stewart's *Calculus* [4]. We then prove these adapted theorems for coarse functions.

## 2. Preliminaries

The following definition, as found in [3], provides the basis for our work with coarse functions.

**Definition 2.1.** Let X and Y be metric spaces, and let $f : X \to Y$ be a map. Then for $x, x' \in X$ we define the following:

(1) The map is *large-scale Lipschitz* if there exist positive constants $c$ and $A$ such that

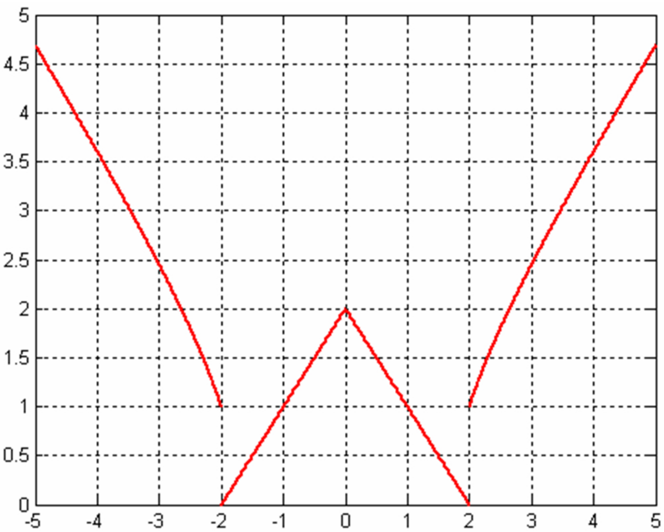
$$d(f(x), f(x')) \leq cd(x, x') + A$$

Note that if $A = 0$, we have that the function satisfies the Lipschitz condition, which in turn implies that $f$ is continuous. However, since our focus will be on discrete functions, then in general we will find that $A \neq 0$. An example of such a map can be found in Figure 3. Let $f$ be defined as follows:

$$f(x) = \begin{cases} \sqrt{x^2 - 3} & \text{for } |x| > 2 \\ -|x| + 2 & \text{for } |x| \leq 2 \end{cases}$$

Then we find that $f$ satisfies the large-scale Lipschitz condition with $c = 2$ and $A = 2$.

FIGURE 3. Large-Scale Lipschitz Mapping




(2) The map $f$ is *bornologous* if for every $R > 0$ there exists an $S > 0$ such that

$$d(x, x') < R \Rightarrow d(f(x), f(x')) < S.$$

Consider the function $f(x) = x^2$ on the set $A = \{1/n \mid n \in \mathbb{Z}^+\}$. Now, let $R > 0$ be given and let $S = R$. So for any $x, x' \in A \subset (0, 1)$, we have by the definition of $f(x)$ that $d(f(x), f(x')) < d(x, x') < R = S$. Clearly then, we have that $f$ is a bornologous function on the discrete set $A$.

(3) The map is *weak bornologous* if there exist $R, S > 0$ such that

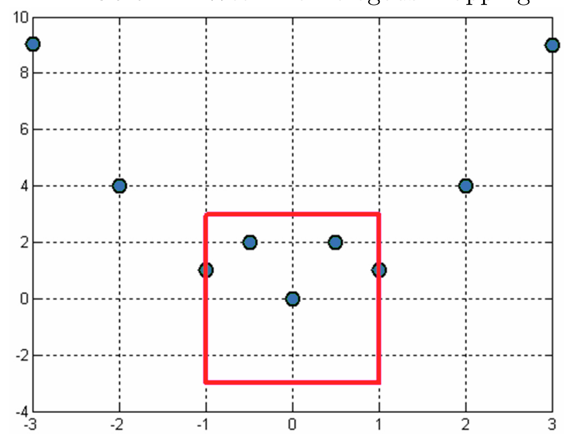
$$d(x, x') < R \Rightarrow d(f(x), f(x')) < S$$

A weak bornologous mapping can be found in Figure 4. Let $f$ be defined as follows:

$$f(n) = \begin{cases} 2 & \text{for } x = \pm\frac{1}{2} \\ n^2 & \text{for } n \in \mathbb{Z} \end{cases}$$

For $R = 1$ and $S = 3$ we find that $f$ satisfies the weak bornologous condition. Note, however, that f is not bornologous.

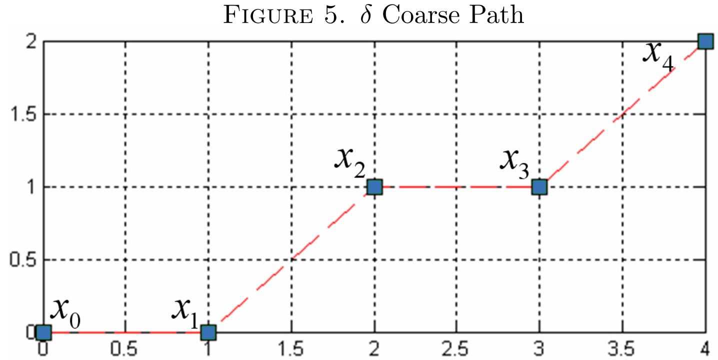
FIGURE 4. Weak Bornologous Mapping




It will be shown that these conditions are in fact equivalent under certain restrictions. Such equivalence allows us to expand several conclusions about continuous functions to those that are coarse and discrete. Before doing so, however, we provide the definitions of those terms we created as tools for expanding these conclusions.

**Definition 2.2.** A *$\delta$-coarse path* $P$ from points $a, b \in X$ with $\delta > 0$ is an ordered set $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ such that $d(x_{i-1}, x_i) < \delta$ for $1 \leq i \leq n$. Now, let $a = (0, 0)$ and $b = (4, 2)$ be points in $\mathbb{R}^2$. Clearly, there are many $\delta$-coarse paths between $a$ and $b$. Figure 5 shows such a path with $\delta = 2$.

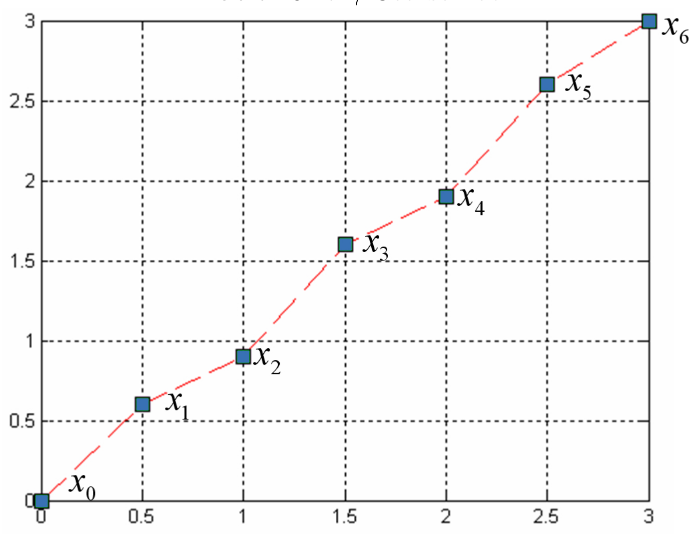
FIGURE 5. $\delta$ Coarse Path




**Definition 2.3.** A *$\delta$-$\gamma$ coarse path* $P$ from $a$ to $b$ is a $\delta$-coarse path from $a$ to $b$ given by $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ such that $\delta - 2\gamma \leq d(x_{i-1}, x_i) < \delta$ for $1 \leq i \leq n$. A $\delta$-$\gamma$ coarse path is dependent on the chosen values of both *delta* and *gamma*. For example, the two previous paths shown in Figure 5 are not $\delta$-$\gamma$ coarse paths for $\delta = 2$ and $\gamma = .5$. However, if we reduce our constraints and let $\delta = 1.5$ and $\gamma = .3$, then the paths satisfy the necessary conditions for a $\delta$-$\gamma$ coarse path. Figure 6 illustrates a $\delta$-$\gamma$ coarse path with $\delta = 1$ and $\gamma = .3$.

**Definition 2.4.** Let $f : X \to \mathbb{R}$ be defined on a metric space $X$. We define a *coarse derivative* function $f'_\delta(x)$ on $X$ as follows. Suppose $P = \{a = x_0, x_1, \ldots, x_i =$

Figure 6. $\delta$-$\gamma$ Coarse Path




$x, x_{i+1}, \ldots, x_{n-1}, x_n = b\}$ is a $\delta$-coarse path from $a$ to $b$ in $X$. Then

$$f'_\delta(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\delta}$$

Note that this definition better mimics the standard definition of a derivative if $P$ is a $\delta$-$\gamma$ course path for $\delta$ very small and $\gamma << \delta$.

**Definition 2.5.** A set $X$ in a metric space is *bounded* if there exists an $R < \infty$ such that $d(x, x') \leq R$ for all $x, x' \in S$.

## 3. The Large Scale Lipschitz Function Theorem for Coarse Functions

We begin by establishing the restrictions under which the weak bornologous condition implies large-scale Lipschitz behavior for discrete functions. In generalizing this relationship for use in a discrete environment, we gain the ability to apply additional properties of coarse geometry already established in [3] .

**Theorem 3.1.** *Let $X$ and $Y$ be metric spaces and $f : X \to Y$ a function such that:*

(1) *$f$ is weak bornologous where $R$ and $S$ are the constants so that $d(x, x') < R$ implies that $d(f(x), f(x')) < S$.*
(2) *There exists an $\epsilon > 0$ such that if $x, y \in X$, $d(x, y) > R$, and $n$ is an the integer so that $\frac{d(x,y)}{n} < R - 2\epsilon$ and $\frac{d(x,y)}{n-1} \geq R - 2\epsilon$, then there is a $\delta$-course path $P = \{x = x_0, x_1, \ldots, x_n = y\}$ with $\delta = R$.*

*Then $f$ is large-scale lipschitz.*

*Proof.* Let $x, y \in X$. Let $R, S$ and $\epsilon$ be given as in (1) and (2). Choose $n$ so that

$$\frac{d(x,y)}{n} < R - 2\epsilon \text{ and } \frac{d(x,y)}{n-1} \geq R - 2\epsilon.$$

Thus we have that $n \leq \frac{d(x,y)}{(R-2\epsilon)} + 1$. Now let $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ be given from (2). Then

$$\begin{aligned} d(f(x), f(y)) &\leq \sum_{i=0}^{n-1} d(f(x_i), f(x_{i+1})) \\ &< nS \\ &\leq \left( \frac{d(x,y)}{R-2\epsilon} + 1 \right) S \\ &= \left( \frac{S}{R-2\epsilon} \right) d(x,y) + S \end{aligned}$$

Thus $d(f(x), f(y)) < \left(\frac{S}{R-2\epsilon}\right) d(x,y) + S$. Therefore, $f$ is large scale Lipschitz.

□

**Corollary 3.2.** *Let $X$ and $Y$ be metric spaces and let $f : X \to Y$ satisfy the restrictions set forth in Theorem 3.1. Then the following statements are equivalent:*

(1) *$f$ is large-scale Lipschitz*
(2) *$f$ is bornologous*
(3) *$f$ is weak bornologous*

*Proof.* We begin by showing that (1) $\Rightarrow$ (2). Suppose $f$ is large-scale Lipschitz. Then there exist positive constants $c$ and $A$ such that

$$d(f(x), f(y))) \leq cd(x,y) + A$$

for all $x, y \in X$. Now let $R > 0$ be given, and choose $S = cR + A$. Thus, if $d(x,y) < R$ we have that

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x,y) + A \\ &< cR + A = S \end{aligned}$$

Now for (2) $\Rightarrow$ (3). Suppose f is bornologous. Thus for every $R > 0$ there exists an $S > 0$ such that

$$d(x, x') < R \Rightarrow d(f(x), f(x')) < S.$$

Clearly, there must then exist some $R > 0$ and $S > 0$ such that the same condition holds. Thus we have that f is weak bornologous.

Finally, (3) $\Rightarrow$ (1) is proved in Theorem 3.1.

□

## 4. The Intermediate Value Theorem for Coarse Functions

**Theorem 4.1.** *Let $X$ be a metric space and $f : X \to \mathbb{R}$ be weak bornologous. Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a $\delta$-coarse path from $a$ to $b$ in $X$ with $\delta = R$. Then, if $y$ is a value strictly between $f(a)$ and $f(b)$, there exists $c \in P$ such that $d(f(c), y) < S$.*

*Proof.* Without loss of generality, suppose that $f(a) \leq f(b)$ and let $A = \{x_i \in P | f(x_i) \leq y\}$. Note that $A \neq \emptyset$ because $x_0 = a \in A$. Also $A \neq P$ because $x_n = b \notin A$. Now, since $P$ is a set of ordered points, so too will $A$ be a set of ordered points. Let $k$ be the maximum index of the $x_i$'s in $A$. Then, clearly, $f(x_k) \leq y < f(x_{k+1})$. Thus

$$\begin{aligned} d(f(x_k), y) &\leq d(f(x_k), f(x_{k+1})) < S \\ d(f(x_k), y) &< S \end{aligned}$$

Therefore, $c = x_k$ is the desired point. □

## 5. The Boundedness Theorem for Coarse Functions

The following theorem is the coarse analog for the Extreme Value Theorem.

**Theorem 5.1.** *Let $f : X \to \mathbb{R}$ be real valued function defined on a bounded set $P$ such that $f$ is large scale Lipschitz. Then the image of $P$ under $f$ is also bounded.*

*Proof.* Let $P$ be a bounded subset of $X$. So for any $x$, $y \in P$, $d(x, y) < R$, where $R < \infty$. Now, since $f$ is large-scale Lipschitz, there exists positive constants $c$ and $A$ such that

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + A \\ &< cR + A \end{aligned}$$

So by definition, the image of P is also bounded. □

## 6. The Mean Value Theorem for Coarse Functions

**Theorem 6.1.** *Let $f : X \to \mathbb{R}$ be defined such that $f'_\delta$ is large scale lipschitz with constants $c$ and $A$. Also, let $P$ be a well-defined $\delta$ coarse path from $x$ to $y$ for $x, y \in X$. Then there exists some $x_i \in P$ such that, for $\beta = \frac{1}{2}(c\delta + A)$:*

$$\left| \frac{f(y) - f(x)}{d(y, x)} - f'_\delta(x_i) \right| \leq \beta$$

*Proof.* Let $x, y \in X$, and let $f$ and $P$ be given as in the hypothesis. Now, assume to the contrary that, for all $x_i \in P$ for which $f'_\delta$ exists, the following holds:

$$\left| \frac{f(y) - f(x)}{d(y, x)} - f'_\delta(x_i) \right| > \beta$$

So we have, for all appropriate $x_i$, that either

$$f'_\delta(x_i) > \frac{f(y) - f(x)}{d(y, x)} + \beta \tag{1}$$

or

$$f'_\delta(x_i) < \frac{f(y) - f(x)}{d(y, x)} - \beta \tag{2}$$

Now, assume without loss of generality that equation (1) holds for $x_1$. It follows that the same equation must hold for all remaining $x_i \in P$. Otherwise, there would exist some $x_i, x_{i-1} \in P$ such that

$$f'_\delta(x_{i-1}) > \frac{f(y) - f(x)}{n\delta} + \beta$$

and

$$f'_\delta(x_i) < \frac{f(y) - f(x)}{n\delta} - \beta$$

This would imply that

$$f'_\delta(x_{i-1}) - f'_\delta(x_i) > 2\beta$$

However, since $f'_\delta$ is large scale lipschitz, then for any $x_i, x_{i-1} \in P$,

$$|f'_\delta(x_i) - f'_\delta(x_{i-1})| \leq cd(x_i, x_{i-1}) + A < c\delta + A = 2\beta$$

This contradiction implies that equation (1) must hold for all $x_i \in P$ if it holds for $x_1$. Now, note that $d(y,x) \leq n\delta$. It follows from equation (1) that

$$\begin{aligned} \sum_{i=0}^{n-1} f'_\delta(x_i) &> (n)\left(\frac{f(y)-f(x)}{d(y,x)} + \beta\right) \\ &\geq (n)\left(\frac{f(y)-f(x)}{n\delta} + \beta\right) \\ &= \frac{f(y)-f(x)}{\delta} + n\beta \end{aligned} \tag{3}$$

By our definition of $f'_\delta(x_i)$, however, we know that

$$\begin{aligned} \sum_{i=0}^{n-1} f'_\delta(x_i) &= \frac{f(x_n) - f(x_{n-1}) + f(x_{n-1}) - \ldots - f(x_1) + f(x_1) - f(x_0)}{\delta} \\ &= \frac{f(x_n) - f(x_0)}{\delta} \\ &= \frac{f(y)-f(x)}{\delta} \end{aligned} \tag{4}$$

Substituting (4) into the left hand side of equation (3) gives us the following:

$$\frac{f(y)-f(x)}{\delta} > \frac{f(y)-f(x)}{\delta} + n\beta$$

and therefore

$$0 > n\beta = \left(\frac{n}{2}\right)(c\delta + A)$$

This is a contradiction, however, as $n, c, \delta$ and $A$ are all nonnegative constants. Thus, it must be the case that there exists some $x_i \in P$ such that:

$$\left|\frac{f(y)-f(x)}{d(y,x)} - f'_\delta(x_i)\right| \leq \beta$$

□

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