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Intrinsically S1 3-Linked Graphs and Other Aspects of S1 Embeddings

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Intrinsically S^1 3-linked graphs and other aspects of S^1 embeddings

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Abstract

An S^1 *embedding* of a graph G is an injective map of the vertices of G into S^1. This paper considers the meaning of *link*, *n-link* and *intrinsically linked* for S^1 embeddings of graphs. Specifically, we are concerned with the minor-minimal set of intrinsically S^1 3-linked graphs. This paper presents a list of known elements of that set, along with methods used to find and verify the list, in hopes of obtaining the complete minor minimal set. Other aspects of S^1 embeddings are also examined.

1 Introduction

Any two disjoint circles embedded in space form a link. This link is said to be *splittable*, or just *split*, if the circles can be pulled apart without passing through each other. More precisely, a link L is split if there is an embedding of a 2-sphere F in $\mathbb{R}^3 - L$ such that each component of $\mathbb{R}^3 - F$ contains at least one component of L. A *cycle* of a graph is a simple closed path with no vertices repeated other than the first/last. A spatial embedding of a graph is *linked* if there is a pair of cycles that form a non-split link. A graph is *intrinsically linked* if every spatial embedding of the graph is linked. Conway and Gordon [5] and Sachs [12] show that the graph K_6 is intrinsically linked.

A graph H is a *minor* of a graph G if it can be obtained from G by a finite sequence of edge deletions, edge contractions, and vertex deletions. In Figure 1 we see a graph G and three possible minors.

A graph G is said to be *minor-minimal* with respect to a property if G has that property but no minor of G has that property. (Note that any

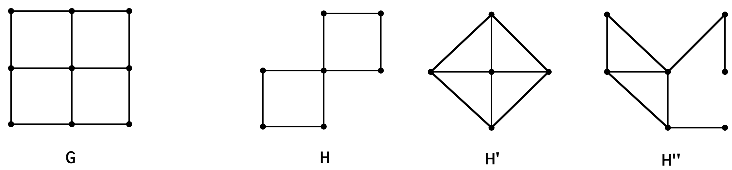

Figure 1: A graph G and three of its minors, H, H', and H''

supergraph of a graph with a certain linking property will have that same property.) One can then consider the *complete minor-minimal set* of graphs with respect to a property, that is, the set of all graphs with a given property such that no minor of these graphs has that property. As a consequence of Robertson and Seymour's result [10], the complete minor-minimal set of graphs with a given property is a finite set of graphs. Moreover, if a graph G has a given property P whenever a minor of G has the property, then the set of all graphs with property P is completely characterized by containing a graph in the (finite) complete minor-minimal set as a minor. If the complete minor-minimal set can be determined, then Robertson and Seymour's results [10] further imply that one can detect whether or not a graph has property P in a polynomial time algorithm. For example, it is shown by Robertson, Seymour, and Thomas in [11] that the Petersen Family of graphs (those obtained by a series of $\triangle$-Y and Y-$\triangle$ exchanges on K_6) form the complete minor-minimal set of intrinsically linked graphs.

In this paper we consider not spatial embeddings of graphs but rather S^1 embeddings of graphs. An S^1 *embedding* of a graph G is an injective map of the vertices of G into S^1. Each distinct S^1 embedding of a graph can be seen as a permutation of the vertices; the order (*mod* rotations and reflections) of the vertices will determine the embedding. Two S^1 embeddings of a graph are equivalent if the order of the vertices is the same in each embedding. To help in describing the arrangement of vertices in an embedding, a vertex a is said to *neighbor* vertex b in an S^1 embedding of a graph G if there is a component of $S^1 - \{a, b\}$ that does not contain any vertices of G. In Figure 2, vertex a neighbors vertices b and h.

We say that two vertices form a *0-sphere* in an S^1 embedding of a graph G if they are the endpoints of a simple path in G. In this paper a 0-sphere is denoted by writing the two vertices of the 0-sphere in set brackets. A path will be denoted by enclosing the ordered list of the vertices that compose the

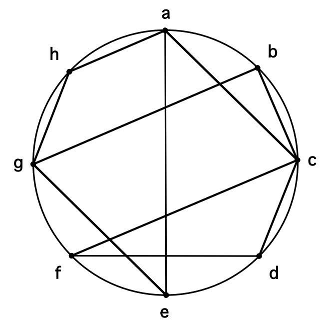

Figure 2: An S^1 embedding of a graph G. The edges are included for convenience and are not part of the S^1 embedding.

path in parentheses. We will say that a simple path *justifies* a 0-sphere if the endpoints of that path are the vertices that compose the 0-sphere. In Figure 2, the 0-sphere $\{a, f\}$ is justified by the path (a, c, f). A pair of 0-spheres is said to be *disjoint* if there exist disjoint simple paths that justify the two 0-spheres. In Figure 2, 0-spheres $\{b, f\}$ and $\{h, e\}$ are disjoint. Although there may be more than one path that justifies a given 0-sphere, we usually only consider a particular path unless we indicate otherwise.

Just as a pair of disjoint cycles forms a link in a spatial embedding, a pair of disjoint 0-spheres forms a link in an S^1 embedding. A link $\{x, y\}$ and $\{z, w\}$ is said to be *split* if x and y lie on the same component of $S^1 - \{z, w\}$. Thus the link is *non-split* if x and y lie on different components of $S^1 - \{z, w\}$. In Figure 2, 0-spheres $\{a, e\}$ and $\{g, h\}$ are split linked while 0-spheres $\{a, e\}$ and $\{c, f\}$ are non-split linked. For S^1 embeddings, the *mod 2 linking number* of two 0-spheres $\{x, y\}$ and $\{z, w\}$, denoted $lk_2(\{x, y\}, \{z, w\})$, is 0 if and only if $\{x, y\}$ and $\{z, w\}$ are split linked and is 1 if and only if $\{x, y\}$ and $\{z, w\}$ are non-split linked.

Just as some graphs are intrinsically linked in space, some graphs are intrinsically S^1 linked. A graph is *intrinsically S^1 linked* if every S^1 embedding contains a non-split link. It is shown by Cicotta et al. [4] that the complete minor-minimal set of intrinsically S^1 linked graphs is K_4 and $K_{3,2}$. At this point, it is worth noting that our definition of 0-sphere is a departure from the definition given in [4], where they define a 0-sphere to be a pair of vertices that form the endpoints of an edge rather than the endpoints of a simple path. Our more flexible definition does not change previous results about intrinsically S^1 linked graphs (See Theorem 2.1), and it allows us to state and prove Lemma 5.2, which is an analog of a lemma concerning intrinsically 3-linked graphs in space.

With the complete set of minor-minimal intrinsically S^1 linked graphs known, we can look at other linking properties. One such property is the S^1 3-linking property. A graph is said to be *intrinsically n-linked* if every spatial embedding has a non-split link of n components. The spatial analog of this property has been examined with limited success. In [7], Flapan, Naimi, and Pommersheim prove that K_{10} is the smallest complete graph to be intrinsically 3-linked. In [3], Bowlin and Foisy show that K_{10} is not minor-minimal by exhibiting two subgraphs that are intrinsically 3-linked. In [9], O'Donnol shows that the complete bipartite graph $K_{2n+1,2n+1}$ is the smallest bipartite graph that is intrinsically n-linked.

In spite of these successes, the problem of identifying intrinsically n-linked graphs in space seems difficult. We hope that the analogous problem for S^1 embeddings is simpler. To this end, we look at n-links in S^1 embeddings of graphs. An S^1 *n-link* in an S^1 embedding of a graph G is a set of n disjoint 0-spheres in the embedding of G (note that the justifying paths are pairwise disjoint). An n-link in an S^1 embedding of a graph G is said to be *split* if there are two points, x and y, on the circle such that both components of $S^1 - \{x, y\}$ contain at least one vertex involved in the n-link and every 0-sphere in the link lies entirely on one component of $S^1 - \{x, y\}$. Otherwise, we say that the n-link is *non-split*. A graph is said to be *intrinsically* S^1 *n-linked* if every S^1 embedding of the graph contains a non-split n-link. The goal of our research was to find the complete set of minor-minimal intrinsically S^1 3-linked graphs. Although we present graphs that are minor-minimal with respect to the intrinsic S^1 3-linking property, many of our methods can be used to find graphs that are intrinsically n-linked for n greater than 3.

In this paper, we will begin by discussing two graph operations, vertex expansion and $\triangle - Y$ exchange. Let g be a vertex of G and A the set of vertices connected to g by an edge. Then a *vertex expansion* on vertex g of G is as follows. Vertex g is deleted from G, vertices g' and g'' are added to G, edge $e_{g',g''}$ is added, and an edge is added between each element of A and either g' or g''. Given the graph G in Figure 3(a), Figure 3(b) depicts a possible vertex expansion on vertex g. Let a, b, and c be vertices of a graph G such that edges $e_{a,b}, e_{a,c}$, and $e_{b,c}$ exist. Then a $\triangle - Y$ *exchange* on a triangle abc of graph G is as follows. Vertex v is added to G, edges $e_{a,b}, e_{a,c}$, and $e_{b,c}$ are deleted, and edges $e_{a,v}, e_{b,v}$, and $e_{c,v}$ are added. Given the graph G in

Figure 3(a), Figure 3(c) depicts the result of $\triangle - Y$ expansion on triangle abc. Later in this paper we will discuss $Y - \triangle$ exchange, the reverse operation.

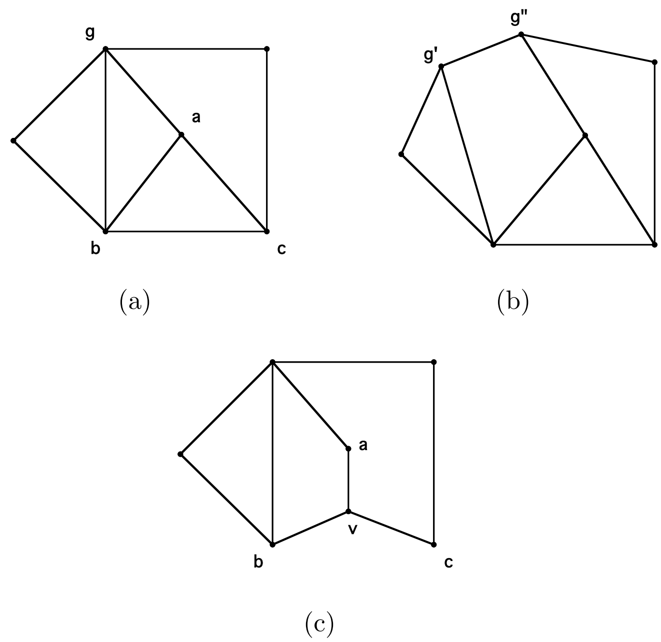

Figure 3: A graph G and the results of graph operations

We will show that both vertex expansion and $\triangle - Y$ exchange preserve certain S^1 linking properties. The Robertson and Seymour result [10] implies that the complete set of minor minimal intrinsically S^1 3-linked graphs, Ω, is finite. Since vertex expansion preserves intrinsic S^1 3-linking, we can say that a graph is intrinsically S^1 3-linked if and only if it contains an element of the finite set Ω as a minor.

Further, we prove several theorems and lemmata that build up to the following theorem:

Theorem 1.1. *The set of twenty-eight graphs represented in Figures 5 and 6 are minor-minimal with respect to intrinsic S^1 3-linking.*

Although we do not know if this set of intrinsically S^1 3-linked graphs is complete, we suspect that it is close to being complete. By comparison, there are thirty-eight minor-minimal non-outer cylindrical graphs [1]. The set of non-outer-cylindrical graphs, examined in [1] by Archdeacon et al., is an interesting set of graphs. Non-outer-cylindrical graphs are defined as graphs that cannot be embedded in the plane so that there are at most two distinct faces whose boundaries together contain all of the vertices. This set of graphs is of interest to us because the complete minor-minimal set of such graphs is somewhat similar to the set of intrinsically S^1 3-linked graphs we found. Further, the general method used in [1] influenced our approach to our problem. Indeed, some of the results in [1] inspired us to prove similar results for our set of graphs; Theorems 5.6 and 5.7 came about in this manner.

Finally we will discuss two other varieties of S^1 linking: intrinsic n-links in $K_{n,n}$ and intrinsic pairwise non-split 3-links.

Note that in this paper we will often use "link" and similar terms such as "3-linked" to mean "non-split link" and so on. Further, in our pictures, we draw chords in the S^1 embeddings to represent edges of the depicted graph. These are just visual reminders and are not part of the S^1 embedding. Also note that although the proofs of Theorems 4.5, 5.7, 5.8, and 7.5 have been omitted. They can be found in our appendix [2]. The appendix has been checked by the authors and the advisor, but it has not been refereed. Theorem 1.1 has also not been rigorously checked by the referee.

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2 Consistency of 0-Sphere Definitions

In [4], a 0-sphere in an S^1 embedding of a graph G was defined to be a pair of vertices connected by an edge in G. In this paper we have extended the definition of 0-sphere to include vertices connected not just by edges, but by simple paths. In order for this new definition to be useful, it is necessary to ensure that our new definition is consistent with the old one.

Theorem 2.1. *A graph G is intrinsically S^1 linked under the old definition if and only if it is intrinsically linked under the new definition.*

Proof. First consider a graph G that is intrinsically S^1 linked under the old definition of 0-sphere. In every S^1 embedding of such a graph, there is a pair of 0-spheres $\{a, b\}$ and $\{c, d\}$ such that a and b lie on different components of $S^1 - \{c, d\}$. Under the new definition, a and b still form a 0-sphere since there is a simple path from a to b, namely the edge connecting them. Similarly, c and d form a $0-$sphere, thus G is intrinsically S^1 linked under the new definition.

Now consider a graph G that is intrinsically S^1 linked under the new definition of 0-sphere. Arbitrarily S^1 embed G. Because G is intrinsically S^1 linked, there exist linking 0-spheres $\{a, b\}$ and $\{x, y\}$ that are justified by disjoint simple paths. Let the simple path from a to b be $(a = v_0, v_1, \ldots, v_{n-1}, v_n = b)$. Let the simple path from x to y be $(x = w_0, w_1, \ldots, w_{m-1}, w_m = y)$. Because $\{a, b\}$ and $\{x, y\}$ form a non-split link, x and y lie in different components of $S^1 - \{a, b\}$.

In order to prove the desired result, we will first introduce a claim: there exists some w_k and w_{k+1} such that the two lie on different components of $S^1 - \{a, b\}$. Because the paths from x to y and from a to b are disjoint, $w_i \neq a$ and $w_i \neq b$ for all i. Thus any two w_i and w_{i+1} either lie on the same or different components of $S^1 - \{a, b\}$. Suppose that w_i and w_{i+1} lie on the same component of $S^1 - \{a, b\}$ for all i, without loss of generality, say the component that contains x. Simple inductive logic shows that w_i lies on the same component of $S^1 - \{a, b\}$ as x for all i. However, this contradicts the fact that x and y lie on different components of $S^1 - \{a, b\}$.

Now, applying this claim, we find some w_k and w_{k+1} such that the two lie on different components of $S^1 - \{a, b\}$. Similarly, we find some v_l and v_{l+1} such that the two lie on different components of $S^1 - \{w_k, w_{k+1}\}$. Because v_l and v_{l+1} are joined by an edge and w_k and w_{k+1} are joined by an edge, $\{v_l, v_{l+1}\}$ and $\{w_k, w_{k+1}\}$ form a non-split link under the old definition. □

3 Vertex Expansion Preserves S^1 Linking Properties

In [4], Cicotta et al. prove that vertex expansion preserves the intrinsic S^1 linking property. Here we generalize their proof using our definition of 0-sphere to all of the intrinsic S^1 n-linking properties.

Theorem 3.1. *The operation vertex expansion preserves the intrinsic S^1 n-linking property (for $n \geq 2$).*

Proof. Let G be a graph with the intrinsic S^1 n-linking property, and let G' be the graph obtained by expanding the vertex v of G into vertices v_1 and v_2. Consider an arbitrary S^1 embedding of G'. We will show that there exists a n-link in this embedding.

Obtain an S^1 embedding of G from this S^1 embedding of G' by deleting vertex v_2, relabeling v_1 as v, and creating an edge to v from each vertex y such that y was adjacent to v_2 in G'. This results in a vertex contraction of v_1 and v_2 to v. We now have an S^1 embedding of G, and it must thus contain a n-link.

Case 1: Suppose that v is not involved in the n-link. Then the same non-split n-link exists in our original S^1 embedding of G'.

Case 2: Suppose $\{w, v\}$ is a 0-sphere involved in the link. Let $(w, w_1, \ldots, w_n, v)$ be the path that justifies $\{w, v\}$. So $(w, w_1, \ldots, w_n, v)$ is disjoint from the paths justifying the other 0-spheres. Expand vertex v to obtain the original S^1 embedding of G'. Vertex w_n is adjacent to either v_1 or v_2 in G'. If w_n is adjacent to v_1, then $(w, w_1, \ldots, w_n, v_1)$ justifies the 0-sphere $\{w, v_1\}$. As v_1 does not move when we expand from G to G', $\{w, v_1\}$ will replace $\{w, v\}$ as part of the n-link. If w_n is adjacent to v_2, $(w, w_1, \ldots, w_n, v_2, v_1)$ justifies the 0-sphere $\{w, v_1\}$. As $(w, w_1, \ldots, w_n, v)$ was disjoint from the paths justifying the other 0-spheres and as v_2 was not part of the S^1 embedding of G, $(w, w_1, \ldots, w_n, v_2, v_1)$ is disjoint from the paths justifying the other 0-spheres. Again, $\{w, v_1\}$ replaces $\{w, v\}$ as part of the n-link. In either case, there is an n-link in the S^1 embedding of G'.

Case 3: Suppose that $\{w_1, w_n\}$, justified by the path $(w_1, \ldots, w_i, v, w_{i+1}, \ldots, w_n)$, is a 0-sphere of the n-link. Expand vertex v to obtain the original

S^1 embedding of G'. There are four possibilities: either v_1 is adjacent to both w_i and w_{i+1}, v_1 is adjacent to w_i and v_2 is adjacent to w_{i+1}, v_1 is adjacent to w_{i+1} and v_2 is adjacent to w_i, or v_2 is adjacent to both w_i and w_{i+1}. In any of these cases, there is a path from w_i to w_{i+1} using at most vertices v_1 and v_2. As v_2 did not exist in the S^1 embedding of G, this path remains disjoint from the the paths justifying the other 0-spheres of the n-link. So $\{w_1, w_n\}$ completes the n-link in the S^1 embedding of G'.

In each case, there exists a n-link in the S^1 embedding of G'. As this S^1 embedding of G' was arbitrary, there is a n-link in every S^1 embedding. Thus, the operation vertex expansion preserves the intrinsic S^1 n-linking property. □

Any linking property is certainly preserved under vertex and edge addition. Because the intrinsic S^1 n-linking property is also preserved under vertex expansion, a graph G is intrinsically S^1 n-linked if and only if G contains a minor-minimal intrinsically S^1 n-linked graph as a minor.

4 $\triangle - Y$ Exchange Preserves S^1 Linking Properties

The operation $\triangle - Y$ exchange is very important in our research. Because, as we will prove, $\triangle - Y$ exchange preserves intrinsic S^1 3-linking, we can construct families of graphs that are related by $\triangle - Y$ exchange. If the first graph (the one on which $\triangle - Y$ exchange is first performed) is intrinsically S^1 3-linked, then the entire family is as well. Here, we prove the more general result that $\triangle - Y$ exchange preserves intrinsic S^1 n-linking. Note that the reverse operation, $Y - \triangle$ exchange, does not necessarily preserve intrinsic S^1 n-linking because a vertex is deleted from the graph. For example, the graph $K_{3,3}$ has the intrinsic S^1 3-linking property, but the graph obtained by a single Y-Triangle exchange, $K_5 - e$, does not have the property.

Theorem 4.1. *The operation $\triangle - Y$ exchange preserves intrinsic S^1 n-linking (for $n \geq 2$).*

Proof. Let G be a graph that is intrinsically S^1 n-linked and contains at least one 3-cycle. Denote the vertices of some 3-cycle as a, b, and c. Let G' be a graph obtained from G by a $\triangle - Y$ exchange on triangle abc. Denote the new vertex as v. We will show that G' is intrinsically S^1 3-linked.

Consider an arbitrary S^1 embedding of G'. Consider the associated S^1 embedding of G that results from deleting vertex v (and consequently edges $e_{a,v}, e_{b,v}$, and $e_{c,v}$) and creating edges $e_{a,b}$,$e_{b,c}$, and $e_{a,c}$. As G is intrinsically S^1 n-linked, there exists a n-link in this S^1 embedding of G.

Case 1: Suppose that none of the paths justifying the 0-spheres of the n-link rely on the existence of any of the edges $e_{a,b}$,$e_{b,c}$, or $e_{a,c}$. Return to the S^1 embedding of G' obtained by reinstating vertex v and edges $e_{a,v}, e_{b,v}, e_{c,v}$ and deleting edges $e_{a,b}$,$e_{b,c}$, and $e_{a,c}$. As none of the paths rely on any of the changed edges the n-link is unchanged by the operation. Thus there is an n-link in the S^1 embedding of G'.

Case 2: Suppose that one of the paths justifying the 0-spheres of the n-link relies on the existence of $e_{a,b}$,$e_{b,c}$, or $e_{a,c}$. Note that, as the paths are disjoint, at most one of these paths can rely on $e_{a,b}$,$e_{b,c}$, or $e_{a,c}$. Thus no other path will be affected by the $\triangle - Y$ exchange. Also, as $e_{a,b}$,$e_{b,c}$, and $e_{a,c}$ form a 3-cycle, this path need not rely on more than one of the three edges. So, without loss of generality, suppose that this path relies on edge $e_{a,b}$. Return to the S^1 embedding of G' obtained by reinstating vertex v and edges $e_{a,v}, e_{b,v}, e_{c,v}$ and deleting edges $e_{a,b}$,$e_{b,c}$, and $e_{a,c}$. The path from the S^1 embedding of G does not exist in this S^1 embedding of G', but there is a new path that justifies the same 0-sphere. It is formed by replacing a, b in the sequence of vertices of the original path by a, v, b. This new path remains disjoint from the paths that justify the other 0-spheres. Thus, the 0-sphere justified by this new path completes the n-link in the S^1 embedding of G'

In both cases there exists a n-link in the S^1 embedding of G'. As the embedding of G' was arbitrary, there exists a n-link in every S^1 embedding of G'. Thus, as abc was an arbitrary 3-cycle, the operation $\triangle - Y$ exchange preserves intrinsic S^1 n-linking. □

It would be convenient if $\triangle - Y$ exchange also preserved minor-minimality with respect to intrinsic S^1 linking properties. However, this is not the case. (Consider the T_7 from Figure 6. It is minor-minimal with respect to the intrinsic S^1 3-linking property. A $\triangle - Y$ exchange on the bottom left triangle results in a supergraph of $K_{3,3}$. Since $K_{3,3}$ is minor-minimal, this new graph is not.) However, the reverse operation, $Y - \triangle$ exchange, does preserve minor-minimality under certain conditions.

Theorem 4.2. *Let Q be a property of a graph that is preserved under $\triangle - Y$ exchange. Let G be a graph that contains at least one degree three vertex and has property Q. Let G' be a graph obtained from G by a $Y - \triangle$ exchange. If G is minor-minimal with respect to Q and if G' has property Q, then G' is also minor-minimal with respect to Q.*

Proof. Suppose G as defined is minor-minimal with respect to property Q and that G' has property Q. Suppose, for the sake of contradiction, that G' is not minor-minimal with respect to property Q. Then there exists a graph H', which is a minor of G' and has property Q.

Case 1: Suppose that the triangle created in G' by a $Y - \triangle$ exchange of G is present in H'. Consider the graph H obtained from H' by $\triangle - Y$ exchange. As property Q is preserved under $\triangle - Y$ exchange and as H' has property Q, H also has property Q. However, H is a minor of G. As G is minor-minimal with respect to Q, H cannot have property Q. This is a contradiction.

Case 2: Suppose that H' is obtained from G' in such a way that the triangle created in G' by a $Y - \triangle$ exchange of G is not wholly present in H'. Denote the vertices of the triangle as a, b, and c. Denote the final vertex of the Y in G as v. We prove the following claim:

Claim: If H' is derived from G' in a way that affects triangle abc, then H' is a minor of G.

Case a) Suppose H' is derived from G' by deleting an edge from triangle abc. Suppose, without loss of generality, that edge $e_{a,b}$ is deleted. Then $H' = G' - e_{a,b}$, which is a contraction of edge $e_{v,c}$ in G.

Case b) Suppose H' is derived from G' by deleting a vertex from triangle abc. Suppose, without loss of generality, that vertex a is deleted. This graph is a minor of $G' - e_{a,b}$ (see case 2a).

Case c) Suppose H' is derived from G' by contracting an edge of triangle abc. Suppose, without loss of generality, that edge $e_{a,b}$ is contracted. Then H' is the same as G with edges $e_{a,v}$ and $e_{b,v}$ contracted.

It follows that if H' is derived from G' in any way that affects triangle abc, then H' is a minor of G.

Now, the supposition of this case is that H' is obtained from G' in such a way that the triangle created in G' by a $Y - \triangle$ exchange of G is not wholly present in H'. Thus, by the claim, H' is a minor of G. However, as G is minor-minimal with respect to Q, H' cannot have property Q. This is a contradiction.

In both case we have reached a contradiction. Thus, G' is minor-minimal with respect to Q. $\square$

This result is interesting in that Q is not necessarily constrained to S^1 properties nor even to linking properties. This theorem is valid for any property preserved by $\triangle - Y$ exchange. Further, before this result, proofs of minor-minimality could only be performed by tedious case checking. Now, if the last graph of a graph family (the graph on which $\triangle - Y$ exchange can no longer be performed) is proved to be minor-minimal with respect to Q, all preceding graphs in that family are also minor-minimal with respect to Q.

5 Preliminary Theorems

During the course of our research, we proved and then used various theorems and lemmata. Lemma 5.2 uses Lemma 5.1 to prove a useful S^1 analog of a linking lemma in S^3, which may be found in [7]. Given two links and a set of conditions, it is possible to determine the existence of a 3-link without specifically knowing how the three link occurs. This lemma was useful in proving that certain graphs had the intrinsic S^1 3-linking property. Later in the paper, we will use this theorem to prove a result about graphs with a cut vertex. It should be noted that this lemma would not hold under the old definition of 0-sphere (see Section 2).

Lemma 5.1. *Given 0-spheres* $\{a, b\}, \{c, d\}, \{c, e\}$, *and* $\{d, e\}$ *in an* S^1 *embedding of graph G with* $\{a, b\}$ *disjoint from each of* $\{c, d\}$, $\{c, e\}$, *and* $\{d, e\}$, $lk_2(\{a, b\}, \{c, e\}) =_{mod\ 2} lk_2(\{a, b\}, \{c, d\}) + lk_2(\{a, b\}, \{d, e\})$.

Proof. Consider an S^1 embedding of G with the given 0-spheres $\{a, b\}, \{c, d\}$, $\{c, e\}$, and $\{d, e\}$. Observe that any 0-sphere disjoint from $\{c, d\}, \{d, e\}$, and $\{c,e\}$, e.g. $\{a,b\}$, that forms a non-split link with at least one of $\{c,d\},\{d,e\}$, and $\{c,e\}$ will form a non-split link with two and only two of $\{c,d\},\{d,e\}$, and $\{c,e\}$. The lemma follows. $\square$

Lemma 5.2. *Consider an S^1 embedding of a graph G with 0-spheres $\{x,y\}$, $\{x,z\}$, $\{s,t\}$, and $\{p,q\}$, such that $\{s,t\},\{x,y\}$, and $\{p,q\}$ are pairwise disjoint, $\{s,t\},\{x,z\}$, and $\{p,q\}$ are pairwise disjoint, and $\{x,y\}$ and $\{x,z\}$ intersect at vertex x. If $lk_2(\{s,t\},\{x,y\}) = 1$ and $lk_2(\{p,q\},\{x,z\}) = 1$, then one of the following is an S^1 3-link:*

1. $\{s,t\},\{x,y\},\{p,q\}$
2. $\{s,t\},\{x,z\},\{p,q\}$
3. $\{s,t\},\{y,z\},\{p,q\}$

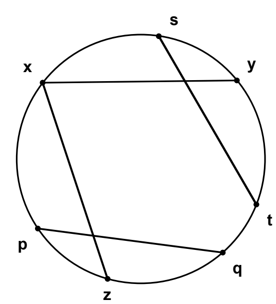

Figure 4: An S^1 embedding illustrating Lemma 5.2

Proof. (Of Lemma 5.2) Consider an S^1 embedding of a graph G with 0-spheres $\{x,y\}$, $\{x,z\}$, $\{s,t\}$, and $\{p,q\}$, such that $\{s,t\},\{x,y\}$, and $\{p,q\}$ are mutually disjoint, $\{s,t\},\{x,z\}$, and $\{p,q\}$ are mutually disjoint, and $\{x,y\}$ and $\{x,z\}$ intersect at vertex x. Assume $lk_2(\{s,t\},\{x,y\}) = 1$, and $lk_2(\{p,q\},\{x,z\}) = 1$.

Suppose that both $\{s,t\},\{x,y\},\{p,q\}$ and $\{s,t\},\{x,z\},\{p,q\}$ do not form 3-links as in Figure 4. Then, $lk_2(\{x,y\},\{p,q\}) = 0$ and $lk_2(\{s,t\},\{x,z\}) = 0$. By Lemma 5.1, $lk_2(\{s,t\},\{y,z\}) = lk_2(\{s,t\},\{x,y\}) + lk_2(\{s,t\},\{x,z\}) \cong (1+0)_{\text{mod }2} \cong 1_{\text{mod }2} \cong 1$. Also, by Lemma 5.1, $lk_2(\{p,q\},\{y,z\}) = lk_2(\{p,q\},\{x,y\}) + lk_2(\{p,q\},\{x,z\}) \cong (0+1)_{\text{mod }2} \cong 1_{\text{mod }2} \cong 1$. Thus $\{s,t\},\{y,z\}$, and $\{p,q\}$ form a 3-link. Hence the result. $\square$

When looking for graphs that are intrinsically S^1 3-linked, it is helpful to be able to quickly identify graphs that are not S^1 3-linked. In essence, determining that a graph is not intrinsically S^1 3-linked relies on finding an S^1 embedding in which a 3-link is not present. The following theorem uses properties of certain kinds of graphs to demonstrate the existence of S^1 3-linkless embeddings.

Theorem 5.3. *Let G' be a graph that is not intrinsically S^1 3-linked and G'' be a graph that is not intrinsically S^1 linked. Let G be the graph formed by pasting G' and G'' at a vertex v. Then G is not intrinsically S^1 3-linked.*

Proof. It will suffice to produce an S^1 embedding of G without a 3-link. Begin with a S^1 3-linkless embedding of G'. Let one of the vertices neighboring v be called x. Now embed G'' in a different component of $S^1 - \{v, x\}$ than the other vertices of $G' - \{v, x\}$ and in such a way that the S^1 embedding of G'' is linkless. This is an S^1 embedding of G. It only remains to be shown that the S^1 embedding is 3-linkless; that is, every set of three disjoint 0-spheres forms a split 3-link.

Consider an arbitrary set of three disjoint 0-spheres in this S^1 embedding. If all three are contained entirely in G', then they do not form a 3-link because the S^1 embedding of G' is 3-linkless. If two 0-spheres are contained in G' and one is contained in G'', or one 0-sphere is contained in G' and two are contained in G'', then they do not form a 3-link because every link containing a 0-sphere in each of G' and G'' is split. Finally, if all three 0-spheres are contained in G'', then there is no 3-link because this S^1 embedding of G'' is linkless.

Thus, consider the case that one of the 0-spheres contains vertices in both $G' - \{v\}$ and $G'' - \{v\}$. Because of disjointness, at most one of the three 0-spheres can contain vertices in both $G' - \{v\}$ and $G'' - \{v\}$. Denote this 0-sphere $\{v_1, v_n\}$ and the path justifying it $(v_1, \ldots, v_i, v, v_{i+1}, \ldots, v_n)$ where $\{v_1, \ldots, v\} \in G'$ and $\{v, \ldots, v_n\} \in G''$. There are two subcases.

First consider the case in which at least one of the other two 0-spheres lies entirely in G''. That 0-sphere forms split links with both $\{v_1, v\}$ (because each link containing a 0-sphere in each of G' and G'' is split) and $\{v, v_n\}$ (because G'' is linkless). Thus, it forms a split link with $\{v_1, v_n\}$. Similarly, that 0-sphere also forms a split link with the remaining 0-sphere. Therefore, the link formed by these three 0-spheres is a split 3-link.

Now consider the case in which the remaining two 0-spheres lie in G'. These 0-spheres do not form a 3-link with $\{v_1, v\}$ because this S^1 embedding of G' is 3-linkless. Further, $\{v, v_n\}$ does not form a link with either 0-sphere. Therefore, there is no 3-link. □

Another similar theorem follows. The proof is omitted due to its similarity with the proof of Theorem 5.3.

Theorem 5.4. *Let G' be a graph that is not intrinsically S^1 3-linked and G'' be a graph that is not intrinsically S^1 linked. Let G be the graph formed by pasting G' and G'' along a single edge $e_{a,b}$, where the edge $e_{a,b}$ is in G' and G''. If there exists a 3-linkless S^1 embedding of G' and a linkless S^1 embedding of G'' such that a and b are neighbors in both embeddings, then G is not intrinsically S^1 3-linked.*

Here we present several other theorems that we proved in the course of our research and were helpful in pursuing our results. We have omitted the proofs of Theorem 5.7 and Theorem 5.8 because they are overly technical and do not provide insight into this problem.

Theorem 5.5. *If a graph G is intrinsically S^1 3-linked and v and w are any two vertices in G, then $G - \{v, w\}$ contains a cycle or a vertex of degree at least three.*

Proof. We will prove the contrapositive. Consider a graph G as defined above. Suppose $G - \{v, w\}$ does not contain a cycle or a vertex of degree 3 or greater. Then $G - \{v, w\}$ can be embedded (the entire graph!) in $\mathbb{R}^1$. As $G - \{v, w\}$ can be embedded in $\mathbb{R}^1$, it can be embedded into S^1 in such a way that any vertex is adjacent only to vertices that it neighbors. (Note that neighboring does not imply adjacency.) Also, as there is no cycle in $G - \{v, w\}$, there exist neighboring vertices a and b such that a and b are not adjacent. Place the vertices v and w into the associated S^1 embedding of $G - \{v, w\}$ as neighbors such that v neighbors a and w, and w neighbors v and b. Any set of three disjoint 0-spheres contains a 0-sphere without v or w in its associated path. Call it $\{p, q\}$. Because each vertex in the justifying path $(p, \dots, q)$ is only adjacent to its neighbors, there is a component of $S^1 - \{p, q\}$ that contains only vertices in the justifying path of $\{p, q\}$, and therefore cannot be involved in a non-split link. Thus, in this embedding, any set of three disjoint 0-spheres forms a split link. Therefore, G is not intrinsically S^1 3-linked. Hence, the result by contrapositive. □

Theorem 5.6. *Let G be a minor-minimal graph with the intrinsic S^1 3-linking property. Then G does not contain a 3-cycle with one of the vertices of the 3-cycle of degree 2.*

Proof. Suppose that G contains a 3-cycle with degree 2 vertex. Let c be one such degree 2 vertex, and let a and b be the other two vertices that form the 3-cycle. Consider an arbitrary S^1 embedding of G. In this embedding, edge $e_{a,b}$ is part of a path justifying a component of a 3-link or it is not. If it is not, then the deletion of edge $e_{a,b}$ from G will not affect the 3-link. Then suppose edge $e_{a,b}$ is part of a path $(x, \ldots, a, b, \ldots, y)$ justifying a component of a 3-link, $\{x, y\}$. Note that the edges $e_{a,c}$ and $e_{b,c}$ are not disjoint from this path and thus cannot be part of paths justifying any other component of the 3-link. Hence, vertex c is not part of any path justifying a component of the 3-link. Delete edge $e_{a,b}$ from G and consider the S^1 embedding of this new graph with the same ordering of vertices. Justify the 0-sphere $\{x, y\}$ in this new graph by the path $(x, \ldots, a, c, b, \ldots, y)$. The 3-link is unchanged. Thus, any S^1 embedding of G with edge $e_{a,b}$ deleted has a 3-link. This is a contradiction as G is minor-minimal with respect to the intrinsic S^1 3-linking property. Thus no minor-minimal intrinsically S^1 3-linked graph contains a 3-cycle with a degree 2 vertex. □

Theorem 5.7. *There are no minor-minimal intrinsically S^1 3-linked graphs with a degree two vertex adjacent to two degree two vertices.*

Theorem 5.8. *No planar graph with six vertices is intrinsically S^1 3-linked.*

The proof for Theorem 5.8 is a demonstration that every minor of K_6 is either non-planar or S^1 3-linkless.

6 The Graphs

Here we present the graphs that we have shown, using the results of sections 5 and 7, to be minor minimal with respect to intrinsic S^1 3-linking (Theorem 1.1). Note that in Figures 5 and 6 the arrows represent $\triangle - Y$ exchange.

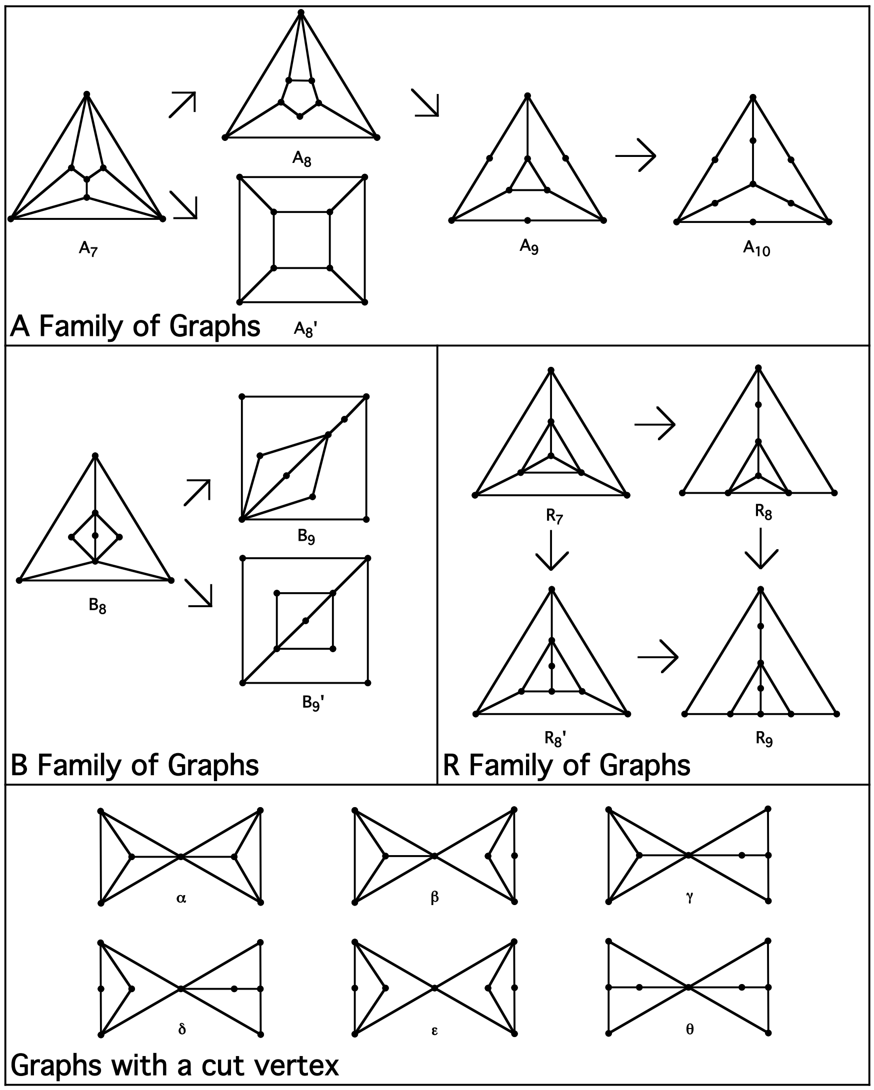

Figure 5: Minor-minimal intrinsically S^1 3-linked graphs

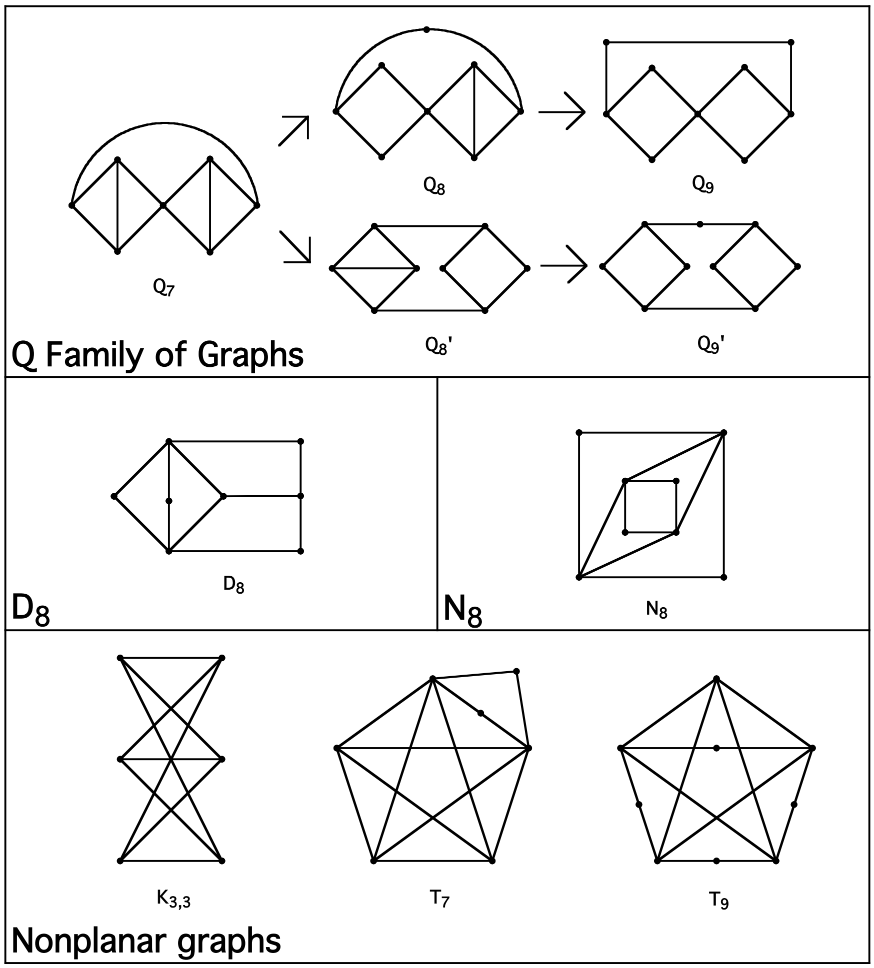

Figure 6: Minor-minimal intrinsically S^1 3-linked graphs, continued

7 Selected Proofs

Although we have rigorously shown that each of the listed graphs is minor minimal intrinsically S^1 3-linked, we do not present every proof here. Many of the proofs are a result of tedious case checking, which is not insightful. We will, however, include the proof that the graph R_7 is intrinsically S^1 3-linked as a short example of such a proof. We also will show that the graphs $\alpha, \beta, \gamma, \delta, \varepsilon$, and θ are intrinsically S^1 3-linked, and that they are the only minor minimal intrinsically S^1 3-linked graphs with a cut vertex.

Refer to Figure 7 for the labeling of the vertices used in the proof that R_7 is intrinsically S^1 3-linked.

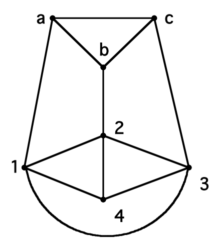

Figure 7: The graph R_7

Theorem 7.1. *The graph R_7 is intrinsically S^1 3-linked.*

Proof. R_7 contains K_4 (vertices 1-4 in Figure 7) as a subgraph. Embed the K_4 in S^1. There is, up to symmetry, only one embedding of the K_4 subgraph. This embedding shown in Figure 8. Then consider the possible placement of vertices a, b, and c. If any of the two of these vertices are in different components of $S^1 - \{1, 2, 3, 4\}$, they will form a 3-link with the 0-spheres $\{1, 3\}$ and $\{2, 4\}$. Now consider the case that all three vertices are in one component of $S^1-\{1, 2, 3, 4\}$. Let x denote the vertex of a, b, c that neighbors both others. This vertex is adjacent to one of vertices 1, 2, 3, or 4. Thus there is an edge $e_{x,y}$ for some $y \in \{1, 2, 3, 4\}$ that forms a non-split link with the edge $e_{w,z}$ where w and z are the neighbors of x. Note also that y is an endpoint of the non-split link formed by $\{1, 3\}$ and $\{2, 4\}$. Then by Lemma 5.2 there is a 3-link. □

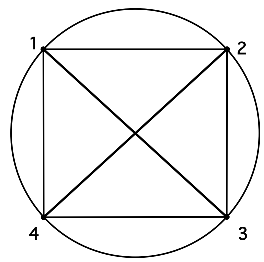

Figure 8: An embedding of vertices 1-4 of R_7

One simple way to form intrinsically S^1 3-linked graphs is by pasting together two intrinsically S^1 linked graphs at one vertex, forming a graph with a cut vertex. The 3-link in the final graph will be composed of the two 2-links in each of the component graphs. This will follow from Lemma 5.2. Because there are only two minor minimal intrinsically S^1 linked graphs, there are only a small number of ways to paste them together to get a minor minimal intrinsically S^1 3-linked graph.

Theorem 7.2. *Let G be a graph formed by pasting together graphs A and B, where A and B are each either a K_4 or $K_{3,2}$, at a vertex. The graph G is intrinsically S^1 3-linked.*

Proof. Given an embedding of G in which the cut vertex is the endpoint of a 0-sphere in a link in both subembeddings of A and B, then G is 3-linked by Lemma 5.2. Thus it will suffice to show that in every S^1 embedding of K_4 and $K_{3,2}$, each vertex can be written as the endpoint of a 0-sphere in a link. For K_4, this follows immediately. For $K_{3,2}$ it requires a little more work. Let the vertices of the $K_{3,2}$ be labeled as $1, 2, a, b$, and c, where the lettered vertices are adjacent to each of the numbered vertices. In each embedding of this graph there is a link. There are two cases. First, suppose, without loss of generality, that $\{1, a\}$ and $\{2, b\}$ form a link. It must be shown that c can be written as the endpoint of a 0-sphere in a link. If c is on the same component of $S^1 - \{2, b\}$ as 1, then $\{c, a\}$ (justified by $(c, 1, a)$) and $\{2, b\}$ form a link. If c is on the same component of $S^1 - \{2, b\}$ as a, then $\{1, c\}$ and $\{2, b\}$ form a link. For the second case, suppose, without loss of generality, that $\{1, a\}$ and $\{b, c\}$ form a link. It must be shown that 2 can be written as the endpoint of a 0-sphere in a link. If 2 is on the same component of $S^1 - \{b, c\}$ as 1, then

$\{2, a\}$ and $\{b, c\}$ form a link. If 2 is on the same component of $S^1 - \{b, c\}$ as a, then $\{1, 2\}$ (justified by $(1, a, 2)$) and $\{b, c\}$ form a link. Thus, we have shown that every vertex of $K_{3,2}$ in an S^1 embedding can be represented as the end point of a link. □

Further, these graphs are minor minimal. This will follow from Theorem 5.3:

Theorem 7.3. *Each of the above graphs with a cut vertex is minor minimal with respect to intrinsic S^1 3-linking.*

Proof. Let the G be the composite graph as defined in the previous theorem. Consider any edge contraction, edge removal, or vertex removal performed on G. Without loss of generality, performing any of those three operations on G to obtain G' is equivalent to performing the operation on A to produce A' and then pasting A' and B to get G'. But because A is minor minimal with respect to intrinsic S^1 linking, A' must have an S^1 embedding without a non-split link. Then, by Theorem 5.3, there exists a 3-linkless S^1 embedding of G'. □

Now it just remains to show that these are the only minor minimal intrinsically S^1 3-linked graphs that have a cut vertex. This will follow from Lemma 5.2 and Theorem 5.3.

Theorem 7.4. *If a graph G is minor minimal intrinsically S^1 3-linked and has a cut vertex v, then G is composed of two minor minimal intrinsically S^1 linked graphs pasted at v.*

Proof. Denote the components of $G - \{v\}$ as $\{X_i\}_{i=1}^n$. Then let H_i be the subgraph of G such that $H_i = G - \sum_{k \neq i} X_k$. Note that each H_i is intrinsically S^1 linked; if H_i were not intrinsically linked, then by Theorem 5.3 $G - H_i$ is intrinsically S^1 3-linked, which is a contradiction. Given that each H_i is intrinsically S^1 linked, any H_i and H_j form an intrinsically S^1 3-linked graph by Lemma 5.2. Therefore, G must be precisely $H_i \amalg H_j$ because G is minor minimal. Therefore G is composed of exactly two intrinsically S^1 linked graphs, denoted H and H'.

It remains to show that both H and H' are minor minimal with respect to the intrinsic S^1 linking property. For the sake of contradiction, assume there is a minor, $\bar{H}$ of H that is intrinsically S^1 linked. Then consider the graph formed by pasting $\bar{H}$ and H' at a vertex. By Lemma 5.2 this graph is intrinsically S^1 3-linked. But this graph is a minor of G, so this is a contradiction. □

We omit the proof of the following theorem because it is overly long and tedious:

Theorem 7.5. *The non-planar graphs shown in Figure 6 form the complete set of non-planar minor minimal intrinsically S^1 3-linked graphs.*

8 n-Component Links in S^1

Another property considered is intrinsic S^1 n-linking. A graph G is intrinsically S^1 n-linked if every S^1 embedding of G contains a non-split link of n-components. We will show that the graph $K_{n,n}$ is intrinsically S^1 n-linked for $n \geq 3$.

Theorem 8.1. *The graph $K_{n,n}$ is intrinsically S^1 n-linked for all $n \geq 3$*

Lemma 8.2. *Given an S^1 embedding of $K_{n,n}$ for some $n > 3$ such that adjacent vertices a and b of $K_{n,n}$ are not neighbors, if $K_{n-1,n-1}$ is intrinsically $(n-1)$-linked, then the 0-sphere $\{a, b\}$ is a component of an n-link in the S^1 embedding of $K_{n,n}$.*

Proof. Consider an S^1 embedding of $K_{n,n}$ for some $n > 3$ such that adjacent vertices a and b of $K_{n,n}$ are not neighbors. Suppose that $K_{n-1,n-1}$ is intrinsically $(n-1)$-linked. Then, the S^1 embedding of $K_{n-1,n-1}$ obtained by deleting a and b from $K_{n,n}$ has a non-split link of $(n-1)$-components, denoted L. If all of the components of L lie in the same component of $S^1 - \{a, b\}$, then a and b are neighbors. As a and b are not neighbors, this is a contradiction. Then there is a component of L that is non-split linked with $\{a, b\}$. Thus the 0-sphere $\{a, b\}$ is a component of an n-component link in the S^1 embedding of $K_{n,n}$.

□

Proof. (Of Theorem 8.1) We will prove by induction for $n \geq 3$.

For the base case, let $n = 3$. Up to symmetry, there are three distinct S^1 embeddings of $K_{3,3}$. As there exists a 3-link in each of these S^1 embeddings, as seen in Figure 9, $K_{3,3}$ is intrinsically S^1 3-linked.

For our induction step, given $n > 3$, we will assume that $K_{n-1,n-1}$ is intrinsically S^1 $(n-1)$-linked.

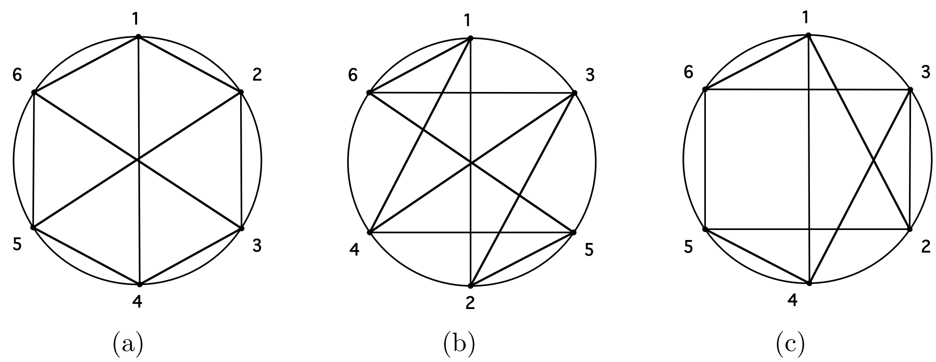

Figure 9: The three S^1 embeddings of $K_{3,3}$, up to symmetry

Consider an arbitrary S^1 embedding of $K_{n,n}$. As each vertex in $K_{n,n}$ has degree n, where $n \geq 3$, each vertex is adjacent to at least one non-neighboring vertex. Then, consider adjacent non-neighboring vertices a and b. By Lemma 8.2, as $K_{n-1,n-1}$ is intrinsically S^1 $(n-1)$-linked (by our assumption), the 0-sphere $\{a, b\}$ is a component of an n-component link in the S^1 embedding of $K_{n,n}$. As this was an arbitrary S^1 embedding of $K_{n,n}$, every S^1 embedding contains a non-split link of n-components. Therefore, $K_{n,n}$ is intrinsically S^1 n-linked.

By the principle of mathematical induction, as our base case and induction step are true, $K_{n,n}$ is intrinsically S^1 n-linked for all $n \geq 3$. □

9 Intrinsically Pairwise Non-Split S^1 3-Linked Graphs

A *pairwise non-split 3-link* is a three component link in which each component of the link is non-split linked with each of the other two components. In [6], Flapan et al. describe a graph that is intrinsically pairwise non-split 3-linked in space. In this section we are concerned with graphs that are pairwise non-split 3-linked in S^1. We will prove that the graphs K_6 and $K_{4,3}$ are intrinsically pairwise non-split S^1 3-linked. From this, we will show that the graphs of the Petersen Family are all intrinsically pairwise non-split S^1 3-linked. Finally, using this result we show that intrinsically linked in space implies pairwise non-split 3-linked in S^1. Note that the converse is not true.

For example, the graph $K_{4,3}$, a subgraph of $K_{3,3,1}$ is intrinsically pairwise nonsplit S^1 3-linked but not intrinsically linked in space.

Theorem 9.1. *The graph K_6 is intrinsically pairwise non-split S^1 3-linked.*

Proof. There is only one distinct S^1 embedding of K_6, as seen in Figure 10. There is a pairwise non-split 3-link in this embedding. Thus, in every S^1 embedding there is a pairwise non-split 3-link. So K_6 is intrinsically pairwise non-split S^1 3-linked. □

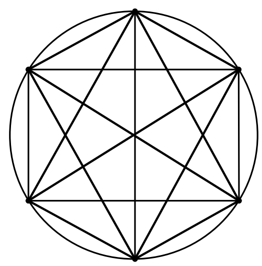

Figure 10: The S^1 embedding of K_6

Not only is K_6 intrinsically pairwise non-split S^1 3-linked, but it is also minor minimal with respect to this property. The proof is simple and we will omit it.

Theorem 9.2. *The graph $K_{4,3}$ is intrinsically pairwise non-split S^1 3-linked.*

Proof. Label the vertices of $K_{4,3}$ such that each of the vertices 1,3,5 and 7 are adjacent to each of 2,4, and 6. Consider the S^1 embeddings possible. There are four cases up to symmetry. Refer to Figure 11. In the first case, S^1 embed $K_{4,3}$ as in Figure 11(a). The 0-spheres $\{1,4\},\{2,5\}$, and $\{3,6\}$ form a pairwise non-split S^1 3-link. In the second case, S^1 embed $K_{4,3}$ as in Figure 11(b). The 0-spheres $\{4,7\}$, $\{2,5\}$, and $\{1,3\}$ (justified by (1,6,3)) form a pairwise non-split S^1 3-link. In the third case, S^1 embed $K_{4,3}$ as in as in Figure 11(c). The 0-spheres $\{1,6\}$, $\{2,5\}$, and $\{4,7\}$ form a pairwise non-split S^1 3-link. In the final case, S^1 embed $K_{4,3}$ as in as in Figure 11(d). The 0-spheres $\{1,6\}$, $\{2,5\}$, and $\{4,7\}$ form a pairwise non-split S^1 3-link. In every case there is a pairwise non-split S^1 3-link. Thus, $K_{4,3}$ is intrinsically pairwise non-split S^1 3-linked. □

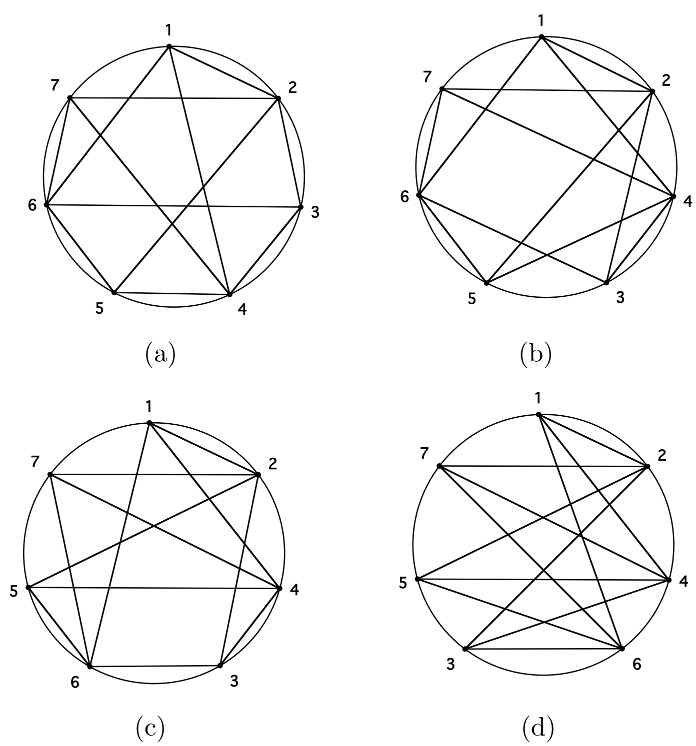

Figure 11: Four S^1 embeddings of $K_{4,3}$

As $K_{3,3,1}$ is a supergraph of $K_{4,3}$, $K_{3,3,1}$ is intrinsically pairwise non-split S^1 3-linked. Now that we have that K_6 and $K_{3,3,1}$ are intrinsically pairwise non-split 3-linked, we are concerned with creating a larger set of graphs with the same property. Indeed, the operation $\triangle - Y$ exchange preserves the intrinsic pairwise non-split S^1 3-linking property; the proof follows similarly to that of Theorem 4.1. With that, the next theorem follows simply:

Theorem 9.3. *The graphs of the Petersen Family are all intrinsically pairwise non-split S^1 3-linked.*

Corollary 9.4. *If a graph G is intrinsically linked in space, then it is intrinsically pairwise non-split 3-linked in S^1.*

Proof. Let G be a graph that is intrinsically linked in space. Then, as the Petersen Family of graphs forms the complete minor minimal set of graphs that are intrinsically S^3 linked, G contains a Petersen graph as a minor. Note that any supergraph of a graph with a certain linking property will have that same property. Then, as G contains a Petersen graph, which is intrinsically pairwise non-split S^1 3-linked by the previous theorem, as a minor, G is intrinsically pairwise non-split S^1 3-linked. □

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