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Stabilizing a Subcritical Bifurcation in a Mapping Model of Cardiac-Membrane Dynamics

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1 Introduction

1.1 Background Information

When the heart beats, individual cardiac cells are initially induced to a lower voltage from their resting potential. Under normal pacing conditions, the time duration required for a cardiac cell to recover to the resting voltage is constant for subsequent evenly spaced stimuli. By contrast, an irregular response termed *alternans* describes recovery periods that alternate between two fixed values under constant pacing frequency. It is believed that alternans may be a precursor to ventricular fibrillation, an arrhythmia causing sudden cardiac death.

Several authors have studied the use of feedback to suppress alternans [2]. Indeed, this has been successfully implemented in small pieces of paced *in vitro* bullfrog heart [2]. In this paper we study theoretically a variant of such control problems.

A full description of the electrical response of the heart to stimulation is difficult, involving a very stiff, many-variable coupled system of time-dependent PDE (of reaction-diffusion type) in an exceedingly complicated 3D geometry. If the spatial dependence in the problem is suppressed (which is approximately valid for small pieces of *in vitro* heart), the description reduces to a system of ODE, often called an *ionic model* since what is being described is the transport of ions across the cell membrane. The most important variable in this system (and the most accessible experimentally) is the transmembrane potential. Typical responses of this variable to a periodic sequence of stimuli are shown in Figure 1. This figure illustrates the property of cardiac tissue that is called *excitability*: i.e., a small stimulus leads to a raised voltage for an extended period of time, far longer than the duration of the stimulus itself, after which the voltage returns to its resting value. Such a time course of the voltage is called an *action potential*. Figure 1a shows a steady situation in which the same response

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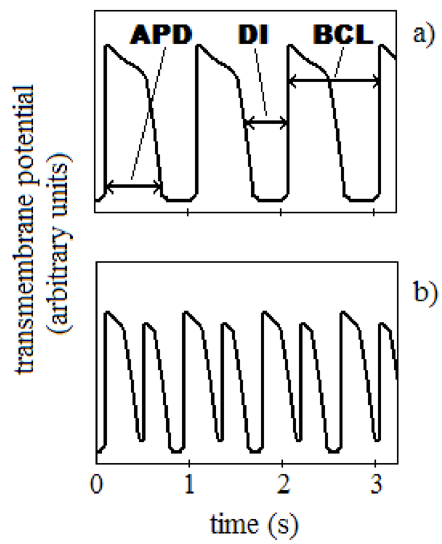

Figure 1: Transmembrane potential response to periodic stimulation. In part a), the stimulation is such that 1:1 response is observed. At more rapid stimulation as in part b), 2:2 response or alternans is observed.

occurs to every stimulus in a periodic sequence, what is called 1:1 response. Such 1:1 response is typical when the stimulus period is large. By contrast, Figure 1b shows an alternans or 2:2, response.

To understand arrhythmias, one needs to examine the dynamics of stimulation and response. Much insight into the dynamics can be gained from a simple model introduced by Nolasco and Dahlen [4] in 1968. In this approximation, each action potential is characterized by a single number, its duration (acronym: APD). More formally, the APD denotes the length of the time interval during which the voltage exceeds a critical voltage. Nolasco and Dahlen proposed the approximation: there is a function F such that the evolution of successive APDs under repeated stimulation is given by

$$APD_{n+1} = F(DI_n) \tag{1}$$

where DI_n, an acronym for *diastolic interval*, equals the time elapsed between the end of the n^{th} action potential and the next stimulus. If the stimuli are applied with period BCL (acronym for basic cycle length), then $DI_n = BCL - APD_n$, so APD_n evolves according to the iterated mapping

$$APD_{n+1} = F(BCL - APD_n) \tag{2}$$

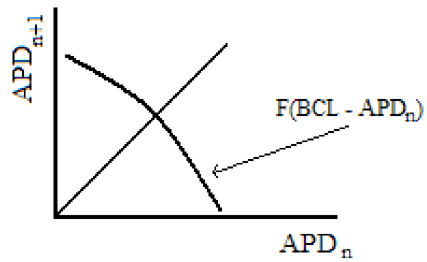

Figure 2: Graph of $F(BCL-APD_n)$ and the line $APD_{n+1} = APD_n$. It can be seen that the sole fixed point APD_* of $F(BCL - APD_n)$ is at its intersection with the line $APD_{n+1} = APD_n$.

Based on experiment, previous authors proposed the function [4]

$$F(DI) = A_{max} - Ce^{-DI/\tau}, \tag{3}$$

where A_{max}, C and τ are constants. Various other forms have also been proposed. In particular, it was shown in [3], using asymptotics, that the behavior of the ODE in a simple ionic model can be approximately described by (1) with

$$F(DI) = \tau_{close} \ln\{\frac{1 - (1 - h_{min})e^{-DI/\tau_{open}}}{h_{min}}\} \tag{4}$$

where τ_{close}, h_{min}, and τ_{open} are parameters derived from the ionic model.

Despite the oversimplification of the approximation (2), it can successfully model alternans. In explaining this, we assume the reader is familiar with the elementary theory of 1D iterated mappings, as discussed for example in Chapter 10 of [5]. Suppose that the function $F(DI)$ is monotone increasing. (Both (3) and (4) have this property.) Then, as illustrated in Figure 2, for any BCL, there is a unique fixed point APD_* such that

$$APD_* = F(BCL - APD_*) \tag{5}$$

If such a fixed point is stable, then $APD_n \to APD_*$ is a possible asymptotic behavior for $n \to \infty$ of a sequence generated by (2). If BCL is large, then the derivative $F'(BCL - APD_*)$ at the fixed point is small, in particular less than unity, so the fixed point is indeed stable [5]. However, as BCL decreases,

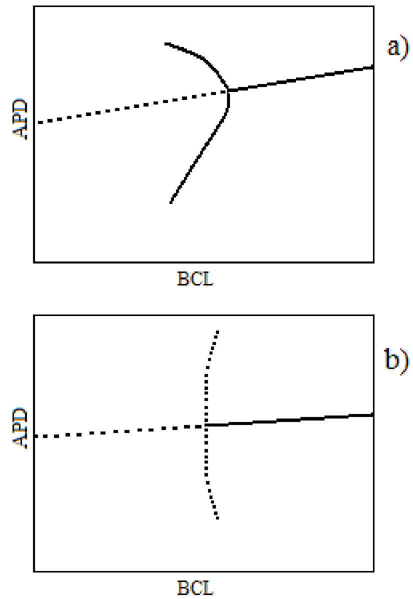

Figure 3: Bifurcation diagram showing APD_* versus BCL for a supercritical bifurcation (a) and a subcritical bifurcation (b). Unstable critical points are shown in dashed lines.

$F'(BCL-APD_*)$ increases, and if $F'(BCL-APD_*)$ passes through unity, the iteration suffers a period-doubling bifurcation.

This behavior is conveniently summarized in a bifurcation diagram (see Figure 3), which for each BCL, plots the limit points of sequences $\{APD_n\}$ generated by (2). Figure 3a illustrates a supercritical bifurcation, a bifurcation into a set of stable critical points, which is the more familiar alternative and the only possibility for the mapping (3) . However, it was shown in [3] that, for certain parameter ranges, (4) can undergo a subcritical bifurcation, a bifurcation into a set of unstable critical points, as illustrated in Figure 3b.

In this paper we are concerned with the case of a subcritical bifurcation. Note that there is a range $BCL_{bif} < BCL < BCL_{crit}$ in which 1:1 response is stable and there exists a small-amplitude, unstable 2:2 response.

1.2 Properties of the Map

To facilitate analysis, we choose a simpler map than (4) that still produces a subcritical bifurcation as shown in Figure 3b. The simplified map used is the following:

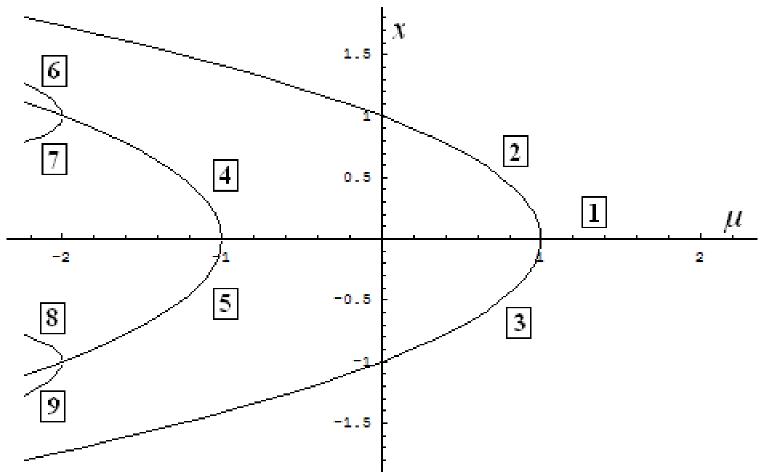

Figure 4: Bifurcation diagram of (6) over the range $-\sqrt{5} < \mu < \sqrt{5}$. Higher period points exist for $\mu < -\sqrt{5}$ but are not shown. Boxed numbers correspond to branches of critical points of the system. These points and their stability are further described in the table at the end of Section 1.2

$$x_{n+1} = f(x_n) = -\mu x_n - x_n^3 \tag{6}$$

Comparing this iterated map to (4), we can consider x_n analogous to APD and μ analogous to BCL. Note the bifurcation diagram of this map in Figure 4.

We focus only on the parameter range $-1 < \mu < 1$ in our study. Over this range, there is a pair of unstable period-two points and a stable fixed point. This is similar to the situation seen in Figure 3b, the bifurcation diagram of (4) under certain parameter settings, except for reversal of orientation of the bifurcation parameter. Outside the range $-1 < \mu < 1$, period-two points either do not exist or occur in the presence of multiple fixed points. In either case, such changes would not hold true to the circumstance of the two-current model. Therefore, we will analyze (6) in the range $-1 < \mu < 1$ as an analogy to subcritical alternans as its simplicity will facilitate analysis while its construction still exhibits the necessary characteristics of the subcritical bifurcation seen in the two-current model.

To allow for a better understanding of the system, we describe the bifurcation diagram illustrated in Figure 4 for all μ, not just the range of special focus, $-1 < \mu < 1$. We identify nine solution branches in figure 4 by numbered boxes.

- In proceeding through the bifurcation diagram from positive to negative μ, points on branch 1 given by x=0 are unstable for $\mu > 1$ and then becomes stable via a subcritical bifurcation into points on branches 2 and 3 (period-two points) at $\mu = 1$. These period-two points are unstable throughout their range.

- As the fixed point on branch 1 (fixed point $x = 0$) loses stability at $\mu = -1$, it bifurcates into points on branches 4 and 5 (fixed points). These fixed points begin stable and lose their stability at $\mu = -2$.
- At $\mu = -2$ points on branches 4 and 5 (fixed points) lose stability and each bifurcates into a pair of stable period-two points. The fixed point on branch 4 bifurcates into period-two points on branches 6 and 7 while the fixed point on branch 5 bifurcates into period-two points on branches 8 and 9. The stable period-two points lose their stability at $\mu = -\sqrt{5}$.
- At $\mu = -\sqrt{5}$ the period-two points on branches 6 & 7 and 8 & 9 each lose stability via a period-doubling bifurcation. Beyond $\mu = -\sqrt{5}$, period-doubling bifurcations continue as μ becomes more negative.

This map has three fixed points and six period-two points. The following table is a summary of their behavior:

Point	Type	Exists for	Stable for	Maps to
1	Fixed point	all μ	$\lvert\mu\rvert < 1$	1
2	Period-two point	$\mu < 1$	Never	3
3	Period-two point	$\mu < 1$	Never	2
4	Fixed point	$\mu < -1$	$-2 < \mu < -1$	4
5	Fixed point	$\mu < -1$	$-2 < \mu < -1$	5
6	Period-two point	$\mu < -2$	$-\sqrt{5} < \mu < -2$	7
7	Period-two point	$\mu < -2$	$-\sqrt{5} < \mu < -2$	6
8	Period-two point	$\mu < -2$	$-\sqrt{5} < \mu < -2$	9
9	Period-two point	$\mu < -2$	$-\sqrt{5} < \mu < -2$	8

1.3 Feedback

Feedback control is simply the use of past information about a system to force future system characteristics. Feedback control is used, for example, when a driver accelerates or decelerates to reach a desired speed. Recall that in Hall and Gauthier [2] feedback is used to suppress alternans. In this paper we analyze an analogous problem: stabilizing a subcritical bifurcation. We would like to be able to devise a method using feedback control to stabilize alternans when theoretically possible but not observable under normal circumstances. Such a method could be used to test predictions of models, some of which predict subcritical bifurcations [3].

To stabilize the period-two points, we wish to perturb iterates of (6) by applying feedback, where an example of applying feedback is below in (7). In Hall and Gauthier [2], feedback is an adjustment of the BCL, which is an argument of

(4). This is an indirect method of adjusting the $n+1st$ iterate. In our analysis, we will adjust the $n+1st$ iterate directly in the following manner:

$$x_{n+1} = f(x_n) + \gamma g(x_n, x_{n-1}, ..., x_{n-k}) \tag{7}$$

Where $g(x_n, x_{n-1}, ..., x_{n-k})$ is a corrective function that adjusts up or down the $n+1st$ iterate directly using information about previous iterates and γ, a constant gain coefficient. In experiment, future APDs would be adjusted indirectly by increasing or decreasing the BCL so as to produce a similar perturbation to that caused by $\gamma g(x_n, x_{n-1}, ..., x_{n-k})$.

In our quest to devise a corrective scheme we make the following requirements:

1. $g(x_n, x_{n-1}, ..., x_{n-k})$ must stabilize the previously unstable period-two points and destabilize the previously stable fixed point. In particular, $g(x_n, x_{n-1}, ..., x_{n-k})$ should vanish at the period-two points to ensure that correction stops after the system becomes steady-state at these points.

2. $g(x_n, x_{n-1}, ..., x_{n-k})$ must use as little information about the function (6) as possible. This is to allow for our corrective scheme to be less sensitive to experimental error and to discrepancies between models and reality.

3. Correction should be used as often as possible. This requirement is necessary to combat the error and noise inherent in experimental measurements and therefore the information used in correction.

A corrective function used by previous authors in a similar context to stabilize an unstable fixed point involves the difference between the nth and the $n-1st$ iterates:

$$x_{n+1} = f(x_n) + \gamma(x_n - x_{n-1}) \tag{8}$$

One can quickly see, however, that this scheme is ill-suited because the corrective function cannot vanish at the period-two points[1].

[1]

To fix this problem for the map (6), one could modify the relation between the nth and $n-1st$ iterates:

$$x_{n+1} = f(x_n) + \gamma(x_n + x_{n-1})$$

Such a function, however, requires that the period-two points be symmetric about zero. If the period-two points weren't symmetric about zero but about some non-zero fixed point, introducing a constant term into the corrective function could force the function to vanish at the period-two points but the fixed point would have to be known. This would require too much information about (6) to be known, thus reducing the generality of the corrective scheme.

In devising a new corrective function, it is helpful to first think about what information is known about (6). Here, we will assume that all that is known about (6) is that it has one stable fixed point and two unstable period-two points. By definition, the period-two points satisfy $f(f(x)) = x$. If the system were to reach equilibrium at these points, the difference between the nth and the $n - 2nd$ iterates would be zero. Therefore, an intuitive suggestion for the corrective function $g(x_n, x_{n-1}, ..., x_{n-k})$ would be the difference between the nth and the $n - 2nd$ iterates:

$$x_{n+1} = f(x_n) + \gamma(x_n - x_{n-2}) \tag{9}$$

Although this seems like a viable option for a corrective scheme, it ultimately fails. A proof that this scheme fails will be provided in section 2.2. In this paper, we prove that the most natural candidates for a corrective scheme scheme do not work and devise a new scheme that is capable of stabilizing unstable period-two points.

2 Corrective Schemes

2.1 Stability Analysis

Before analyzing particular corrective schemes, we need to understand how to diagnose their stability as well as the stability of the map without feedback. The fixed points of an iterated map will be stable iff the eigenvalues λ_i of the Jacobian satisfy the following requirement [1]:

$$|\lambda_i| < 1, \forall i \tag{10}$$

Note that the period-two points of $f(x_n)$ are fixed points of the map composed with itself

$$h(x) = f(f(x)) \tag{11}$$

For $-1 < \mu < 1$ the location of the period-two points can be solved analytically:

$$x^{(1)} = \sqrt{1-\mu},\ x^{(2)} = -\sqrt{1-\mu} \tag{12}$$

Linearizing (11) and evaluating at the alternans points yields

$$h'(\pm\sqrt{1-\mu}) = (9 - 12\mu + 4\mu^2)^2 = (f'(\sqrt{1-\mu})^2 = f'(-\sqrt{1-\mu})^2 \tag{13}$$

Note that

$$f'(\pm\sqrt{1-\mu}) - 1 = 4(\mu-1)(\mu-2) \tag{14}$$

It can be seen from (14) that $h'(\pm\sqrt{1-\mu}) > 1$ for $-1 < \mu < 1$ because $f'(\pm\sqrt{1-\mu}) > 1$ for $-1 < \mu < 1$.

In this paper, we will propose various correction schemes, which we will later classify as one-step, two-step and three-step correction schemes. The one-step scheme will simply be (9), where the difference between the nth and the $n-2nd$ iterates is used as a corrective function multiplied by a constant gain parameter γ.

The two-step scheme will use a corrective function of the same form on each iterate, the difference between the nth and the $n-2nd$ iterates, but will switch between two constant gain parameters γ_1 and γ_2. The two-step scheme can be stated as follows:

$$x_{n+1} = f(x_n) + \gamma_1(x_n - x_{n-2})$$

$$x_{n+2} = f(x_{n+1}) + \gamma_2(x_{n+1} - x_{n-1})$$

where n is even. Note that the one-step scheme is a special case of the two-step scheme with $\gamma_1 = \gamma_2$. The two-step scheme can be rewritten in vector notation:

$$\left| \begin{array}{c} x_{n+2} \\ x_{n+1} \\ x_n \end{array} \right| = m(x_n,\ x_{n-1},\ x_{n-2}) = \left| \begin{array}{c} -\mu Y_n - Y_n^3 + \gamma_2(Y_n - x_{n-1}) \\ Y_n \\ x_n \end{array} \right| \tag{15}$$

where $Y_n = -\mu x_n - x_n^3 + \gamma_1(x_n - x_{n-2})$

Next, we setup a method for diagnosing the stability of the period-two points under the two-step scheme. Note that the two-step scheme is a function of x_n, x_{n-1}, and x_{n-2} that generates the next two iterates in the sequence. Rather than analyze this as a multi-step scheme, we may look at the two-step scheme as a function that takes an ordered triple and advances the indices by two (15). The linearization of the two-step scheme evaluated at the period-two points (12) is then shown below:

$$\left| \begin{array}{c} x_{n+2} \\ x_{n+1} \\ x_n \end{array} \right| = \left| \begin{array}{ccc} \xi & -\gamma_2 & 3\gamma_1 - \gamma_1\gamma_2 - 2\gamma_1\mu \\ -3+\gamma_1+2\mu & 0 & -\gamma_1 \\ 1 & 0 & 0 \end{array} \right| \left| \begin{array}{c} x_n \\ x_{n-1} \\ x_{n-2} \end{array} \right| \tag{16}$$

where the 1,1-entry is given by

$$\xi = 9 - 233\gamma_1 - 3\gamma_2 + \gamma_1\gamma_2 - 12\mu + 2\gamma_1\mu + 2\gamma_2\mu + 4\mu^2$$

The sequence of iterates produced by the two-step scheme will converge to the period-two points if the eigenvalues of (16) satisfy (10). Note that period-two points of $f(x_n)$ are fixed points of $m(x_n,\, x_{n-1},\, x_{n-2})$.

Next, we will determine a set of requirements on the entries of (16) to allow us to determine if this matrix satisfies (10). Note that the characteristic polynomial of (16), a 3x3 matrix, will be cubic:

$$-\lambda^3 - a\lambda^2 - b\lambda - c \tag{17}$$

We can place requirements on the real coefficients of (17) so that the eigenvalues of the Jacobian (16) satisfy (10). The requirements are outlined in the table below:

Requirement	Description
1. a + b + c > -1	Real eigenvalues will not become larger than 1
2. a - b + c < 1	Real eigenvalues will not become less than -1
3. c(a - c) - b > -1	Complex eigenvalues will not have magnitude greater than 1

A proof of these requirements is included as an appendix to this paper. They are sufficient for the stability of any 3x3 linear mapping. Thus if (16) meets requirements 1 - 3 outlined in the above table then the two-step scheme is capable of stabilizing the period-two points.

We will now address the three-step scheme and show that the same requirements can be used to analyze its Jacobian. The three-step scheme will also use a corrective function of the same form on each iterate, the difference between the nth and the $n-2nd$ iterates, but will alternate between three constant gain parameters γ_1, γ_2 and γ_3. The three-step scheme can be stated as follows:

$$\begin{aligned} x_{n+1} &= f(x_n) + \gamma_1(x_n - x_{n-2}) \\ x_{n+2} &= f(x_{n+1}) + \gamma_2(x_{n+1} - x_{n-1}) \\ x_{n+3} &= f(x_{n+2}) + \gamma_3(x_{n+2} - x_n) \end{aligned} \tag{18}$$

where n is a multiple of three. The three-step scheme can also be rewritten in vector form:

$$(x_{n+3},\, x_{n+2},\, x_{n+1}) = L(x_n,\, x_{n-1},\, x_{n-2}) \tag{19}$$

Note that because the three-step scheme is a function of x_n, x_{n-1}, and x_{n-2} that generates the next three iterates in the sequence, we may look at the corrective scheme as a function that takes an ordered triple and advances the indices by three (19). The linearization of the three-step scheme is then shown below (actual entries of the Jacobian are omitted because they are too complicated):

$$\left|\begin{array}{c} x_{n+3} \\ x_{n+2} \\ x_{n+1} \end{array}\right| = \left|\begin{array}{ccc} \frac{\partial x_{n+3}}{\partial x_n} & \frac{\partial x_{n+3}}{\partial x_{n-1}} & \frac{\partial x_{n+3}}{\partial x_{n-2}} \\ \frac{\partial x_{n+2}}{\partial x_n} & \frac{\partial x_{n+2}}{\partial x_{n-1}} & \frac{\partial x_{n+2}}{\partial x_{n-2}} \\ \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial x_{n-1}} & \frac{\partial x_{n+1}}{\partial x_{n-2}} \end{array}\right| \left|\begin{array}{c} x_n \\ x_{n-1} \\ x_{n-2} \end{array}\right| \tag{20}$$

Because $f(x^*) = -x^*$ for period-two points x^* of this particular mapping, the period-two points of $f(x_n)$ will be period-two points of the three-step scheme and fixed points of the three-step scheme composed with itself. Thus, the eigenvalues of the corrective scheme composed with itself must satisfy (10) for the period-two points to be stable. This is only true, however, when the eigenvalues of (20) satisfy (10). Thus, if it is shown that if the eigenvalues of (20) satisfy (10), the period-two points can be stabilized using the three-step scheme. In addition, because this matrix is 3x3, we can use the stability requirements outlined in the above table to diagnose the stability of the period-two points under this scheme.

Also note that the one-step scheme is a special case of the three-step scheme but the two-step scheme is not a special case of the three-step scheme. This is because there is no way to alternate between two different gain parameters in the three-step scheme.

2.2 The One-step and Two-step Schemes Fail

Because the one-step scheme is a special case of the two-step scheme, we show that both schemes fail by proving that the two-step scheme fails. In this effort, we will show that the eigenvalues of (16) are unstable as defined in (10) for $-1 < \mu < 1$. The Jacobian of (16) has the following characteristic polynomial:

$$-\lambda^3 - (-9 + 3\gamma_1 + 3\gamma_2 - \gamma_1\gamma_2 + 12\mu - 2\gamma_1\mu - 2\gamma_2\mu - 4\mu^2)\lambda^2$$

$$-(-3\gamma_1 - 3\gamma_2 + 2\gamma_1\gamma_2 + 2\gamma_1\mu + 2\gamma_2\mu)\lambda + \gamma_1\gamma_2 \tag{21}$$

To analyze the two-step scheme without correction, we set $\gamma_1 = \gamma_2 = 0$. The eigenvalues of (16) with these requirements can be read off the diagonal: $\lambda_1 = 9 - 12\mu + 4\mu^2$, $\lambda_2 = 0$, and $\lambda_3 = 0$. As previously mentioned, $\lambda_1 > 1$ for $-1 < \mu < 1$, the range of the subcritical bifurcation. Thus, before implementing correction, one eigenvalue of (16) is larger than 1.

We will now show that even with nonzero gain parameters (16) will have at least one eigenvalue larger than 1 for $-1 < \mu < 1$. We prove this assertion by showing that the coefficients of (21) violate stability requirement 1. Requirement 1 from our stability requirement table is the following:

$$a + b + c > -1 \tag{22}$$

where a, b, and c are the coefficients of the characteristic polynomial of a 3x3 matrix as designated in (17). Violation of this requirement means that at least one of the eigenvalues of (16) is greater than positive one. To determine whether or not requirement (22) is met, we plug the coefficients of (21) into their corresponding places in (22). The resulting inequality is

$$-9 + 12\mu - 4\mu^2 > -1 \tag{23}$$

Thus, if the above inequality is true for a particularly value of μ, none of the eigenvalues of (16) would be larger than one for that value of μ. We see that this inequality is not satisfied for $-1 < \mu < 1$, the range of our model of the subcritical bifurcation, meaning the subcritical bifurcation cannot be stabilized by the two-step scheme. Note that (23), a stability requirement, does not depend on γ_1 or γ_2 and therefore cannot be manipulated by correction. In other words, perturbations that depend on γ_1 and γ_2 have no effect on keeping all eigenvalues within the positive one boundary. Therefore, the two-step scheme fails because it is unable to keep all eigenvalues of the mapping less than 1. Because the one-step scheme is a special case of the two-step scheme, the failure of the two-step scheme necessarily means that the one-step scheme also fails.

2.3 The Three-step Scheme is Successful Under Certain Parameter Settings

We now analyze the stability of the three-step scheme (19) for $-1 < \mu < 1$. We start by finding the eigenvalues of the Jacobian (20) of the three-step scheme without correction. Setting $\gamma_1 = \gamma_2 = \gamma_3 = 0$, we see that the eigenvalues before correction are: $\lambda_1 = 8\mu^3 - 36\mu^2 + 54\mu - 27$, $\lambda_2 = 0$, and $\lambda_3 = 0$. We note that $\lambda_1 < -1$ for $-1 < \mu < 1$, the range of our subcritical bifurcation.

Next we address whether or not nonzero gain parameters can be used in the three-step scheme to stabilize the period-two points. Due to the fact that the x_{n+3} term in (19) is an order 27 polynomial in x_n, the entries of the Jacobian and the coefficients of the characteristic polynomial are quite complicated and have been omitted. Nevertheless, it is possible to analyze the general three-step scheme using computations similar to those performed in section 2.2. The inequalities that define the set of all possible $(\gamma_1, \gamma_2, \gamma_3)$ combinations are available on-line in a Mathematica notebook file at:

www.duke.edu/~maf15/Research/ThreeStepScheme/

Here we analyze only a minimal case of the three-step scheme that applies feedback every third iterate, i.e. $\gamma_1 = \gamma_2 = 0$ in (19). This scheme may be rewritten as:

$$y = f(x_n)$$

$$z = f(y)$$

$$x_{n+1} = f(z) + \gamma_3(z - x_n) \tag{24}$$

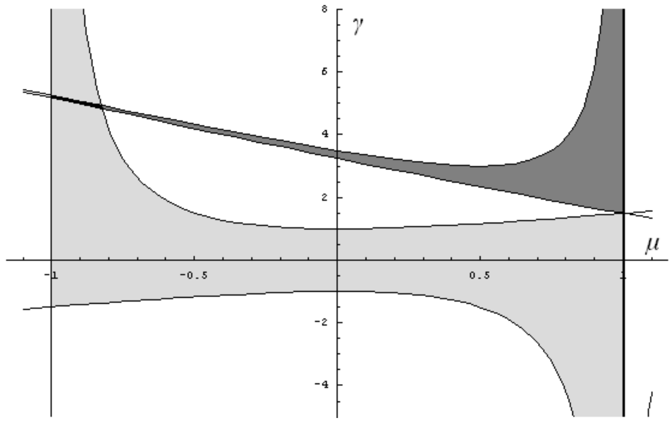

Figure 5: Range of γ_3 for $-1 < \mu < 1$. The dark gray region corresponds to values of γ_3 that stabilize the period-two points. Above this region, the eigenvalue at the period-two points is greater than 1 and below this region the eigenvalue is less than -1. The light gray region corresponds to values of γ_3 that stabilize the fixed point. Above this region, the eigenvalue at the fixed point is greater than 1 and below this region the eigenvalue is less than -1.

where (24) is a subsequence of the iterates of a sequence constructed using the three-step scheme (19) with $\gamma_1 = \gamma_2 = 0$. This subsequence corresponds to elements whose subscript is divisible by 3. Note that in (24), x_{n+1} depends only on x_n. Because x_{n+1} is determined by a single variable map, the Jacobian of such a mapping is a 1x1 matrix whose lone entry is its eigenvalue. Thus, γ_3 should be chosen such that the following holds:

$$-1 < \frac{\partial x_{n+1}}{\partial x_n}|_{x_n = \pm\sqrt{1-\mu}} < 1 \tag{25}$$

These two inequalities determine the range of γ_3 that stabilize the period-two points in terms of μ. This range of γ_3 is shown in dark gray in Figure 5. γ_3 values above the dark gray area result in eigenvalues larger than 1 whereas γ_3 values below the dark gray area result in eigenvalues less than -1. Note that as μ decreases, the range of γ_3 that stabilizes the period-two points decreases. Thus, the farther away from the bifurcation point, the smaller the range of effective gain parameters. This analysis suggests that stabilizing a subcritical bifurcation in experiment with the three-step scheme will be easier closer to the bifurcation point.

Another point of interest is the stability of the fixed point $x = 0$. This solution will be stable as long as the following holds:

$$-1 < \frac{\partial x_{n+1}}{\partial x_n}|_{x_n = 0} < 1 \tag{26}$$

The fixed point will be unstable otherwise. These two inequalities determine the range of γ_3 that preserves the stability of the fixed point for particular values of μ. This range of possible γ_3 is shown in light gray in Figure 5. γ_3 values above the light gray area result in eigenvalues larger than 1 whereas γ_3 values below the light gray area result in eigenvalues less than -1. In determining a scheme to stabilize a period-two point, we want the fixed point to be unstable to ensure that the system is driven to equilibrium at the period-two points rather than at the fixed point. Therefore, we will want to choose γ_3 outside of the light gray region.

Fortunately, for most μ the range of γ_3 that stabilizes the period-two points and the range of γ_3 that preserves the stability of the fixed point do not overlap. There exists a range of μ for which values of γ_3 that stabilize the period-two points also preserve the stability of the fixed point. This range of μ is approximately $-1 < \mu < -.826155$ and the corresponding region is shaded black. In this range of μ, a set of period-six points appears, with two of the period-six points lying between the period-two points and the fixed point. These period-six points are not critical points of the original map and are created by implementing the three-step scheme. These new critical points are also unstable throughout the range $-1 < \mu < -.826155$.

3 Conclusions

In this paper we devised a method of stabilizing a subcritical bifurcation using feedback. We determined that a minimum of a three-step scheme is required when using the difference between the nth and $n-2nd$ iterates as the corrective function. Although a minimal case of the three-step scheme is analyzed in this paper, feedback may be applied on each iterate, which is in accordance with the guideline set in section 1.3. In addition to stabilizing the period-two points, the three-step scheme destabilizes the fixed point, uses little information about the system and has a corrective function that vanishes at the period-two points. Thus, it meets all the guidelines we established before formulating a feedback scheme.

Our analysis suggests that this scheme may be most effectively implemented closer to the bifurcation point. The range of successful gain parameters far from the bifurcation point is very small. Also, closer to the bifurcation point convergence to the period-two points is less sensitive to the initial value x_0 of the sequence. Further studies should seek to increase the robustness of the corrective scheme by decreasing its sensitivity to initial x_0 of the sequence.

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References

[1] Guckenheimer, J. and P. Holmes. Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. (1983) Springer-Verlag, New York

[2] Hall, G. M., and D. J. Gauthier (2002). Experimental Control of Cardiac Muscle Alternans. *Phys. Rev. Lett.* 88, 198102.

[3] Mitchell, C. C., and D. G. Schaeffer (2003). A Two-Current Model for the Dynamics of Cardiac Membrane. *Bulletin Math Bio.* 65, 767-793.

[4] Nolasco, J. and R. Dahlen (1968). A Graphic Method for the Study of Alternation in Cardiac Action Potentials. *J. Appl. Physiol.* 25, 191-196.

[5] Strogatz, S. H. *Nonlinear Dynamics and Chaos.* (1994) Westview Press, Massachusetts

Appendix: Proof of Stability Requirements

Theorem: The eigenvalues of a real 3x3 matrix A are contained in the open unit disk in the complex plane if the coefficients of its characteristic polynomial $p(\lambda)$ below satisfy

1. $a + b + c > -1$
2. $a - b + c < 1$
3. $c(a - c) - b > -1$

where $det(A - \lambda I) = p(\lambda) = -\lambda^3 - a\lambda^2 - b\lambda - c$.

Proof. If $a = b = c = 0$, each $\lambda = 0$. Note that inequalities 1-3 are satisfied when all three coefficients vanish. As coefficients of the characteristic polynomial are varied, the roots move continuously. The roots will not exit the unit disk unless

i. A real root passes through $\lambda = 1$.

ii. A real root passes through $\lambda = -1$.

iii. A pair of complex conjugate roots crosses the unit circle off the real axis.

Conditions 1, 2 and 3 derive from excluding possibilities i, ii, and iii, respectively.

If $\lambda = 1$ is a root, then $a + b + c = -1$. Thus if a, b and c are varied away from zero but $a + b + c$ remains greater than -1, no real root can pass through $\lambda = 1$.

Similarly if $\lambda = -1$ is a root, then $a - b + c = 1$. Thus inequality 2 prevents a real root from passing through $\lambda = -1$.

If $p(\lambda)$ has complex conjugate roots on the unit circle, then $p(\lambda)$ may be factored as:

$$(\lambda - e^{i\theta})(\lambda - e^{-i\theta})(\lambda - r) = \lambda^3 - (2cos\theta + r)\lambda^2 + (2rcos\theta + 1)\lambda - r$$

where r$\in \mathbb{R}$. Matching coefficients in

$$\lambda^3 - (2cos\theta + r)\lambda^2 + (2rcos\theta + 1)\lambda - r$$

and

$$-\lambda^3 - a\lambda^2 - b\lambda - c,$$

we obtain the equations

$$a = -2cos\theta - r$$

$$b = 2rcos\theta + 1$$

$$c = -r$$

Eliminating r and θ from these equations we deduce that $c(a - c) - b = -1$. Thus inequality 3 prevents complex conjugate roots from crossing the unit circle.