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Abstract

In this paper we discuss some topological and geometrical properties of terms in a sequence of Cantor fractals and the limit of the sequence in order to obtain an exact relation between positive real numbers and Hausdorff dimensions of fractals of Euclidean spaces.

1 Introduction

Georg Cantor(1845-1918), the founder of axiomatic set theory, studied many interesting sets. He was very interested in infinite sets, in particular, those with strange properties. In 1883, he published a description of a set, called Cantor set in his own honor. The Cantor set C is the set of all $x \in [0,1]$ with the ternary expansion $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, and $a_n = 0, 2$ for all $n \in \mathbb{N}$. This set is one of the classical examples of fractals in the fractal geometry and is usually referred to as Cantor fractal.
In this paper, we replace the base 3 by an arbitrary odd positive integer other than 1, with even coefficients in expansion. For every positive integer s, let:

$$\Gamma(s) = \left\{ x \in [0,1] : \ x = \sum_{n=1}^{\infty} \frac{a_n}{q^n}, \ q = 2s+1, \ a_n = 0, 2, 4, \cdots, (q-1) \right\}$$

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We call the set above a Cantor fractal of the middle $q'th$ and of the order s. We show that for every positive integer s, the $\Gamma(s)$ has the same properties as Cantor fractal $C = \Gamma(1)$.

2 Basic Topological and Geometrical Results

There is another useful way to define $\Gamma(s)$, see e.g.[1],[2]. To define $\Gamma(s)$ in the unit interval $[0,1]$, let $q = 2s+1$ for $s = 1,2,3,\cdots$; then let us consider the affine maps:

$$T_{s,i}(x) = (\tfrac{1}{q})x + (\tfrac{2i}{q}) \qquad i = 0,1,\cdots,s$$

Now let $I_{s,0} = [0,1]$ and let $n = 1,2,\cdots$, then we define $I_{s,n}$ inductively as :

$$I_{s,n} = \bigcup_{i=0}^{s} T_{s,i}(I_{s,n-1})$$

Since $T_{s,i}$'s are continuous and closed, they take each closed subinterval of $[0,1]$ in to a closed subinterval of $[0,1]$, each map $T_{s,i}$ sends $(s+1)^{n-1}$ disjoint closed intervals to $(s+1)^{n-1}$ disjoint closed intervals, hence, their union which is $I_{s,n}$ is the disjoint union of $(s+1)^n$ disjoint closed intervals $I_{s,nj} = [a_{s,nj}, b_{s,nj}]$, each has a length of $(\frac{1}{q})^n$, so the total length of $I_{s,n}$ is $(\frac{s+1}{q})^n$. By constructing $I_{s,n}$'s for each $n = 1,2,\cdots$ and $i = 1,2,\cdots,n$, we have $T_{s,i}(I_{s,n-1}) \subseteq I_{s,n-1}$, hence $I_{s,n} = \bigcup_{i=0}^{s} T_{s,i}(I_{s,n-1}) \subseteq I_{s,n-1}$ this implies :

$$I_{s,0} \supseteq I_{s,1} \supseteq I_{s,2} \supseteq \cdots$$

and each $I_{s,n}$ is a closed set and thus a compact subset of $[0,1]$. Since this collection has the finite intersection property and $[0,1]$ is compact, they have a nonempty intersection, then $\Gamma(s)$ is defined as:

$$\Gamma(s) = \bigcap_{n=0}^{\infty} I_{s,n}$$

Since $\Gamma(s)$ is an intersection of closed subsets of $[0,1]$, it is a bounded and closed subset of $[0,1]$, thus by using the Heine-Borel Theorem in $\mathbb{R}$, $\Gamma(s)$ is compact for every $s = 1,2,\cdots$.

In the first theorem, we obtain an explicit formula for $\Gamma(s)$.

Theorem 1. Let $\Gamma(s)$ be a Cantor fractal of the middle $q'th$ and of the order s, then

$$\Gamma(s) = [0,1] - \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{q^{m-1}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^m}, \frac{qk+2r}{q^m}\right) \tag{2.1}$$

for every $s = 1, 2, 3, \cdots$.

Proof . Before we introduce the two lemmas that are needed to prove this theorem, let us have the following notation:

$$I^{\star}_{s,n} = I_{s,0} - I_{s,n}$$
for $n = 0, 1, 2, \cdots$ and $s = 1, 2, \cdots$.

Lemma 1. With the above definitions and notations, the equality

$$I^{\star}_{s,n} = I^{\star}_{s,1} \bigcup \left(\bigcup_{i=0}^{s} T_{s,i}(I^{\star}_{s,n-1})\right)$$

holds for every $n = 1, 2, \cdots$ and $s = 1, 2, 3, \cdots$.

Proof. By the definition of $I^{\star}_{s,n}$:
$I^{\star}_{s,n} = I_{s,0} - I_{s,n} = I_{s,0} - (\bigcup_{i=0}^{s} T_{s,i}(I_{s,n-1})) = \bigcap_{i=0}^{s}(I_{s,0} - T_{s,i}(I_{s,n-1})) = \bigcap_{i=0}^{s}(I_{s,0} - T_{s,i}(I_{s,0} - I^{\star}_{s,n-1})) = \bigcap_{i=0}^{s}(I_{s,0} - (T_{s,i}(I_{s,0}) - T_{s,i}(I^{\star}_{s,n-1}))) = \bigcap_{i=0}^{s}((I_{s,0} - T_{s,i}(I_{s,0})) \bigcup T_{s,i}(I^{\star}_{s,n-1})) = I^{\star}_{s,1} \bigcup (\bigcup_{i=0}^{s} T_{s,i}(I^{\star}_{s,n-1}))$. □

Lemma 2. With the above definitions and notations, the equality:

$$I^{\star}_{s,n} = \bigcup_{m=1}^{n} \bigcup_{k=0}^{q^{m-1}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^m}, \frac{qk+2r}{q^m}\right)$$

holds for $n = 1, 2, \cdots$ and $s = 1, 2, \cdots$.

Proof. We prove the assertion by induction. It is clear that it holds for $n = 1$. Let it hold for positive integer $n-1$, then:

$$\begin{aligned}
I^{\star}_{s,n} &= I^{\star}_{s,1} \bigcup \left(\bigcup_{i=0}^{s} T_{s,i}(I^{\star}_{s,n-1}\right) \\
&= I^{\star}_{s,1} \bigcup \left(\bigcup_{i=0}^{s} \bigcup_{m=1}^{n-1} \bigcup_{k=2iq^{m-1}}^{(2i+1)q^{m-1}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^{m+1}}, \frac{qk+2r}{q^{m+1}}\right)\right) \\
&= \left(I^{\star}_{s,1} \bigcup \left(\bigcup_{i=1}^{s} \bigcup_{m=1}^{n-1} \bigcup_{k=(2i-1)q^{m-1}}^{2iq^{m-1}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^{m+1}}, \frac{qk+2r}{q^{m+1}}\right)\right)\right) \bigcup \\
&\quad \bigcup_{i=0}^{s} \bigcup_{m=1}^{n-1} \bigcup_{k=2iq^{m-1}}^{(2i+1)q^{m-1}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^{m+1}}, \frac{qk+2r}{q^{m+1}}\right) \\
&= I^{\star}_{s,1} \bigcup \left(\bigcup_{m=1}^{n-1} \bigcup_{k=0}^{q^{m}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^{m+1}}, \frac{qk+2r}{q^{m+1}}\right)\right) \\
&= \bigcup_{m=1}^{n} \bigcup_{k=0}^{q^{m-1}-1} \bigcup_{r=1}^{s} \left(\frac{qk+(2r-1)}{q^{m}}, \frac{qk+2r}{q^{m}}\right).
\end{aligned}$$
□

Now by virtue of the Lemma 2, we prove Theorem 1:

$\Gamma(s) = \bigcap_{n=0}^{\infty} I_{s,n} = \bigcap_{n=0}^{\infty}(I_{s,0} - I_{s,n}^{\star}) = I_{s,0} - \bigcup_{n=0}^{\infty} I_{s,n}^{\star} =$
$[0,1] - \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{q^{m-1}-1} \bigcup_{r=1}^{s} (\frac{qk+(2r-1)}{q^m}, \frac{qk+2r}{q^m})$ for every $s = 1, 2, 3, \cdots$. □

In the next theorem we study some remarkable and interesting topological properties of these fractals whose proof shows that description of these sets can be made without using digit expansion viewpoint.

Theorem 2. The $\Gamma(s)$ is
(i) nowhere dense
(ii) totally disconnected
(iii) perfect
(iv) uncountable
(v) with Lebesgue measure zero
for every $s = 1, 2, 3, \cdots$.

Proof. (i) To prove the nowhere densness of $\Gamma(s)$ it is sufficient to show that it does not contain any open interval (γ, δ) of $[0, 1]$. By using Archimedean property, there exists positive integer m such that $q^{-m} < \frac{\delta-\gamma}{2q}$, which implies $(2q)q^{-m} + \gamma < \delta$. Now let $k_0 = min\{k \in Z_0^+ \mid \gamma < \frac{qk+1}{q^m}\}$, hence, $\gamma q^m < qk_0 + 1$. If $\delta < \frac{qk_0+2}{q^m}$, then $(2q)q^{-m} + \gamma < \frac{qk_0+2}{q^m}$ or $(2q-1) + \gamma q^m < qk_0 + 1$. This contradicts what k_0 represents. Hence $\frac{qk_0+2}{q^m} \leq \delta$ and thus $(\frac{qk_0+1}{q^m}, \frac{qk_0+2}{q^m}) \subseteq (\gamma, \delta)$. Now by using the equation (2.1) we conclude that $\Gamma(s)$ does not contain $(\frac{qk_0+1}{q^m}, \frac{qk_0+2}{q^m})$ which, in turn, proves that $\Gamma(s)$ does not contain (γ, δ).

(ii) Suppose $x_1, x_2 \in \Gamma(s)$ such that $x_1 < x_2$. Since $\Gamma(s)$ does not contain the interval (x_1, x_2), so there exists $x_1 < y < x_2$ such that $y \notin \Gamma(s)$. Now $(\Gamma(s) \bigcap [0, y)) \bigcup (\Gamma(s) \bigcap (y, 1])$ is a disconnectedness of $\Gamma(s)$ such that $x_1 \in \Gamma(s) \bigcap [0, y)$ and $x_2 \in \Gamma(s) \bigcap (y, 1]$.

(iii) Since $\Gamma(s)$ is closed, it is sufficient to prove that every element of $\Gamma(s)$ is its limit point. For every $s = 1, 2, 3, \cdots$ and $n = 0, 1, 2, \cdots$ put $L_{s,n} = \bigcup_{j=1}^{(s+1)^n} \{a_{s,nj}, b_{s,nj}\}$ and consider the set $L(s) = \bigcup_{n=1}^{\infty} L_{s,n}$. By considering the definition of $L(s)$, it is obvious that $L(s) \subseteq \Gamma(s)$. Next we prove that $L(s)$ is dense in $\Gamma(s)$. Let $x \in \Gamma(s)$ and $\epsilon > 0$, there exists positive integer n such that $q^{-n} < \epsilon$. Since $x \in I_{s,n}$, there exists a unique $1 \leq j \leq (s+1)^n$ such that $x \in I_{s,nj}$. Now $I_{s.nj} \subseteq (x - \epsilon, x + \epsilon)$ implies that $a_{s,nj}, b_{s,nj} \in (x - \epsilon, x + \epsilon)$, thus $\Gamma(s)$ clusters at x, so it is perfect.

(iv) Let $x \in \Gamma(s) = \bigcap_{n=0}^{\infty} I_{s,n}$, so $x \in I_{s,n}$ for every $n \in \mathbb{N}$. Further more, $I_{s,n} = \bigcup_{j=1}^{(s+1)^n} I_{s,nj}$ where $I_{s,nj}$'s are disjoint closed intervals. Thus, there exists a unique positive integer $1 \leq j_n \leq (s+1)^n$ such that $x \in I_{s,nj_n}$ for $n \in \mathbb{N}$ (in fact $I_{s,1j_1} \supseteq I_{s,2j_2} \supseteq \cdots \supseteq I_{s,nj_n} \supseteq \cdots$ and $x = \bigcap_{n=1}^{\infty} I_{s,nj_n}$). Now by virtue of division algorithm for each j_n there exist unique integers k_n, r_n such that $j_n = (s+1)k_n + r_n$ where $r_n = 0, 1, \cdots, s$. Thus for each $x \in \Gamma(s)$ there exists a unique sequence $\{r_n\}_{n=1}^{\infty}$ where $r_n = 0, 1, \cdots, s$ for all $n \in \mathbb{N}$. Hence, if we put $W_s = \{0, 1, \cdots, s\}$, then there exists a function $\Phi_s : \Gamma(s) \to W_s^{\mathbb{N}}$defined by $\Phi_s(x) = \{r_n\}_{n=1}^{\infty}$. Next an application of Cantor's intersection theorem shows the surjection of Φ_s, this completes the proof.

(v) By considering the construction of $\Gamma(s)$, it is clear that for every $n \in \mathbb{N}$, $\Gamma(s) \subseteq I_{s,n}$. So $m(\Gamma(s)) \leq m(I_{s,n})$; nevertheless, the sequence $\{(\frac{s+1}{2s+1})^n\}_{n=0}^{\infty}$ has zero limit, thus $m(\Gamma(s)) = 0$. □

Before starting to state and proof the next Lemma, we should use a notation. We use the symbols ind, dim_H to represent the topological and Hausdorff dimensions of sets, respectively. According to the Mandelbrot definition, the following Lemma shows that the $\Gamma(s)$ is an example of fractals.

Lemma 3.
(i) $\mathrm{ind}\Gamma(s) = 0$
(ii) $\dim_H \Gamma(s) = \frac{\ln(s+1)}{\ln(2s+1)}$
for every $s = 1, 2, 3, \cdots$.

Proof. (i) We consider the set $\Lambda_s = \{\Gamma(s) \bigcap I_{s,nj} | n = 0, 1, 2, \cdots and \ \ j = 1, 2, \cdots, (s+1)^n\}$. The Λ_s constitutes an open base for $\Gamma(s)$ with clopen elements. This shows that $\mathrm{ind}\Gamma(s) = 0$.
(ii) This statement is a special case of Example 4.5 of [3]. □

For any inquisitive reader, a discussion on the convergence of the sequence of fractals $\{\Gamma(s)\}_{s=1}^{\infty}$ may be interesting . We consider $\lim_{s\to\infty} \Gamma(s)$ in the sense employed in [4].

Theorem 3. The $\lim_{s\to\infty} \Gamma(s)$ does not exist.

Proof. First, Suppose that $s = 1, 2, 3, \cdots$, then there exists a minimum odd positive integer s_0 such that $s_0 \geq s$. An application of induction shows that

:

$\bigcap_{t=s_0}^{u} I_{t,1} \subseteq [0,1] - (\frac{s_0}{2u+1}, \frac{2u+1-s_0}{2u+1})$ for $u = s_0, s_0+1, \cdots$

Hence, $\bigcap_{t=s_0}^{\infty} I_{t,1} = \{0,1\}$ so $\bigcap_{t=s}^{\infty} I_{t,1} = \{0,1\}$. Now by definition of $\Gamma(t)$, we conclude that $\bigcap_{t=s}^{\infty} \Gamma(t) = \{0,1\}$, thus $\bigcup_{s=1}^{\infty} \bigcap_{t=s}^{\infty} \Gamma(t) = \{0,1\}$ or

$$\liminf_{s\to\infty} \Gamma(s) = \{0,1\} \tag{2.2}$$

Secondly, suppose $q = 2s+1 (s = 1, 2, \cdots)$. By using induction, it is clear that $I_{(\frac{q-1}{2}),2n} \subseteq I_{(\frac{q^2-1}{2}),n}$ $n = 0, 1, 2, \cdots$, so $\bigcap_{n=0}^{\infty} I_{(\frac{q-1}{2}),2n} \subseteq \bigcap_{n=0}^{\infty} I_{(\frac{q^2-1}{2}),n}$ which implies $\Gamma(\frac{q-1}{2}) \subseteq \Gamma(\frac{q^2-1}{2})$, thus by virtue of induction, we obtain:

$$\Gamma(\tfrac{q-1}{2}) \subseteq \Gamma(\tfrac{q^2-1}{2}) \subseteq \cdots \subseteq \Gamma(\tfrac{q^{2^m}-1}{2}) \subseteq \cdots \quad \text{for } q = 3, 5, 7, \cdots. \tag{2.3}$$

This implies $\bigcup_{s=1}^{\infty} \Gamma(s) = \bigcup_{s=1}^{\infty} \Gamma(\frac{q^{2^m}-1}{2})$ for $m = 1, 2, 3, \cdots$. Next for each $s = 1, 2, 3, \cdots$, there exists a positive integer m such that $s \leq \frac{3^{2^m}-1}{2}$ so $\bigcup_{t=s}^{\infty} \Gamma(t) \supseteq \bigcup_{t=1}^{\infty} \Gamma(\frac{q^{2^m}-1}{2})$ or $\bigcup_{t=s}^{\infty} \Gamma(t) \supseteq \bigcup_{s=1}^{\infty} \Gamma(s)$; thus $\bigcup_{t=s}^{\infty} \Gamma(t) = \bigcup_{s=1}^{\infty} \Gamma(s)$, hence, $\bigcap_{s=1}^{\infty} \bigcup_{t=s}^{\infty} \Gamma(t) = \bigcup_{s=1}^{\infty} \Gamma(s)$ or

$$\limsup_{s\to\infty} \Gamma(s) = \bigcup_{s=1}^{\infty} \Gamma(s) \tag{2.4}$$

Now by using (2.2)and (2.4), we have $\liminf_{s\to\infty} \Gamma(s) \neq \limsup_{s\to\infty} \Gamma(s)$. This proves the desired result. □

As we saw in the previous theorem, the $\lim_{s\to\infty} \Gamma(s)$ does not exist; however, we wish to discuss some properties of $\liminf_{s\to\infty} \Gamma(s)$ and $\limsup_{s\to\infty} \Gamma(s)$. To begin with, by virtue of equation (2.2) the discussion about $\liminf_{s\to\infty} \Gamma(s)$ is obvious and in particular it is not a fractal. Secondly, for the set $\limsup_{s\to\infty} \Gamma(s)$, we have:

(i) density. The $\limsup_{s\to\infty} \Gamma(s)$ is dense in $[0,1]$. To prove this, let $x \in [0,1] - \limsup_{s\to\infty} \Gamma(s)$ and $\epsilon > 0$ be given. There exists an odd positive integer q such that $\frac{3}{q} < \epsilon$, hence there exists at least one odd positive integer j so that $a_{s,1j} \in (x-\epsilon, x+\epsilon)$, since $a_{s,1j} \in \limsup_{s\to\infty} \Gamma(s)$, we conclude the desired result.

(ii) nonperfectness. Since $\limsup_{s\to\infty}\Gamma(s)$ is a countable union of sets with Lebesgue measure zero so it has Lebesgue measure zero. If it is perfect in unit interval, then it has Lebesgue measure one which contradicts the previous result.

(iii)noncompactness. The nonperfectness of $\limsup_{s\to\infty}\Gamma(s)$ implies that it is not closed so, by using Heine-Borel theorem we conclude that it is not compact(However it is locally compact).

(iv)total disconectedness. For each $s = 1, 2, 3, \cdots$, the $[0, 1] - \Gamma(s)$ is an open dense subset of $[0, 1]$, hence by virtue of the Bair's theorem, in the version of [5], $\bigcap_{s=1}^{\infty}([0, 1] - \Gamma(s)) = [0, 1] - \limsup_{s\to\infty}\Gamma(s)$ is a dense subset of$[0, 1]$. Now let $x_1, x_2 \in \limsup_{s\to\infty}\Gamma(s)$ such that $0 \leq x_1 < x_2 \leq 1$, hence there exists $x_1 < y < x_2$ so that $y \notin \limsup_{s\to\infty}\Gamma(s)$ so $(\limsup_{s\to\infty}\Gamma(s)\bigcap[0, y))\bigcup(\limsup_{s\to\infty}\Gamma(s)\bigcap(y, 1])$ is a disconnectedness of $\limsup_{s\to\infty}\Gamma(s)$ so that $x_1 \in \limsup_{s\to\infty}\Gamma(s)\bigcap[0, y)$ and $x_2 \in \limsup_{s\to\infty}\Gamma(s)\bigcap(y, 1]$.

(v)The $\mathbf{limsup_{s\to\infty}\Gamma(s)}$ is a fractal. Notice that for $s = 1, 2, 3, \cdots$, $\Gamma(s) \subseteq \limsup_{s\to\infty}\Gamma(s)$ and $\operatorname{ind}\Gamma(s) = 0$, so by virtue of [6], we conclude that $\operatorname{ind}(\limsup\limits_{s\to\infty}\Gamma(s)) = 0$. Next, we have:

$$\dim_H(\limsup_{s\to\infty}\Gamma(s)) = \sup_{1\leq s<\infty}(\dim_H\Gamma(s)) = 1$$

Hence, by comparing *ind* and *dim*$_H$ for this set, the proof is complete.

Notice that most fractals in the real line have Hausdorff dimension less than 1 and every subset of the real line which contains an open subset has Hausdorff dimension 1 . Here we constituted an example of a fractal in the real line with Hausdorff dimension 1 which does not contain any open subset of the real line. In fact, there is a continuum fractals of this type.

In the proof of Theorem 3, we obtained the sequence of fractals$\{\Gamma(\frac{q^{2^m}-1}{2})\}_{m=0}^{\infty}$ for $q = 3, 5, 7, \cdots$. We want to find a relation between these sequences and the main sequence$\{\Gamma(s)\}_{s=1}^{\infty}$. Since $\{\Gamma(\frac{q^{2^m}-1}{2})\}_{m=0}^{\infty}$ is a subsequence of $\{\Gamma(s)\}_{s=1}^{\infty}$, so all its elements have the same properties that we proved for elements of $\{\Gamma(s)\}_{s=1}^{\infty}$. Furthermore, with respect to equation (2.3) we have $\lim_{m\to\infty}\Gamma(\frac{q^{2^m}-1}{2}) = \bigcup_{m=0}^{\infty}\Gamma(\frac{q^{2^m}-1}{2})$. So a similar discussion that we had about $\limsup_{s\to\infty}\Gamma(s)$ still holds for the $\lim_{m\to\infty}\Gamma(\frac{q^{2^m}-1}{2})$, in particular, it is a fractal with Hausdorff dimension 1.

3 The Hausdorff dimension Theorem

In this section we intend to obtain an exact relation between Hausdorff dimensions of fractals of Euclidean spaces and positive real numbers. In the previous section we showed that there are denumerable fractals with Hausdorff dimension 1.This result is a special case of the next Lemma.

Lemma 4. For any positive integer n, there is a continuum fractals with Hausdorff dimension n in n-dimensional Euclidean space.

Proof . Let $\{q_n\}_{n=1}^{\infty}$ be the increasing sequence of prime numbers, for any $n \in \mathbb{N}$ the set $\lim_{m\to\infty} \Gamma(\frac{{q_n}^{2^m}-1}{2})$ is a fractal with Hausdorff dimension 1. Now for any non-empty subset $\mathcal{A}$ of $\mathbb{N}$ put:

$$F = \bigcup_{n\in\mathcal{A}}(\lim_{m\to\infty} \Gamma(\frac{{q_n}^{2^m}-1}{2})) \tag{3.1}$$

All fractals of type (3.1) have all the properties of $\limsup_{s\to\infty} \Gamma(s)$ and in particular, the Hausdorff dimension 1. So the uncountability of subsets of $\mathbb{N}$ proves the Lemma for case $n = 1$. Next, each F is a Borel set and has topological dimension zero, so for any $n \geq 2$ taking n^{th} Cartesian products $\prod_{i=1}^{n} F_i$ of pairwise distinct above fractals with $\dim_H(\prod_{i=1}^{n} F_i) = \sum_{i=1}^{n} \dim_H(F_i) = n$ and $\mathrm{ind}(\prod_{i=1}^{n} F_i) = 0$ gives the desired result. □

In [7], it has been proved that for each $0 < r < 1$, there is a continuum fractals in $\mathbb{R}$ with Hausdorff dimension r. Furthermore, for each positive integer n, there is a continuum fractals with Hausdorff dimension r in $\mathbb{R}^n (r < n)$. To obtain a little more exact result, we end our work with:

Theorem 4 (The Hausdorff dimension Theorem). For any real $r > 0$, there is a continuum fractals with Hausdorff dimension r in n-dimensional Euclidean space($(-[-r]) \leq n$).

Proof. It is sufficient to prove the theorem for any non-integer $r > 1$. Since we have $r = [r] + (r)$ by virtue of Lemma 4 and [7] there are continuum fractals with Hausdorff dimensions $[r]$ and (r) in $\mathbb{R}^{[r]}$ and $\mathbb{R}$, respectively, say G_1 and G_2. Taking the Cartesian products $G_1 \times G_2$ completes the proof.□

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References

[1] R. Kraft, Intersections of thick Cantor sets, Mem. Amer. Math. Soc. 97 (1992), no. 468

[2] J. Li, F.Nekka, Intersections of triadic Cantor sets with their translates. (2). Hausdorff measure spectrum function and its introduction for the classification of Cantor sets. Chaos, Solitons & Fractals. 19(2004), no 1. 35-46

[3] K. J. Falconer. Fractal Geometry-Mathematical Foundations and Applications, wiley, chichester, 1990

[4] J. H. Dshalalow. Real Analysis-An Introduction to the Theory of Real Functions and Integration, Chapman & Hall/CRC, Boca Ration, Florida, 2001

[5] C. C. Pugh. Real Mathematical Analysis, Springer-verlage, New-York, Inc, 2002

[6] G. A. Edgar. Measure,Topology and Fractal Geometry, Springer-verlage, New-York, 1992

[7] S and V.Sharapov. Dimensions of Some Generalized Cantor Sets, preprint:
(http://classes.yale.edu/fractals/Frac And Dim/cantorDims/CantorDims.html)