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A Special Case of Selberg's Integral

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A SPECIAL CASE OF SELBERG'S INTEGRAL

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Abstract. We evaluate the integral

$$\int_0^\infty \cdots \int_0^\infty \prod_{1\le i<j\le n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n$$

using orthogonal polynomials and techniques from linear algebra.

1. Introduction

The integral evaluated in this paper arose in the study of a class of invariant differential operators of a matrix argument. Briefly, these polynomials can be described by considering invariant polynomials on a cross section of diagonal matrices that is essentially $\mathbb{R}^n$ ([BHR1] [BHR2]). The way in which the relevant measure is pushed down to this cross section can be determined by computing the integral

$$\int_0^\infty \cdots \int_0^\infty \prod_{1\le i<j\le n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n. \tag{1.1}$$

Since the integrand is a polynomial, and using the fact that, for $k \in \mathbb{N}$, the Euler Gamma function satisfies

$$\Gamma(k+1) = \int_0^\infty x^k e^{-x} dx = k!$$

we see that the value of this integral must be an integer that depends on n. We denote the value of the integral (1.1) by $I(n)$.

Somewhat later we discovered that the above integral is a special case of Selberg's integral ([AAR], hence the title of this paper) and can be evaluated using published formulae with $\alpha = 1$ and $\gamma = 1$.

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Specifically,

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{1\le i<j\le n} |x_j - x_i|^{2\gamma} e^{-x_1} dx_1 \cdots e^{-x_n} dx_n$$
$$= \prod_{j=1}^{n} \left(\frac{\Gamma(\alpha + (j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} \right)$$

The direct computation of this integral demonstrates the use of algebraic techniques to answer a question that at first appears to be strictly analytic.

2. Preliminaries

We first computed this integral for manageable values of n using a computer algebra system and obtained the following results:

n	$I(n)$
2	2
3	24
4	3465
5	9953280

After some consideration we realized that we could write these values as:

n	I(n)
2	$2!$
3	$3!\,(2!)^2$
4	$4!\,(3!\,2!)^2$
5	$5!\,(4!\,3!\,2!)^2$

This led us to conjecture that, for $n \in \mathbb{N}$

$$I(n) = n![(n-1)!(n-2)!\cdots(2)(1)]^2.$$

It was then suggested that we consider Laguerre polynomials. The Laguerre polynomial in the variable x of degree k can be defined using Rodrigues' formula by

$$L_k(x) = e^x \frac{d^k}{dx^k}(x^k e^{-x}).$$

Note that the leading term will be x^k. Direct computation using integration by parts shows that

$$\int_0^\infty L_k(x)L_m(x)e^{-x}dx = \begin{cases} (k!)^2 & \text{k=m} \\ 0 & \text{k} \neq \text{m} \end{cases}$$

so that the Laguerre polynomials are orthogonal with respect to the inner product

$$\langle f\,,\,g\rangle = \int_0^\infty f(x)g(x)e^{-x}dx.$$

It is well known that the Laguerre polynomials are an orthogonal basis for the vector space of polynomials in x.

Note also that products of Laguerre polynomials in the variable x and Laguerre polynomials in the variable y will be a basis for the vector space of polynomials in the two variables x and y, orthogonal with respect to the inner product given by

$$\langle f\,,\,g\rangle = \int_0^\infty \int_0^\infty f(x,y)\,g(x,y)\,e^{-x}\,e^{-y}\,dx\,dy.$$

In particular, by Fubini's Theorem we have

$$\langle L_n(x)L_m(y)\,,\,L_n(x)L_m(y)\rangle = \langle L_n(x)\,,\,L_n(x)\rangle\,\langle L_m(y)\,,\,L_m(y)\rangle = (n!)^2(m!)^2.$$

This idea extends to polynomials in n variables. The products of Laguerre polynomials in the variables $x_1\cdots x_n$ form a basis for the vector space of polynomials in these variables, orthogonal with respect to the inner product

$$\langle f\,,\,g\rangle = \int_0^\infty \cdots \int_0^\infty f(x_1,\cdots,x_n)g(x_1,\cdots,x_n)e^{-x_1}dx_1\cdots e^{-x_n}dx_n$$

with

$$\begin{aligned}\langle L_{k_1}(x_1)\cdots L_{k_n}(x_n)\,,\,L_{k_1}(x_1)\cdots L_{k_n}(x_n)\rangle&\\ = \langle L_{k_1}(x_1)\,,\,L_{k_1}(x_1)\rangle\cdots\langle L_{k_n}(x_n)\,,\,L_{k_n}(x_n)\rangle&\\ = (k_1!)^2\cdots(k_n!)^2.&\end{aligned}$$

Our first step will be to expand the integrand of (1.1) using Laguerre polynomials.

3. Expansion of the integrand using Laguerre polynomials

Making further investigations using a computer algebra system, we expanded $\prod_{1\le i<j\le n}(x_j-x_i)$ for $n=3$ and then subtracted off successive Laguerre polynomials in x_1, x_2, and x_3 by matching the highest order terms. For example, we have

$$(x_2-x_1)(x_3-x_1)(x_3-x_2) = x_2x_3^2 - x_2^2x_3 + x_2^2x_1 - x_1x_3^2 + x_1^2x_3 - x_1^2x_2.$$

Since the first term in the left hand side is $x_2x_3^2$, we first subtract $L_1(x_2)L_2(x_3)$. We then continue in this manner until the result is zero (this is guaranteed to occur since the products of Laguerre polynomials form a basis) and obtain

$$\begin{aligned}(x_2-x_1)(x_3-x_1)(x_3-x_2) =& L_1(x_2)L_2(x_3)-L_1(x_3)L_2(x_2)\\ &+L_1(x_1)L_2(x_2)-L_1(x_1)L_2(x_3)\\ &+L_1(x_3)L_2(x_1)-L_1(x_2)L_2(x_1).\end{aligned}$$

This led us to conjecture the following formula:

$$\prod_{1\le i<j\le n}(x_j-x_i)=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})\cdots L_{n-1}(x_{\sigma(n)}) \tag{3.1}$$

where S_n is the symmetric group of degree n and $\operatorname{sgn}(\sigma)=\pm 1$, depending on whether the permutation σ is even or odd. We remark that the left hand side is the well known Vandermonde determinant.

To prove (3.1), we first show that the polynomial defined by

$$P(x_1,x_2,\cdots,x_n)=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})L_2(x_{\sigma(3)})\cdots L_{n-1}(x_{\sigma(n)}).$$

is alternating. That is, we show that the polynomial changes sign whenever two variables are interchanged. Let $(i.j)$ be a transposition in S_n and let $(i,j).P(x_1,x_2,\cdots,x_n)$ denote the action of this transposition that interchanges the i^{th} and j^{th} variables in $P(x_1,x_2,\cdots,x_n)$. Then we have

$$(i,j).P(x_1,x_2,\cdots,x_n)=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)L_0(x_{(i,j)\sigma(1)})L_1(x_{(i,j)\sigma(2)})\cdots L_{n-1}(x_{(i,j)\sigma(n)}).$$

Now let $\tau=(i,j)\sigma$, so that $\operatorname{sgn}(\sigma)=-\operatorname{sgn}(\tau)$. Then we have

$$\begin{aligned}(i,j).P(x_1,x_2,\cdots,x_n)&=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)L_0(x_{\tau(1)})L_1(x_{\tau(2)})\cdots L_{n-1}(x_{\tau(n)})\\ &=\sum_{\tau\in S_n}-\operatorname{sgn}(\tau)L_0(x_{\tau(1)})L_1(x_{\tau(2)})\cdots L_{n-1}(x_{\tau(n)})\\ &=-P(x_1,x_2,\cdots,x_n).\end{aligned}$$

Therefore, P is alternating.

Now set $P_n(x_n)=P(x_1,x_2,\cdots,x_n)$. That is, consider P_n as a polynomial in x_n, with the coefficients being polynomials in $x_1,x_2,\cdots,x_{n-1}$. Since P is alternating, we have $P_n(x_i)=0$ for $i\neq n$. Thus, by the Fundamental Theorem of Algebra, each $x_1,x_2,\cdots,x_{n-1}$ is a root of

$P_n(x_n)$ and so for each $i \neq n$, $(x_n - x_i)$ is a factor of $P_n(x_n)$. Therefore

$$\begin{aligned} P_n(x_n) &= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) A_{n-1}(x_1, \cdots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) A_{n-1}(x_1, \cdots, x_{n-1}) \end{aligned} \tag{3.2}$$

for some polynomial $A_{n-1}(x_1, \cdots, x_{n-1})$. Note that $A_{n-1}(x_1, \cdots, x_{n-1})$ is the coefficient of x_n^{n-1} in $P_n(x_n)$.

Next we split the sum (3.1) over those elements of S_n which leave n fixed (a subgroup of S_n isomorphic to S_{n-1}), and those which do not.

$$\begin{aligned} P_n(x_n) = &\sum_{\sigma(n)=n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)}) \\ &+ \sum_{\sigma(n)\neq n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)}) \\ &= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-2}(x_{\sigma(n-1)}) L_{n-1}(x_n) \\ &+ \text{(lower degree terms)} \\ &= L_{n-1}(x_n) \left[\sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-2}(x_{\sigma(n-1)}) \right] \\ &+ \text{(lower degree terms).} \end{aligned}$$

Equating the leading coefficient of x_n^{n-1} above with that in (3.2) we see that

$$\begin{aligned} A_{n-1}(x_1, x_2, \cdots, x_{n-1}) &= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-2}(x_{n-1}) \\ &= P_{n-1}(x_{n-1}). \end{aligned}$$

Combining these results we have

$$\begin{aligned} P_n(x_n) &= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) A_{n-1}(x_1, \cdots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) A_{n-1}(x_1, \cdots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) P_{n-1}(x_{n-1}). \end{aligned}$$

Now by the same argument used earlier, we have that $x_1, x_2, \cdots, x_{n-2}$ are roots of $P_{n-1}(x_{n-1})$, so

$$P_n(x_n) = \prod_{i=1}^{n-1}(x_n - x_i)P_{n-1}(x_{n-1})A_{n-2}(x_1, x_2, \cdots, x_{n-2})$$
$$= \prod_{i=1}^{n-1}(x_n - x_i)\prod_{j=1}^{n-2}(x_{n-1} - x_j)A_{n-2}(x_1, x_2, \cdots, x_{n-2})$$

where A_{n-2} is the coefficient of x_{n-1}^{n-2}.

Now, for $n = 2$ we can show that

$$\sum_{\sigma \in S_2} \operatorname{sgn}(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)}) = L_0(x_1)L_1(x_2) - L_0(x_2)L_1(x_1) = x_2 - x_1$$

and so the result follows by induction. Namely,

$$\prod_{0 \leq i < j \leq n}(x_j - x_i) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)})$$

as required.

4. Evaluating the integral as an inner product

Recall that we have an inner product on the vector space of polynomials on $\mathbb{R}^n$ given by

$$\langle f\, ,\, g\rangle = \int_0^\infty \cdots \int_0^\infty f(x_1, \cdots, x_n)g(x_1, \cdots, x_n)e^{-x_1}dx_1 \cdots e^{-x_n}dx_n.$$

Therefore we can treat our integral

$$I(n) = \int_0^\infty \cdots \int_0^\infty \prod_{1 \leq i < j \leq n}(x_j - x_i)^2 e^{-x_1}dx_1 \cdots e^{-x_n}dx_n \tag{4.1}$$

as the inner product

$$I(n) = \langle \prod_{1 \leq i < j \leq n}(x_j - x_i)\, ,\, \prod_{1 \leq i < j \leq n}(x_j - x_i)\rangle.$$

By (3.1), we have

$$I(n) =$$
$$\langle \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)})\, ,\, \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)})\rangle.$$

But each of the terms in the above sums are mutually orthogonal, *i.e.*

$$\langle L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\,,\, L_0(x_{\tau(1)})\cdots L_{n-1}(x_{\tau(n)})\rangle = \begin{cases}(0!)^2(1!)^2(2!)^2\cdots(n-1!)^2 & \sigma=\tau,\\ 0 & \sigma\neq\tau.\end{cases}$$

Evaluating our integral in this context yields the required result:

$$\begin{aligned} I(n) =& \\ &\langle\sum_{\sigma\in S_n} sgn(\sigma)L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\,,\,\sum_{\sigma\in S_n} sgn(\sigma)L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\rangle \\ &= \sum_{\sigma\in S_n}\langle\ sgn(\sigma)L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\,,\ \ sgn(\sigma)L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\rangle \\ &= \sum_{\sigma\in S_n}(0!)^2(1!)^2(2!)^2\cdots(n-1!)^2 \\ &= n![(n-1)!\cdots(2)!(1)!]^2. \end{aligned}$$

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