Rose-Hulman Undergraduate Mathematics Journal

Volume 6
Issue 1

Article 3

Positive Solutions to a Diffusive Logistic Equation with Constant Yield Harvesting

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Tammy Ladner, Anna Little, Ken Marks, Amber Russell

Abstract

We consider a reaction diffusion equation which models the constant yield harvesting of a spatially heterogeneous population which satisfies a logistic growth. In particular, we study the existence of positive solutions subject to a class of nonlinear boundary conditions. We also provide results for the case of Neumann and Robin boundary conditions. We obtain our results via a quadrature method and *Mathematica* computations.

1 Introduction

In [1], the Dirichlet boundary value problem

$$\begin{cases} -\triangle u(x) = au - bu^2 - c\,h(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

arising in the study of population dynamics was studied. Here $\triangle$ is the Laplacian operator, Ω is a bounded domain of $\mathbb{R}^n$; $n \geq 1$ with $\partial\Omega \in C^2$, u is the population density, $au - bu^2$ represents the logistic growth, where a and b are both positive constants, and $c\,h(x)$ represents the constant yield harvesting rate, where $c \geq 0$ is a constant.

In [2], more detailed results were obtained for the case $n = 1$ and $h(x) \equiv 1$. In particular, for the boundary value problem

$$\begin{cases} -u''(x) = au - bu^2 - c, & x \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.1}$$

the following results were established via the quadrature method and *Mathematica* computations:

[A] Let $a \leq \pi^2$. Then (1.1) has no positive solutions. (Note that the principal eigenvalue of $-u''$ with Dirichlet boundary conditions is π^2.)

[B] Let $a > \pi^2$. Then there exists a $c_0(a,b)$ such that if $c \leq c_0$, then (1.1) has at least one positive solution. If $c > c_0$, then (1.1) has no positive solutions. That is, the bifurcation diagram resembles Figure 1.1.

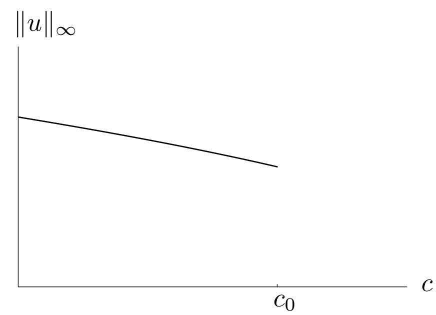

Figure 1.1: Bifurcation Diagram for $a > \pi^2$

[C] Let $a \in (\pi^2,\ 4\pi^2)$. (Note that $4\pi^2$ is the second eigenvalue of $-u''$ with Dirichlet boundary conditions.) Then there exists a $c_1(a, b)$ such that if $0 < c < c_1$, then (1.1) has exactly two positive solutions. If $c = c_1$, then (1.1) has a unique positive solution. For $c > c_1$, there are no positive solutions to (1.1). That is to say, when $a \in (\pi^2, 4\pi^2)$ the bifurcation diagram is exactly Figure 1.2.

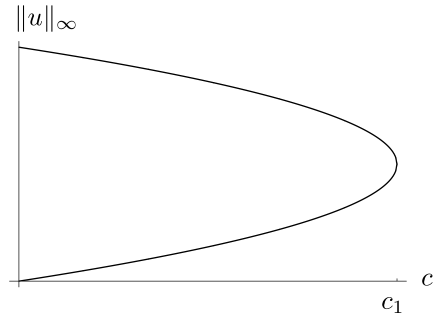

Figure 1.2: Bifurcation diagram for $\pi^2 < a < 4\pi^2$

In this paper, we extend this study to other types of boundary conditions in population dynamics. We first focus on boundaries where α, the fraction of individuals who do not cross the boundary, is a function of the population density itself. This leads to the nonlinear boundary condition

$$\alpha(u, x) = \frac{u}{u + [-d\nabla u \cdot \boldsymbol{\eta}]}, \tag{1.2}$$

where d is a positive constant and $\boldsymbol{\eta}$ is the outward unit normal on the boundary. Such boundary conditions were recently proposed in population dynamics by Cantrell and Cosner in [3]. Also, see [4], a recent book by Cantrell and Cosner on spatial ecology via reaction-diffusion equations. Specifically, we consider the case where $b = 1$, $d = 1$, and

$$\alpha(u, x) = \begin{cases} 0; & x = 0 \\ \dfrac{u}{a}; & x = 1 \end{cases}$$

That is, we study the nonlinear boundary value problem

$$\begin{cases} -u''(x) = a\ u - u^2 - c, \quad x \in (0, 1), \\ u(0) = 0 \\ -u(1)\ u'(1) + [u(1) - a]\ u(1) = 0. \end{cases} \tag{1.3}$$

Studying (1.3) is clearly equivalent to studying the following two problems

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0, 1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.4}$$

and

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0, 1), \\ u(0) = 0, \\ u'(1) = u(1) - a. \end{cases} \tag{1.5}$$

As noted earlier, (1.4) has been analyzed in [2] (see [A]-[C], Figures 1.1 and 1.2). We will focus on the analysis of (1.5).

Also, in this paper we analyze the Neumann boundary value problem which arises by letting $\alpha(u, x) = \alpha(x) = 1$ for $x \in \{0, 1\}$, namely

$$\begin{cases} -u'' = au - u^2 - c, \quad x \in (0, 1), \\ u'(0) = 0 = u'(1), \end{cases} \tag{1.6}$$

and the Robin boundary value problem by assuming $\alpha(u,x) = \alpha(x)$ where $\alpha(0) = 0$ and $0 < \alpha(1) < 1$, namely

$$\begin{cases} -u'' = au - u^2 - c, \quad x \in (0,1), \\ u(0) = 0, \\ u'(1) + \left(\dfrac{1}{\alpha(1)} - 1\right) u(1) = 0. \end{cases} \tag{1.7}$$

In Section 2, we develop a quadrature method for (1.5). In Section 3, we provide *Mathematica* results on positive solutions to (1.5). In Section 4, we provide results on positive solutions to (1.3). In Sections 5 and 6, we analyze the Neumann boundary value problem, (1.6), and the Robin boundary value problem, (1.7), respectively. We conclude with remarks on related boundary conditions in Section 7.

2 Quadrature Method for the Boundary Value Problem (1.5)

In this section we develop a quadrature method to study positive solutions $u(x)$ less than a for $x \in (0,1)$ to (1.5), namely

$$\begin{cases} -u''(x) = au - u^2 - c = f(u), \quad x \in (0,1), \\ u(0) = 0, \\ u'(1) = u(1) - a. \end{cases} \tag{2.1}$$

Since this is an autonomous differential equation, if $u(x)$ is a positive solution to (2.1) with the property $u'(x_0) = 0$ for some $x_0 \in (0,1)$, then both $v(x) = u(x_0 + x)$ and $w(x) = u(x_0 - x)$ satisfy the initial value problem

$$\begin{cases} -z'' = f(z), \\ z(0) = u(x_0), \\ z'(0) = 0, \end{cases} \tag{2.2}$$

for $x \in [0,d)$, where $d = \min\{x_0,\ 1 - x_0\}$. This implies that $u(x_0 + x) = u(x_0 - x)$ for $x \in [0,d)$ by the Picard's Existence and Uniqueness Theorem. We can use the boundary conditions, symmetry about x_0 and concavity prescribed by f to determine that a positive solution $u(x)$ of (2.1) that is less than a will resemble 2.1.

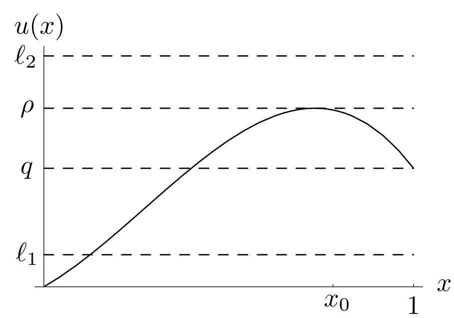

Figure 2.1: General Shape of Solution to (2.1)

Here, $\ell_1(a,c) = \dfrac{a - \sqrt{a^2 - 4c}}{2}$ and $\ell_2(a,c) = \dfrac{a + \sqrt{a^2 - 4c}}{2}$ are the roots of $f(u)$ (see Figure 2.2) and $x_0 \in [\frac{1}{2}, 1]$. This implies that $c < \frac{a^2}{4}$ is necessary for such a positive solution to exist. Let $\rho = u(x_0) = \|u\|_\infty$ and let $q = u(1)$. Then clearly $\ell_1 < \rho < \ell_2$, $0 \le q < \ell_2$, $u'(x) \ge 0$ for $x \in [0, x_0]$, and $u'(x) \le 0$ for $x \in [x_0, 1]$. Now define $F(u) = \displaystyle\int_0^u f(s)\,ds$, which must resemble Figure 2.3.

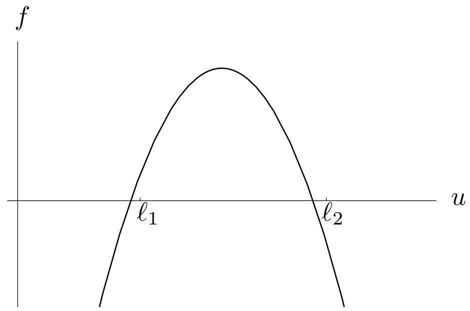

Figure 2.2: Graph of $f(u)$ ($c < \frac{a^2}{4c}$)

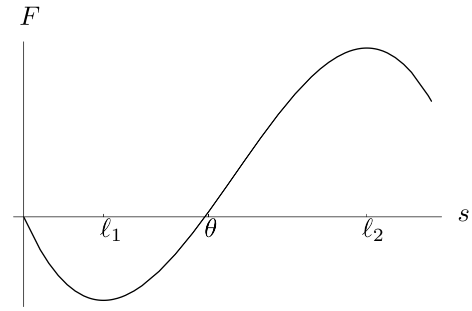

Figure 2.3: Graph of $F(s)$

Multiplying the differential equation in (2.1) by u' and integrating we obtain

$$\frac{-(u')^2}{2} = F(u) + K. \tag{2.3}$$

Using $u'(x_0) = 0$ along with the second boundary condition in (2.1) we can arrive at

$$F(\rho) = -K = F(q) + \frac{(q-a)^2}{2}. \tag{2.4}$$

Further, solving for u' in (2.3) we obtain

$$u' = \sqrt{2(F(\rho) - F(u))}, \quad x \in [0, x_0] \tag{2.5}$$

and

$$u' = -\sqrt{2(F(\rho) - F(u))}, \quad x \in [x_0, 1]. \tag{2.6}$$

Integrating these two equations and using the first boundary condition we have

$$\int_0^{u(x)} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2}x, \quad x \in [0, x_0] \tag{2.7}$$

and

$$\int_\rho^{u(x)} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = -\sqrt{2}(x - x_0), \quad x \in [x_0, 1]. \tag{2.8}$$

Now substituting $x = x_0$ into (2.7) and $x = 1$ into (2.8) we obtain

$$\int_0^{\rho} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2}x_0 \tag{2.9}$$

and

$$\int_\rho^{q} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = -\sqrt{2}(1 - x_0). \tag{2.10}$$

Further, subtracting (2.10) from (2.9), we have

$$\int_0^{\rho} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds + \int_q^{\rho} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2}. \tag{2.11}$$

We note that $\int_0^{\rho} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds$ exists if and only if $\rho \in (\theta, \ell_2)$, where $\theta = \frac{3a - \sqrt{9a^2 - 48c}}{4}$ is the zero of F in (ℓ_1, ℓ_2) (this implies that $c < \frac{3a^2}{16}$ is necessary for the existence of such a positive solution). Note that since $f(\rho) > 0$ on this interval, this improper integral converges. Also, since x_0 is fixed for a given ρ, (2.4) must be satisfied for a unique $q = q(\rho) \le \rho$.

Now consider

$$H(x) = F(x) + \frac{(x-a)^2}{2} = \frac{ax^2}{2} - \frac{x^3}{3} - cx + \frac{(x-a)^2}{2}.$$

Then $H(0) = \frac{a^2}{2}$, $H'(0) = -c - a < 0$, and $H'(x) = -x^2 + (a+1)x - (c+a)$. Thus, in order to find a unique q satisfying (2.4), we must have $H'(x) > 0$ for some $x > 0$. This implies $(a+1)^2 - 4(c+a) > 0$, or equivalently, $c < \frac{(a-1)^2}{4}$. Further, in order to obtain a unique q for each ρ, ρ must be such that $F(\rho) > \frac{a^2}{2}$.

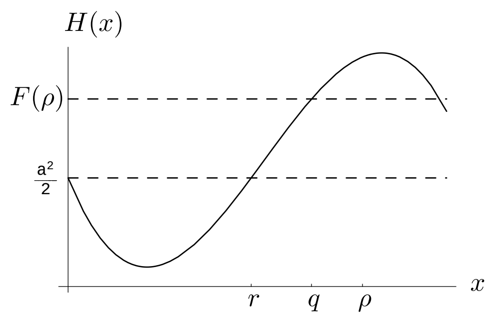

Figure 2.4: Graph of $H(x)$

Clearly, no such ρ will exist unless $F(\ell_2(a,c), c) > \frac{a^2}{2}$. But $\dfrac{dF}{dc} = \dfrac{\partial F}{\partial \ell_2}\dfrac{d\ell_2}{dc} + \dfrac{\partial F}{\partial c} = \dfrac{\partial F}{\partial c} = -\ell_2 < 0$. Hence, a necessary condition for such a solution to exist will be $F(\ell_2(a,0),0) = \frac{a^3}{6} > \frac{a^2}{2}$. In fact, we have the following results:

Theorem 1. *Let $a \le 3$. Then (2.1) has no positive solutions u less than a.*

Theorem 2. *Let $a > 3$. Then there exists $c_0(a) \le \min\left\{\frac{3a^2}{16}, \frac{(a-1)^2}{4}\right\}$ such that for $c > c_0$ there are no positive solutions to (2.1) such that $u(x) < a$. Here $c_0(a)$ is the unique root of $F(\ell_2(c), c) = \frac{a^2}{2}$.*

Theorem 3. *Let $a > 3$ and $c \le c_0$. Then there exists a unique $r(a,c) \in (\theta, \ell_2)$ such that $F(r) = \frac{a^2}{2}$. Further, if $(\rho, c) \in S(a) = \{(\rho, c) |\ r \le \rho < \ell_2, 0 \le c \le c_0\}$, then*

$$G(\rho, c) = \int_0^\rho \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds + \int_q^\rho \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds$$

is well defined. Here, $q = q(\rho) \le \rho$ is the unique point where $F(\rho) = H(q)$.

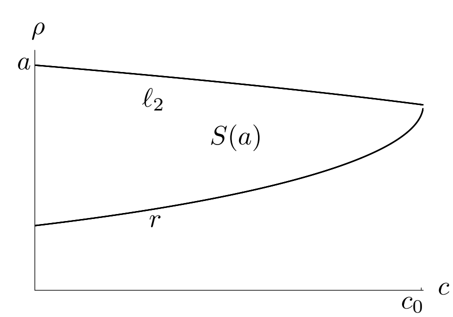

Figure 2.5: Graph of the set $S(a)$

We can now state and prove our main result in this section.

Theorem 4. *Let $a > 3$. Then (2.1) has a positive solution $u(x) < a$ if and only if $G(\rho, c) = \sqrt{2}$ for $(\rho, c) \in S(a)$.*

Proof. It has already been clearly shown in our discussion that if such a positive solution exists, then $G(\rho, c) = \sqrt{2}$ for some $(\rho, c) \in S(a)$. Now suppose that $G(\rho, c) = \sqrt{2}$ for some $(\rho, c) \in S(a)$. We will show that the function $u(x)$ defined by the following integrals

$$\begin{aligned} &\int_0^u \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2}x, \quad x \in [0, x_0], \\ &\int_\rho^u \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = -\sqrt{2}(x - x_0), \quad x \in [x_0, 1], \end{aligned} \tag{2.12}$$

is a positive solution to (2.1). The turning point, x_0, is given by $x_0 = \frac{1}{\sqrt{2}} \int_0^\rho \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds$.

Clearly, $\frac{1}{\sqrt{2}} \int_0^u \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds$ is a differentiable function of u which is strictly increasing from 0 to x_0 as u increases from 0 to ρ. Hence for each $x \in [0,\ x_0]$, there exists a unique $u(x)$ such that $\int_0^{u(x)} \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2}x$. Further, by the Implicit Function Theorem, u is differentiable with respect to x, so we can write $u'(x) = \sqrt{2[F(\rho) - F(u(x))]}$ for $x \in [0,\ x_0]$. Similarly, u decreases for $x \in [x_0,\ 1]$, so $u'(x) = -\sqrt{2[F(\rho) - F(u(x))]}$ for $x \in [x_0,\ 1]$. Differentiating again, we easily see that u satisfies (2.1) for $x \in [0,\ 1]$.

Note that $u(0) = 0$, and hence the first boundary condition is satisfied. Now since $G(\rho, c) = \sqrt{2}$, we have $u(1) = q(\rho)$, and since $F(\rho) = H(q(\rho)) = F(q) + \frac{(q-a)^2}{2}$, $u'(1) = -\sqrt{2[F(\rho) - F(q)]} = -\sqrt{(q-a)^2}$. Thus $u'(1) = q - a = u(1) - a$, and the second boundary condition is satisfied. □

In the next section, we will use this theorem combined with *Mathematica* computations to analyze the existence of positive solutions less than a to (2.1).

3 Positive Solutions to (1.5)

In this section, we provide computational results of positive solutions to (1.5), namely

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0, 1), \\ u(0) = 0, \\ u'(1) = u(1) - a. \end{cases}$$

In particular, recalling Theorem 4 in Section 2, for $a > 3$, we use *Mathematica* computations to analyze the level sets $G(\rho, c) - \sqrt{2} = 0$ within $S(a)$.

Our computations indicate the following results:

[I] For $a \in [8.291,\ 9.464]$, there exists $c^*(a) \le c_0(a)$ such that for all $c < c^*(a)$, (1.5) has exactly two positive solutions; for $c = c^*(a)$, (1.5) has a unique positive solution and for $c > c^*(a)$, (1.5) has no positive solutions. (See Figures 2.2-3.2.)

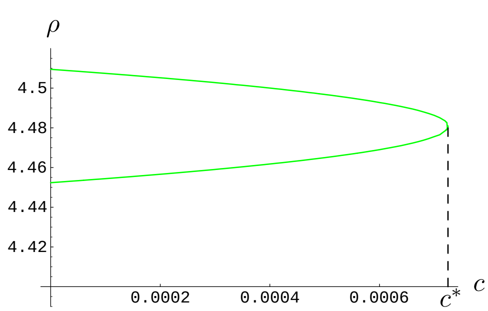

Figure 3.1: ρ vs. c for $a = 8.291$

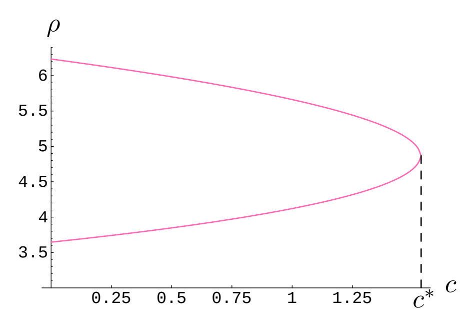

Figure 3.2: ρ vs. c for $a = 9$

[II] For $a \in (9.464,\ 16.318)$, there exists $c^*(a)$ and $\tilde{c}(a)$ where $\tilde{c}(a) \le c^*(a) \le c_0(a)$ such that for all $c \in [\tilde{c}(a),\ c^*(a))$, (1.5) has exactly two positive solutions, and for $c \in [0,\ \tilde{c}(a)) \cup \{c^*(a)\}$, (1.5) has a unique positive solution and for $c > c^*(a)$, (1.5) has no positive solutions. (See Figure 3.3.)

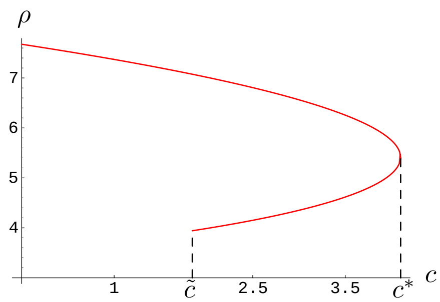

Figure 3.3: ρ vs. c for $a = 10$

[III] For $a \ge 16.318$, there exists $c^*(a)$ such that for all $c \le c^*(a)$, (1.5) has a unique positive solution and for $c > c^*(a)$, (1.5) has no positive solutions. (See Figures 3.4-3.5.)

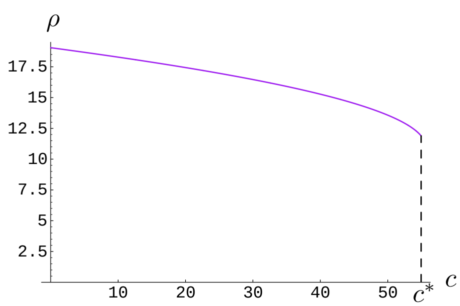

Figure 3.4: ρ vs. c for $a = 20$

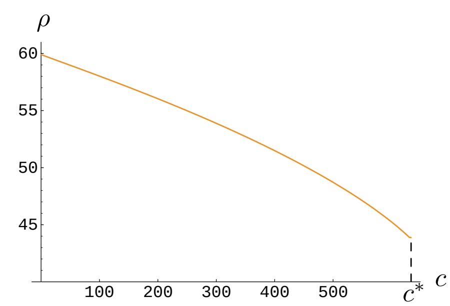

Figure 3.5: ρ vs. c for $a = 60$

Note that, from Figures 3.6-3.7, we can conclude there exists a c_1 such that:

[IV] If $0 \le c < c_1$, there exists $a_0, \tilde{a}$ such that if

- $a_0 < a < \tilde{a}$, (1.5) will have two solutions.
- $a > \tilde{a}$, (1.5) will have a unique solution.

[V] If $c \ge c_1$, there exists a_0 such that if $a \ge a_0$, (1.5) will have a unique solution.

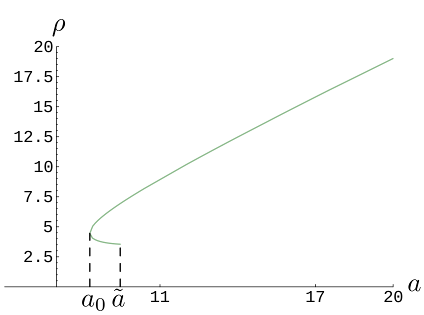

Figure 3.6: ρ vs. a for $c = 0$

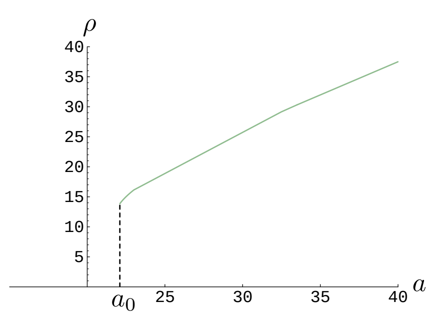

Figure 3.7: ρ vs. a for $c = 70$

In Figures 3.8-3.12, we provide samples of the level sets, $G(\rho,\ c) - \sqrt{2} = 0$, within $S(a)$. These computations produce the following results:

[VI] As $a \to \infty$, $c^*(a) \to c_0(a)$, and $\rho \to \ell_2$ where $\rho = \|u\|_\infty$ and $u(x)$ is the unique positive solution to (1.5).

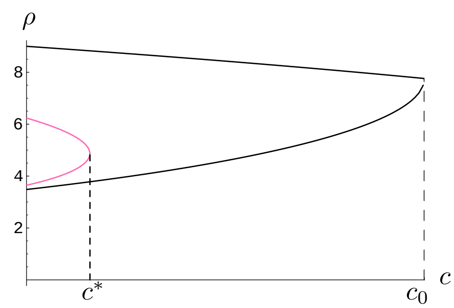

Figure 3.8: ρ vs. c for $a = 9$ in $S(9)$

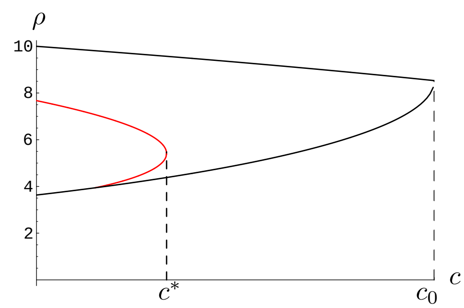

Figure 3.9: ρ vs. c for $a = 10$ in $S(10)$

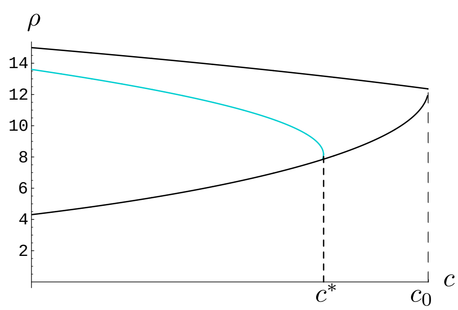

Figure 3.10: ρ vs. c for $a = 15$ in $S(15)$

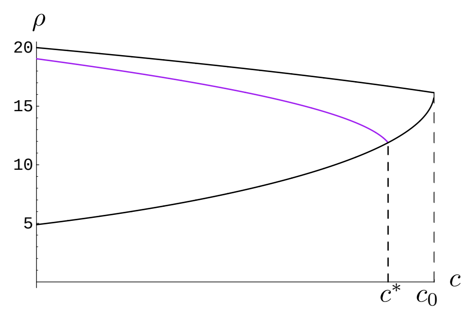

Figure 3.11: ρ vs. c for $a = 20$ in $S(20)$

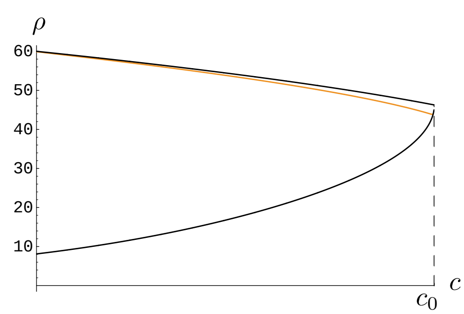

Figure 3.12: ρ vs. c for $a = 60$ in $S(60)$

Finally, we analyze the smaller solution of $\underline{u}(x)$ at $\tilde{c}(a)$ for $a \geq 8.291$. (Note that for $a \geq 16.318$, $\tilde{c}(a) = c^*$ and $\underline{u}(x)$ is the unique solution for (1.5).) In Figure 3.13-3.16, we provide sample computations of these solutions, $u(x)$.

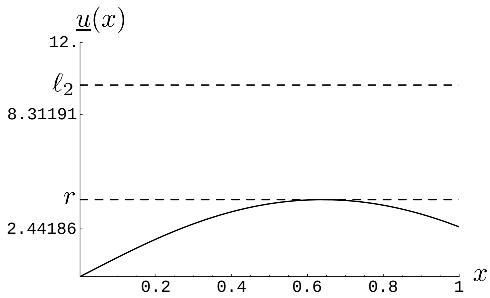

Figure 3.13: $\underline{u}(x)$ for $a = 10, c = 1.8455$

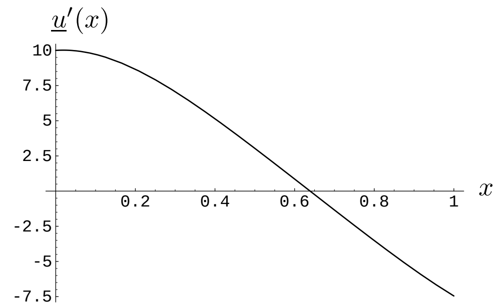

Figure 3.14: $\underline{u}'(x)$ for $a = 10, c = 1.8455$

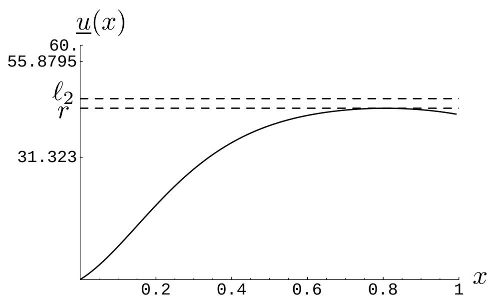

Figure 3.15: $\underline{u}(x)$ for $a = 60, c = 633.59$

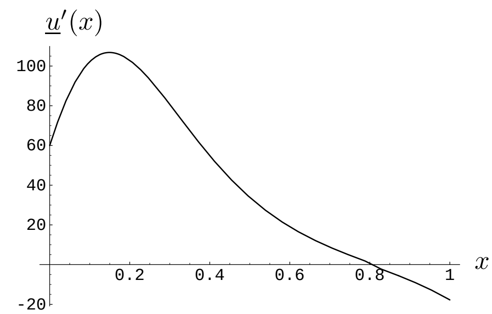

Figure 3.16: $\underline{u}'(x)$ for $a = 60, c = 633.59$

4 Positive Solutions to (1.3)

Here we combine our results for (1.5) from the previous section with the known results (see ([2]) for the Dirichlet problem (1.4). (See Figures 4.1-4.3.) In particular, we obtain the following results for positive solutions to (1.3).

[VIII] If $c = 0$, there exists a_1, a_2 such that if

- $a_1 < a < a_2$, (1.3) will have at least two solutions.
- $a_2 < a < \pi^2$, (1.3) will have at least one solution.
- $a > \pi^2$, (1.3) will have at least two solutions.

There exists some positive c_1 such that:

[IX] If $0 < c < c_1$, there exists a_1, a_2, a_3, a_4 such that if

- $a_1 < a < a_2$, (1.3) will have at least two solutions.
- $a_2 < a < a_3$, (1.3) will have at least one solution.
- $a_3 < a < a_4$, (1.3) will have at least three solutions.
- $a > a_4$, (1.3) will have at least two solutions.

[X] If $c > c_1$, there exists a_1, a_2, a_3 such that if

- $a_1 < a < a_2$, (1.3) will have at least one solution.
- $a_2 < a < a_3$, (1.3) will have at least three solutions.
- $a > a_3$, (1.3) will have at least two solutions.

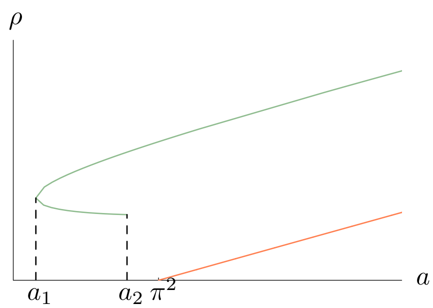

Figure 4.1: Solutions for $c = 0$ for (1.5) (top), (1.4) (bottom)

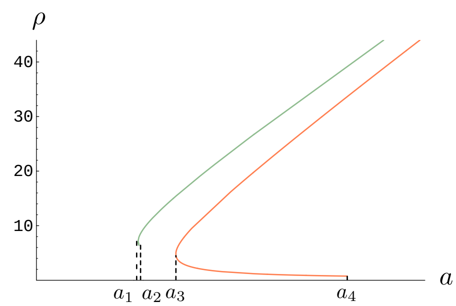

Figure 4.2: Solutions for $c = 15$ for (1.5) (top), (1.4) (bottom)

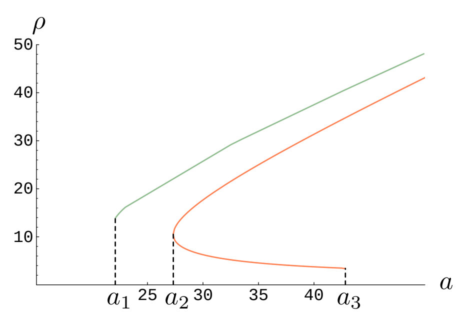

Figure 4.3: Solutions for $c = 70$ for (1.5) (top), (1.4) (bottom)

5 Neumann Boundary Conditions

First we describe a quadrature method to study positive solutions (1.6), namely

$$\begin{cases} -u'' = au - u^2 - c = f(u), & x \in (0,1), \\ u'(0) = 0 = u'(1). \end{cases} \tag{5.1}$$

Clearly, the roots of $f(u)$, $u \equiv \ell_1$ and $u \equiv \ell_2$, are constant solutions that exist for all $c < c_0 = \frac{a^2}{4}$. However we are interested in studying nonconstant solutions. It is clear that there are no positive solutions where $||u||_\infty > \ell_2$ and every solution must be such that $\int_0^1 f(u)\,dx = 0$. So nonconstant solutions cannot be such that $u(x) \leq \ell_1$ nor such that $\ell_1 \leq u(x) \leq \ell_2$. This clearly implies that positive nonconstant solutions cannot exist when $c = 0$, since $\ell_1 = 0$. We begin our study by analyzing solutions that have the shape of Figure 5.1.

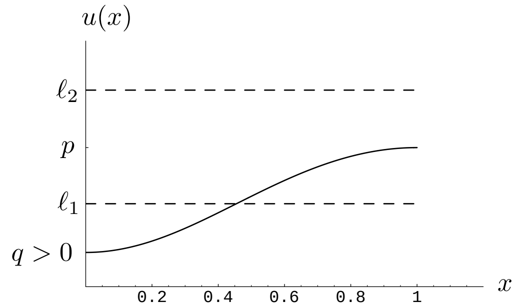

Figure 5.1: Positive Solution to the Neumann BVP

Here $u(0) = q$ and $u(1) = p$.

We note that if $u(x)$ is a solution, then $u(1-x)$ is also a solution. So for every nonconstant solution we find of the form above, we also get a second solution which has the shape of Figure 5.2.

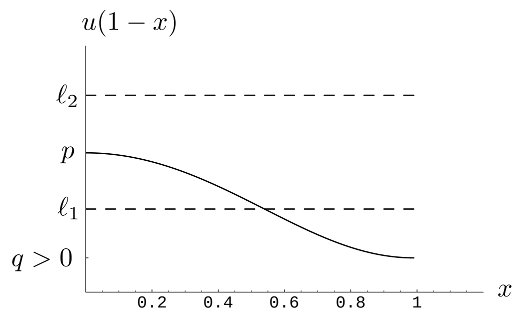

Figure 5.2: Positive Solution to the Neumann BVP

One can also study oscillatory solutions. Some examples of these types of solutions are shown in Figures 5.3 and 5.4.

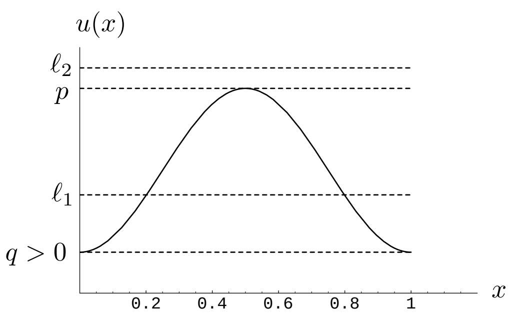

Figure 5.3: Solutions to (5.1) on $[0,\ \frac{1}{2}]$

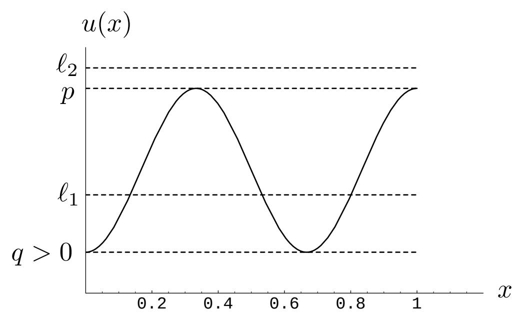

Figure 5.4: Solutions to (5.1) on $[0,\ \frac{1}{3}]$

However, this is equivalent to studying solutions of the form illustrated in Figure 5.1 in the intervals $[0, \frac{1}{2}]$, $[0, \frac{1}{3}]$, $[0, \frac{1}{4}]$, etc.

5.1 Quadrature Method for the Neumann Boundary Value Problem

Here we describe a quadrature method to study solutions of the form shown in Figure 5.1. First, suppose such a solution exists. Multiply (5.1) by $u'(x)$ and integrate to obtain

$$\frac{-(u')^2}{2} = F(u) + K, \tag{5.2}$$

where $F(u) = \int_0^u f(s)\,ds$. Now by applying the Neumann boundary conditions to (5.2) we easily obtain

$$F(p) = F(q). \tag{5.3}$$

We further obtain

$$u'(x) = \sqrt{2[F(p) - F(u)]}, \quad x \in [0, 1] \tag{5.4}$$

and

$$\int_q^{u(x)} \frac{ds}{\sqrt{F(p) - F(s)}} = \sqrt{2}x, \quad x \in [0, 1]. \tag{5.5}$$

Now setting $x = 1$, we get

$$\int_q^p \frac{ds}{\sqrt{F(p) - F(s)}} = \sqrt{2}. \tag{5.6}$$

Now we need to consider shapes of F that will allow ranges of p where (5.3) is satisfied and $F(p)-F(s) \geq 0$ for all $s \in [q,\ p]$. Typical F which meet these conditions are show in Figures 5.5 and 5.6.

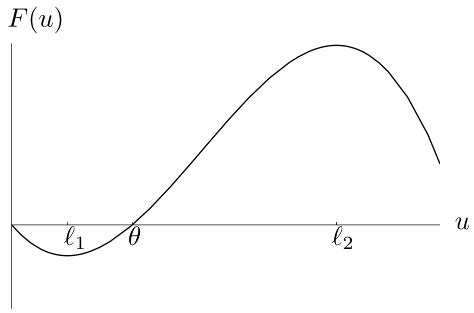

Figure 5.5: Shape of F when $c \leq \frac{3a^2}{16}$

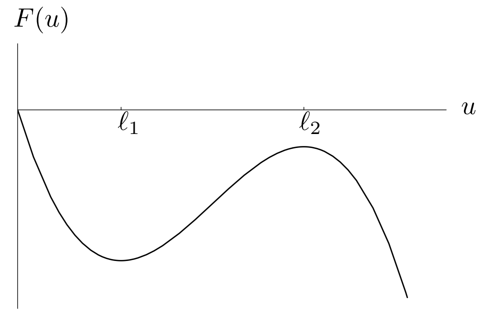

Figure 5.6: Shape of F when $\frac{3a^2}{16} < c < c_0$

So we require that ℓ_1 and ℓ_2 are real and distinct. Therefore, if $c \geq c_0 = \frac{a^2}{4}$, then there are no positive solutions. We can define an upper bound on p as a function of a and c,

$$r(a, c) = \begin{cases} \theta, & c \leq \frac{3a^2}{16}, \\ \ell_2, & \frac{3a^2}{16} < c < c_0. \end{cases}$$

If $c < c_0$, then for $p \in (\ell_1,\ r)$ there is a unique $q = q(p) < p$ such that $F(p) = F(q)$ and

$$G(p) = \int_q^p \frac{ds}{\sqrt{F(p) - F(s)}}$$

is well defined.

By using an argument similar to that used in Section 2 we obtain the following theorem.

Theorem 5. *Let $a > 0$ and $c < c_0$, then* (5.1) *has a positive nonconstant solution described in Figure 5.1 if and only if $G(p) = \sqrt{2}$ for $p \in (\ell_1,\ r)$.*

5.2 Computational Results for Neumann Boundary Value Problem

We now provide computational results to (5.1). With Theorem 5, we use *Mathematica* computations to analyze the roots of $G(p) = \sqrt{2}$.

Our computations (see sample bifurcation diagrams of p vs. c for fixed a in Figures 5.7 - 5.10) indicate the following results:

[XI] For $a \leq \pi^2$ we were unable to find nonconstant positive solutions of the form shown in Figure 5.1.

[XII] For $a > \pi^2$, there exist c_1 and c_2 such that if $c \in [c_1,\ c_2)$, then there is a nonconstant solution to (5.1) of the form shown in Figure 5.1.

[XIII] The branch of nonconstant solutions branches off of the constant solution ℓ_1 at $c = c_2$. Near this value of c, both p and q are close to ℓ_1. See Figure 5.17 for such a recovered $u(x)$.

[XIV] If $c_1 \le \frac{3a^2}{16}$, then as $c \to c_1$, the nonconstant solutions move towards the limiting case where $p \to \theta$ and $q \to 0$. Such a nonconstant solution has been recovered in Figure 5.11.

[XV] If $c_1 > \frac{3a^2}{16}$, then as $c \to c_1$, the nonconstant solutions move towards the limiting case where $p \to \ell_2$ and q is the solution of $F(q) = F(\ell_2)$. Such a nonconstant solution has been recovered in Figure 5.12.

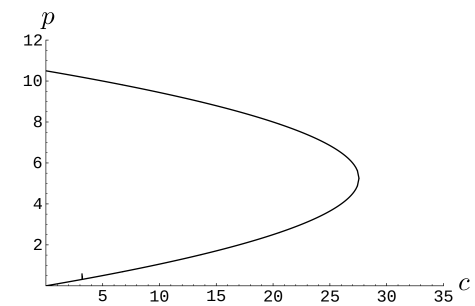

Figure 5.7: Bifurcation diagram for $a = 10.5$

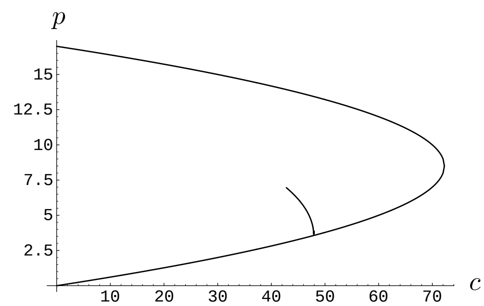

Figure 5.8: Bifurcation diagram for $a = 17$

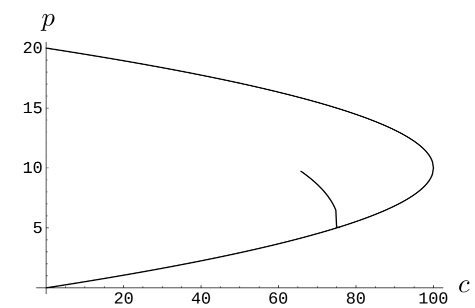

Figure 5.9: Bifurcation diagram for $a = 20$

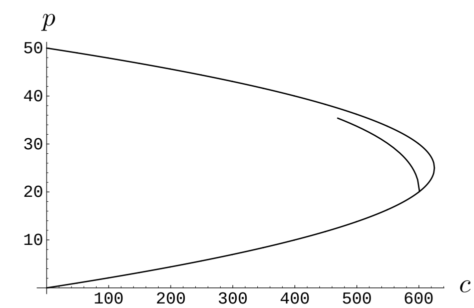

Figure 5.10: Bifurcation diagram for $a = 50$

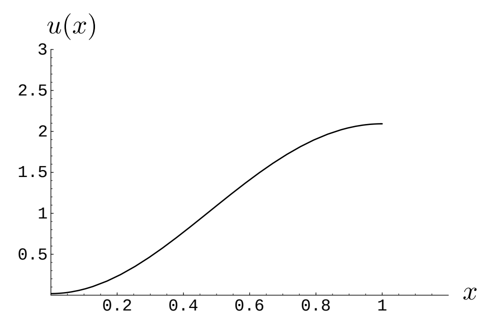

Figure 5.11: Recovered $u(x)$ for $a = 12, c = 11.2$

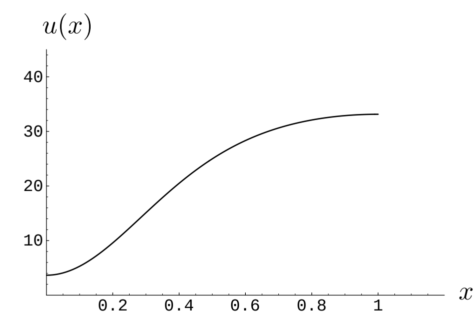

Figure 5.12: Recovered $u(x)$ for $a = 50c = 509$

By fixing values of c and varying a, we obtain similar results (see Figures 5.13 - 5.16):

[XVI] For any c, there exists a_1 and a_2 such that if $a \in (a_1,\ a_2]$, then there is a nonconstant solution to (5.1) of the form shown in Figure 5.1.

[XVII] The branch of nonconstant solutions branches off of the constant solution ℓ_1 at $a = a_1$. Near this value of a, both p and q are close to ℓ_1. See Figure 5.18 for such a recovered $u(x)$.

[XVIII] If $a > 4\sqrt{\frac{c}{3}}$, then as $a \to a_2$, the nonconstant solutions move towards the limiting case where $p \to \theta$ and $q \to 0$. If $a < 4\sqrt{\frac{c}{3}}$, then as $a \to a_2$, the nonconstant solutions move towards the limiting case where $p \to \ell_2$ and q is the solution of $F(q) = F(\ell_2)$.

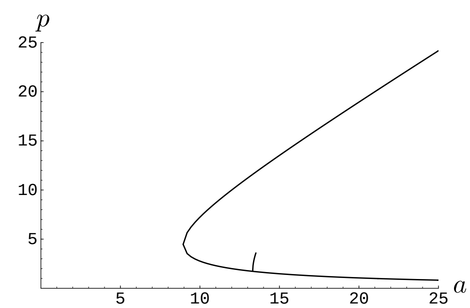

Figure 5.13: Bifurcation diagram for $c = 10$

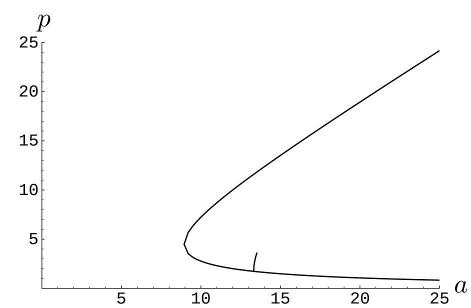

Figure 5.14: Bifurcation diagram for $c = 20$

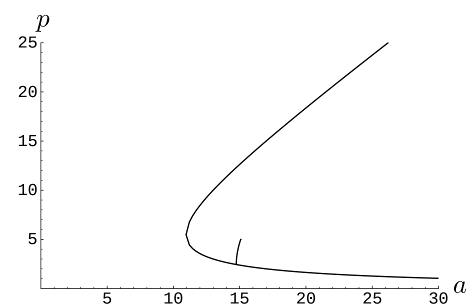

Figure 5.15: Bifurcation diagram for $c = 30$

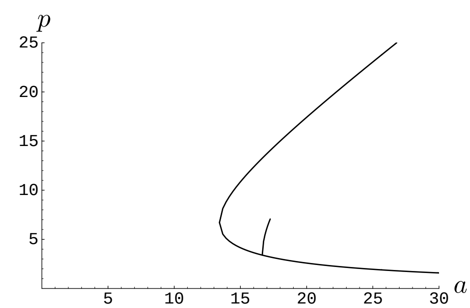

Figure 5.16: Bifurcation diagram for $c = 45$

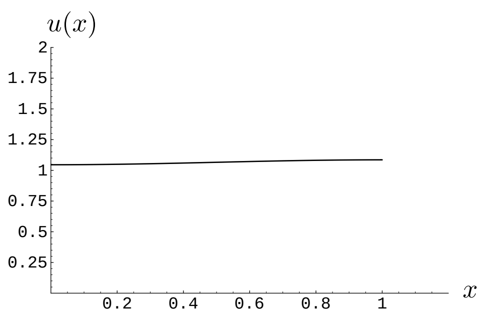

Figure 5.17: Recovered $u(x)$ for $a = 12$, $c = 11.65$, $\ell_1 = 1.065$

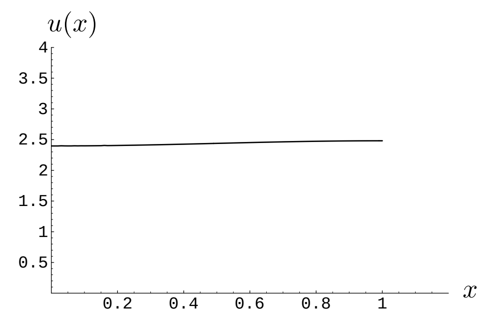

Figure 5.18: Recovered $u(x)$ for $a = 14.74$, $c = 30$, $\ell_1 = 2.349$

Remark: These bifurcation diagrams are not complete. They do not include the type of solutions shown in Figures 5.3 and 5.4. The study of such solutions is currently in progress.

6 Robin Boundary Conditions

In this section we study the positive solutions to (1.7), namely

$$\begin{cases} -u'' = au - u^2 - c = f(u), \quad x \in (0,1), \\ u(0) = 0, \\ u'(1) + \beta u(1) = 0, \end{cases} \tag{6.1}$$

where $\beta = \dfrac{1}{\alpha(1)} - 1$.

First we establish a non-existence result. Let $\lambda = \lambda_1(\beta) > 0$ be the principal eigenvalue of

$$\begin{cases} -\phi'' = \lambda\phi, \quad x \in (0,1), \\ \phi(0) = 0, \\ \phi'(1) + \beta\phi(1) = 0. \end{cases} \tag{6.2}$$

In fact, the eigenvalues λ of (6.2) are positive and satisfy the equation $-\tan\sqrt{\lambda} = \dfrac{\sqrt{\lambda}}{\beta}$.

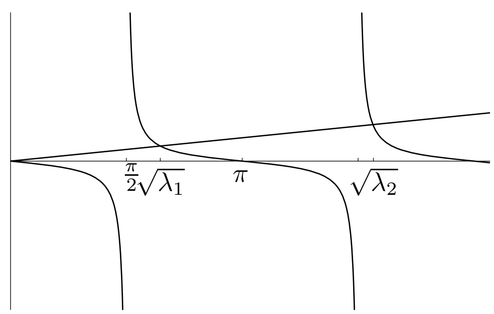

Figure 6.1: Graph of $-tan(\sqrt{\lambda})$ and $\frac{\sqrt{\lambda}}{\beta}$

Remark: $\lambda_1(\beta) \to \pi^2$ as $\beta \to \infty$ and $\lambda_1(\beta) \to \frac{\pi^2}{4}$ as $\beta \to 0$.
Then we establish:

Theorem 6. *Let $a \leq \lambda_1(\beta)$. Then (6.1) has no positive solutions.*

Proof. Multiplying (6.1) by $\phi_1 = \sin(x\sqrt{\lambda_1})$ (eigenfunction corresponding to λ_1) and integrating we have

$$\int_0^1 -u'' \sin(x\sqrt{\lambda_1})dx = \int_0^1 (au - u^2 - c)\sin(x\sqrt{\lambda_1})dx. \tag{6.3}$$

Now integrating by parts twice and applying the boundary condition at $x = 0$ in (6.1) we have

$$\begin{aligned} \int_0^1 -u'' \sin(x\sqrt{\lambda_1})dx &= \int_0^1 u\lambda_1 \sin(x\sqrt{\lambda_1})dx + u(1)[\beta \sin\sqrt{\lambda_1} + \sqrt{\lambda_1}\cos(\sqrt{\lambda_1})] \\ &= \int_0^1 u\lambda_1 \sin(x\sqrt{\lambda_1})dx, \end{aligned} \tag{6.4}$$

since by the boundary condition at $x = 1$ in (6.2), $\beta \sin\sqrt{\lambda_1} + \sqrt{\lambda_1}\cos(\sqrt{\lambda_1}) = 0$. From (6.3) and (6.4) we have

$$\int_0^1 u(a - \lambda_1)\sin(x\sqrt{\lambda_1})dx = \int_0^1 (u^2 + c)\sin(x\sqrt{\lambda_1})dx. \tag{6.5}$$

Clearly the right hand side of (6.5) is positive, and hence for a positive solution u to exist $a > \lambda_1$ is a necessary condition for the existence of a solution of (6.1). $\square$

6.1 Quadrature Method for the Robin Boundary Value Problem

Here we describe a quadrature method to study the positive solutions to (1.7). It is easy to see that positive solutions must look like

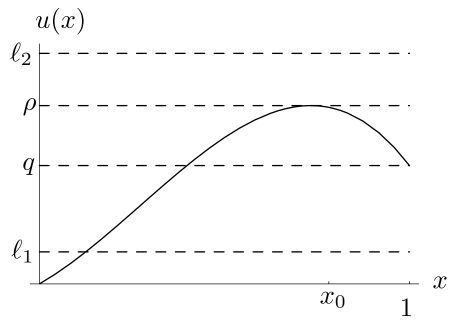

Figure 6.2: General shape of solution

In particular we study the case when $q > 0$. Here, $\ell_1 = \dfrac{a-\sqrt{a^2-4c}}{2}$ and $\ell_2 = \dfrac{a+\sqrt{a^2-4c}}{2}$ are the positive zeroes of f, and clearly $c < \frac{a^2}{4}$ is a necessary condition for such solutions to exist. We proceed by following arguments similar to those in Section 2, with the exception that ρ and q are now related by the equation

$$F(\rho) = F(q) + \frac{\beta^2 q^2}{2}. \tag{6.6}$$

Let $\tilde{H}(x) = F(x) + \dfrac{\beta^2 x^2}{2}$; then $\tilde{H}(x) = \left(\dfrac{a+\beta^2}{2}\right)x^2 - \dfrac{x^3}{3} - cx.$

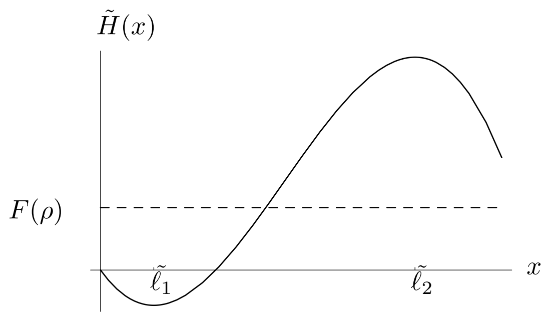

Figure 6.3: Graph of $\tilde{H}(x)$

Here $\tilde{\ell_1} = \dfrac{(a+\beta^2)-\sqrt{(a+\beta^2)^2-4c}}{2}$ and $\tilde{\ell_2} = \dfrac{(a+\beta^2)+\sqrt{(a+\beta^2)^2-4c}}{2}$ are the roots of $\tilde{H}'(x)$, and $\tilde{H}$ increases in $(\tilde{\ell_1}, \tilde{\ell_2})$. Again following arguments similar to those found in Section 2, the existence of a unique $q = q(\rho) \le \rho$ satisfying (6.6) requires $F(\rho) > 0$. Thus $c < \frac{3a^2}{16}$ is a necessary condition, and $\rho \in (\theta, \ell_2)$, where $\theta = \dfrac{3a^2+\sqrt{9a^2-48c}}{2}$ is the first positive root of F.

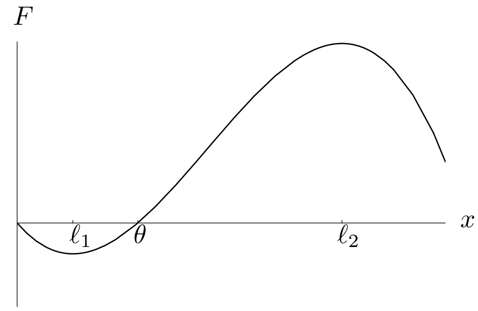

Figure 6.4: Graph of $F(x)$

In particular, the following results hold:

Theorem 7. *Let $a > \lambda_1(\beta)$ and $c \geq \frac{3a^2}{16}$. Then (6.1) has no positive solutions as described in Figure 6.2.*

Theorem 8. *Let $a > \lambda_1(\beta)$ and $c < \frac{3a^2}{16}$. Then (6.1) has a positive solution as described in Figure 6.2 if and only if $G(\rho, c) = \sqrt{2}$ for some $(\rho, c) \in S(a)$ where*

$$G(\rho, c) = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(ds)}},$$

$S(a) = \left\{(\rho, c) : \theta < \rho < \ell_2, 0 \leq c < \dfrac{3a^2}{16}\right\}$, *and $q = q(\rho)$ is the unique number such that $0 < q < \rho$ and $F(\rho) = \tilde{H}(q)$. Further, $u(x)$ is defined by*

$$\int_0^u \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2}x, \quad x \in (0, x_0),$$

and

$$\int_\rho^u \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = -\sqrt{2}(x - x_0), \quad x \in (x_0, 1),$$

where

$$x_0 = \frac{1}{\sqrt{2}} \int_0^\rho \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds.$$

6.2 Computational Results for Robin Boundary Value Problem

In this section we analyze the level sets $G(\rho, c) - \sqrt{2} = 0$ within $S(a)$ via *Mathematica* computations. Our computations lead to the following results:

[XIX] There exists $a^*(\beta) \in (\lambda_1(\beta), \lambda_2(\beta))$ such that for every $a \in (\lambda_1(\beta), a^*(\beta)]$ there exists $c^*(a, \beta) \leq \frac{3a^2}{16}$ such that (6.1) has exactly two positive solutions for $c \in (0, c^*)$, a unique positive solution for $c \in \{0, c^*\}$, and no positive solutions for $c > c^*$. Here $\lambda_2(\beta)$ is the second eigenvalue of (6.2). Recall that for the Dirichlet problem (1.1) such a bifurcation diagram persists for c small and $a \in (\lambda_1 = \pi^2, \lambda_2 = 4\pi^2)$. But here, for the Robin boundary value problem, $a^*(\beta) < \lambda_2(\beta)$ with $a^*(\beta)$ an increasing function such that $a^*(\beta) \to 4\pi^2$ as $\beta \to \infty$. (See Figure 6.5 for the description of $\lambda_1(\beta)$, $a^*(\beta)$, and $\lambda_2(\beta)$ as β varies, and Figures 6.6-6.7 for sample bifurcation diagrams for ρ vs. c of fixed a.)

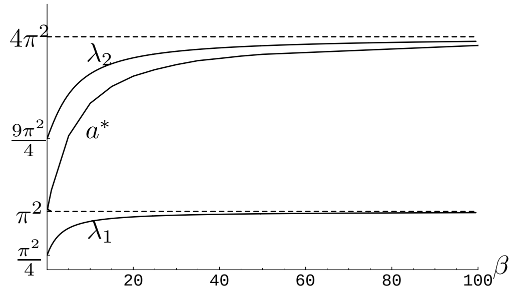

Figure 6.5: Top: β vs. $\lambda_2(\beta)$, Middle: β vs. $a^*(\beta)$, Bottom: β vs. $\lambda_1(\beta)$

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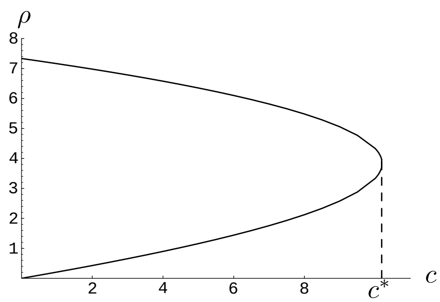

Figure 6.6: Graph of ρ vs. c for $a = 10$ and $\beta = 0.5$

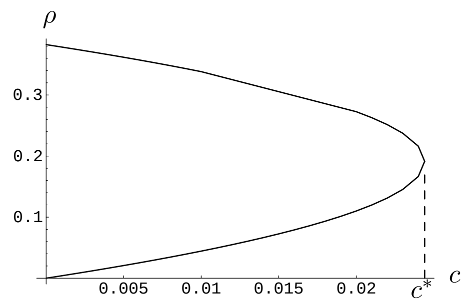

Figure 6.7: Graph of ρ vs. c for $a = 10$ and $\beta = 100$

[XX] There exists a^{**} such that for $a > a^*(\beta)$, there exist $\tilde{c}(a, \beta)$ and $c^*(a, \beta)$ (both less than or equal to $\frac{3a^2}{16}$) such that (6.1) has a unique positive solution for $c \in [0, \tilde{c}) \bigcup \{c^*\}$, exactly two positive solutions for $c \in [\tilde{c}, c^*)$, and no positive solutions for $c > c^*$. (See Figures 6.8-6.10 for sample bifurcation diagrams of ρ vs c for fixed a.)

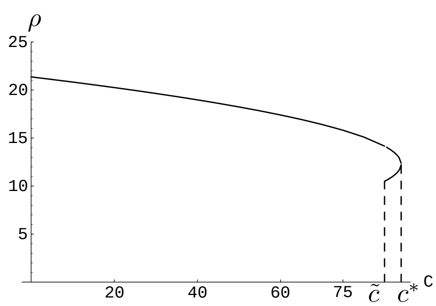

Figure 6.8: Graph of ρ vs. c for $a = \lambda_2(\beta)$ and $\beta = 0.5$

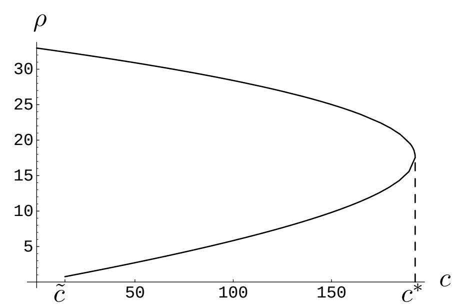

Figure 6.9: Graph of ρ vs. c for $a = \lambda_2(\beta)$ and $\beta = 100$

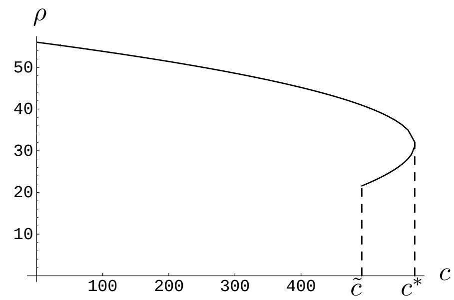

Figure 6.10: Graph of ρ vs. c for $a = 60$ and $\beta = 100$

[XXI] For $a > a^{**}(\beta)$, there exists $c^*(a,\beta) < \frac{3a^2}{16}$ such that for every $c < c^*$, (6.1) has a unique positive solution, and for $c > c^*$, (6.1) has no positive solutions. (See Figure 6.11 for a conjectured picture of this occurrence.)

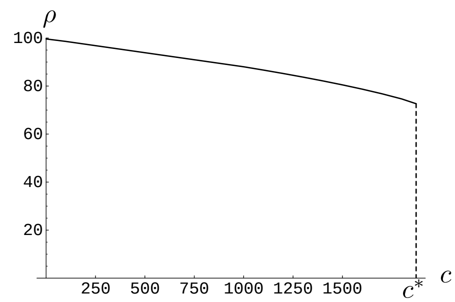

Figure 6.11: Graph of ρ vs. c for $a = 100$ and $\beta = 0.5$.

[XXII] Given $0 < c < \frac{3a^2}{16}$, there exists an $a_0(c,\beta)$ and $\tilde{a}(c,\beta)$, both greater than $\lambda_1(\beta)$, such that for $a \in (a_0, \tilde{a})$ we have exactly two positive solutions, and for $a = a_0$ and $a > \tilde{a}$ there is a unique solution. Further, as $a \to \infty$ this unique solution is such that $\rho \to a$ for any fixed c. For $c = 0$, (6.1) has a unique positive solution for $a > \lambda_1(\beta)$. (See Figures 6.12-6.16 for sample bifurcation diagrams of ρ vs. a for fixed c.)

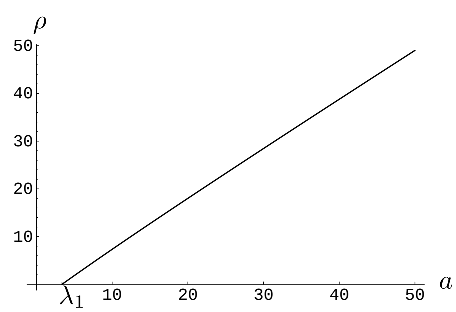

Figure 6.12: Graph of ρ vs. a for $c = 0$

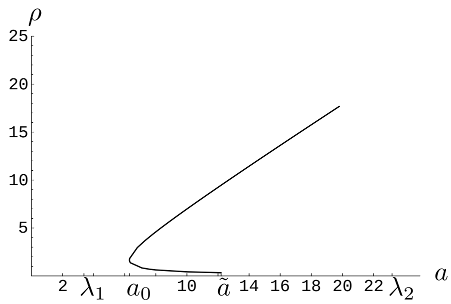

Figure 6.13: Graph of a vs. ρ for $c = 2$ and $\beta = 0.5$

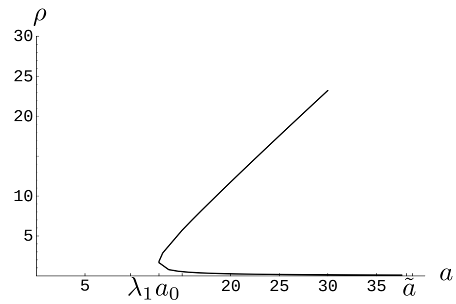

Figure 6.14: Graph of a vs. ρ for $c = 0$ (left) and $c = 2$ (right), where $\beta = 100$

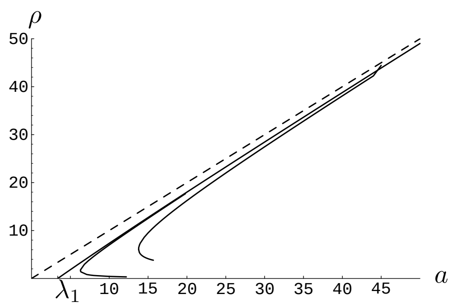

Figure 6.15: Graph of $\rho = a$ (dotted) and ρ vs. a for (left to right) $c = 0$, $c = 2$, and $c = 25$, where $\beta = 0.5$

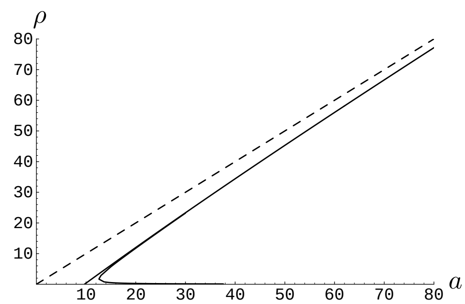

Figure 6.16: Graph of $\rho = a$ (dotted) and ρ vs. a for $c = 0$ (left) and $c = 2$ (right), where $\beta = 100$

[XXIII] As expected, as $\beta \to 0$ our solutions to (6.1) resemble solutions of the Dirichlet boundary value problem on $[0, 2]$, and as $\beta \to \infty$ our solutions resemble those of the Dirichlet problem on the interval $[0, 1]$. (See sample computation of the smaller solution $\underline{u}(x)$ at $\tilde{c}$ in Figure 6.17 for a small β and Figure 6.18 for a large β.

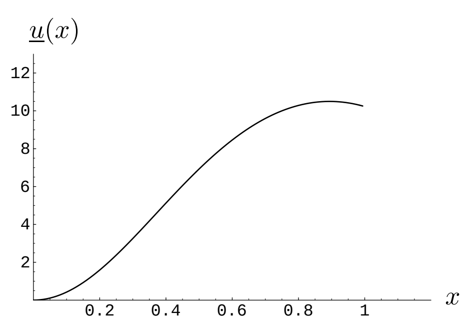

Figure 6.17: Graph of the solution $\underline{u}(x)$ for $a = \lambda_2(\beta)$, $c = 85$, $\rho = 10.49$, and $\beta = 0.5$

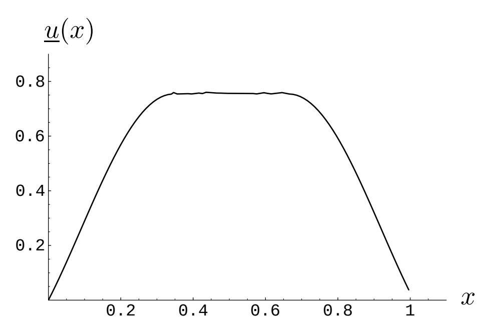

Figure 6.18: Graph of the solution $\underline{u}(x)$ for $a = \lambda_2(\beta)$, $c = 10$, $\rho = 0.7539$, and $\beta = 100$

7 Related Boundary Conditions

In this section we discuss boundary conditions related to those studied in Section 2.

7.1 Single Nonhomogeneous Boundary Condition

First, we consider the boundary conditions where

$$\alpha(u,x) = \begin{cases} \dfrac{u}{a}; & x = 0 \\ 0; & x = 1 \end{cases}$$

That is, we study the nonlinear boundary value problem

$$\begin{cases} -u''(x) = a\ u - u^2 - c, \quad x \in (0,1), \\ -u(0)\ u'(0) + [a - u(0)]\ u(0) = 0, \\ \ \ u(1) = 0. \end{cases} \tag{7.1}$$

Note that this problem is equivalent to studying (1.4) and

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0,1), \\ \ \ u'(0) = a - u(0), \\ \ \ u(1) = 0. \end{cases} \tag{7.2}$$

In fact, if $u(x)$ is a solution of (1.5), then $v(x) = u(1-x)$ is a solution of (7.2). Thus the analysis of (1.5) is also applicable to (7.2).

7.2 Two Nonhomogeneous Boundary Conditions

We also consider the boundary condition

$$\alpha(u,x) = \frac{u}{a}; \quad x \in \{0,1\}. \tag{7.3}$$

That is, we study the nonlinear boundary value problem

$$\begin{cases} -u''(x) = a\ u - u^2 - c, \quad x \in (0,1), \\ -u(0)\ u'(0) + [a - u(0)]\ u(0) = 0, \\ -u(1)\ u'(1) + [u(1) - a]\ u(1) = 0. \end{cases} \tag{7.4}$$

Studying (7.4) is clearly equivalent to studying the following four problems

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0,1), \\ \ \ u(0) = 0 = u(1), \end{cases} \tag{7.5}$$

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0,1), \\ \ \ u(0) = 0, \\ \ \ u'(1) = u(1) - a, \end{cases} \tag{7.6}$$

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0,1), \\ \ \ u'(0) = a - u(0), \\ \ \ u(1) = 0, \end{cases} \tag{7.7}$$

and

$$\begin{cases} -u''(x) = au - u^2 - c, \quad x \in (0,1), \\ \ \ u'(0) = a - u(0), \\ \ \ u'(1) = u(1) - a. \end{cases} \tag{7.8}$$

Note that (7.5) has been studied by [2], (7.6) has been analyzed in Section 2, and this analysis also provides results for (7.7).

We can develop a quadrature method similar to the one in Section 2 to study solutions to (7.8). We focus on the analysis of positive solutions $u(x)$ less than a. Let $u(0) = m$ and $u(1) = q$. It can be easily shown that if $u(x)$ is a solution to (7.8), then $u(1-x)$ is also a solution with $u(0) = q$ and $u(1) = m$. Thus is it sufficient to obtain results for solutions where $u(0) \leq u(1)$. Because we are requiring our solutions to be less than a, typical solutions will resemble Figure 7.1.

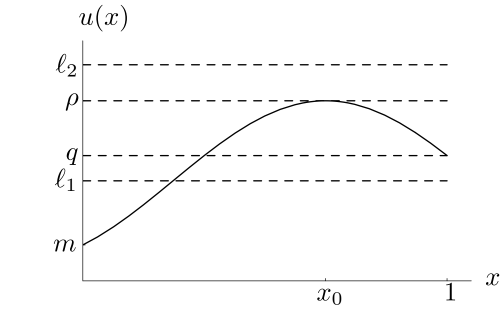

Figure 7.1: Graph of $u(x)$

Namely, there must once again exist an $x_0 \in (0,1)$ such that $u'(x_0) = 0$ and $u(x_0) = \rho = \|u\|_\infty$. Some calculations easily show that for such a solution to exist the equation $F(\rho) = F(z) + \frac{(z-a)^2}{2}$ (where $F(z)$ is as defined in Section 2) must hold for $z = m$ and $z = q$. Thus once again we define the function $H(z) = F(z) + \frac{(z-a)^2}{2}$, and the roots of $H'(z)$, r_1 and r_2, must be real and distinct. A typical graph of $H(z)$ is as follows:

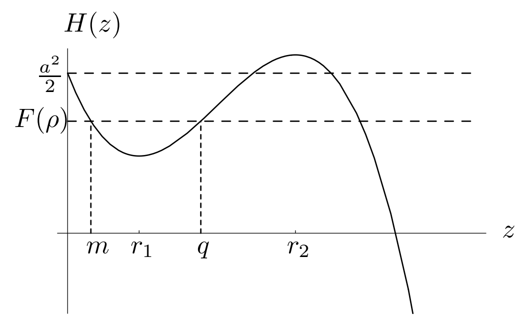

Figure 7.2: Graph of $H(z)$

Noting that $F(\rho) \le H(\rho)$, we see that $\rho \in [\underline{\rho}, \overline{\rho}] \cap (\theta, \ell_2)$. Here $\underline{\rho} \le \overline{\rho}$ are such that $F(\underline{\rho}) = \max\{H(r_1), 0\}$ and $F(\overline{\rho}) = \min\left\{\frac{a^2}{2}, H(r_2)\right\}$. In particular, our analysis yields the following result.

Theorem 9. *There exists a positive solution $u(x)$ to* (7.8) *as described by Figure 7.1 if and only if*

$$G(\rho, c) = \int_m^\rho \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds + \int_q^\rho \frac{1}{\sqrt{F(\rho) - F(s)}}\, ds = \sqrt{2},$$

for $(\rho, c) \in S(a) = \{(\rho, c) \mid \rho \in [\underline{\rho}, \overline{\rho}] \cap (\theta, \ell_2),\ 0 \le c \le c_0\}$, where $c_0 < \min\left\{\frac{3a^2}{16}, \frac{(a-1)^2}{4}\right\}$. Here $m \le q$ are the roots of $F(\rho) = H(z)$ as described in Figure 7.2.

Using this theorem, one can systematically search the domain $S(a)$ to determine the existence of solutions to (7.8).

Acknowledgements

This work was performed by the NSF REU students at Mississppi State University during the Summer of 2004. We would like to extend a special thanks to Dr. Ratnasingham Shivaji for his guidance throughout this research. We would also like to thank Mark Riggs and Yijun Sun for their additional assistance and guidance.

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References

[1] Shobha Oruganti, Junping Shi, Ratnasingham Shivaji, *Diffusive Logistic Equation with Constant Yield Harvesting, I: Steady States*, Trans. Amer. Math. Soc. Vol 354(2002) No. 9, pp. 3601-3619.

[2] A. Collins, M. Gilliland, C. Henderson, S. Koone, L. McFerrin, E. K. Wampler, *Population Models with Diffusion and Constant Yield Harvesting*, Rose-Hulman Undergraduate Mathematics Journal Vol 5(2004) Issue 2.

[3] Robert Cantrell and Chris Cosner, *Density Dependent Behavior at Habitat Boundaries and the Allee Effect*, AIMS' Fifth International Conference on Dynamical Systems and Differential Equations. June 2004, Pomona, CA.

[4] Robert Cantrell and Chris Cosner, *Spatial Ecology via Reaction-Diffusion Equation*. Wiley, West Sussex, England, 2003.

[5] Theodore Laetsch. *The Number of Solutions of a Nonlinear Two Point Boundary Values Problem*, Indiana Univ. Math. J. Vol 20(1920) No. 1, pp. 1-13.