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TIGHT SUBDESIGNS OF THE HIGMAN-SIMS DESIGN

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Abstract. The Higman-Sims design is an incidence structure of 176 points and 176 blocks of cardinality 50 with every two blocks meeting in 14 points. The automorphism group of this design is the Higman-Sims simple group. We demonstrate that the point set and the block set of the Higman-Sims design can be partitioned into subsets $X_1, X_2, \dots, X_{11}$ and $B_1, B_2, \dots, B_{11}$, respectively, so that the substructures $(X_i, B_i), i = 1, 2, \dots, 11$, are isomorphic symmetric $(16, 6, 2)$-designs.

1. Introduction

One of the most basic concepts in the theory of combinatorial designs is that of an *incidence structure*. Essentially, it indicates two finite sets and indicates how they are related.

Definition 1.1. A (finite) *incidence structure* is a triple $\mathbf{D} = (X, \mathcal{B}, I)$, where X and $\mathcal{B}$ are nonempty finite sets and $I \subseteq X \times B$. The sets X and $\mathcal{B}$ are called the *point set* and *block set* of $\mathbf{D}$, respectively, and their elements are called *points* and *blocks*. The set I is called the *incidence relation*. If $(x, B) \in I$, we will say that point x and block B are incident and that (x, B) is a *flag*. The number of points incident with a block B is called the *size* or the *cardinality* of B and denoted by $|B|$. The number of blocks incident with a point x is called the *replication number* of x and denoted by $r(x)$. For distinct points x and y, $\lambda(x, y)$ denotes the number of blocks incident with both x and y. An *incidence matrix* of $\mathbf{D}$ is a $(0, 1)$ matrix whose rows are indexed by the points of $\mathbf{D}$, columns are indexed by the blocks of $\mathbf{D}$, and the (x, B)-entry is equal to 1 if and only if $(x, B) \in I$.

Imposing certain conditions on an incidence structure yields a (v, b, r, k, λ)-design.

Definition 1.2. A (v, b, r, k, λ)-design is an incidence structure $\mathbf{D} = (X, \mathcal{B}, I)$ satisfying the following conditions: (i) $|X| = v$; (ii) $|\mathcal{B}| = b$; (iii) $r(x) = r$ for all $x \in X$; (iv) $|B| = k$ for all $B \in \mathcal{B}$; (v) $\lambda(x, y) = \lambda$ for all distinct $x, y \in X$, (vi) if $I = \emptyset$ or $I = X \times \mathcal{B}$, then $v = b$.

The following result is well known from [6].

Proposition 1.3. *Let $\mathbf{D} = (X, \mathcal{B}, I)$ be an incidence structure satisfying conditions (i)-(iv) and (vi) of Definition 1.2. Suppose further that there exists a nonnegative integer λ such that $(v-1)\lambda = r(k-1)$. If any two points of D are contained in at most λ blocks or any two points of $\mathbf{D}$ are contained in at least λ blocks, then $\mathbf{D}$ is a (v, b, r, k, λ)-design.*

When a (v, b, r, k, λ)-design has certain symmetric properties, we have a symmetric (v, k, λ)-design. Two examples of symmetric designs are the Fano Plane and the $(16, 6, 2)$-design.

Example 1.4. The *Fano Plane* is a $(7, 3, 1)$ design, and is the smallest non-trivial example of a symmetric design. It is shown in Figure 1 at the end of the paper.

Example 1.5. The following matrix is an incidence matrix of a symmetric $(16, 6, 2)$-design.

$$\begin{pmatrix}
0000 & 1100 & 1010 & 1001 \\
0000 & 1100 & 0101 & 0110 \\
0000 & 0011 & 1010 & 0110 \\
0000 & 0011 & 0101 & 1001 \\
\\
1100 & 0000 & 1001 & 1010 \\
1100 & 0000 & 0110 & 0101 \\
0011 & 0000 & 0110 & 1010 \\
0011 & 0000 & 1001 & 0101 \\
\\
1010 & 1001 & 0000 & 1100 \\
0101 & 0110 & 0000 & 1100 \\
1010 & 0110 & 0000 & 0011 \\
0101 & 1001 & 0000 & 0011 \\
\\
1001 & 1010 & 1100 & 0000 \\
0110 & 0101 & 1100 & 0000 \\
0110 & 1010 & 0011 & 0000 \\
1001 & 0101 & 0011 & 0000
\end{pmatrix} \tag{1}$$

Remark 1.6. There are three non-isomorphic $(16, 6, 2)$-designs which may be distinguished by the rank of their incidence matrices over the field of two elements. All designs we study are isomorphic to the one whose incidence matrix is presented in (1).

Definition 1.7. A (v, b, r, k, λ)-design where $v = b$ and $r = k$ is referred to as a symmetric (v, k, λ)-design.

Another incidence structure that will be considered in this paper is a 5-(v, k, λ) design. In a t-(v, k, λ) design, there are v points, all blocks have size k, and every subset of t points is incident with exactly λ blocks. The Witt design on 24 points, $\mathbf{W_{24}}$, is a 5-$(24, 8, 1)$ design. It is interesting to study $\mathbf{W_{24}}$ because its automorphism group is one of the 26 sporadic simple groups. It is also known that any two designs $\mathbf{W_{24}}$ are isomorphic.

Definition 1.8. The Witt design on 24 points, $\mathbf{W_{24}}$, is an incidence structure $\mathbf{W_{24}} = (Q, \mathcal{S})$ satisfying the following conditions:

(i)$|Q| = 24$;
(ii) $|B| = 8$ for every $B \in \mathcal{S}$;
(iii) every set of 5 points is incident with exactly one block.

The following two theorems are known from [1].

Theorem 1.9. *Let λ_i denote the number of blocks of $\mathbf{W_{24}}$ containing a given set of i points. Then we have the following:*
$\lambda_5 = 1, \lambda_4 = 5, \lambda_3 = 21, \lambda_2 = 77, \lambda_1 = 253, \lambda_0 = 759.$

Definition 1.10. An *intersection number* of an incidence structure is the size of the intersection of two distinct blocks.

Theorem 1.11. *The intersection numbers of* $\mathbf{W_{24}}$ *are* $0, 2,$ *and* 4*; given a block* A *of* $\mathbf{W_{24}}$*, there are 30 blocks that are disjoint from* A*, 448 blocks that meet* A *in two points, and* 280 *blocks that meet* A *in four points.*

The list of blocks for $\mathbf{W_{24}}$ can be found in [3].

For the following theorem, refer to [8].

Theorem 1.12. *Let* a *and* b *be distinct points of* $\mathbf{W_{24}}$*. Let* $\mathcal{A}$ *be the set of all blocks of* $\mathbf{W_{24}}$ *that contain* a *and do not contain* b*, and let* $\mathcal{B}$ *be the set of blocks of* $\mathbf{W_{24}}$ *that contain* b *and do not contain* a*. Let* $\mathbf{H}$ *denote the incidence structure* $(\mathcal{A}, \mathcal{B}, I)$ *with* $(A, B) \in I$ *if and only if* $|A \cap B| = 2$*. Then* $\mathbf{H}$ *is a symmetric* $(176, 126, 90)$*-design.*

The design constructed in this theorem, along with its complementary design are called the Higman-Sims designs. The automorphism group of the Higman-Sims design is also a sporadic simple group, known as the Higman-Sims simple group. Throughout this paper, we refer to the symmetric $(176, 126, 90)$-design, $\mathbf{H}$, as the Higman-Sims design.

When studying an incidence structure, it may be useful to look at its substructures. Some of these substructures are themselves symmetric designs referred to as *tight subdesigns*.

Definition 1.13. A subdesign of a symmetric design $\mathbf{D} = (X, \mathcal{B}, I)$ is a symmetric design $\mathbf{D_1} = (X_1, \mathcal{B}_1, I_1)$ such that $X_1 \subseteq X, \mathcal{B}_1 \subseteq \mathcal{B}$, and, for $x \in X_1$ and $B \in \mathcal{B}_1$, the point x and the block B are incident in $\mathbf{D_1}$ if and only if they are incident in $\mathbf{D}$. If $\mathbf{D_1}$ is a symmetric (v_1, k_1, λ_1)-design, we will refer to it as a (v_1, k_1, λ_1)-subdesign of $\mathbf{D}$.

Further, a (v_1, k_1, λ_1)-subdesign $\mathbf{D_1} = (X_1, \mathcal{B}_1, I_1)$ of a symmetric (v, k, λ)-design $\mathbf{D} = (X, \mathcal{B}, I)$ is called a *tight subdesign* if $v_1 < v$ and there is an integer k_2 such that $|B \cap X_1| = k_2$ for all blocks $B \in \mathcal{B} \setminus \mathcal{B}_1$.

We have the following theorem from [7]:

Theorem 1.14. *Let* $\mathbf{D}_1$ *be a* (v_1, k_1, λ_1)*-subdesign of a symmetric* (v, k, λ)*-design* $\mathbf{D}$ *with* $v_1 < v$ *and let* $k_2 = \frac{v_1(k-k_1)}{(v-v_1)}$*. Then* $\mathbf{D}_1$ *is tight if and only if* $k - \lambda = (k_1 - k_2)^2$*.*

Corollary 1.15. *A* $(16, 6, 2)$*-subdesign of the Higman-Sims design is a tight subdesign.*

Remark 1.16. The notion of a tight subdesign was introduced by Haemers and M.S. Shrikhande [4]. The original concept and definition of tight subdesigns uses the technique of interlacing of eigenvalues developed earlier by Haemers [5]. The original definition and the definition presented in this paper were proved equivalent by Jungnickel [7].

Throughout the paper, we fix two points a and b of $\mathbf{W_{24}} = (Q, \mathcal{S})$ and assume that $\mathbf{H}$ is the Higman-Sims design described in Theorem 1.12.

2. The $(16, 6, 2)$-subdesign

Theorem 2.1. *Let* c *and* d *be distinct points of* $\mathbf{W_{24}}$ *other than* a *or* b*. Let* $\mathcal{A}_0 \subseteq \mathcal{A}$ *be the set of all blocks of* $\mathbf{W_{24}}$ *which contain* a*,* c*, and* d *and do not contain* b*; and let* $\mathcal{B}_0 \subseteq \mathcal{B}$ *be the set of all blocks of* $\mathbf{W_{24}}$ *which contain* b*,* c*, and* d *and do not contain* a*. Let* $\mathbf{H}_0$ *denote the substructure* $(\mathcal{A}_0, \mathcal{B}_0)$ *of* $\mathbf{H}$*. Then* $\mathbf{H}_0$ *is a tight* $(16, 6, 2)$ *subdesign of* $\mathbf{H}$*.*

We prove this theorem through a series of lemmas.

Lemma 2.2. $|\mathcal{A}_0| = |\mathcal{B}_0| = 16$*; each element of* $\mathcal{A}_0$ *is incident with exactly 6 elements of* $\mathcal{B}_0$*, and each element of* $\mathcal{B}_0$ *is incident with exactly 6 elements of* $\mathcal{A}_0$*.*

Proof. The cardinality of $\mathcal{A}_0$ and the cardinality of $\mathcal{B}_0$ are each equal to $\lambda_3 - \lambda_4 = 16$.

Fix $C \in \mathcal{A}_0$ and let $X = C \setminus \{c, d\}$. For $i = 0, 2$, let n_i denote the number of blocks $D \in \mathcal{B}_0$ such that $|D \cap X| = i$. Then

$$n_0 + n_2 = 16,$$

Counting pairs (D, x) where $x \in C \cap D$ yields:

$$2n_2 = 5(\lambda_4 - \lambda_5) = 20.$$

Therefore $n_2 = 10$ and $n_0 = 6$. So each element of $\mathcal{A}_0$ is incident with exactly 6 elements of $\mathcal{B}_0$. A similar argument shows that each element of $\mathcal{B}_0$ is incident with exactly 6 elements of $\mathcal{A}_0$. □

Remark 2.3. For the purposes of the following lemmas, we will allow $C, D \in \mathcal{A}_0$ with $C \neq D$. Notice that C and D each contain the points $\{a, c, d\}$ by construction. Since the intersection numbers of $\mathbf{W_{24}}$ are 0, 2, and 4 by Theorem 1.8, we must have $|C \cap D| = 4$. Let $C \cap D = \{a, c, d, e\}$. Let $C \setminus D = \{t_1, t_2, t_3, t_4\}$ and $D \setminus C = \{u_1, u_2, u_3, u_4\}$.

There are exactly $\lambda_4 - \lambda_5 = 4$ blocks in $\mathbf{W_{24}}$ which contain $\{b, c, d, e\}$ and do not contain $\{a, b, c, d, e\}$. Denote such blocks as B_1, B_2, B_3, B_4. We know $|B_i \cap C| = |B_i \cap D| = 4$, so without loss of generality we may assume that $B_i \cap C = \{c, d, e, t_i\}$ and $B_i \cap D = \{c, d, e, u_i\}$ for $i = 1, 2, 3, 4$. Since $|B_i \cap B_j| = 4$ for distinct $i, j \in \{1, 2, 3, 4\}$, we may assume that $B_i = \{b, c, d, e, t_i, u_i, v_i, w_i\}$.

Lemma 2.4. *If C and D are distinct blocks of $\mathcal{A}_0$ containing a, c, and d, then there are at most six blocks $B \in \mathcal{B}_0$ such that $|B \cap C| = |B \cap D| = 4$.*

Proof. Let m denote the number of blocks $B \in \mathcal{B}_0$ which contain two elements of the set $\{t_1, t_2, t_3, t_4\}$ and two elements of the set $\{u_1, u_2, u_3, u_4\}$. Begin by noting that if $t_i \in B$ then $u_i \notin B$ and vice versa. This is because there is a unique block in $\mathbf{W_{24}}$ which contains $\{b, c, d, e, t_i, u_i\}$ (as $\lambda_5 = 1$) and this block is the block B_i described above.

We will assume that $m > 2$ and let B_5, B_6, and B_7 be distinct blocks, each of which contains two elements of the set $\{t_1, t_2, t_3, t_4\}$ and two elements of the set $\{u_1, u_2, u_3, u_4\}$. By the pigeonhole principle, two of these blocks share a common point from $\{t_1, t_2, t_3, t_4\}$, say, without loss of generality, B_5 and B_6. Also without loss of generality assume that $B_5 \ni t_1, t_2$ and $B_6 \ni t_1, t_3$. This implies that $B_5, B_6 \ni u_4$ for the reasons stated at the beginning of this proof. But this means that $B_5 \cap B_6 = \{b, c, d, t_1, u_4\}$, a contradiction to the intersection numbers of $\mathbf{W_{24}}$. So $m \leq 2$.

Recall from Remark 2.3 that there are four blocks $B_1, B_2, B_3, B_4 \in \mathcal{B}_0$ which contain $\{b, c, d, e\}$. For each of these blocks, we have $|B_i \cap C| = |B_i \cap D| = 4$. Now we consider the remaining blocks in $\mathcal{B}_0$ which contain $\{b, c, d\}$ but do not contain $\{a, e\}$. If there is a block B among these remaining blocks such that $|B \cap C| = |B \cap D| = 4$, B must contain two elements of the set $\{t_1, t_2, t_3, t_4\}$ and two elements of the set $\{u_1, u_2, u_3, u_4\}$. We have just shown that there are no more than two such blocks. This completes the proof. □

Lemma 2.5. *Let C, D be distinct blocks of $\mathcal{A}_0$ containing a, c, and d. Then there are exactly two blocks $B \in \mathcal{B}_0$ such that $|B \cap C| = |B \cap D| = 2$.*

Proof. There are $\lambda_3 - 2\lambda_4 + \lambda_5 = 12$ blocks $B \in \mathcal{B}_0$ such that B contains $\{b, c, d\}$ and B is disjoint from $\{a, e\}$. We classify these blocks with the help of Lemma 2.4.

For $i, j = 2, 4$, let k_{ij} denote the number of blocks $B \in \mathcal{B}_0$ such that $|B \cap C| = i$ and $|B \cap D| = j$. We obtain the following equations by simple observation and from Lemma 2.2:

$$\begin{aligned} k_{22} + k_{24} + k_{42} + k_{44} &= 16, \\ k_{22} + k_{24} &= 6, \\ k_{22} + k_{42} &= 6. \end{aligned}$$

Solving this system gives the equation

$$k_{22} + 4 = k_{44}. \tag{2}$$

Lemma 2.4 tells us that $k_{44} \leq 6$. This means $k_{22} \leq 2$ from Eq. (2). From Proposition 1.3, $k_{22} \leq 2$ implies $k_{22} = 2$, which concludes the proof. □

3. Another $(16, 6, 2)$ Subdesign

Definition 3.1. For two disjoint sets $X, Y \subseteq Q$, we define the following set:

$$\mathcal{P}(X, Y) = \{B \in \mathcal{S} | X \subseteq B \text{ and } Y \cap B = \emptyset\}$$

Theorem 3.2. *Consider a block $B = \{a, b, x, x', y, y', z, z'\}$ of $\mathbf{W_{24}}$. Let $\mathcal{A}_1 = \mathcal{P}(\{a, x, x'\}, \{b, z, z'\}) \cup \mathcal{P}(\{a, y, y'\}, \{b, z, z'\})$ and $\mathcal{B}_1 = \mathcal{P}(\{b, x, x'\}, \{a, z, z'\}) \cup \mathcal{P}(\{b, y, y'\}, \{a, z, z'\})$. Let $\mathbf{H}_1$ denote the substructure $(\mathcal{A}_1, \mathcal{B}_1)$ of $\mathbf{H}$. Then $\mathbf{H_1}$ is a tight $(16, 6, 2)$-subdesign of $\mathbf{H}$.*

We prove this theorem through a series of lemmas.

Lemma 3.3. *The cardinalities of $\mathcal{A}_1$ and $\mathcal{B}_1$ are each 16.*

Proof. For any distinct u, u', v, v', w, w' such that there is a block $D \in \mathbf{W_{24}}$ with $D = \{a, b, u, u', v, v', w, w'\}$, we have $|\mathcal{P}(\{u, v, v'\}, \{u', w, w'\})| = \lambda_3 - 3\lambda_4 + 3\lambda_5 - 1 = 8$. Therefore, $|\mathcal{A}_1| = |\mathcal{B}_1| = 16$. □

We will prove that each element of $\mathcal{A}_1$ is incident with exactly 6 elements of $\mathcal{B}_1$, and each element of $\mathcal{B}_1$ is incident with exactly 6 elements of $\mathcal{A}_1$ with the help of several lemmas.

Lemma 3.4. *For any block $A \in \mathcal{P}(\{a, x, x'\}, \{b, z, z'\})$, there are two blocks $B \in \mathcal{P}(\{b, x, x'\}, \{a, z, z'\})$ such that $|A \cap B| = 2$.*

Proof. For $i = 2, 4$, let n_i denote the number of blocks $B \in \mathcal{P}(\{b, x, x'\}, \{a, z, z'\})$ such that $|B \cap A| = i$. Then

$$n_2 + n_4 = 8$$

Counting pairs (B, s) where $s \in A \cap B$ and $s \neq y, y'$ yields:

$$2n_4 = 5(\lambda_4 - \lambda_5) - 2(\lambda_4 - \lambda_5) = 12$$

Therefore $n_4 = 6$ and $n_2 = 2$. □

Lemma 3.5. *For any block $A \in \mathcal{P}(\{a, x, x'\}, \{b, z, z'\})$, there are four blocks $B \in \mathcal{P}(\{b, y, y'\}, \{a, z, z'\})$ such that $|A \cap B| = 2$.*

Proof. For $i = 2, 4$, let m_i denote the number of blocks $B \in \mathcal{P}(\{b, y, y'\}, \{a, z, z'\})$ such that $|B \cap A| = i$. Then

$$m_2 + m_4 = 8$$

Counting pairs (B, t) where $t \in A \cap B$ and $t \neq x, x'$ yields:

$$2m_4 = 4(\lambda_4 - \lambda_5) - 2(\lambda_4 - \lambda_5) = 8$$

Therefore $m_4 = 4$ and $m_2 = 4$. □

From Lemmas 3.4 and 3.5, we see that every block $A \in \mathcal{A}_1$ is incident with $n_2 + m_2 = 6$ blocks in $\mathcal{B}_1$. A similar argument shows that every block $B \in \mathcal{B}_1$ is incident with 6 blocks in $\mathcal{A}_1$.

Lemma 3.6. *The substructure $(\mathcal{P}(\{a, x, x', y\}, \{b, z, z'\}), \mathcal{B}_1)$ of $(\mathcal{A}_1, \mathcal{B}_1)$ admits the following incidence matrix:*

$$\begin{pmatrix} 0000 & 1100 & 1010 & 1001 \\ 0000 & 1100 & 0101 & 0110 \\ 0000 & 0011 & 1010 & 0110 \\ 0000 & 0011 & 0101 & 1001 \end{pmatrix}$$

Proof. We begin by only considering the blocks in $\mathcal{P}(\{a, x, x', y\}, \{b, z, z'\})$. We may assume without loss of generality that the four blocks in this set have the following form:

$$\begin{aligned} A_1 &= \{a, x, x', y\} \cup T \\ A_2 &= \{a, x, x', y\} \cup U \\ A_3 &= \{a, x, x', y\} \cup V \\ A_4 &= \{a, x, x', y\} \cup W \end{aligned}$$

where $T = \{t_1, t_2, t_3, t_4\}$, $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, and $W = \{w_1, w_2, w_3, w_4\}$; and T, U, V, and W are pairwise disjoint.

None of the blocks in $\mathcal{P}(\{b, x, x', y\}, \{a, z, z'\})$ are incident to the blocks in $\mathcal{P}(\{a, x, x', y\}, \{b, z, z'\})$. Now we consider the blocks $\{B_1, B_2, B_3, B_4\} = \mathcal{P}(\{b, x, x', y\}, \{a, z, z'\})$. Each of these blocks contains two points each from two of the sets T, U, V, and W. If this is not the case, we have a contradiction to the intersection numbers with A_1, A_2, A_3 and A_4. Without loss of generality we may assume these blocks have the following form:

$$\begin{aligned} B_1 &= \{b, x, x', y', t_1, t_2, w_3, w_4\} \\ B_2 &= \{b, x, x', y', u_1, u_2, v_3, v_4\} \\ B_3 &= \{b, x, x', y', v_1, v_2, u_3, u_4\} \\ B_4 &= \{b, x, x', y', w_1, w_2, t_3, t_4\} \end{aligned}$$

Now we consider the blocks in $\mathcal{P}(\{b, x, y, y'\}, \{a, z, z'\}) = \{B_5, B_6, B_7, B_8\}$ and $\mathcal{P}(\{b, x', y, y'\}, \{a, z, z'\}) = \{B_9, B_{10}, B_{11}, B_{12}\}$. We know that each A_i is incident to two blocks from each of these sets from Lemmas 3.4 and 3.5. We may assume without loss of generality that $t_1, t_3 \in B_5$ and $t_2, t_4 \in B_6$. This implies that $w_2, w_4 \in B_7$, $w_1, w_3 \in B_8$, and without loss of generality that $t_2, t_4 \in B_9$ and $t_2, t_3 \in B_{10}$. Further, this implies that $w_i \notin B_5, B_6, B_9, B_{10}$ for $i \in \{1, 2, 3, 4\}$. So we may conclude that $w_2, w_4 \in B_7$, $w_1, w_3 \in B_8$, $w_2, w_3 \in B_{11}$, and $w_1, w_4 \in B_{12}$. So now we may assume without loss of generality that $u_1, u_3 \in B_5$ and $v_1, v_3 \in B_6$. This implies that $u_2, u_4 \in B_7$ and $v_2, v_4 \in B_8$. Since $w_i \notin B_9, B_{10}$, we may assume $u_1, u_4 \in B_9$ and $v_1, v_4 \in B_{10}$. This implies that $u_2, u_3 \in B_{11}$ and $v_2, v_3 \in B_{12}$. This construction gives the desired incidence matrix as stated above. □

In a similar manner, one can show that the substructures

$$\begin{aligned} &(\mathcal{P}(\{a, x, x', y'\}, \{b, z, z'\}), \mathcal{B}_1), \\ &(\mathcal{P}(\{a, x, y, y'\}, \{b, z, z'\}), \mathcal{B}_1), \text{ and} \\ &(\mathcal{P}(\{a, x', y, y'\}, \{b, z, z'\}), \mathcal{B}_1) \end{aligned}$$

have the following incidence matrices:

$$\begin{pmatrix} 1100 & 0000 & 1001 & 1010 \\ 1100 & 0000 & 0110 & 0101 \\ 0011 & 0000 & 0110 & 1010 \\ 0011 & 0000 & 1001 & 0101 \end{pmatrix}$$

$$\begin{pmatrix} 1010 & 1001 & 0000 & 1100 \\ 0101 & 0110 & 0000 & 1100 \\ 1010 & 0110 & 0000 & 0011 \\ 0101 & 1001 & 0000 & 0011 \end{pmatrix}$$

and

$$\begin{pmatrix} 1001 & 1010 & 1100 & 0000 \\ 0110 & 0101 & 1100 & 0000 \\ 0110 & 1010 & 0011 & 0000 \\ 1001 & 0101 & 0011 & 0000 \end{pmatrix}$$

respectively.

Corollary 3.7. *The incidence structure* $(\mathcal{A}_1, \mathcal{B}_1)$ *admits matrix* (1) *as an incidence matrix, and therefore it is a tight* $(16, 6, 2)$*-subdesign of the Higman-Sims design.*

Proof. We need only consider the following substructures:

$$\begin{gathered} (\mathcal{P}(\{a, x, x', y'\}, \{b, z, z'\}), \mathcal{B}_1) \\ (\mathcal{P}(\{a, x, y, y'\}, \{b, z, z'\}), \mathcal{B}_1) \\ (\mathcal{P}(\{a, x', y, y'\}, \{b, z, z'\}), \mathcal{B}_1). \end{gathered}$$

Since

$$\begin{aligned} \mathcal{A}_1 = & \mathcal{P}(\{a, x, x', y\}, \{b, z, z'\}) \\ & \cup \mathcal{P}(\{a, x, x', y'\}, \{b, z, z'\}) \\ & \cup \mathcal{P}(\{a, x, y, y'\}, \{b, z, z'\}) \\ & \cup \mathcal{P}(\{a, x', y, y'\}, \{b, z, z'\}), \end{aligned}$$

we have the stated result. □

4. Finding Independent $(16, 6, 2)$ Subdesigns

Definition 4.1. Two subdesigns $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ of a symmetric designs are said to be *independent* if $X_1 \cap X_2 = \emptyset$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.

Now that we have constructed a $(16, 6, 2)$-subdesign of the Higman-Sims design, we would like to find 11 independent $(16, 6, 2)$-subdesigns, which use each point and each block of the Higman-Sims design exactly once. In order to do this, we will first rename the points in Q by partitioning the 22 points of $\mathbf{W_{24}}$ other than a or b into the following pairs:

$$Q = \{a, b, 1, 1', 2, 2', \ldots, 10, 10', 11, 11'\}.$$

Remark 4.2. We would like to construct these pairs such that each pair of pairs is contained in a unique block which also contains a and b. For example, if there is a block B such that $\{a, b, 1, 1', 2, 2'\} \subseteq B$, then $\mathcal{P}(\{a, 1, 1', 2, 2'\}, \{b\}) = \emptyset$, and we may use Theorem 2.1 to construct two independent subdesigns: $(\mathcal{P}(\{a, 1, 1'\}, b), \mathcal{P}(\{b, 1, 1'\}, a))$ and $(\mathcal{P}(\{a, 2, 2'\}, b), \mathcal{P}(\{b, 2, 2'\}, a))$.

The following theorem constructs seven independent subdesigns.

Theorem 4.3. *It is possible to construct seven pairwise independent tight* $(16, 6, 2)$*-subdesigns of the Higman-Sims design as described in Theorem 2.1.*

Proof. There are $\lambda_4 = 5$ blocks $B \in \mathbf{W_{24}}$ such that $a, b, 1, 1' \in B$. We may call these blocks $C_1, C_2, \ldots, C_5$. Without loss of generality we may assume these blocks are as follows:

$$\begin{aligned}
C_1 &= \{a, b, 1, 1', 2, 2', 3, 3'\} \\
C_2 &= \{a, b, 1, 1', 4, 4', 5, 5'\} \\
C_3 &= \{a, b, 1, 1', 6, 6', 7, 7'\} \\
C_4 &= \{a, b, 1, 1', 8, 8', 9, 9'\} \\
C_5 &= \{a, b, 1, 1', 10, 10', 11, 11'\}.
\end{aligned}$$

There are 4 remaining blocks which contain $\{a, b, 2, 2'\}$. Call these blocks D_1, D_2, D_3, D_4. Again without loss of generality we may assume these blocks are as follows:

$$\begin{aligned}
D_1 &= \{a, b, 2, 2', 4, 4', 6, 6'\} \\
D_2 &= \{a, b, 2, 2', 5, 5', 7, 7'\} \\
D_3 &= \{a, b, 2, 2', 8, 8', 11, 11'\} \\
D_4 &= \{a, b, 2, 2', 9, 9', 10, 10'\}.
\end{aligned}$$

There are 4 remaining blocks which contain $\{a, b, 3, 3'\}$. Call these blocks E_1, E_2, E_3, E_4. The construction of blocks $C_1, \ldots, C_5$ and $D_1, \ldots, D_4$ imply that these blocks are as follows:

$$\begin{aligned}
E_1 &= \{a, b, 3, 3', 4, 4', 7, 7'\} \\
E_2 &= \{a, b, 3, 3', 5, 5', 6, 6'\} \\
E_3 &= \{a, b, 3, 3', 8, 8', 10, 10'\} \\
E_4 &= \{a, b, 3, 3', 9, 9', 11, 11'\}.
\end{aligned}$$

Now we claim that there is not another block which contains two or more pairs of pairs of points as described above. Assume there is a block which contains $\{a, b, 4, 4'\}$. This block cannot contain the pairs $\{5, 5'\}$, $\{6, 6'\}$, or $\{7, 7'\}$, because these contradict the sizes of the intersections with C_2, D_1, and E_1 respectively. Also, any pair of pairs from the sets $\{8, 8'\}, \{9, 9'\}, \ldots, \{11, 11'\}$ is already contained in some $C_i, D_i,$ or E_i. Thus, there is no block which contains three pairs of pairs of points. Further, note that if there is a block which contains $\{a, b, 4, 4', 8, 8'\}$, the remaining two points cannot be any of the following: $\{1, 1', 2, 2', 3, 3', 5, 5', 6, 6', 7, 7', 8, 8', 9, 9', 10, 10', 11, 11'\}$ because in each of these cases we have a contradiction to the intersection sizes with previously constructed blocks. A similar contradiction arises for blocks which contain $\{a, b, 4, 4', 9, 9'\}$, $\{a, b, 4, 4', 10, 10'\}$, or $\{a, b, 4, 4', 11, 11'\}$. Thus, the blocks constructed above are the only blocks which contain $\{a, b\}$ and two pairs of points $\{1, 1'\}, \ldots, \{11, 11'\}$.

Thus, by Theorem 2.1, we may construct the following tight subdesigns:

$$\begin{aligned}
\mathbf{D_1} &= (\mathcal{P}(\{a, 1, 1'\}, \{b\}), \mathcal{P}(\{b, 1, 1'\}, \{a\})) \\
\mathbf{D_2} &= (\mathcal{P}(\{a, 2, 2'\}, \{b\}), \mathcal{P}(\{b, 2, 2'\}, \{a\})) \\
\mathbf{D_3} &= (\mathcal{P}(\{a, 3, 3'\}, \{b\}), \mathcal{P}(\{b, 3, 3'\}, \{a\})) \\
\mathbf{D_4} &= (\mathcal{P}(\{a, 4, 4'\}, \{b\}), \mathcal{P}(\{b, 4, 4'\}, \{a\})) \\
\mathbf{D_5} &= (\mathcal{P}(\{a, 5, 5'\}, \{b\}), \mathcal{P}(\{b, 5, 5'\}, \{a\})) \\
\mathbf{D_6} &= (\mathcal{P}(\{a, 6, 6'\}, \{b\}), \mathcal{P}(\{b, 6, 6'\}, \{a\})) \\
\mathbf{D_7} &= (\mathcal{P}(\{a, 7, 7'\}, \{b\}), \mathcal{P}(\{b, 7, 7'\}, \{a\})).
\end{aligned}$$

□

Theorem 4.4. *The substructures* $\mathbf{D_1}, \mathbf{D_2}, \ldots, \mathbf{D_7}$ *of Theorem 4.3 and the following two substructures are nine independent tight* $(16, 6, 2)$*-subdesigns of* $\mathbf{H}$.

$$\begin{aligned}\mathbf{D_8} = &(\mathcal{P}(\{a, 8, 8'\}, \{b, 1, 1'\}) \cup \mathcal{P}(\{a, 9, 9'\}, \{b, 1, 1'\}),\\ &\mathcal{P}(\{b, 8, 8'\}, \{a, 1, 1'\}) \cup \mathcal{P}(\{b, 9, 9'\}, \{a, 1, 1'\}))\\ \mathbf{D_9} = &(\mathcal{P}(\{a, 10, 10'\}, \{b, 1, 1'\}) \cup \mathcal{P}(\{a, 11, 11'\}, \{b, 1, 1'\}),\\ &\mathcal{P}(\{b, 10, 10'\}, \{a, 1, 1'\}) \cup \mathcal{P}(\{b, 11, 11'\}, \{a, 1, 1'\}))\end{aligned}$$

Proof. We will show that $\mathcal{P}(\{a, 8, 8'\}, \{b, 1, 1'\})$ is disjoint from the point set of the first seven subdesigns, and the remaining cases follow by the same method.

Assume there is a block $B \in \mathcal{P}(\{a, 8, 8'\}, \{b, 1, 1'\})$ such that $\{x, x'\} \subseteq B$ for $x = 1, 2, \ldots, 7$. We know $x \neq 1, 2, 3$ because of the blocks C_4, D_3, E_3 of the proof of Theorem 4.3. Further, if $x = 4$, any block containing $\{a, 4, 4', 8, 8'\}$ also contains one point each from $\{1, 1'\}, \{2, 2'\}, \{3, 3'\}$, a contradiction to our assumption that $1, 1' \notin B$. We arrive at the same contradiction for $x = 5, 6, 7$. Thus, the points and blocks described above are independent from the points and blocks in previously constructed designs.

By Theorem 3.2, $\mathbf{D_8}$ and $\mathbf{D_9}$ are independent $(16, 6, 2)$-subdesigns. □

At this point, we may classify the remaining points and blocks of the Higman-Sims design which have not been used in any of $\mathbf{D_1}, \mathbf{D_2}, \ldots, \mathbf{D_9}$. The proofs of the following two theorems can be found in [1].

Theorem 4.5. *If* A *and* B *are blocks of* $\mathbf{W_{24}}$ *such that* $|A \cap B| = 4$*, then* $A \triangle B$ *is a block of* $\mathbf{W_{24}}$

Theorem 4.6. *If* A *and* B *are disjoint blocks of* $\mathbf{W_{24}}$*, then* $Q \setminus \{A \cup B\}$ *is a block of* $\mathbf{W_{24}}$.

Proposition 4.7. *The following sets of blocks of the Higman-Sims design are disjoint from the point sets of* $\mathbf{D_1}, \mathbf{D_2}, \ldots, \mathbf{D_9}$*:*

- **i:** *Eight blocks* B *of* $\mathbf{W_{24}}$ *such that* $8, 8', 9, 9', 10, 10', 11, 11' \notin B$ *and* $a \in B$.
- **ii:** *Eight blocks* C *of* $\mathbf{W_{24}}$ *such that* $4, 4', 5, 5', 6, 6', 7, 7' \notin C$ *and* $a \in C$.
- **iii:** *Four blocks* D *of* $\mathbf{W_{24}}$ *such that* $a, 8, 8' \in D$.
- **iv:** *Four blocks* E *of* $\mathbf{W_{24}}$ *such that* $a, 9, 9' \in E$.
- **v:** *Four blocks* F *of* $\mathbf{W_{24}}$ *such that* $a, 10, 10' \in F$.
- **vi:** *Four blocks* G *of* $\mathbf{W_{24}}$ *such that* $a, 11, 11' \in G$

Proof. We consider each of the above items:

i: For this proof, we consider the set of blocks $\mathcal{C} = \{C_1, C_2, C_3, D_1, D_2, E_1, E_2\}$ constructed in the proof of Theorem 4.3. Also, we consider the block $R = \{8, 8', 9, 9', 10, 10', 11, 11'\}$. Such a block exists by considering blocks C_4 and C_5, and Theorem 4.5. From Theorem 1.11, we know that there are 30 blocks in $\mathbf{W_{24}}$ which are disjoint from R. Seven of these blocks are mentioned at the beginning of this proof in the set $\mathcal{C}$. By Theorem 4.6, we know that for all $X \in \mathcal{C}$, the blocks $Q \setminus (X \cup R)$ are also blocks of $\mathbf{W_{24}}$ which are disjoint from R. Now we show that there is not another block T such that $T \supset \{a, b\}$ and $R \cap T = \emptyset$.

Assume such a block T exists. T cannot contain any of the pairs of points $\{1, 1'\}, \ldots, \{7, 7'\}$. Assume T contains some pair y, y' and consider some other point $x \in T$. Then $T \cap C \supset \{a, b, y, y', x\}$, a contradiction to the intersection numbers $\mathbf{W_{24}}$. Since T cannot contain any of these pairs of points, by the pigeonhole principle, T must contain some 3 points other than a and b which are contained in some block in $C \in \mathcal{C}$, a contradiction to the intersection numbers of $\mathbf{W_{24}}$.

Since there are seven blocks T such that $T \supset \{a, b\}$ and $T \cap R = \emptyset$, there are also seven blocks U such that $a, b \notin U$ and $U \cap R = \emptyset$ by Theorem 4.6. Recall that there are 30 blocks in $\mathbf{W_{24}}$ that are disjoint from R. For any block $V \in \mathbf{W_{24}}$ such that $a \in V$, $b \notin V$, and $V \cap R = \emptyset$, there is a block $W = Q \setminus \{V \cup R\}$ such that $a \notin W$, $b \in W$, and $W \cap R = \emptyset$ by Theorem 4.6.

Thus, there must be eight blocks $B \in \mathbf{W_{24}}$ such that $B \cap R = \emptyset$, $a \in B$, and $b \notin B$.

ii: This result uses the same logic as in **i**.

iii: From the proof of Theorem 4.3, we know there is no block which contains the points $\{a, b, 4, 4', 8, 8'\}$, but we also know there is a unique block which contains the points $\{a, 4, 4', 8, 8'\}$. The same argument holds for blocks containing $\{a, 5, 5', 8, 8'\}$, $\{a, 6, 6', 8, 8'\}$, and $\{a, 7, 7', 8, 8'\}$. Further, we use eight blocks which contain $\{a, 8, 8'\}$ in $\mathbf{D_8}$. Lemma 2.2 tells us there are 16 blocks in $\mathbf{W_{24}}$ which contain $\{a, 8, 8'\}$ and do not contain b, and we have only used 12 of them.

iv: Similar to the proof of **iii**.

v: Similar to the proof of **iii**.

vi: Similar to the proof of **iii**.

□

Proposition 4.8. *The following blocks of the Higman-Sims design are independent of the block sets of* $\mathbf{D_1}, \mathbf{D_2}, \ldots, \mathbf{D_9}$*:*

i: *Eight blocks H of $\mathbf{W_{24}}$ such that $8, 8', 9, 9', 10, 10', 11, 11' \notin H$ and $b \in H$.*

ii: *Eight blocks J of $\mathbf{W_{24}}$ such that $4, 4', 5, 5', 6, 6', 7, 7' \notin J$ and $b \in J$.*

iii: *Four blocks K of $\mathbf{W_{24}}$ such that $b, 8, 8' \in K$.*

iv: *Four blocks L of $\mathbf{W_{24}}$ such that $b, 9, 9' \in L$.*

v: *Four blocks M of $\mathbf{W_{24}}$ such that $b, 10, 10' \in M$.*

vi: *Four blocks N of $\mathbf{W_{24}}$ such that $b, 11, 11' \in N$*

Proof. The proof of this Proposition uses the same logic as the proof of Proposition 4.7. □

Using Propositions 4.7 and 4.8, we obtain the following result.

Theorem 4.9. *The following are the points and blocks of the Higman-Sims design which have not been considered in* $\mathbf{D_1}, \ldots, \mathbf{D_9}$*:*

$$\begin{aligned}
A_1 &= \{a,1,2,3',4,5',6,7\} & A_2 &= \{a,1,2,3',4',5,6',7'\}\\
A_3 &= \{a,1,2',3,4,5',6',7'\} & A_4 &= \{a,1,2',3,4',5,6,7\}\\
A_5 &= \{a,1',2,3,8,9',10,11\} & A_6 &= \{a,1',2,3,8',9,10',11'\}\\
A_7 &= \{a,1',2',3',8,9',10',11'\} & A_8 &= \{a,1',2',3',8',9,10,11\}\\
A_9 &= \{a,1',4',5',8',9',11,11'\} & A_{10} &= \{a,1',4,5,8,9,11,11'\}\\
A_{11} &= \{a,1',4',5',8,9,10,10'\} & A_{12} &= \{a,1',4,5,8',9',10,10'\}\\
A_{13} &= \{a,1',6,7',9,9',10,11'\} & A_{14} &= \{a,1',6',7,9,9',10',11\}\\
A_{15} &= \{a,1',6',7,8,8',10,11'\} & A_{16} &= \{a,1',6,7',8,8',10',11\}\\
A_{17} &= \{a,1',2',3',4,5,6',7\} & A_{18} &= \{a,1',2,3,4',5',6',7\}\\
A_{19} &= \{a,1',2,3,4,5,6,7'\} & A_{20} &= \{a,1',2',3',4',5',6,7'\}\\
A_{21} &= \{a,1,2,3',8,9,10,11'\} & A_{22} &= \{a,1,2',3,8,9,10',11\}\\
A_{23} &= \{a,1,2',3,8',9',10,11'\} & A_{24} &= \{a,1,2,3',8',9',10',11\}\\
A_{25} &= \{a,1,4',5,8,8',10',11'\} & A_{26} &= \{a,1,4,5',8,8',10,11\}\\
A_{27} &= \{a,1,4,5',9,9',10',11'\} & A_{28} &= \{a,1,4',5,9,9',10,11\}\\
A_{29} &= \{a,1,6',7',8',9,10,10'\} & A_{30} &= \{a,1,6,7,8,9',10,10'\}\\
A_{31} &= \{a,1,6',7',8,9',11,11'\} & A_{32} &= \{a,1,6,7,8',9,11,11'\},
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= \{b,1,2,3,4',5',6,7'\} & B_2 &= \{b,1,2,3,4,5,6',7\}\\
B_3 &= \{b,1,2',3',4',5',6',7\} & B_4 &= \{b,1,2',3',4,5,6,7'\}\\
B_5 &= \{b,1',2,3',8,9,10',11\} & B_6 &= \{b,1',2,3',8',9',10,11'\}\\
B_7 &= \{b,1',2',3,8',9',10',11\} & B_8 &= \{b,1',2',2,8,9,10,11'\}\\
B_9 &= \{b,1',4,5',8,8',10',11'\} & B_{10} &= \{b,1',4',5,8,8',10,11\}\\
B_{11} &= \{b,1',4,5',9,9',10,11\} & B_{12} &= \{b,1',4',5,9,9',10',11'\}\\
B_{13} &= \{b,1',6,7,8',9,10,10'\} & B_{14} &= \{b,1',6',7',8,9',10,10'\}\\
B_{15} &= \{b,1',6',7',8',9,11,11'\} & B_{16} &= \{b,1',6,7,8,9',11,11'\}\\
B_{17} &= \{b,1',2,3',4,5',6',7'\} & B_{18} &= \{b,1',2',3,4',5,6',7'\}\\
B_{19} &= \{b,1',2',3,4,5',6,7\} & B_{20} &= \{b,1',2,3',4',5,6,7\}\\
B_{21} &= \{b,1,2',3',8,9',10,11\} & B_{22} &= \{b,1,2,3,8',9,10,11\}\\
B_{23} &= \{b,1,2,3,8,9',10',11'\} & B_{24} &= \{b,1,2',3',8',9,10',11'\}\\
B_{25} &= \{b,1,6',7,8,8',10',11\} & B_{26} &= \{b,1,6,7',8,8',10,11'\}\\
B_{27} &= \{b,1,6',7,9,9',10,11'\} & B_{28} &= \{b,1,6,7',9,9',10',11\}\\
B_{29} &= \{b,1,4',5',8',9',10,10'\} & B_{30} &= \{b,1,4,5,8,9,10,10'\}\\
B_{31} &= \{b,1,4,5,8',9',11,11'\} & B_{32} &= \{b,1,4',5',8,9,11,11'\}.
\end{aligned}$$

From Theorem 4.9, we obtain the following result:

Theorem 4.10. *Consider the following sets of points and blocks from Theorem 4.9:*

$$\begin{aligned}
\mathcal{A}_{10} &= \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}\} \\
\mathcal{A}_{11} &= \{A_{17}, A_{18}, A_{19}, A_{20}, A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, A_{26}, A_{27}, A_{28}, A_{29}, A_{30}, A_{31}, A_{32}\} \\
\mathcal{B}_{10} &= \{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}\} \\
\mathcal{B}_{11} &= \{B_{17}, B_{18}, B_{19}, B_{20}, B_{21}, B_{22}, B_{23}, B_{24}, B_{25}, B_{26}, B_{27}, B_{28}, B_{29}, B_{30}, B_{31}, B_{32}\}
\end{aligned}$$

The substructures $(\mathcal{A}_{10}, \mathcal{B}_{10})$ *and* $(\mathcal{A}_{11}, \mathcal{B}_{11})$ *form tight* $(16, 6, 2)$*-subdesigns of the Higman-Sims Design.*

Proof. This theorem is easily verified by constructing the incidence matrices of the two designs. □

5. Designs with Tight $(16, 6, 2)$ Subdesigns

In this section, we wish to classify all (v, k, λ)-designs $\mathbf{D}$ which have the $(16, 6, 2)$-design as a tight subdesign.

The following proposition is known from [6].

Proposition 5.1. *If a non-trivial symmetric* (v_1, k_1, λ_1)*-design with* $\lambda_1 \neq 0$ *is a tight subdesign of a symmetric* (v, k, λ)*-design, then there exist positive integers* $d, t, u,$ *and* u_1 *such that* $du_1 = v_1$*,* $d \geq 2, t \leq d - 1$*,*

$$u = \frac{(v_1 - 1)((k_1 - tu_1)^2 - (k_1 - \lambda_1))}{tu_1(d - t)} \tag{3}$$

$$v = d(u_1 + u) \tag{4}$$

$$k = k_1 + tu \tag{5}$$

and

$$\lambda = k - (k_1 - tu_1)^2 \tag{6}$$

Proposition 5.1 gives us the following:

Theorem 5.2. *If a symmetric* (v, k, λ)*-design has a tight* $(16, 6, 2)$*-subdesign, then* $(v, k, \lambda) = (64, 36, 20)$*,* $(v, k, \lambda) = (176, 126, 90)$*, or* $(v, k, \lambda) = (1248, 1161, 1080)$*.*

Proof. We manipulate (3) to obtain the following:

$$tu = \frac{15(tu_1 - 8)(tu_1 - 4)}{16 - tu_1} \tag{7}$$

and

$$tu = \frac{15((tu_1 - 16)(tu_1 - 2) + 6tu_1)}{16 - tu_1} \tag{8}$$

Eq. (7) tells us $8 < tu_1 < 16$, and from (8) we have:

$$tu = 15(2 - tu_1) + \frac{90tu - 1}{16 - tu_1}$$

We need $\frac{90tu_1}{16 - tu_1} \in \mathbb{Z}^+$, and examining all possible values for tu_1 gives $tu_1 \in \{10, 11, 12, 13, 14, 15\}$.

Examining these values with respect to eqs. (7), (5), (6), and (4) shows that the only possible values for (v, k, λ) with integer solutions are $(64, 36, 20)$, $(176, 126, 90)$, and $(1248, 1161, 1080)$. □

Remark 5.3. There exists a symmetric $(64, 36, 20)$-design with a tight $(16, 6, 2)$-subdesign (see [6]), and the $(176, 126, 90)$-design is the Higman-Sims design. We do not know whether a $(1248, 1161, 1080)$ design exists.

6. Other Tight Subdesigns of the Higman-Sims Design

In this section, we would like to classify non-trivial tight subdesigns of the Higman-Sims design.

Theorem 6.1. *If a symmetric* (v_1, k_1, λ_1)*-design is a tight subdesign of the* $(176, 126, 90)$ *Higman-Sims design, then* $(v_1, k_1, \lambda_1) = (16, 6, 2)$*,* $(v_1, k_1, \lambda_1) = (22, 21, 20)$*, or* $(v_1, k_1, \lambda_1) = (36, 21, 12)$*.*

Proof. We know from [6] that a tight (v_1, k_1, λ_1)-subdesign of a (v, k, λ)-design has the property that

$$k - \lambda = (k_1 - k_2)^2 \tag{9}$$

where $k_2 = \frac{v_1(k-k_1)}{v-v_1}$. Further, from [6] that if $\mathbf{D}$ is a symmetric (v, k, λ)-design, then

$$\lambda(v-1) = k(k-1) \tag{10}$$

Manipulation of (9) gives the following equation:

$$\frac{176k_1 - 126v_1}{176 - v_1} = \pm 6 \tag{11}$$

Now we must consider two cases:

Case 1:

$$\begin{aligned} 176k_1 - 126v_1 &= -6(176 - v_1) \\ &\implies 4(k_1 + 6) = 3v_1 \end{aligned}$$

This tells us that $3|k_1$, and we can easily check through all cases for $k_1 < 126$ to verify that the only integer solutions to (11) are $(16, 6, 2)$, $(36, 21, 12)$, and $(76, 51, 34)$. However, we may rule out the $(76, 51, 34)$ design because [1] tells us that in a symmetric (v, k, λ)-design in which v is even, $k - \lambda$ must be a square.

Case 2:

$$\begin{aligned} 176k_1 - 126v_1 &= 6(176 - v_1) \\ &\implies 22(k_1 - 6) = 15v_1 \end{aligned}$$

This tells us that $22|v_1$. Again we check through all cases for $v_1 < 176$ with respect to the equations from Proposition 5.1 and (10) to verify that the only integer solution to (11) is $(22, 21, 20)$.

□

Remark 6.2. We have already found a tight $(16, 6, 2)$ subdesign of the Higman-Sims design; but, although we know that there are designs with parameters $(36, 21, 12)$ and $(22, 21, 20)$, we cannot say whether there are tight subdesigns of the Higman-Sims design with these parameters.

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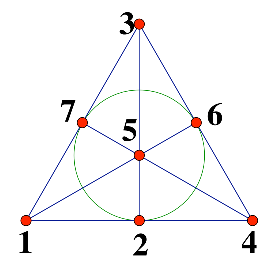

Figure 1