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Population Models with Diffusion and Constant Yield Harvesting

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Population Models with Diffusion and Constant Yield Harvesting

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Abstract

In this paper we discuss reaction-diffusion equations arising in population dynamics with constant yield harvesting in one dimension. We focus on the mathematical models of the logistic growth, the strong Allee effect, and the weak Allee effect and their influence on the existence of positive steady states as well as global bifurcation diagrams. We analyze the equations using the quadrature method and the method of sub-super solutions.

1 Introduction

First, we consider the steady state reaction diffusion equation with Dirichlet boundary conditions in population dynamics:

$$\begin{cases} -u''(x) = au - bu^2 - ch(x) & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{1.1}$$

Here, u is the population density, $au - bu^2$ represents the logistic growth where a and b are positive constants, and $ch(x)$ represents the constant yield harvesting rate. We assume h(x) is a smooth function such that for $x \in [0,1]$, $h(x) \geq 0$ and $\|h\|_\infty = 1$. In a recent paper [1] the authors studied the higher dimensional case:

$$\begin{cases} -\triangle u(x) = au - bu^2 - ch(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1.2}$$

Here, $\triangle$ is the Laplacian operator, and Ω is a bounded domain of $\mathbb{R}^n$; $n \geq 1$ with $\partial\Omega$ in class C^2. They established:

[A] If $a \leq \lambda_1$, where λ_1 is the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions, then (1.2) has no positive solutions.

[B] Let $a > \lambda_1$. Then there exists $c_1(a,b)$ such that if $c > c_1$ then (1.2) has no positive solutions.

[C] Let $a > \lambda_1$. Then there exists $c_0(a,b)$ such that if $c < c_0$ then (1.2) has a positive solution.

[D] There exists $\varepsilon > 0$ such that for $a \in (\lambda_1, \lambda_1 + \varepsilon)$ the global bifurcation diagram of positive solutions of (1.2) is exactly the following:

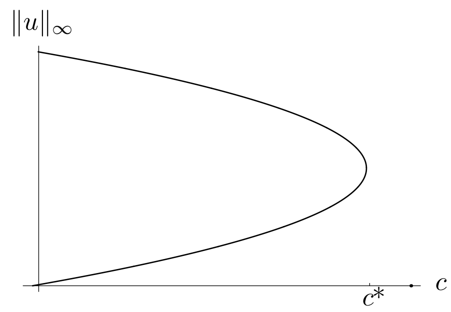

Figure 1.1: Global Bifurcation Diagram

That is, for $0 < c < c^*$ (1.2) has exactly two positive solutions, for $c = c^*$ (1.2) has a unique positive solution, and for $c > c^*$ (1.2) has no positive solutions.

We will study the single dimensional case (n=1) of (1.2), which reduces to (1.1). In particular, in Section 2 we will discuss the case $h(x) \equiv 1$ using a modified version of the quadrature method obtained from [2], whereas previously in [1] the solutions have been studied via sub-super solutions and bifurcation theory. By the quadrature method we will reestablish [A], [B], [C], and extend [D] considerably. As depicted by Figure 1.1 above, the bifurcation diagrams developed by Mathematica computations persist for $a \in (\lambda_1, \lambda_2)$ where the $\lambda_1 = \pi^2$ and $\lambda_2 = 4\pi^2$ are the first and second eigenvalues respectively of the operator $-u''$ with Dirichlet boundary conditions on the domain (0,1). Also, for $a \in (\pi^2, 2\pi^2)$ and small c, we prove analytically that there exists at least two positive solutions as illustrated by Figure 1.2.

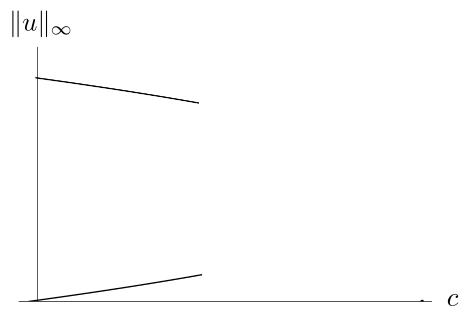

Figure 1.2: Bifurcation Diagram for $a \in (\pi^2, 2\pi^2)$.

Relying again on the computational aide of Mathematica, we conjecture that for $a > 4\pi^2$ and c small there exists only one positive solution. Here the bifurcation diagram resembles:

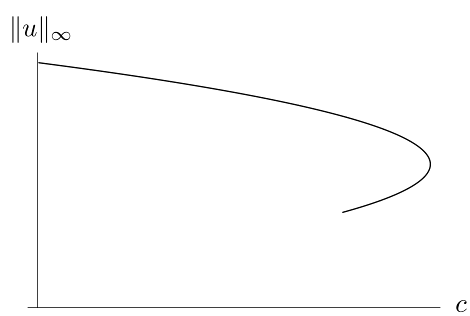

Figure 1.3: Global Bifurcation Diagram for $a > 4\pi^2$.

We next analyze problems incorporating the strong and weak Allee effect. Namely,

$$\text{Strong}\begin{cases} -u''(x) = u(u-a)(b-u) - c & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{1.3}$$

and

$$\text{Weak}\begin{cases} -u''(x) = u(u+a)(b-u) - c & x \in (0,1) \\ u(0) = 0 = u(1). \end{cases} \tag{1.4}$$

In each case, we assume $0 < a < b$. Due to the complex nature of the cubics involved, we utilized Mathematica to computationally analyze the equation via the quadrature method. Furthermore, we establish sufficient conditions for the existence of positive solutions by means of the sub-super solutions method in the weak Allee effect case. We will describe the quadrature method in Section 2, and the method of sub-super solution in Section 3. We study the logistic growth case in Section 4, the strong Allee effect case in Section 5, and the weak Allee effect in Section 6. We note that it is well documented that the study of positive solutions to such classes of problems where the reaction term is negative at the origin (semipositone problems) is nontrivial (see [3] - [4]). Also see [5] for a review on semipositone problems.

2 Quadrature Method

In this section we present a brief overview of the quadrature method, which we use extensively in the sections that follow.

Suppose we have a two point boundary value problem of the form:

$$\begin{cases} -u''(x) = f(a,b,c,u(x)) & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{2.1}$$

where f is a positive continuously differentiable function.

Let u be a positive solution to (2.1) with the property that $u'(x_0) = 0$ with $x_0 \in (0,1)$. Observe that both $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ satisfy the following initial value problem:

$$-z''(x) = f(z(x))$$

$$z(0) = u(x_0)$$

$$z'(0) = 0$$

for $x \in [0,c)$ where $c = min(x_0, 1 - x_0)$. This implies that $u(x_0 + x) = u(x_0 - x) \quad \forall\, x \in [0,c)$. Thus, since u is a positive solution to (2.1), it must be symmetric around $x = \frac{1}{2}$, at which point it has a maximum. Let this maximum of $u(x)$ be denoted by ρ.

Multiplying (2.1) by $u'(x)$:

$$-\left(\frac{[u'(x)]^2}{2}\right)' = [F(u(x))]' \tag{2.2}$$

where $F(s) = \int_0^s f(a,b,c,z)dz$.

Integrate both sides twice to obtain:

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2}; \quad x \in \left[0, \frac{1}{2}\right) \tag{2.3}$$

then

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{2}x; \quad x \in \left[0, \frac{1}{2}\right]. \tag{2.4}$$

Since $u(\frac{1}{2}) = \rho$, we have:

$$G(\rho) := \int_0^\rho \frac{dt}{\sqrt{F(\rho) - F(t)}} = \frac{1}{\sqrt{2}}. \tag{2.5}$$

Therefore, if there exists a solution u such that $u(\frac{1}{2}) = \rho$, then ρ must be such that it satisfies the equation $G(\rho) = \frac{1}{\sqrt{2}}$. If we have such a ρ, we can define u by

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{2}x; \quad x \in \left[0, \frac{1}{2}\right].$$

By the Implicit Function Theorem, u is differentiable and hence

$$u'(x) = \sqrt{2[F(\rho) - F(u(x))]}.$$

Differentiating again we obtain

$$-u''(x) = f(a, b, c, u(x)).$$

Thus u is a positive solution to (2.1) with

$$u(\frac{1}{2}) = \rho \quad iff \quad G(\rho) = \frac{1}{\sqrt{2}}.$$

Note that in the above discussion, f is assumed to be always positive. However, in the following sections, such is not the case. Therefore, in order to use the quadrature method we must work only within an interval where $f(\rho)$ is indeed positive and $F(\rho) > F(z) \, \forall \, z \in [0, \rho)$. In our discussion below, this interval is from the first positive root of F to the next local maximum of F. For further literature on the quadrature method refer to [2] and [6].

3 Sub-Super Solution Method

Consider the following differential equation:

$$\begin{cases} -u''(x) = f(u(x)) & \forall x \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{3.1}$$

where f is a continuously differentiable function.
First we define the terms *sub-solution* and *super-solution*.

A function, ψ, is said to be a *sub-solution* of (3.1) if it satisfies the following differential inequalities.

$$\begin{cases} -\psi''(x) \leq f(\psi(x)) & \forall x \in (0,1) \\ \psi(0) \leq 0, \ \psi(1) \leq 0 \end{cases} \tag{3.2}$$

A function, ϕ, is said to be a *super-solution* of (3.1) if it satisfies the reversed inequalities, namely:

$$\begin{cases} -\phi''(x) \geq f(\phi(x)) & \forall x \in (0,1) \\ \phi(0) \geq 0, \ \phi(1) \geq 0. \end{cases} \tag{3.3}$$

Lemma. *If there exists a sub-solution, ψ, and a super-solution, ϕ, satisfying $\psi(x) \leq \phi(x) \quad \forall x \in (0,1)$, then (3.1) has at least one solution, u such that $\psi(x) \leq u(x) \leq \phi(x) \quad \forall x \in (0,1)$. Here we assume that f is a nondecreasing continuous function on* $[\min \psi, \max \phi]$.

An outline of the proof is shown below. See [7] for more literature.

Proof. Let G be Green's function defined as follows:

$$G(x,y) = \begin{cases} (1-y)\,x & if\ 0 \le x \le y \le 1 \\ (1-x)\,y & if\ 0 \le y \le x \le 1. \end{cases}$$

Define an operator $T : C[0,1] \to C[0,1]$, by $T[u(x)] = \int_0^1 G(x,y)\ f(u(y))\ dy$. It can be shown that solving (3.1) is equivalent to finding a fixed point of T. That is, a function u is a solution to (3.1) if and only if it satisfies $u(x) = \int_0^1 G(x,y)\ f(u(y))\ dy$.

Let $\psi_0 \equiv \psi$. Now recursively define ψ_n by $\psi_n(x) = T(\psi_{n-1}(x)) \quad \forall\ n \in \mathbb{N}$. Similarly $\phi_0 \equiv \phi$ and ϕ_n is defined by $\phi_n(x) = T(\phi_{n-1}(x)) \quad \forall\ n \in \mathbb{N}$. It can be shown via the Maximum Principle that $\psi_0 \le \psi_1 \le \ldots\psi_{n-1} \le \psi_n \le \ldots\phi_n \le \phi_{n-1} \le \phi_1 \le \phi_0$ for each $n \in \mathbb{N}$ and every $x \in (0,1)$.

Let $x_0 \in (0,1)$ be arbitrary. $\{\psi_n(x_0)\}_{n=1}^{\infty}$ is a bounded, monotone sequence in $\mathbb{R}$, so it converges. Define $u : (0,1) \to \mathbb{R}$ by letting $\{\psi_n(x_0)\}_{n=1}^{\infty} \to u(x_0)$ for each $x_0 \in (0,1)$.

It can be shown that the operator T is completely continuous(compact). Therefore $\{T(\psi_n)\}_{n=1}^{\infty}$ must have a convergent subsequence $\{T(\psi_{n_k})\}_{k=1}^{\infty}$. Moreover, $\{T(\psi_{n_k})\}_{k=1}^{\infty}$ converges in $C[0,1]$, so call the limit function v. Now by observing that $\{T(\psi_{n_k})\}_{k=1}^{\infty}$ is the sequence $\{\psi_{n_k+1}\}_{k=1}^{\infty}$, we see that $u(x) = v(x)\ \forall\ x \in (0,1)$.

Finally we have that $u = \lim\limits_{k\to\infty} T(\psi_{n_k}) = T(\lim\limits_{k\to\infty} \psi_{n_k}) = T(u)$ by continuity of the operator T.

$\square$

Remark: The method of sub-super solutions easily extends for nonautonomous equations of the form

$$\begin{cases} -u''(x) = f(x, u(x)); (0,1) \\ \quad u(0) = 0 = u(1) \end{cases} \tag{3.4}$$

where f is continuous and $\dfrac{\partial f}{\partial u}$ is bounded from below on $[0,1] \times [\min \psi, \max \phi]$.

4 Logistic Growth

In this section we will refer to equation (1.1). To determine the number of positive solutions to our boundary value problem. We utilize the following functions obtained from the quadrature method:

$$F(u,c) = \int_0^u f(s,c)ds \tag{4.1}$$

$$G(\rho,c) = \int_0^\rho \frac{dz}{\sqrt{F(\rho,c) - F(z,c)}} \tag{4.2}$$

and

$$G'(\rho,c) = \int_0^1 \frac{H(\rho,c) - H(\rho v,c)}{[F(\rho,c) - F(\rho v,c)]^{\frac{3}{2}}} \tag{4.3}$$

where

$$H(\rho,c) = F(\rho,c) - \frac{\rho}{2} f(\rho,c) \tag{4.4}$$

and

$$f(u,c) = au - bu^2 - c. \tag{4.5}$$

We will then find the values of ρ such that $G(\rho,c) = \frac{1}{\sqrt{2}}$. This will give us the range of a, b, and c, for which there are positive solutions to (1.1).

4.1 Range of ρ within Set S

As stated within section 2, there are certain conditions necessary for the quadrature method to apply, which may be defined by the following set S.

$$S = \{\rho | f(\rho, c) > 0 \ \& \ F(\rho, c) > F(z, c) \forall \ z \in [0, \rho)\}$$

To find the region for which $f(u, c)$ is positive we obtain the roots at

$$u = \frac{a \pm \sqrt{a^2 - 4bc}}{2b} \quad \text{where } c < \frac{a^2}{4b}. \tag{4.6}$$

Since the function is concave down, $f(u, c) > 0$ between these two points.
Next we must find the values of u such that $F(u, c) > F(z, c) \ \forall \ z \in [0, u)$. The real roots of this function are

$$u = \frac{3a \pm \sqrt{9a^2 - 48bc}}{4b} \text{ and } u = 0 \text{ where } c < \frac{3a^2}{16b}. \tag{4.7}$$

Throughout this section we will assume $c < \dfrac{3a^2}{16b}$.
From the roots and its derivative, the general shape of $F(u, c)$ as shown in Figure 4.1.

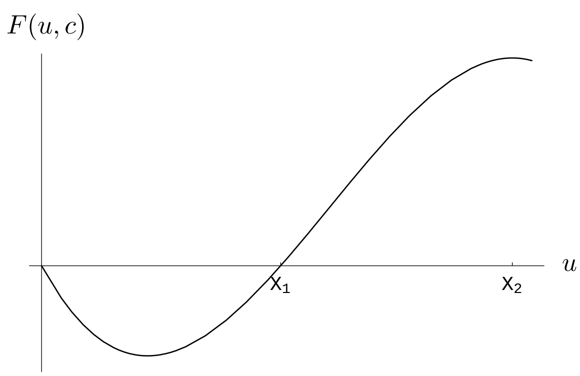

Figure 4.1: General shape of $F(u, c)$

Clearly

$$S = (X_1, X_2) = \left(\frac{3a - \sqrt{9a^2 - 48bc}}{4b}, \frac{a + \sqrt{a^2 - 4bc}}{2b} \right) \quad \textit{with} \quad c < \frac{3a^2}{16b}.$$

Note that when c satisfies this condition, it may be easily shown that $X_1 < X_2$.

4.2 Behavior of $G(\rho, c)$

Now that we know the range of values for ρ, we now analyze $G(\rho, c)$. From the above equation for $G'(\rho, c)$ we may see that it relies heavily on the function $H(\rho, c)$. For our specific function $f(u, c)$,

$$H(u, c) = \frac{bu^3 - 3cu}{6}$$
$$H'(u, c) = \frac{bu^2 - c}{2}$$

Note that this function does not depend on the parameter a. The roots of $H(u, c)$ are $u = \pm\sqrt{\dfrac{3c}{b}}$, and the roots of $H'(u, c)$ are $u = \pm\sqrt{\dfrac{c}{b}}$. By looking at the graph of $H(u, c)$ in Figure 4.2 we may further understand the graph of $G(\rho, c)$.

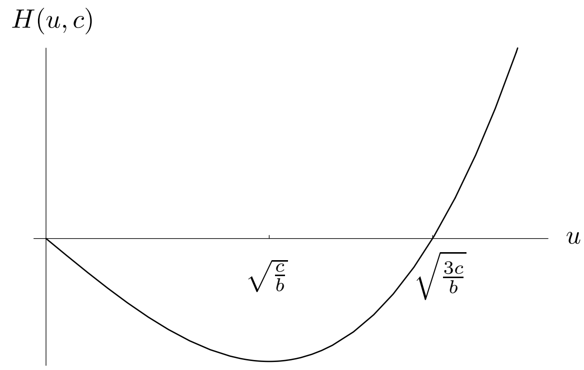

Figure 4.2: General shape of $H(u,c)$

Clearly $H(u,c)$ decreases until the value $\sqrt{\dfrac{c}{b}}$ and increases beyond $\sqrt{\dfrac{c}{b}}$. Further, $H(u,c)$ is positive beyond $\sqrt{\dfrac{3c}{b}}$. This means that for

$$\begin{array}{ll} \rho \in (0, \sqrt{\frac{c}{b}}] & G' < 0 \\ \rho \in (\sqrt{\frac{c}{b}}, \sqrt{\frac{3c}{b}}] & G' \text{ uncertain} \\ \rho \in (\sqrt{\frac{3c}{b}}, X_2] & G' > 0 \text{ where } \sqrt{3c} < X_2. \end{array}$$

Thus we are assured that $G(\rho,c)$ will be initially decreasing and will eventually reach a turning point where it will increase to the value of X_2. Here $X_2 > \sqrt{\dfrac{3c}{b}}$ when $c < \dfrac{3a^2}{16}$.

4.3 Bifurcation Diagrams when $a > \pi^2$ and $b = 1$

We found for values of a close to the first eigenvalue π^2, c must be very small for there to exist positive zeros of

$$\widetilde{G}(\rho,c) = G(\rho,c) - \frac{1}{\sqrt{2}}. \tag{4.8}$$

As shown in Figure 4.3, the values of both ρ and c are very close to zero. Note that when the birth rate is small, $a \leq \pi^2$, the system is not able to support harvesting. We will later establish this analytically. To see the progression of the values of c, refer to Figures 4.3-4.8 where c_0 is the maximum value of c such that $\widetilde{G}(\rho,c)$ still has a zero contained within set S. These values increase from .0043 up to 52.64.

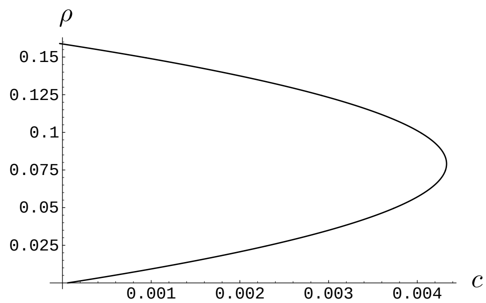

Figure 4.3: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 10$

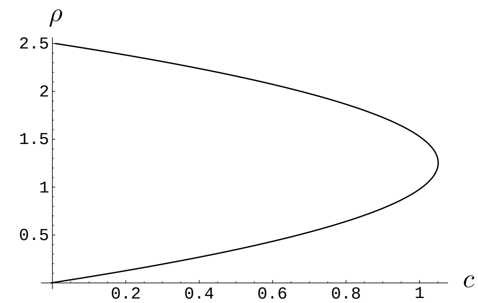

Figure 4.4: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 12$

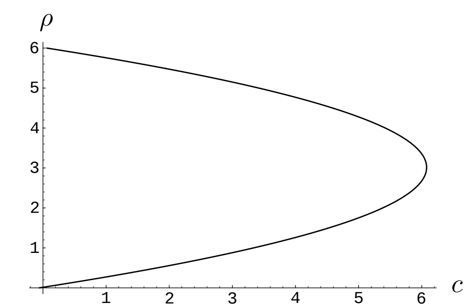

Figure 4.5: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 15$

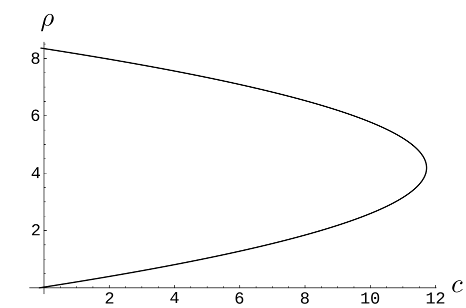

Figure 4.6: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 17$

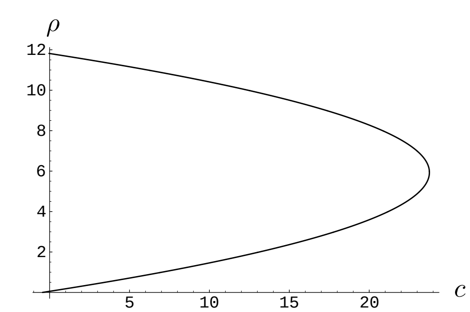

Figure 4.7: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 20$

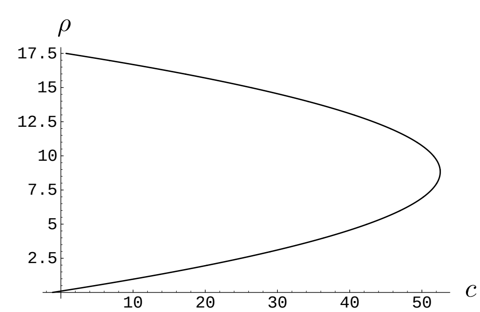

Figure 4.8: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 25$

4.4 Upper and Lower Bounds for $\widetilde{G}(\rho, c)$

We can find upper and lower bounds for $\widetilde{G}(\rho, c)$, namely,

$$L(\rho, c) \leq \widetilde{G}(\rho, c) \leq U(\rho, c)$$

where

$$\begin{aligned} L(\rho, c) &= \frac{\pi}{2\sqrt{\frac{a}{2} - \frac{b\rho}{3} - \frac{c}{2\rho}}} - \frac{1}{\sqrt{2}} \\ \widetilde{G}(\rho, c) &= \int_0^\rho \frac{dz}{\sqrt{(\rho - z)[\frac{a}{2}(\rho + z) - \frac{b}{3}(\rho^2 + \rho z + z^2) - c]}} - \frac{1}{\sqrt{2}} \quad \textit{and} \\ U(\rho, c) &= \frac{\pi}{2\sqrt{\frac{a}{2} - \frac{2b\rho}{3} - \frac{c}{\rho}}} - \frac{1}{\sqrt{2}}. \end{aligned}$$

To find the lower bound, we may do the following calculations:

$$\begin{aligned} \widetilde{G}(\rho, c) &\geq \int_0^\rho \frac{dz}{\sqrt{(\rho - z)[\frac{a}{2}(\rho + z) - \frac{b\rho}{3}(\rho + z) - c(\frac{\rho+z}{2\rho})]}} - \frac{1}{\sqrt{2}} \\ &= \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)[\frac{a}{2} - \frac{b\rho}{3} - \frac{c}{2\rho}]}} - \frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{\frac{a}{2} - \frac{b\rho}{3} - \frac{c}{2\rho}}} - \frac{1}{\sqrt{2}}. \end{aligned}$$

Likewise, we may find the upper bound by:

$$\begin{aligned} \widetilde{G}(\rho, c) &\leq \int_0^\rho \frac{dz}{\sqrt{(\rho - z)[\frac{a}{2}(\rho + z) - \frac{b}{3}(\rho^2 + \rho z + \rho z + z^2) - c(\frac{\rho+z}{\rho})]}} - \frac{1}{\sqrt{2}} \\ &\leq \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)[\frac{a}{2} - \frac{b}{3}(\rho + z) - \frac{c}{\rho}]}} - \frac{1}{\sqrt{2}} \\ &\leq \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)[\frac{a}{2} - \frac{2b\rho}{3} - \frac{c}{\rho}]}} - \frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{\frac{a}{2} - \frac{2b\rho}{3} - \frac{c}{\rho}}} - \frac{1}{\sqrt{2}}. \end{aligned}$$

As shown by Figure 4.9, when the upper and lower bounds exist and both have zeros within our range of set S, we can show that $\widetilde{G}(\rho, c)$ has two zeros.

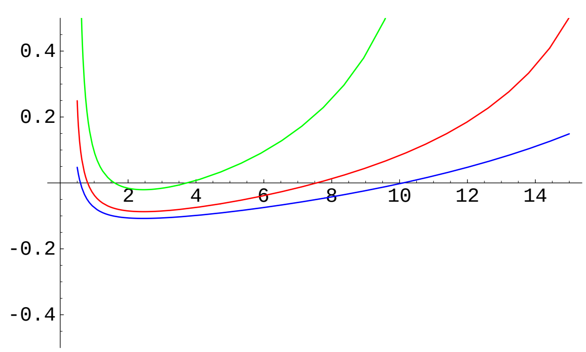

Figure 4.9: top $= U(\rho, c)$, middle $= \widetilde{G}(\rho, c)$, bottom $= L(\rho, c)$ for $a = 17$ and $c = 4$

4.5 Main Analysis

Our next result establishes [A] by means of the quadrature method.

Theorem 1. *If $a \le \lambda_1 (= \pi^2)$ then (1.2) has no positive solutions.*

Proof. Consider

$$G(\rho, c) = \int_0^\rho \frac{dz}{\sqrt{(\rho - z)[\frac{a}{2}(\rho + z) - \frac{b}{3}(\rho^2 + \rho z + z^2) - c]}}$$

where the values of $F(\rho, c)$ and $F(z, c)$ have been substituted and factored. Then

$$\begin{aligned}
\widetilde{G}(\rho, c) = G(\rho, c) - \frac{1}{\sqrt{2}} &> \int_0^\rho \frac{dz}{\sqrt{(\rho - z)[\frac{a}{2}(\rho + z)]}} - \frac{1}{\sqrt{2}} \\
&= \sqrt{\frac{2}{a}} \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)}} - \frac{1}{\sqrt{2}} \\
&= \sqrt{\frac{2}{a}} \frac{\rho^2}{\rho} \int_0^1 \frac{dz}{\sqrt{(\rho^2 - \rho^2 z^2)}} - \frac{1}{\sqrt{2}} \\
&\ge \sqrt{\frac{2}{a}} \int_0^1 \frac{dz}{\sqrt{(1 - z^2)}} - \frac{1}{\sqrt{2}} \\
&= \sqrt{\frac{2}{a}} \frac{\pi}{2} - \frac{1}{\sqrt{2}} \ge 0 \text{ when } a \le \pi^2.
\end{aligned}$$

So when $a \le \pi^2$, $\widetilde{G}(\rho, c)$ is always positive. This means that $\widetilde{G}(\rho, c)$ will have no zeros and therefore (1.1) has no solutions. ☐

Through the maximum principle we also establish [B].

Theorem 2. *There exists $c_0 = c_0(a, b)$ such that if $c > c_0$ then (1.2) has no positive solutions.*

Proof. There exist c_0 for a given a and b such that $f(u, c) < 0$ for all $u > 0$. Thus, by the maximum principle, (1.2) can have no solutions for $c > c_0$. ☐

Next by comparing the zeros of $L(\rho, c)$ and $U(\rho, c)$ to the bounds of $S = (X_1, X_2)$ in Theorem 3, we prove the existence of a solution to (1.2) and establish [C] via the quadrature method. Further, in Theorem 4, we establish the existence of a second positive solution for small c and $a \in (\pi^2, 2\pi^2)$.

Theorem 3. *Let $a \in (\pi^2, 3\pi^2)$ and $c < c_1 = \min\left\{\frac{3\,(a - \pi^2)^2}{32\,b}, \frac{3\,(3a + \pi^2)\,(a - \pi^2)}{64\,b}, \frac{6\,\pi^4}{b}\right\}$. Then (1.2) has at least one positive solution.*

Proof. Step 1: We will first show that $\widetilde{G}(\rho, c)$ becomes negative at some point within the interval (X_1, X_2).

It is sufficient to show that $U(\rho, c)$ has a zero within set S, thereby forcing $\widetilde{G}(\rho, c)$ to become negative because it is strictly less than $U(\rho, c)$. Since it can be shown that $U(\rho, c)$ is concave up over the entire domain, it is only necessary to show that $U(\rho, c) = 0$ at one point within the interval. First note that $U(\rho, c) = 0$ when $\rho = \frac{3(a - \pi^2) \pm \sqrt{-96bc + 9(a - \pi^2)^2}}{8b}$. These zeros are real when $c < \frac{3\,(a - \pi^2)^2}{32\,b}$. Define U_{avg} to be the average of the zeros of $U(\rho, c)$, namely $U_{\text{avg}} = \frac{3(a - \pi^2)}{8b}$. Finally, define

$$r(a, C) = 8b(U_{\text{avg}} - X_1) = -3a - 3\pi^2 + 2\sqrt{9a^2 - 48bc} \quad \text{where } C = bc.$$

The zeros of this function occur when $a = \frac{\pi^2 + 2\sqrt{\pi^4 + 16bc}}{3}$. But $c < \frac{3(3a + \pi^2)(a - \pi^2)}{64b}$ implies that $a > \frac{\pi^2 + 2\sqrt{\pi^4 + 16bc}}{3}$.

The first derivative of $r(a,C)$ with respect to a is

$$r_a(a,C) = -3 + \frac{18a}{\sqrt{9a^2 - 48bc}} \geq -3 + \frac{18a}{\sqrt{9a^2}} = 3 > 0.$$

Hence $r(a,C) \geq 0$, i.e. $U_{\text{avg}} \geq X_1$.

Also, $U_{\text{avg}} \leq X_2$ since

$$\frac{3(a-\pi^2)}{8b} \leq \frac{3a}{8b} \leq \frac{a}{2b} \leq \frac{a + \sqrt{a^2 - 4bc}}{2b}.$$

Therefore, if $c < c_1$, U_{avg} is in S, forcing $\widetilde{G}(\rho, c)$ to be negative within S.

Step 2: Now we must prove that $\widetilde{G}(\rho, c)$ becomes positive within S and thus has a zero. Since $\widetilde{G}(\rho, c)$ is strictly greater than $L(\rho, c)$, it is sufficient to show that the upper zero of $L(\rho, c)$, L_2, is within the interval (X_1, X_2) and thus forces $\widetilde{G}(\rho, c)$ to be positive.

For L_2 to lie within the interval, it is necessary that

$$X_2 - L_2 = \frac{a + \sqrt{a^2 - 4bc}}{2b} - \frac{3(a-\pi^2) + \sqrt{9(a-\pi^2)^2 - 24bc}}{4b} = \frac{r(a,C)}{4b} > 0$$

where

$$r(a,C) = -a + 2\sqrt{a^2 - 4C} + 3\pi^2 - \sqrt{9(a-\pi^2)^2 - 24C} \text{ with } C = b\,c.$$

We first show that $a = \dfrac{C + 9\pi^4}{3\pi^2}$ is a zero of r.

$$\begin{aligned} r\left(\frac{C}{3\pi^2} + 3\pi^2, C\right) &= -\frac{C}{3\,\pi^2} - 3\,\pi^2 + 3\,\pi^2 + 2\sqrt{\left(\frac{C}{3\,\pi^2} + 3\,\pi^2\right)^2 - 4\,C} - \sqrt{9\left(\frac{C}{3\,\pi^2} + 3\,\pi^2 - \pi^2\right)^2 - 24\,C} \\ &= -\frac{C}{3\,\pi^2} + 2\sqrt{\left(3\,\pi^2 - \frac{C}{3\,\pi^2}\right)^2} - \sqrt{\left(6\,\pi^2 - \frac{C}{\pi^2}\right)^2} \\ &= 0 \text{ for } C < 6\,\pi^4. \end{aligned}$$

To show that $r(a,C)$ is positive for values of a up to $3\pi^2$, it is sufficient to show that the derivative r_a is negative when $a < \dfrac{C + 9\pi^4}{3\pi^2}$. The derivative of $r(a,C)$ is

$$r_a(a,C) = \frac{2a}{\sqrt{a^2 - 4C}} - \frac{9(a-\pi^2)}{\sqrt{9(a-\pi^2)^2 - 24C}} - 1.$$

But since $C < \dfrac{3}{16}\,a^2$, we have

$$r_a(a,C) < \frac{2a}{\sqrt{a^2 - \frac{3a^2}{4}}} - \frac{9(a-\pi^2)}{3(a-\pi^2)} - 1 = 4 - 3 - 1 = 0$$

Hence, since the upper zero of $L(\rho, c)$ is contained within our interval, $\widetilde{G}(\rho, c)$ has a zero when $a \in (\pi^2, 3\pi^2)$. $\square$

Theorem 4. *Let $a \in (\pi^2, 2\pi^2)$ and $c < c_1 = \min\left\{\dfrac{3\,(a-\pi^2)^2}{32\,b}, \dfrac{3\,(3a+\pi^2)\,(a-\pi^2)}{64\,b}, \dfrac{3\,\pi^4}{2\,b}\right\}$. Then (1.2) has at least two positive solutions.*

Proof. Define a function $r(a,C) = 4b(L_1 - X_1)$ with $C = bc$ where L_1 is the lower zero of the lower bound function and X_1 is the lower bound of set S. So

$$r(a,C) = -3\pi^2 + \sqrt{9a^2 - 48bc} - \sqrt{9(a-\pi^2)^2 - 24bc} > 0.$$

The zero of this function occurs at $a = \dfrac{2(bc+3\pi^4)}{3\pi^2}$ which is proven analytically below.

$$\begin{aligned}
r\left(\frac{2(bc+3\pi^4)}{3\pi^2}, C\right) &= -3\pi^2 - \sqrt{9\left(\frac{2(bc+3\pi^4)}{3\pi^2} - \pi^2\right)^2 - 24bc} + \sqrt{9\left(\frac{2(bc+3\pi^4)}{3\pi^2}\right)^2 - 48bc} \\
&= -3\pi^2 - \sqrt{\left(\frac{2bc}{\pi^2} + 3\pi^2\right)^2 - 24bc} + \sqrt{\left(\frac{2bc}{\pi^2} + 6\pi^2\right)^2 - 48bc} \\
&= -3\pi^2 - \sqrt{\left(3\pi^2 - \frac{2bc}{\pi^2}\right)^2} + \sqrt{\left(6\pi^2 - \frac{2bc}{\pi^2}\right)^2} \\
&= 0 \text{ for } bc < \frac{3\pi^4}{2}
\end{aligned}$$

Then for $r(a, C)$ to be positive prior to this zero, it is sufficient to prove that the derivative of $r(a, C)$ is negative for values of $a < \dfrac{2(bc+3\pi^4)}{3\pi^2}$. Now

$$\begin{aligned}
r_a(a, C) &= \frac{9a}{\sqrt{9a^2 - 48bc}} - \frac{9(a-\pi^2)}{\sqrt{9(a-\pi^2)^2 - 24bc}} \\
&= \frac{9a\sqrt{9(a-\pi^2)^2 - 24bc} - 9(a-\pi^2)\sqrt{9a^2 - 48bc}}{\sqrt{9a^2 - 48bc}\sqrt{9(a-\pi^2)^2 - 24bc}} \\
&= \frac{\sqrt{81a^2(9(a-\pi^2)^2 - 24bc)} - \sqrt{81(a-\pi^2)^2(9a^2 - 48bc)}}{\sqrt{9a^2 - 48bc}\sqrt{9(a-\pi^2)^2 - 24bc}}, \text{ and}
\end{aligned}$$

for $a \in \left(\pi^2, \dfrac{2(bc+3\pi^4)}{3\pi^2}\right)$,

$$\begin{aligned}
&81a^2(9(a-\pi^2)^2 - 24bc) - 81(a-\pi^2)^2(9a^2 - 48bc) \\
&= 81a^2(-24bc) - 81(a-\pi^2)^2(-48bc) \\
&= 81[-24a^2bc - (a-\pi^2)^2(-48bc)] \\
&= 81[24bc(-a^2 + 2(a-\pi^2)^2)] \\
&= 81[24bc(-a(4\pi^2 - a) + 2\pi^4)] \le 0.
\end{aligned}$$

Hence $r(a, C) > 0$ when $a \in (\pi^2, 2\pi^2)$. This tells us that for $a \in (\pi^2, 2\pi^2)$ our lower limit has a zero within our set S. Combining this with the previous result gives the theorem.

□

Conjecture: Theorem 4 establishes that there exists two solutions for logistic equation (1.1) when $a \in (\pi^2, 2\pi^2)$ for c small enough. We conjecture that this will persist for $a \in (\pi^2, 4\pi^2)$.

5 Strong Allee Effect

Here we discuss the positive solutions to the Strong Allee effect problem (1.3) where

$$f(u, c) = u(u-a)(b-u) - c = -u^3 + (b+a)u^2 - abu - c.$$

Then it follows that

$$F(u, c) = -\frac{u^4}{4} + \frac{(b+a)u^3}{3} - \frac{abu^2}{2} - cu,$$

and,

$$\widetilde{G}(\rho, c) = \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)\left(-\frac{\rho^2+z^2}{4} + \frac{(b+a)(\rho^2+\rho z+z^2)}{3(\rho+z)} - \frac{ab}{2} - \frac{c}{\rho+z}\right)}} - \frac{1}{\sqrt{2}}.$$

5.1 Bifurcation Diagrams for Strong Allee Effect

In this section we will present several bifurcation diagrams for the equation (1.3) developed by computational analysis via Mathematica and the quadrature method. The bifurcation diagrams will represent the ρ vs c values when $\widetilde{G}(\rho, c) = 0$. Notice that our computations indicate the existence of two positive solutions for small c. Also notice that the bifurcation curves for positive solutions are not connected for certain values of a, b, and c.

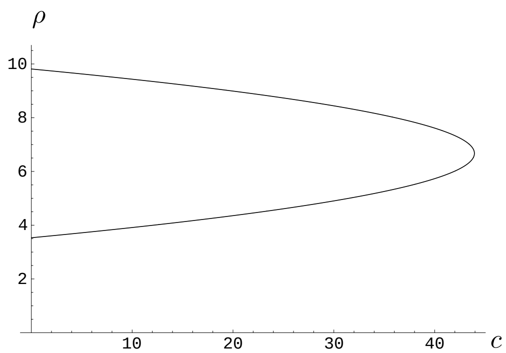

Figure 5.1: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 10$ and $b = 1$

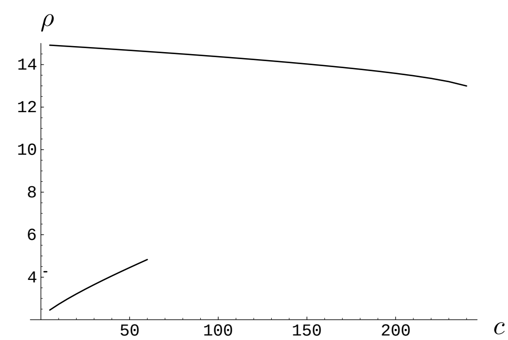

Figure 5.2: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 15$ and $b = 1$

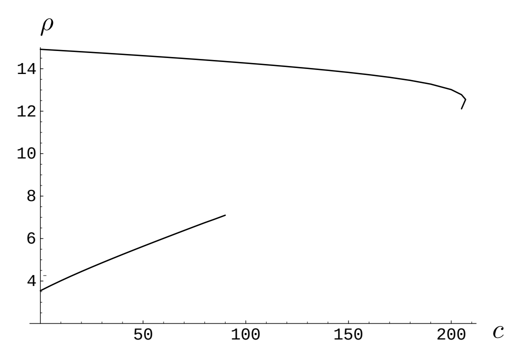

Figure 5.3: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 15$ and $b = 2$

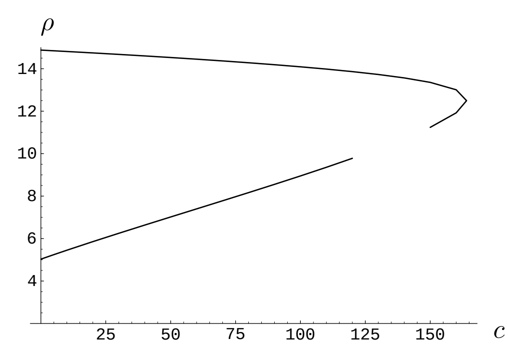

Figure 5.4: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 15$ and $b = 3$

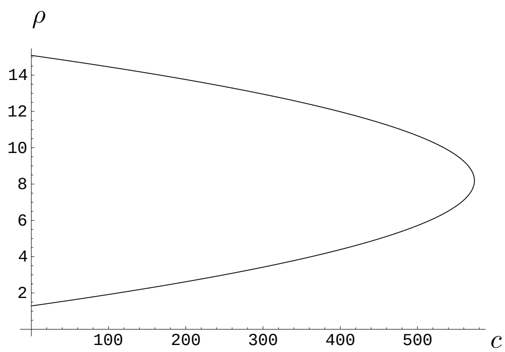

Figure 5.5: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 15$ and $b = 4$

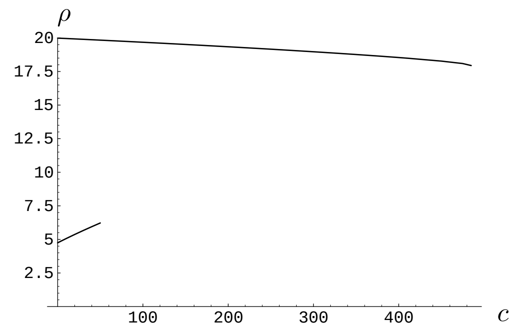

Figure 5.6: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 20$ and $b = 3$

5.2 Non-Existence Results

Here we establish non-existence results similar to [A] and [B] in Theorems 5 and 6 respectively.

Theorem 5. *Let a and b satisfy $b(b + a) \leq \frac{3}{4}\pi^2$. Then (1.3) does not have a positive solution.*

Proof.

$$\begin{aligned}
\widetilde{G}(\rho, c) &> \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)(\frac{(b+a)(\rho^2+\rho z+z^2)}{3(\rho+z)})}} - \frac{1}{\sqrt{2}} \\
&> \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)(\frac{(b+a)(\rho+z)}{3})}} - \frac{1}{\sqrt{2}} \\
&> \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)(\frac{2(b+a)\rho}{3})}} - \frac{1}{\sqrt{2}} \\
&\geq \frac{\pi}{2\sqrt{\frac{2(b+a)\rho}{3}}} - \frac{1}{\sqrt{2}}.
\end{aligned}$$

Observe that $\rho(= \|u\|_\infty) < \rho$ via the maximum principle, and so $\widetilde{G}(\rho, c) > 0$ when $b(b + a) \leq \frac{3}{4}\pi^2$. Hence there are no positive solutions. □

Theorem 6. *There exists $c_0 = c_0(a, b)$ such that if $c > c_0$ then (1.3) has no positive solution.*

Proof. Clearly we can choose c_0 such that $f(u) = u(u - a)(b - u) - c_0 < 0 \quad \forall \quad u > 0$. Thus, by the maximum principle, (1.3) cannot have positive solutions for $c > c_0$.

□

6 Weak Allee Effect

Here we discuss the positive solutions to the weak Allee effect problem (1.4) where

$$f(u, c) = u(u + a)(b - u) - c = -u^3 + (b - a)u^2 + abu - c.$$

Then it follows that

$$F(u,c) = -\frac{u^4}{4} + \frac{(b-a)u^3}{3} + \frac{abu^2}{2} - cu$$

and

$$\widetilde{G}(\rho,c) = \int_0^\rho \frac{dz}{\sqrt{(\rho^2 - z^2)(-\frac{\rho^2+z^2}{4} + \frac{(b-a)(\rho^2+\rho z+z^2)}{3(\rho+z)} + \frac{ab}{2} - \frac{c}{\rho+z})}} - \frac{1}{\sqrt{2}}.$$

6.1 Bifurcation Diagrams for Weak Allee Effect

In this section we present several bifurcation diagrams for the equation (1.4). These graphs represent values of c and ρ such that $\widetilde{G}(\rho,c) = 0$. In these cases, unlike the strong Allee effect case, we do not always get two solutions for small c. Again notice that at some points the bifurcation curve of positive solutions are not connected for certain values of a, b, and c.

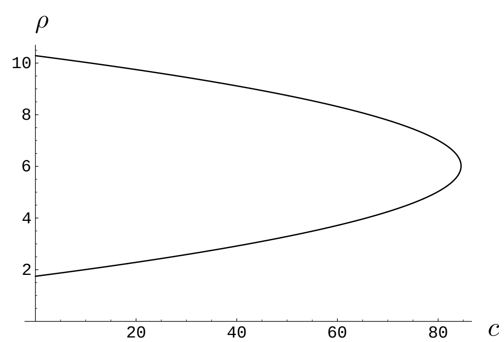

Figure 6.1: ρ vs. c such that $\widetilde{G}(\rho,c) = 0$ with $a = 1$ and $b = 10$

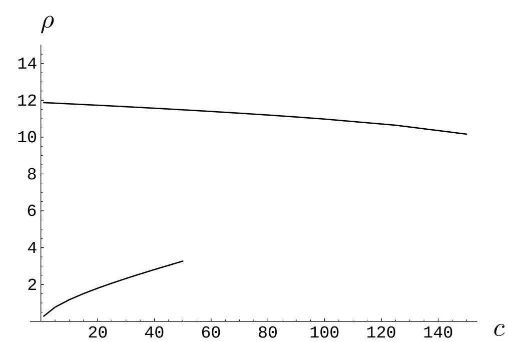

Figure 6.2: ρ vs. c such that $\widetilde{G}(\rho,c) = 0$ with $a = 1$ and $b = 12$

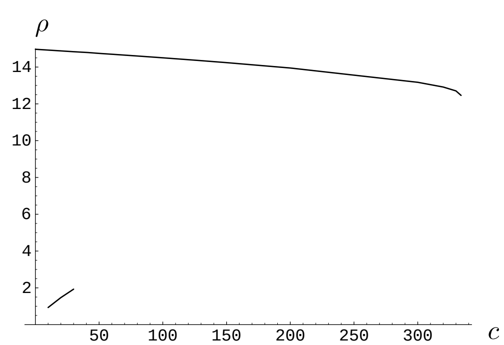

Figure 6.3: ρ vs. c such that $\widetilde{G}(\rho,c) = 0$ with $a = 1$ and $b = 15$

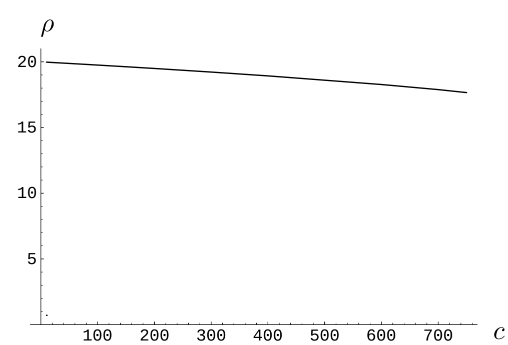

Figure 6.4: ρ vs. c such that $\widetilde{G}(\rho,c) = 0$ with $a = 1$ and $b = 20$

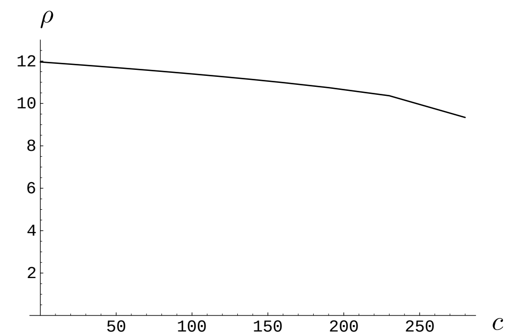

Figure 6.5: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 5$ and $b = 12$

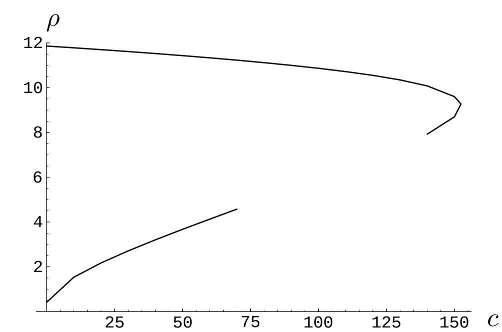

Figure 6.6: ρ vs. c such that $\widetilde{G}(\rho, c) = 0$ with $a = 0.5$ and $b = 12$

6.2 Non-existence Results

Here we establish nonexistence results similar to [A] and [B] in Theorems 7 and 8 respectively.

Theorem 7. *Let a and b satisfy $b(4b - a) \le 3\pi^2$. Then (1.4) does not have a positive solution.*

Proof. We approximate $\widetilde{G}(\rho, c)$ from below:

$$\widetilde{G}(\rho, c) \ge \frac{\pi}{2\sqrt{\frac{2(b-a)\rho}{3} + \frac{ab}{2}}} - \frac{1}{\sqrt{2}},$$

and find that if

$$\frac{2\rho(b-a)}{3} + \frac{ab}{2} \le \frac{\pi^2}{2}$$

then $\widetilde{G}(\rho, c) > 0$. We observe via the maximum principle that $\rho < b$, and thus if

$$b(4b - a) \le 3\pi^2,$$

there can be no positive solution as $\widetilde{G}(\rho, c)$ has no zeros.

$\square$

Theorem 8. *There exists $c_0 = c_0(a, b)$ such that if $c > c_0$ then (1.4) has no positive solution.*

Proof. Again we can choose c_0 such that $f(u) = u(u+a)(b-u) - c_0 < 0 \quad \forall \quad u > 0$. Thus, by the maximum principle, (1.4) cannot have positive solution for $c > c_0$.

$\square$

6.3 Existence via Sub- Super- Solutions Method

Consider the following differential equation under the assumptions $a, b, c > 0$ and $ab > \pi^2$.

$$\begin{cases} -u'' = u(a+u)(b-u) - ch(x) \\ u(0) = 0 = u(1) \end{cases} \tag{6.1}$$

We will show that equation (6.1) has a positive solution, for a sufficiently small values of c, using the method of sub-super solutions.

Theorem 9. *Let $a, b > 0$ with $ab > \pi^2$. There exists $c_0 = c_0(a,b) > 0$, such that (6.1) has at least one positive solution whenever $c < c_0$.*

Proof. First a non-negative sub-solution will be established. Choose $\lambda^* \in (\pi^2, \min\{ab, 4\pi^2\})$ and consider the boundary value problem:

$$\begin{cases} -z'' = \lambda^* z - 1 \\ z(0) = 0 = z(1) \end{cases} \tag{6.2}$$

Solving (6.2) we see that

$$z(t) = \frac{\cos(\sqrt{\lambda^*}) - 1}{\lambda^* \sin(\sqrt{\lambda^*})} \sin(t\sqrt{\lambda^*}) - \frac{1}{\lambda^*} \cos(t\sqrt{\lambda^*}) + \frac{1}{\lambda^*}.$$

This is clearly positive on (0, 1) since $\pi^2 \leq \lambda^* \leq 4\pi^2$ (see also [8]).
Let $\psi(x) = k\,c\,z$. For ψ to be a sub-solution, k and c must be found such that:

$$\begin{aligned} k\,c\,(-z'') &\leq k\,c\,z(a + k\,c\,z)(b - k\,c\,z) - c \\ k\,c\,(\lambda^*\,z - 1) &\leq a\,b\,(k\,c\,z) + (b-a)\,(k\,c\,z)^2 - (k\,c\,z)^3 - c \\ 0 &\leq (a\,b - \lambda^*)\,(k\,c\,z) + (b-a)\,(k\,c\,z)^2 - (k\,c\,z)^3 + c\,(k-1) \end{aligned} \tag{6.3}$$

Let $H(y) = (a\,b - \lambda^*)\,y + (b-a)\,y^2 - y^3 + c(k-1)$ with $k > 1$. Since $H(0)$, $H'(0)$, and $H''(0)$ are all positive and the second derivative changes sign only once. It follows that H must have exactly one positive zero. Hence if $H(\beta) > 0$ for some $\beta > 0$, then $H(y) > 0$ for all $y \in (0, \beta)$.

Let $\alpha = \|z\|_\infty$. If k and c are chosen such that $H(k\,c\,\alpha) > 0$, then (6.3) is satisfied for all $x \in (0,1)$ and hence, $\psi(x) = k\,c\,z$ is a sub-solution. But

$$H(kc\alpha) = (a\,b - \lambda^*)\,(k\,c\,\alpha) + (b-a)\,(k\,c\,\alpha)^2 - (k\,c\,\alpha)^3 + c\,(k-1) \geq 0$$

provided

$$(a\,b - \lambda^*)\,(k\,\alpha) - (k\,\alpha)^3 + (k-1) \geq 0. \tag{6.4}$$

$$\text{Let } c_0 = \sup_{k>1} \left\{ \frac{(a\,b - \lambda^*)\,(k\,\alpha) + (k-1)}{(k\,\alpha)^3} \right\}^{\frac{1}{2}}.$$

Then for $c < c_0$ there exist $k > 1$ such that (6.4) is satisfied. Hence $H(k\,c\,\alpha) > 0$ and ψ is a sub-solution.

Observe that any large positive constant is a super-solution. Hence for $c < c_0$, (6.1) has a positive solution.

∎

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