Properties of Divisor Graphs

Christopher Frayer
Grand Valley State University, cfrayer@ms.uky.edu

Properties of Divisor Graphs

Christopher Frayer

Department of Mathematics
Grand Valley State University
Allendale, MI 49401

Abstract

For a finite, nonempty set S of positive integers, the divisor graph $G(S)$ of S has vertex set S, and two distinct vertices i and j are adjacent if and only if $i|j$ or $j|i$, while the divisor digraph $D(S)$ of S has vertex set S and (i,j) is an arc of $D(S)$ if and only if $i|j$. A graph G is a divisor graph if there exists a set S of positive integers such that G is isomorphic to $G(S)$. It is shown that for a divisor graph G with a transitive vertex, $G \times H$ is a divisor graph if and only if $E(H) = \emptyset$. For $m, n \in \mathbb{N}$, $P_m \times P_n$ is a divisor graph. Also, given $m, n \in \mathbb{N}$ with $m \geq 5$ there exists a non-divisor graph G, of order $m + n$, that has m neighborhoods that are divisor graphs and n neighborhoods that are not divisor graphs.

1 Introduction

In 2000 Singh and Santhosh [2] defined the concept of a divisor graph. They defined a *divisor graph* G as an ordered pair (V, E) where $V \subset \mathbb{Z}$ and for all $u, v \in V$, $u \neq v$, $uv \in E$ if and only if $u|v$ or $v|u$. Singh and Santhosh showed that every odd cycle of length five or more is not a divisor graph while all even cycles, complete graphs, and caterpillars are divisor graphs.

In 2001, Chartrand, Muntean, Saenpholphant and Zhang [1] also studied divisor graphs. They let S be a finite, nonempty set of positive integers. Then, the *divisor graph* $G(S)$ of S has S as its vertex set, and vertices i and j are adjacent if and only if either $i|j$ or $j|i$. A graph G is a divisor graph

if $G \cong G(S)$ for some nonempty, finite set S of positive integers. Hence, if G is a divisor graph, then there exists a function $f : V(G) \rightarrow \mathbb{N}$, called a *divisor labeling* of G, such that $G \cong G(f(V(G)))$. Then the *divisor digraph* $D(S)$ of S has the vertex set S and (i, j) is an arc of $D(S)$ if and only if $i|j$. The results of [2] are confirmed in [1], where it was shown that trees and bipartite graphs are divisor graphs and a characterization of all divisor graphs was given.

In this paper, we will define divisor graphs in terms of a nonempty finite set of positive integers as in [1]. See [3] and [4] for basic definitions. For the graph G given in Figure 1, the vertices can be labeled with $S = \{2, 3, 4, 6\}$ to produce the divisor graph $G(S)$ and the divisor digraph $D(S)$. The divisor labeling $f : V(G) \longrightarrow \mathbb{N}$ is defined as

$$f(a) = 2, f(b) = 6, f(c) = 3, f(d) = 4.$$

Observe that the labeling $f' : V(G) \longrightarrow \mathbb{N}$ with $f'(x) = cf(x)$ for $c \in \mathbb{N}$ is also a divisor labeling of G. Therefore, any divisor graph G has infinitely many divisor labelings.
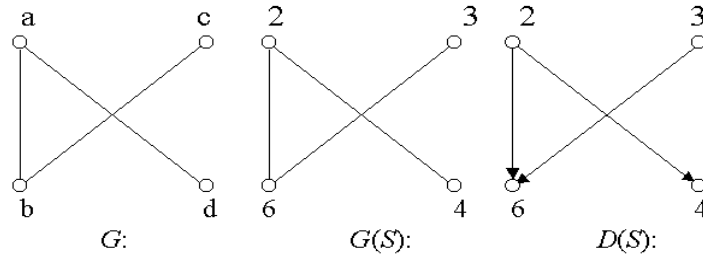

Figure 1: A graph G, a divisor labelling of G, the resulting digraph.

The goal of this paper is to explore several families of divisor graphs, as well as looking at divisor graphs and their relationship with their neighborhoods, their complements, and products.

2   Classification of Divisor Graphs

In this section, we state the characterization of divisor graphs given in [1], and give some results based upon this characterization. Let D be a labeled

divisor digraph. A *transmitter* of D is a vertex having indegree 0 and a *receiver* of D is a vertex having outdegree 0. For a vertex u of D, let

$$N^+(u) = \{x|(u,x) \in E(D)\} \quad \text{and} \quad N^-(u) = \{x|(x,u) \in E(D)\}.$$

A vertex v in D is *transitive* if its outdegree and indegree are both greater than zero, and for every $x \in N^-(u)$ and $y \in N^+(u)$, necessarily $(x,y) \in E(D)$. In [1] it was shown that:

Theorem A *Let G be a graph. Then G is a divisor graph if and only if there exists an orientation D of G such that every vertex of D is a transmitter, a receiver, or a transitive vertex.*

Using theorem A one can easily verify that all odd cycles of length greater than three are not divisor graphs. Hence, the five cycle, C_5, is not a divisor graph.

Theorem 2.1 *An orientation of a divisor graph, G, contains no transitive vertices if and only if G is a bipartite graph.*

Proof: Let G be a divisor graph. If an orientation of G contains a transitive vertex, G must contain a 3-cycle and therefore could not be bipartite. Conversely, if an orientation of G does not contain a transitive vertex then each vertex is a transmitter or a receiver, so G is a bipartite graph. ∎

The following can be proven as a corollary to Theorem 3.3 and Proposition 2.3 in [1]; however, we present the following independent proof.

Theorem 2.2 *Let G be a divisor graph with an orientation that contains a transitive vertex. The graph $G \times H$ is a divisor graph if and only if $E(H) = \emptyset$.*

Proof: Let G be a divisor graph with an orientation that contains a transitive vertex. If H is a graph of order $n \in \mathbb{N}$, such that $E(H) = \emptyset$, then $G \times H$ is composed of n individual copies of G and is a divisor graph.

Assume, to the contrary, $G \times H$ with orientation D and induced labeling $f : V(G \times H) \to \mathbb{N}$ is a divisor graph and $E(H) \neq \emptyset$. Since $E(H) \neq \emptyset$, let $h_1 h_2 \in E(H)$. Also, since G contains a transitive vertex, say g_1, each copy of G in $G \times H$ contains a transitive vertex, (g_1, h_i). Then, in the first copy of G, there exist vertices (g_2, h_1) and (g_3, h_1) such that $f(g_2, h_1)|f(g_1, h_1)$ and

$f(g_1, h_1)|f(g_3, h_1)$, where, since $E(H) \neq \emptyset$, (g_1, h_1) is adjacent to (g_1, h_2). We must consider two cases.

Case 1: We assume $f(g_1, h_1)|f(g_1, h_2)$. Therefore $f(g_2, h_1)|f(g_1, h_2)$. However (g_2, h_1) is not adjacent to (g_1, h_2) and $G \times H$ is not a divisor graph.

Case 2: We assume $f(g_1, h_2)|f(g_1, h_1)$. Therefore $f(g_1, h_2)|f(g_3, h_1)$. However (g_1, h_2) is not adjacent to (g_3, h_1) and $G \times H$ is not a divisor graph. ∎

Corollary 2.3 $K_n \times K_2$ *is not a divisor graph for each* $n \geq 3$.

Corollary 2.4 $K_n \times G$ *is a divisor graph for each* n *if and only if* G *contains no edges.*

3 Some General Results

In this section we will look at several results concerning divisor graphs. We begin by making the following observation that appears in [1]. Since our proof is different and shorter, we include it.

Theorem 3.1 *Every bipartite graph is a divisor graph.*

Proof: Let G be a bipartite graph with partite sets U and W. Without loss of generality we let any isolated vertices of G be in W. Label each vertex $w \in W$ with a distinct prime, and every vertex $u \in U$ with the product of the labels of its adjacent vertices. Multiply the label of each vertex of U with a prime distinct from the labels of the vertices in W. No two vertex labels from the same partite set divide each other since they have distinct prime factorizations with at least one uncommon prime. Vertex labels divide each other if and only if they correspond to an edge of G. Thus this is a divisor labeling of G, and every bipartite graph must be a divisor graph. ∎

Singh and Santhosh showed that all graphs of order less than five are divisor graphs. We extend this result by showing that C_5 is the unique graph of order less than six that is not a divisor graph.

Theorem 3.2 *All graphs of order less than six are divisor graphs with the exception of* C_5.

Proof: It has already been shown that all graphs of order less than five are divisor graphs. To extend to order 6, we simply must show that all graphs of order 5 are divisor graphs except C_5. Actually we only need to look at the connected graphs of order five. Any non-connected graph or order 5 must be a divisor graph, since all graphs of order less then 5 are divisor graphs. For the connected components a divisor labelling must exist, and then label any isolated vertices with distinct primes not seen in any of the other labels. Figures 2 and 3 illustrate that all connected graphs of order five are divisor graphs (We let p, q, r, s, and t be distinct primes. For a table of all graphs of order five see [3].). Therefore all graphs of order less than six are divisor graphs with the exception of C_5. ∎
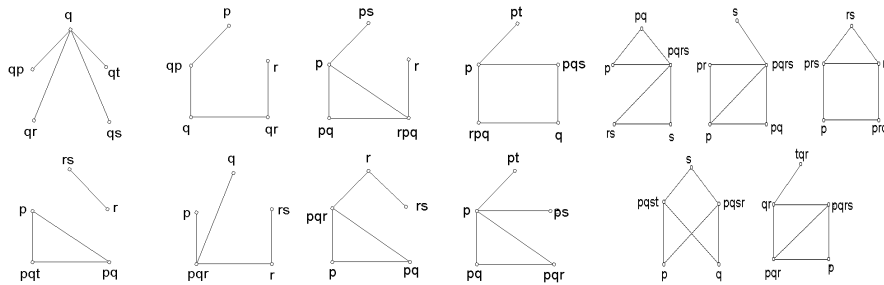

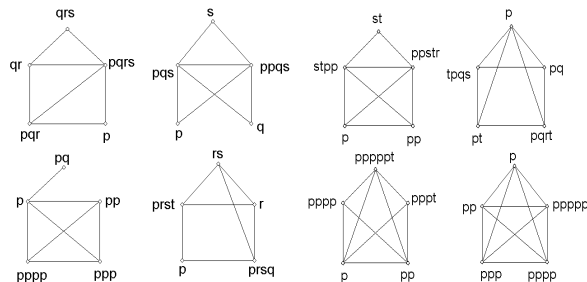Figure 2: Order five and size 4, 5, and 6.

Figure 3: Order five and size 7, 8, 9, and 10.

We now turn our attention to look at products of paths.

Theorem 3.3 $P_m \times P_n$ *is a divisor graph.*

5

Proof: We clearly see that $P_m \times P_n$ is a bipartite graph and therefore is a divisor graph. ∎

We also present an alternative proof.

Proof: Construct $P_m \times P_n$ and think of $P_m \times P_n$ as a m by n grid graph. Start in the top left hand corner and label this vertex $v_{1,1}$. In a similar fashion label the vertex in the ith row and j^{th} column as $v_{i,j}$. We now introduce an orientation D. For every vertex $v_{i,j}$ such that $i + j = 2n$ for some $n \in \mathbb{N}$ make $v_{i,j}$ a transmitter. Therefore every vertex $v_{i,j}$ such that $i + j = 2n + 1$ for some $n \in \mathbb{N}$ must be a receiver. See Figure 4 for the resulting orientation, D, of $P_4 \times P_4$. The orientation D is such that the only directed edges are from a transmitter to a receiver. Therefore, by Theorem A, $P_m \times P_n$ is a divisor graph. ∎
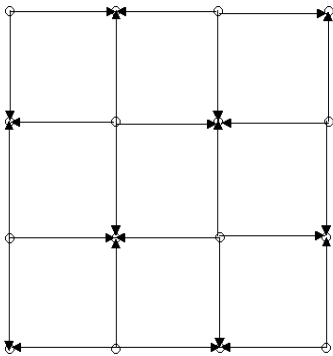

Figure 4: An example of the orientation given to $P_4 \times P_4$.

Theorem 3.4 *Let G be a divisor graph. Construct H from G by adding an edge between any two vertices whose distance apart is an even number greater than 2, then H is not a divisor graph.*

Proof: Let G be a divisor graph. Construct H from G by adding an edge between any two vertices whose distance apart is an even number greater than 2, say v_1 and v_2. Then the induced subgraph containing v_1 and v_2 along with the odd number of vertices corresponding to the even number of edges on the path connecting v_1 and v_2 is a cycle of odd length greater than three, and therefore H is not a divisor graph. ∎

4 Complements and Neighborhoods

For a graph G, we denote the complement of G by $\overline{G}$. Some results arise when we ask about the relationship that exists between a graph and its complement with regards to divisor graphs. Consider the following corollary of Theorem 3.2.

Corollary 4.1 *Except for C_5, all graphs of order less than six and their complements are divisor graphs. Furthermore, C_5 and its complement are non-divisor graphs.*

Note that when we look at graphs of order six this property does not hold. By Corollary 2.3, $K_3 \times K_2$ is not a divisor graph, but $\overline{K_3 \times K_2} = C_6$ is a divisor graph.

We also have the following:

Theorem 4.2 *$K_n \times K_2$ is not a divisor graph, but $\overline{K_n \times K_2}$ is a divisor graph for each $n \geq 3$.*

Proof: From Corollary 2.3 we know that $K_n \times K_2$ is not a divisor graph for each $n \geq 3$. Since $\overline{K_n \times K_2}$ is a bipartite graph, it is a divisor graph. ∎

Note that since $\overline{K_m} = nK_1$ and $\overline{K_{m,n}} = K_m \cup K_n$, if $G = K_n$ or $G = K_{m,n}$ then G and $\overline{G}$ are both divisor graphs. On the other hand, C_5 is not a divisor graph and its complement, $\overline{C_5}$, is not a divisor graph. However, some graphs, for example, $\overline{K_n \times K_2}$, from Theorem 4.2, are divisor graphs but their complements are not.

We now focus on neighborhoods of a vertex. For $v \in V(G)$, we define the neighborhood of v in G as, $N(v) = \{u | uv \in E(G)\}$. The induced neighborhood of v, denoted $\langle N(v) \rangle$, is the induced subgraph of $N(v)$. When talking about an induced neighborhood we will simply refer to it as a neighborhood. Note that if G is a graph such that the neighborhood of every vertex is a divisor graph, then G does not have to be a divisor graph. Clearly C_5 is such a graph. We also can have a graph for which the neighborhood of no vertex is a divisor graph.

Theorem 4.3 *For each $n \geq 10$ there exists a graph G of order n with the property, for all $v \in V(G)$, $\langle N(v) \rangle$ is not a divisor graph.*
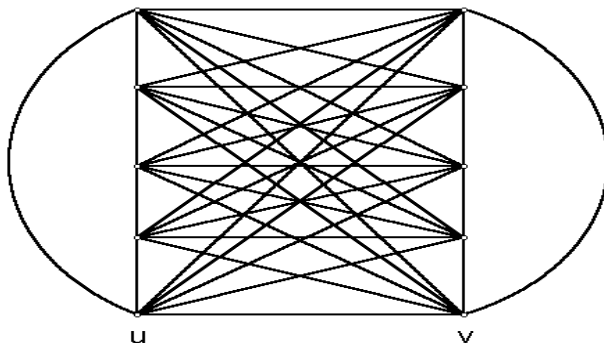
Figure 5: G is such that for each $v \in V$, $\langle N(v) \rangle$ is not a divisor graph.

Proof: We first show that this is true for a graph G of order 10. Let U and V be the partite sets of $K_{5,5}$, and G be the graph obtained by adding 5 edges in each U and V to form a copy of C_5 in both U and V. See Figure 5. We see that the neighborhood of each vertex of G contains the induced cycle C_5 so is not a divisor graph.

Let $n > 10$ and consider G as defined above. Add $n - 10$ vertices into the set U without interfering with the 5-cycle that has already been constructed, making each new vertex adjacent to each vertex in the set V, and call the graph G'. We once again see that the induced neighborhood of each of the n vertices in G' contains an induced 5-cycle and thus is not a divisor graph. ∎

Theorem 4.4 *Given $m, n \in \mathbb{N}$ with $m \geq 5$, there exists a non-divisor graph G that has m induced neighborhoods which are divisor graphs and n induced neighborhoods which are not divisor graphs.*

Proof: Let $m, n \in \mathbb{N}$ with $m \geq 5$. We will build a graph of order $m + n$ that has the property that m neighborhoods are divisor graphs and n neighborhoods are not divisor graphs. First consider $K_{m,n}$. Then select five vertices in the partite set of order m, and add 5 edges so that the subgraph induced by those five vertices is C_5. See Figure 6. Then every vertex in the partite set of order m has a finite collection of isolated vertices as a neighborhood, so we have m induced neighborhoods that are divisor graphs. Also, the neighborhood of every vertex in the partite set of order n contains the induced subgraph C_5 and so is not a divisor graph. Therefore we have

8

a non-divisor graph G that has m induced neighborhoods which are divisor graphs and n induced neighborhoods which are not divisor graphs. ∎
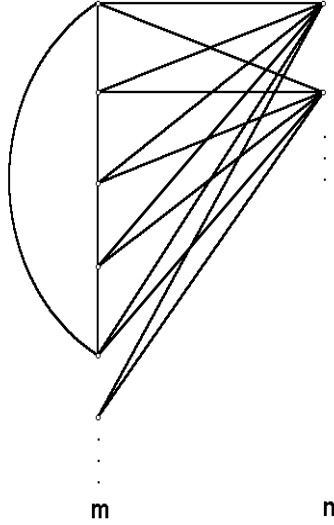

Figure 6: G has m induced neighborhoods which are divisor graphs and n induced neighborhoods which are not divisor graphs.

5 Open Questions

Early on in [2] it was shown that C_{2n+1} is not a divisor graph for each $n \in \mathbb{N}$ with $n \geq 2$. However not much more is known about non-divisor graphs, with the exception of several families based upon the cross product of graphs, Theorem 2.2. It would be nice to know more families of non-divisor graphs. Is there a characterization of non-divisor graphs not depending upon odd cycles? Similarly, is there a characterization of all graphs which are divisor graphs but their complements are not? Is there a characterization of all graphs which are not divisor graphs and their complements are not divisor graphs? For a given integer $n \geq 5$ what is the maximum integer m for which there is a non-divisor graph of order n and size m. These appear to be difficult problems.

I would like to thank Dr. Donald VanderJagt for all of his time, help, and suggestions.

References

[1] G. Chartrand, R. Muntean, V. Saenpholphat, and P. Zhang *Which Graphs Are Divisor Graphs?* Congr. Numer. 151(2001) 189-200.

[2] G. S. Singh and G. Santhosh, *Divisor Graphs - I.* Preprint.

[3] F. Harary *Graph Theory.* Addison-Wesley Publishing, 1969.

[4] G. Chartrand and O. R. Oellermann *Applied and Algorithmic Graph Theory.* McGraw-Hill publishing 1993.