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KNOTS IN FOUR DIMENSIONS AND THE FUNDAMENTAL GROUP

JEFF BOERSEMA AND ERICA J. TAYLOR

1. Introduction

The theory of knots is an exciting field of mathematics. Not only is it an area of active research, but it is a topic that generates considerable interest among recreational mathematicians (including undergraduates). Knots are something that the lay person can understand and think about. However, proving some of the most basic results about them can involve some very high-powered mathematics. Classical knot theory involves thinking about knots in 3-dimensional space, a space for which we have considerable intuition. In this paper, we wish to push your intuition as we consider knots in 4-dimensional space.

The purpose of this paper is twofold. The first is to provide an introduction to knotted 2-spheres in 4-dimensional space. We found that while many people have studied knots in four dimensions and have written about them, there is no suitable introduction available for the novice. This is addressed in Sections 2 - 5, which could stand alone. By way of introducing knots in 4 dimensions, we also provide a brief overview of knots in 3 dimensions (classical knots). We encourage a reader who wants a full introduction to classical knot theory to see [11, 2, 10, 16]. This part of the paper also discusses ways of viewing 4-dimensional space and a property of knots called *local flatness*. The focus of this paper is knots that satisfy this property.

In what follows we are concerned with an important invariant for knots called the *fundamental group*. This approach is highly motivated by a powerful result by Freedman in 1983 [4], which tells us that for locally flat embeddings of S^2 in S^4, the only knot that has the same fundamental group as the unknot is the unknot itself. So, while the fundamental group is still not a perfect invariant, it does allow us to distinguish between those locally flat embeddings that are knotted and those that are not.

With this in mind, we discuss a method for computing the fundamental group of a knot. One such method is an algorithm first presented by Fox in [3]; we call this The Fox Presentation and give a statement

and proof of this algorithm in Section 6.2. This algorithm, however, is unnecessarily tedious, and we present in Algorithm 6.5 a modified version, which is a slicker simpler tool to use in calculating fundamental groups.

These two heavy-hitting results, Freedman's Theorem and the Fox Presentation, appear together for the first time in this paper. Their combination forms a powerful tool in the analysis of knots in 4-dimensions. We will use them to study many examples. Specifically, we have an example of a locally flat embedding that is knotted and another embedding that is unknotted despite the fact that one of its 3-dimensional cross-sections is a knotted embedding of S^1. Both these examples have appeared before, but the proofs are made simpler by the new approach. Appearing for the first time in this paper is also an example of an embedding, not locally flat, that is knotted but has the same fundamental group as the unknot. This demonstrates that Freedman's theorem cannot be extended to embeddings that are not locally flat. We also apply this tool and Freedman's result in Section 7 to determine the minimum number of critical points required for an embedding to be knotted.

Before we proceed, we must give a few definitions.

Definition 1.1. *The n-dimensional sphere, S^n, is defined by $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.*

Intuitively, S^1 denotes a circle, and S^2 denotes a standard sphere, as in the surface of a ball in three dimensions. Note that S^2, with one point removed, can be flattened onto the plane $\mathbb{R}^2$. Thus, we can think of S^2 as the union of $\mathbb{R}^2$ and one other point, usually thought of as ∞. This is called the *one-point compactification* of $\mathbb{R}^2$. Similarly, S^3 can be thought of as the one-point compactification of $\mathbb{R}^3$. So, locally, we can think of S^3 as $\mathbb{R}^3$. Visualizing S^4, the one-point compactification of $\mathbb{R}^4$, is a bit more tricky; we will discuss how to do this in Section 2.

Definition 1.2. *A continuous function $f : X \longrightarrow Y$ is an embedding if and only if f is injective and open.*

An embedding defines a homeomorphism between X and its image $f(X)$ as a subspace of Y. In general, knot theory is the study of embeddings of a k-sphere into an n-sphere ($k < n$). So we can think of a knot as a copy of S^k living in (embedded in) S^n.
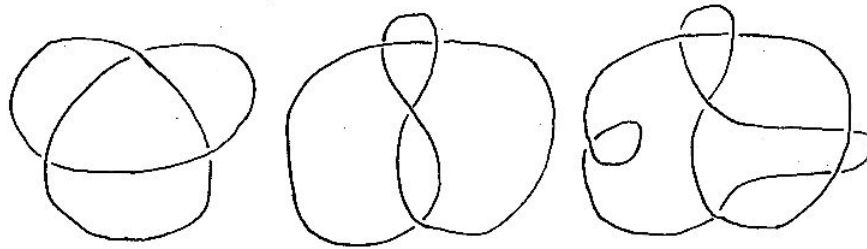
2. Classical Knots

By a classical knot we mean an embedding of S^1 into S^3. The trivial example of such an embedding is the unknot, which is shown in Figure 1.

FIGURE 1
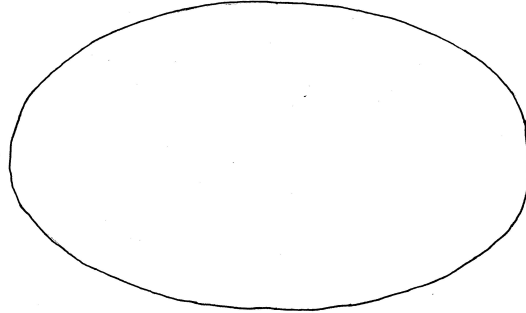
Of course, some embeddings such as those shown in Fibure 2 can be much more complex and interesting.

FIGURE 2

Notice that each of the last two examples shown in Figure 2 can be changed into the other just by shifting the image of the embedding around in space. When one knot can be deformed into the other in this way while leaving the string connected and without passing one arc through another, we say that the two knots are *equivalent*. We call such a deformation an *ambient isotopy*. The principal problem in knot theory is the problem of determining if two given embeddings are equivalent.

The usual strategy for dealing with this problem is to assign to each knot an algebraic object such as a polynomial or a group. These assignments should be made in such a way that if two knots are equivalent, then they are assigned the same object. We call such a mapping an *invariant*. Examples of invariants are the fundamental group of the

complement [16] and the Jones polynomial [10]. We will be talking about the former in Section 6.

These invariants are useful, but none are perfect. That is, it is possible for two inequivalent knots to be assigned the same object.

3. 4-DIMENSIONAL SPACE

The subject of this paper is knots in S^4. Since most of us cannot visualize four dimensions at once, the easiest way to think about something in S^4 is to think about cross-sections. If the fourth dimension is t, then for every time t, we can look at a 3-dimensional cross-section of our 4-dimensional object. This is analogous to representing a 3-dimensional object as a series of 2-dimensional cross-sections.

As an example, what would $S^3 = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 = 1\}$ look like? Let's see what the cross-sections C_t look like for a few values of t. If $|t| > 1$, $C_t = \emptyset$. For some values of t in the interval $[-1, 1]$ we have

$$
\begin{aligned}
C_{-1} &= \{(x, y, z) : x^2 + y^2 + z^2 + (-1)^2 = 1\} \\
&= \{(0, 0, 0)\} \\
C_{-\frac{1}{2}} &= \{(x, y, z) : x^2 + y^2 + z^2 + (-\frac{1}{2})^2 = 1\} \\
&= \{(x, y, z) : x^2 + y^2 + z^2 = \frac{3}{4}\} \\
C_0 &= \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \\
C_{\frac{1}{2}} &= \{(x, y, z) : x^2 + y^2 + z^2 = \frac{3}{4}\} \\
C_1 &= \{(0, 0, 0)\}
\end{aligned}
$$

So the series of cross-sections begins at $t = -1$ with a point at $(0, 0, 0, -1)$. This point becomes a 2-sphere that grows to radius 1 as t goes to 0. This sphere shrinks again to the point $(0, 0, 0, 1)$ as t moves from 0 to 1. We can visualize this by drawing a series of pictures shown in Figure 3. Each diagram there is a picture of the intersection of S^3 with the 3-dimensional space $\{(x, y, z, t) : x, y, z \in \mathbb{R}\}$ where t is fixed.

Now we can consider knots in S^4. Our first inclination is to consider embeddings of S^1. However, all piecewise linear embeddings of S^1 into S^4 are equivalent to the unknot. (A *piecewise linear* embedding is an embedding that is comprised of a finite number of straight line segments.) To see this, consider any such embedding. Because it is piecewise linear it can be projected into three dimensions, say $t = 0$,

FIGURE 3

$t=-1$ $t=-\frac{1}{2}$ $t=0$ $t=\frac{1}{2}$ $t=1$
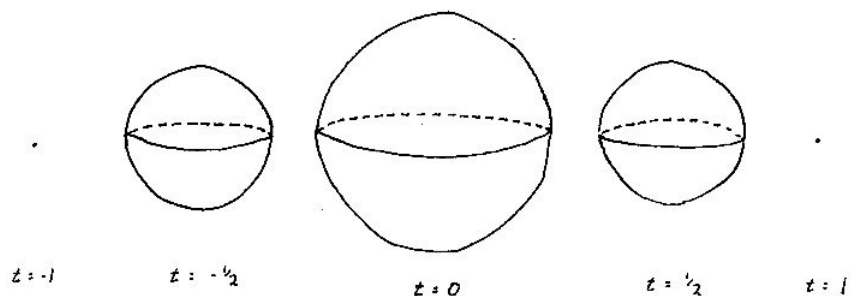
and again into a 2-dimensional plane except for at a finite number
of crossing points. We now have the type of projection discussed in
classical knot theory. Recall that a knot can be unknotted by changing
a finite number of crossings; that is, by changing an overcrossing to an
undercrossing or vice versa. We will describe how, in four dimensions,
a crossing can be changed without passing one arc through another.

Say we have an arc α that crosses under an arc σ. The idea is to
pull α into the fourth dimension and then let it pass by σ. It will not
intersect σ because the two arcs no longer lie in the same cross section
with respect to t. Once α has passed by σ it can be pushed back into
the projection $t = 0$. The result is that α now crosses over σ rather
than under.

Pictorially, the series of pictures in Figure 4 represents what happens
to α in this process. The dotted lines represent the $t = 1$ cross-section,
while the solid lines represent the $t = 0$ cross-section.

FIGURE 4
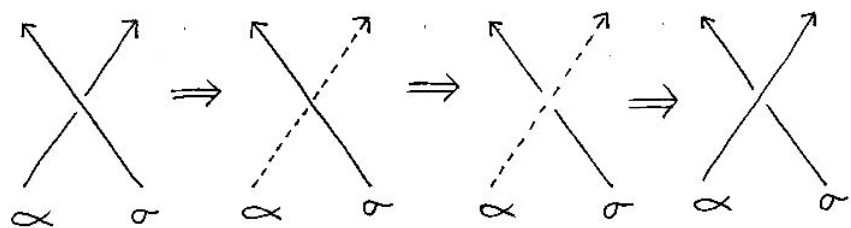
This procedure can be described a bit more rigorously. Assume our knot lies in the 3-space with t fixed at 0. Arrange it so that the crossing occurs at $x = 0$ and $y = 0$. The segment α crosses at the level $z = -\frac{1}{2}$ along the line $x = y$, and σ crosses above it at the level $z = 0$ along $x = -y$. So in a neighborhood of the crossing, α is the straight line from $(-\epsilon, -\epsilon, -\frac{1}{2}, 0)$ to $(\epsilon, \epsilon, -\frac{1}{2}, 0)$ and σ is the straight line from $(-\epsilon, \epsilon, 0, 0)$ to $(\epsilon, -\epsilon, 0, 0)$.

The α arc can be described by the function $f_0 : [0, 1] \longrightarrow S^4$.

$$f_0(t) = \begin{cases} (-\epsilon, -\epsilon, -\frac{1}{2}, 0) & 0 \leq t \leq \frac{3}{8} \\ (-\epsilon + 8\epsilon(t - \frac{3}{8}), -\epsilon + 8\epsilon(t - \frac{3}{8}), -\frac{1}{2}, 0) & \frac{3}{8} \leq t \leq \frac{5}{8} \\ (\epsilon, \epsilon, -\frac{1}{2}, 0) & \frac{5}{8} \leq t \leq 1 \end{cases}$$

We want to manipulate f_0 to get a function f_1 that describes an arc α' passing over σ.

$$f_1(t) = \begin{cases} (-\epsilon, -\epsilon, -\frac{1}{2}, 0) & 0 \leq t \leq \frac{1}{8} \\ (-\epsilon, -\epsilon, -\frac{1}{2} + 8(t - \frac{1}{8}), 0) & \frac{1}{8} \leq t \leq \frac{2}{8} \\ (-\epsilon, -\epsilon, \frac{1}{2}, 0) & \frac{2}{8} \leq t \leq \frac{3}{8} \\ (-\epsilon + 8\epsilon(t - \frac{3}{8}), -\epsilon + 8\epsilon(t - \frac{3}{8}), \frac{1}{2}, 0) & \frac{3}{8} \leq t \leq \frac{5}{8} \\ (\epsilon, \epsilon, \frac{1}{2}, 0) & \frac{5}{8} \leq t \leq \frac{6}{8} \\ (\epsilon, \epsilon, \frac{1}{2} - 8(t - \frac{6}{8}), 0) & \frac{6}{8} \leq t \leq \frac{7}{8} \\ (\epsilon, \epsilon, -\frac{1}{2}, 0) & \frac{7}{8} \leq t \leq 1 \end{cases}$$

When shifting f, it will pass through two intermediate stages (analogous to the two intermediate stages in Figure 4). These two stages are described by $f_{\frac{1}{3}}$ and $f_{\frac{2}{3}}$.

$$f_{\frac{1}{3}}(t) = \begin{cases} (-\epsilon, -\epsilon, -\frac{1}{2}, 8t) & 0 \leq t \leq \frac{1}{8} \\ (-\epsilon, -\epsilon, -\frac{1}{2}, 1) & \frac{1}{8} \leq t \leq \frac{3}{8} \\ (-\epsilon + 8\epsilon(t - \frac{3}{8}), -\epsilon + 8\epsilon(t - \frac{3}{8}), -\frac{1}{2}, 1) & \frac{3}{8} \leq t \leq \frac{5}{8} \\ (\epsilon, \epsilon, -\frac{1}{2}, 1) & \frac{5}{8} \leq t \leq \frac{7}{8} \\ (\epsilon, \epsilon, -\frac{1}{2}, 1 - 8(t - \frac{7}{8})) & \frac{7}{8} \leq t \leq 1 \end{cases}$$

$$f_{\frac{2}{3}}(t) = \begin{cases} (-\epsilon, -\epsilon, -\frac{1}{2}, 8t) & 0 \leq t \leq \frac{1}{8} \\ (-\epsilon, -\epsilon, -\frac{1}{2} + 8(t - \frac{1}{8}, 1)) & \frac{1}{8} \leq t \leq \frac{2}{8} \\ (-\epsilon, -\epsilon, \frac{1}{2}, 1) & \frac{2}{8} \leq t \leq \frac{3}{8} \\ (-\epsilon + 8\epsilon(t - \frac{3}{8}), -\epsilon + 8\epsilon(t - \frac{3}{8}), \frac{1}{2}, 1) & \frac{3}{8} \leq t \leq \frac{5}{8} \\ (\epsilon, \epsilon, \frac{1}{2}, 1) & \frac{5}{8} \leq t \leq \frac{6}{8} \\ (\epsilon, \epsilon, \frac{1}{2} - 8(t - \frac{6}{8}), 1) & \frac{6}{8} \leq t \leq \frac{7}{8} \\ (\epsilon, \epsilon, -\frac{1}{2}, 1 - 8(t - \frac{7}{8})) & \frac{7}{8} \leq t \leq 1 \end{cases}$$

By linearly interpolating between these four functions, we can produce a function $f_s(t)$, continuous in both variables, taking the arc α to

the arc α' without intersecting σ. This function is tedious to write out but can be produced easily by the reader.

We can repeatedly apply this procedure to as many crossings as necessary in order to render the knot untied. Thus we have the desired function showing us how to manipulate a piecewise linear embedding of S^1 in S^4.

4. EMBEDDINGS OF 2-SPHERES

We have seen that embeddings of S^1 in S^4 can always be untied. The knots that we consider in S^4 are embeddings of 2-spheres. We will view these knots by taking cross-sections for different values of t, as discussed above. We will take these in such a way that, for each value of t, the hyperplane will intersect the 2-sphere in one or more closed curves.

First consider the unknot, represented by a 2-sphere with radius 2, $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 4\}$. If we view this as a series of 3-dimensional slices, then for each value of t, we will see either nothing, a single point, or an unknotted circle lying in 3-dimensional space, as shown in Figure 5. By mentally interpolating between the cross-sections, we can visualize the whole sphere (so to speak).

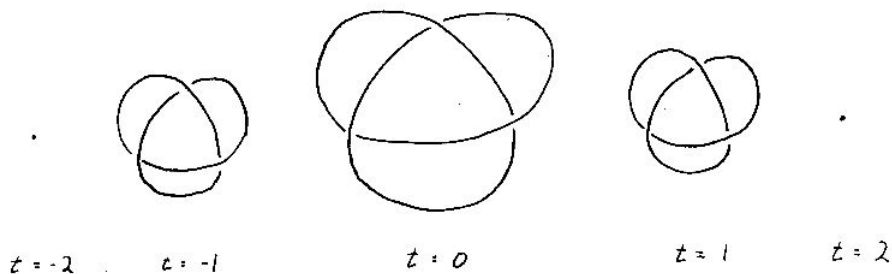In this case, since we know what the 2-sphere looks like, we can understand how these circles are actually connected.

FIGURE 5

$t : -2$ $t : -1$ $t : 0$ $t : 1$ $t : 2$
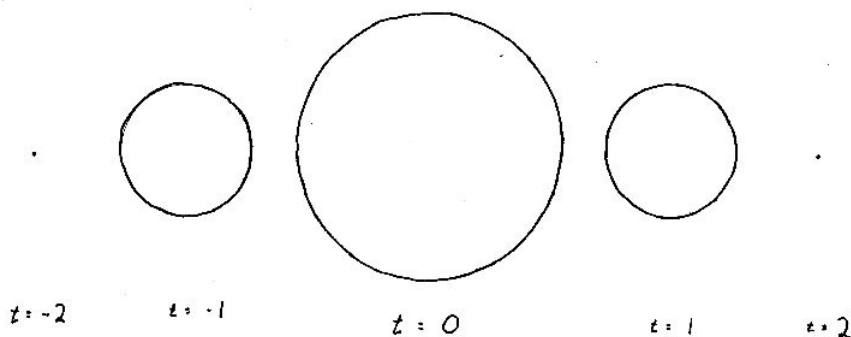
So how might cross-sections of a 2-sphere look if it were knotted? One example is in Figure 6. We see a point for $|t| = 2$, a small trefoil shown for $|t| = 1$, and a larger trefoil for $t = 0$.

FIGURE 6

$t = -2$ $t = -1$ $t = 0$ $t = 1$ $t = 2$

This particular knot is called the *suspension* of the trefoil. Another way to describe this knot is to consider the trefoil in a 3-dimensional plane, and connect each of its points with line segments to a point above and a point below in the 4th dimension, as in Figure 7. (In 4-dimensional space, these lines will not intersect.) In general, there is a suspension of every classical knot.

In one sense, the suspension of a knot is particularly nice. Other than at the endpoints, each cross-section is the same type of classical knot: they differ only in size. Not every knot will have this property. For example, Figure 8 shows another example of a knotted 2-sphere.

Notice that in Figure 6, the knot has a minimum at $t = -2$, has one component for $|t| < 2$, and has a maximum at $t = 2$. In Figure 8, the knot has two minima at $t = -4$, which become two components. These combine into one component at $t = -1$ through what we call a *saddle point transformation*, and split through another saddle point at $t = 1$. This knot has two maxima at $t = 4$.

The maxima, minima, and saddle points are called *critical points*. At a maximum a new component emerges and at a minimum a component disappears. A saddle point transformation will affect the cross-sections in a knot diagram by increasing or reducing the number of components. Figure 9 demonstrates the coming together of the arcs at a saddle point.

How do we know that our diagrams represent embeddings of spheres? We do this by analyzing components in the sequence of cross-sections as above. We then compare these schemes to cross-sections of a 2-sphere in 3-space. Figure 6 compares to a standard 2-sphere, while Figure 8 compares to the distorted sphere shown in Figure 9. To be sure that a series of diagrams actually represents an embedding of S^2 in S^4, we should be able to compare the number of components and
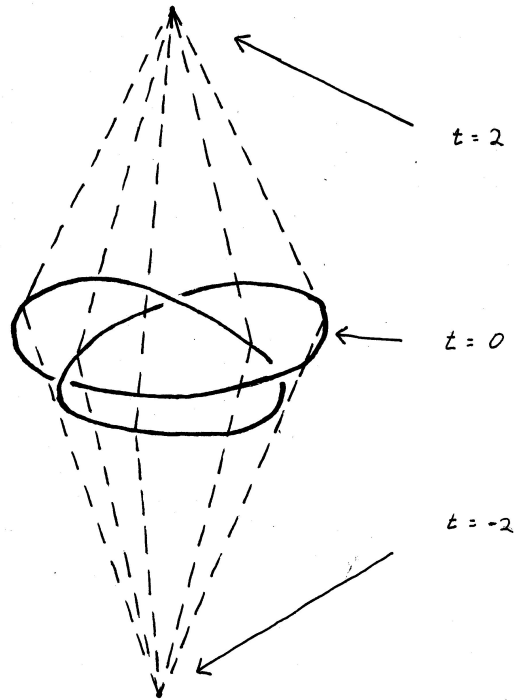
FIGURE 7

$t = 2$

$t = 0$

$t = -2$

FIGURE 8

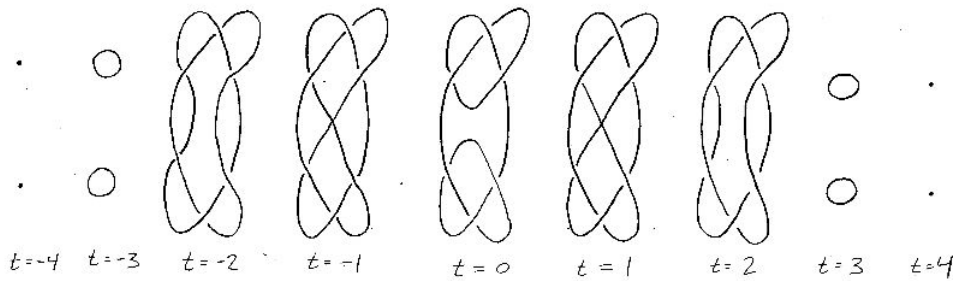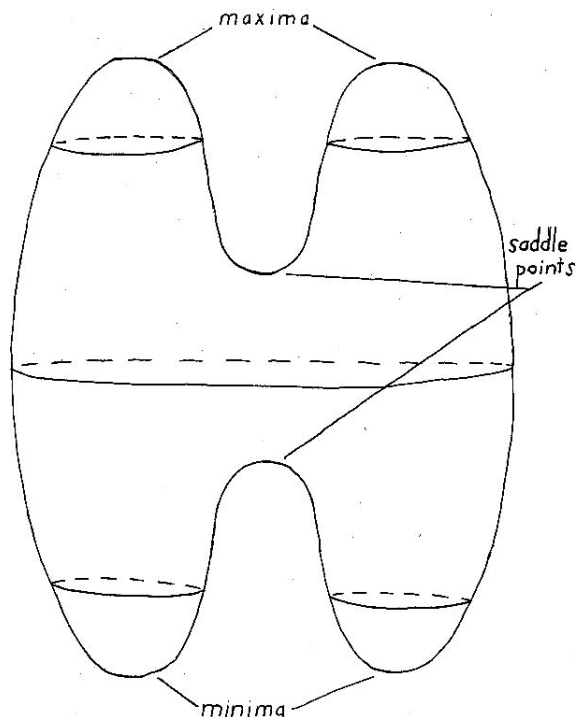$t = -4$ $t = -3$ $t = -2$ $t = -1$ $t = 0$ $t = 1$ $t = 2$ $t = 3$ $t = 4$

location of saddle points with cross-sections of a distorted 2-sphere in S^3. Think of this distorted 2-sphere as the domain of the embedding. It can show us how many components we will see in a corresponding slice of 3-space and how they will connect, but not how they will be arranged or knotted within the slice.

FIGURE 9

A knotted 2-sphere is *piecewise linear* (PL) if it can be composed of a finite number of triangles. We restrict ourselves to PL knots, to be assured that we need only a finite number of cross-sections to understand what is happening in the knot. You may object that our pictures of knots appear smooth and nonlinear. However, what is of concern is equivalence classes of knots. Every knot that will appear in this paper is equivalent via ambient isotopy to a piecewise linear knot.

5. Locally Flat Embeddings

In dealing with classical knots, restricting ourselves to PL embeddings is helpful in that it prevents local knotting. Each point of a PL classical knot has a neighborhood that looks like a straight line segment, or a line segment with one bend. Thus, the PL classical knot can be globally knotted, but cannot be locally knotted.

PL is not sufficient for this assurance in 4-space. We can have a 2-sphere made out of a finite number of triangles and still have knotting within any neighborhood of a point. Such points are called *non-locally flat points*. For example, the points on the top and bottom of

FIGURE 10
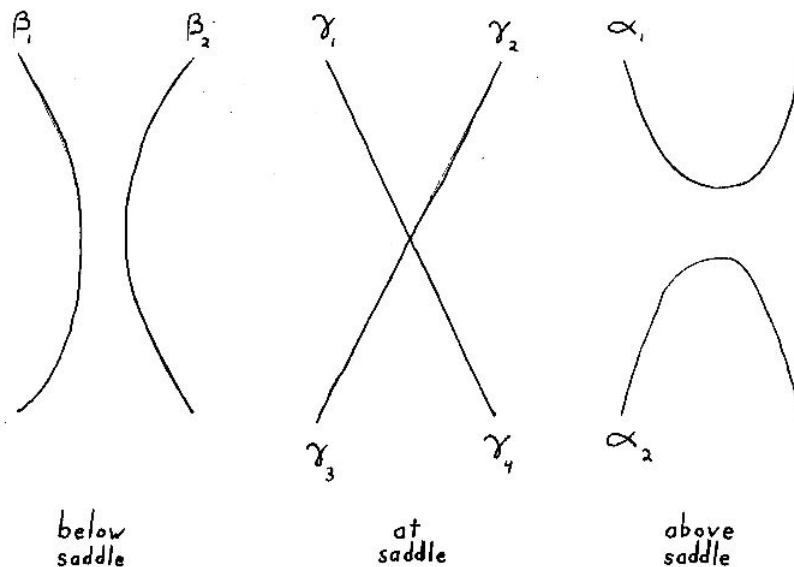
the trefoil suspension are non-locally flat points: any sufficiently close cross-section is a trefoil knot. All other points in the suspension are locally flat.

We call on Rolfson for a technical definition of locally flat ([16]).

Definition 5.1. *A knot Σ in S^4 is locally flat at x if there is a closed neighborhood N of x in S^4 such that $(N, \Sigma \cap N)$ is homeomorphic, as a pair, with the standard ball pair (B^4, B^2).*

By B^n, we mean the *n-dimensional ball* $\{x \in \mathbb{R}^n : |x| \leq 1\}$. *Homeomorphic as a pair* means that there is a homeomorphism that maps N to B^4 and also maps $\Sigma \cap N$ to B^2. This means that the portion of the sphere immediately close to the point can be flattened out, to appear as a disk in B^4.

If the knot Σ is locally flat at x for every $x \in \Sigma$, we say that Σ is *locally flat*. For example, the knot pictured in Figure 8 is a locally flat knot. We would like to be able to tell if a knot is locally flat by examining the cross-sections. It has been shown in [7] that every PL locally flat 2-sphere is ambient isotopic to a sphere in *normal form*. This means that it can be arranged so that the only critical points are the maxima, minima, and saddle points; and that when each component first appears, it is unknotted and unlinked. The only other changes

that can occur between diagrams are Reidemeister moves. For example, notice that in Figure 8, the two components are unlinked until after the saddle point transformation at $|t| = 1$. Also, each additional component must join with another through a saddle point before it disappears. Otherwise, it is not connected to the whole, and represents part of a separate sphere.

So far, we have given no good explanation as to why these embeddings of 2-spheres in 4-space are actually knots. We will say that an embedding of a 2-sphere Σ is knotted, or *non-trivial* if there is no ambient isotopy of 4-space that brings Σ to the unknot, or standard 2-sphere. This notion is analogous to knotting for classical knots in S^3.

There are two ways that an embedding can be knotted: locally and globally. If an embedding of a 2-sphere has a non-locally flat point, then it is knotted. Since the region near the non-locally flat point cannot be smoothed out, the entire sphere cannot be deformed into a trivial sphere. These 2-spheres are locally knotted.

We have not yet shown how to determine if an embedding of a locally flat sphere is globally knotted. Our instinct might suggest that a locally flat knot is non-trivial if one of its cross-sections is non-trivial. However, this is not the case – we will see a counter example in Section 6. To discuss global knotting, we will employ the fundamental group.

5.1. **Combining Non-Locally Flat Points.** In Section 6, we will discuss an algorithm to compute the fundamental group associated with a knot. This algorithm will only directly apply to knots that have at most one non-locally flat point. When we refer to the fundamental group of a knot, we are actually dealing with a property of the complement of the knot. It turns out that, if we have a knot with more than one non-locally flat point, we can alter this knot in order to identify all such points without changing the complement. That is, we describe a new knot, with only one non-locally flat point, but with a complement homeomorphic to the original.

Suppose we have a piecewise linear knot that has more than one non-locally flat point. First, we should notice that because it is piecewise linear, it can only have a finite number of such points. The points in the interior of the triangles, and on the interior of the line segments of their boundaries, are all locally flat. The only points that could be non-locally flat are the vertices: a vertex is a non-locally flat point if the triangles that emanate from it form a knot along their boundary, as in a suspension.

Suppose we have found two such points. We need to find an arc that joins these points and find a function that will shrink the arc to a point,

while being a homeomorphism of the complement of the arc. We can use the coordinates defined in Figure 11 for each straight segment in this arc, with end points at $(0,0)$ and $(0,1)$. We will first define the function $g : [0,2] \longrightarrow [0,2]$ which shrinks this segment to a point.

$$g(y) = \begin{cases} 0 & 0 \leq y \leq 1 \\ 2y - 2 & 1 \leq y \leq 2 \end{cases}$$

Now, we need to define a function on S^4 that will shrink the segment and be a homeomorphism of its complement. In the plane ($z = 0$, $t = 0$), define the function f to be the identity mapping outside the square $(-1 < x < 1, 0 < y < 2)$, while inside, define it as

$$f(x,y) = (x, (1 - |x|)g(y) + |x|y)$$

The reader can check that this function is continuous, onto, and injective except for the segment; thus, it is a homeomorphism on the complement of the segment. This shrinking can be extended to four dimensions by a similar formula. We can repeat this for all segments in the arc, so that the two non-locally flat points are identified. We can then repeat this process until we have connected each of the the non-locally flat points to our first one, so that we are left with only one such point.

FIGURE 11
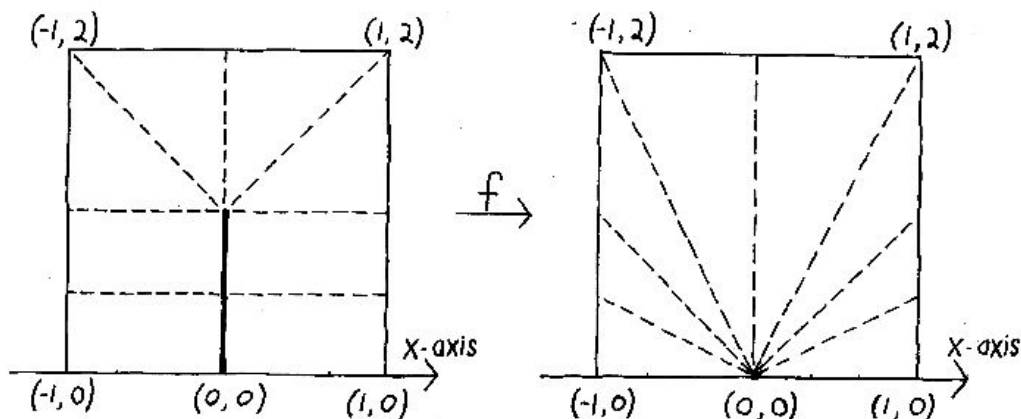
We now have a new knot with only one non-locally flat point, but with a complement homeomorphic to our original complement. If we take a cross-section very close to this point, the cross-section will be a

knotted embedding of S^1 called a *slice knot*. It turns out that not all classical knot can appear as a slice knot.

If we are first considering an arbitrary knot with more than one non-locally flat point, we will not be able to predict what the new cross-sections will be once we have combined these points. But because we know that we can combine these points without changing the complement, the fundamental group can't tell the difference. Hence we know that considering all knots with one non-locally flat point suffices for considering all non-locally flat knots — at least this is true when our mode of study is the fundamental group.

6. The Fundamental Group

The fundamental group is an algebraic invariant assigned to a topological space. For a good introduction to fundamental groups the reader is referred to [15]. Briefly, given a space X and a point x (called the base point) in that space we define a loop to be a continuous function $f : [0,1] \longrightarrow X$ such that $f(0) = f(1) = x$. Two loops f_1 and f_2 are equivalent if f_1 can be continuously deformed into f_2 keeping the endpoints fixed. The set of equivalence classes of loops are the elements in the group.

The operation of the group is juxtaposition, thought of as the process of forming a loop by tracing the path of one loop followed by another. Juxtaposition of equivalence classes, it turns out, is well defined in terms of juxtaposition of representative loops. These equivalence classes form a group under the operation juxtaposition. If X is a path connected space, then the group does not depend on the choice of our base point x. So the fundamental group of the space X is unique up to isomorphism. We denote the fundamental group of X by $\pi(X)$.

Because the fundamental groups of two homeomorphic spaces are isomorphic, this group provides us with certain information about the topological properties of a space. It doesn't provide us complete information, however: it is possible for two non-homeomorphic spaces to have the same (isomorphic) fundamental group.

In knot theory, we like to look at the *complement* of a knot. If there is an ambient isotopy between two knots, then there is a homeomorphism between their respective complements. As we have seen, the converse is not true. We have seen in Section 5 that a knot with several non-locally flat points and a knot with only one non-locally flat point may have homeomorphic complements. Thus, the complement is not a complete invariant for a knot in 4 dimensions.

It turns out that if we restrict ourselves to the class of locally flat knots, then the complement is a complete invariant. This is true for both knots in 3-dimensions and knots in 4-dimensions. Note that in 3 dimensions, the condition of being locally flat is the same as the condition of being piecewise linear.

Then the fundamental group of a space is an invariant for a space and the complement of a knot is an invariant for the knot; therefore the fundamental group of the complement of a knot is an invariant for a knot. We will use this to distinguish between two given knots: if the fundamental groups of the complements of the two knots are non-isomorphic, then the knots are not equivalent.

Although the converse is not true in general, we have the following partial result for the case $\pi = \mathbb{Z}$.

Theorem 6.1. *For $n = 3$ or 4, a locally flat embedding of S^{n-2} into S^n is unknotted iff $\pi(S^n - \Sigma) \cong \mathbb{Z}$.*

This result was completed by Freedman in 1983 [4] for $n = 4$, the case we're interested in. (For the $n = 3$ case, see [2, 16].) Because of this result, we can determine whether or not a locally flat embedding is knotted precisely by determining whether or not the fundamental group of the complement is isomorphic to $\mathbb{Z}$.

There is another important aspect of the fundamental group of a space that involves a few more definitions:

Definition 6.1. *Let A be a subset of a topological space X. A continuous function $r : X \longrightarrow A$ is a retraction if $r(a) = a$ for all $a \in A$.*

Definition 6.2. *A retraction $r : X \longrightarrow A$ is a strong deformation retraction if it is homotopic to the identity map. That is, there exists a continuous function $f : X \times [0,1] \longrightarrow X$ such that for all $x \in X$, $f(x,0) = x$ and $f(x,1) = r(x)$ and for all $a \in A$ and all $t \in [0,1]$, $f(a,t) = a$.*

We have the following theorem concerning the effect a strong deformation retraction has on the fundamental group. A complete discussion (including proof) can be found in [13, 17].

Theorem 6.2. *If $r : X \longrightarrow A$ is a strong deformation retraction, then $\pi(X)$ and $\pi(A)$ are isomorphic.*

This theorem is very useful for finding the fundamental group of a space. When calculating the fundamental group of a space X, it is often convenient to find a strong deformation retraction of X to a subspace A, and then calculate the fundamental group of A. We will use this

method when we calculate the Fox presentation of the fundamental group of the complement of a knotted two-sphere in S^4.

6.1. **The Wirtinger Presentation.** We'll now return to classical knots in S^3 for a moment to see how we calculate the fundamental groups in this case. The standard method produces a presentation (called the Wirtinger presentation) of the group. Recall that a *group presentation* is a way of representing a finitely generated group by listing the generators and certain relationships (called *relations*) between the generators. The reader is referred to [16, 17] for a complete introduction to the Wirtinger presentation and to an undergraduate abstract algebra text for an introduction to group presentations in general.

Suppose that Σ is a classical knot given by a projection showing n crossings. There is an easy algorithm for reading the generators and relations from this projection. The n crossings separate Σ into n arcs $\alpha_1, \ldots, \alpha_n$, where each arc begins and ends at an undercrossing. Give the knot an orientation – the particular orientation does not affect the group. Pick a base point x in $S^3 - \Sigma$ well above the plane of the projection. (Think of it as being identified with your eye as your look at the presentation.) Then associated with each arc, α_k, is a loop a_k in the complement that goes around that arc in the right-hand direction (see Figure 12). The fundamental group of $S^4 - \Sigma$ is generated by $(a_1, \ldots, a_n)$.

FIGURE 12
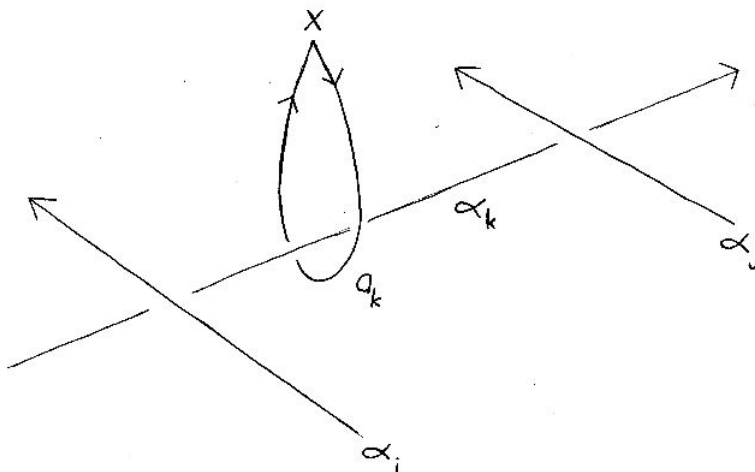
The relations are acquired by looking more closely at each cross-ing. A loop l that passes around the crossing point below the arcs is homotopic to the trivial element. But l can also be described as a juxtaposition of the four loops around the four arcs that l passes under at the crossing point. As seen in Figure 13, the loop l represents the element $a_i a_j^{-1} a_i^{-1} a_{j+1}$. This gives us the relation $a_i a_j^{-1} a_i^{-1} a_{j+1} = 1$. We can repeat this for each crossing point to get n relations $r_1, \ldots, r_n$. It turns out that one of the relations is redundant. The group $\pi(S^4 - \Sigma)$ has the Wirtinger presentation $(a_1, \ldots, a_n : r_1, \ldots, r_{n-1})$. A proof that this group presentation is really the group of the complement of Σ can be found in [16, 2, 17].

FIGURE 13
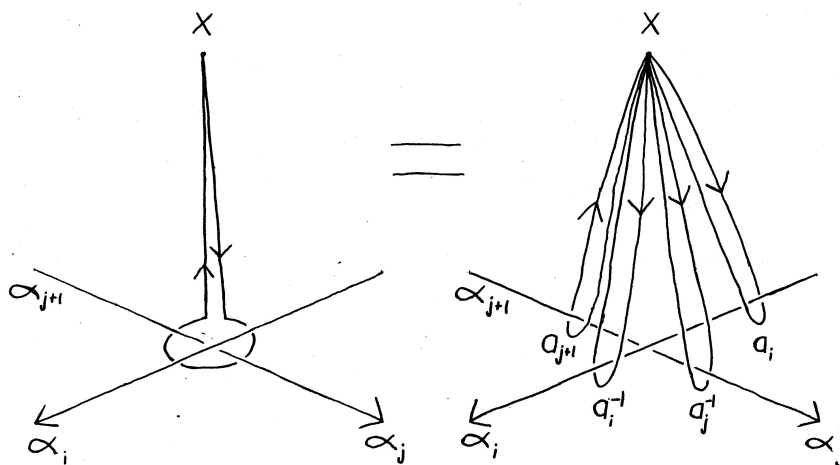
Example 1: The square knot in Figure 14 has six crossings; so we label the six arcs defined by the crossings: a, b, c, d, e, f. By an abuse of notation, we will also use these variables to refer to the respec-tive generators of the fundamental group. Then the relations we get from the six crossings are $ac^{-1}a^{-1}b = 1$, $acb^{-1}c^{-1} = 1$, $df^{-1}e^{-1}f = 1$, $f^{-1}aea^{-1} = 1$, $efe^{-1}a^{-1} = 1$, and $cbd^{-1}b^{-1} = 1$. (After this, we will denote the relation $r = 1$ simply by r.) We may drop the last relation, because it is a consequence of the others. Then using the first, third, and fourth relations we can express the generators c, e, and d in terms of the generators a, b, and f as follows: $c = a^{-1}ba$, $d = f^{-1}a^{-1}faf$, and $e = a^{-1}fa$. Thus, when we write out the remaining two relations

in terms of our remaining three generators, our presentation can be written as $(a, b, f : aba = bab, afa = faf)$. Although not obvious, it turns out that this group is not the same group as the fundamental group of the unknot.

It is often difficult to understand groups when they are given by a presentation such as this. In fact, two groups given by completely different presentations may turn out to be isomorphic! One can show that the group $(a, b, f : aba = bab, afa = faf)$ is not isomorphic to $\mathbb{Z}$ by showing it is not commutative. We won't carry out the details in this example, but we will do so in Example 2 in Section 6.2.

FIGURE 14
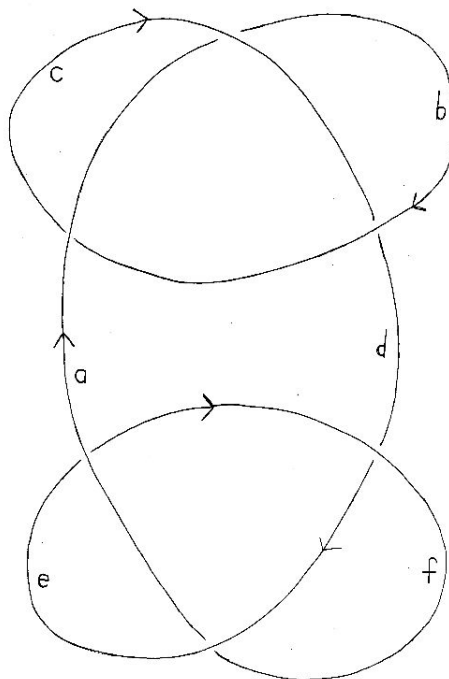
One note must be made concerning the fundamental group of the complement of a knot. Assume that k_1 and k_2 are two distinct projections of equivalent knots. Since the two projections are different, their Wirtinger presentations, W_1 and W_2, are different. On the other hand, since the knots are equivalent, the groups represented by these Wirtinger presentations are isomorphic. This means that any element in one group can be represented by a word in the other group. In particular if α is an arc in k_2, any loop around α in the complement can be written as an element in W_1. This also holds true for relations. If

r_2 is a relation in W_2, we can translate that relation into a relation r_1 in terms of the generators of W_1; then r_1 can be derived in W_1 based on the original relations of W_1.

6.2. **The Fox Presentation.** Now, we will turn to the problem of the calculation of the fundamental group of the complement of a knotted 2-sphere Σ in S^4. We will be elaborating on the procedure described in Fox and Kinoshita [3, 9].

We will describe two algorithms for calculating the fundamental group of the complement of a knot. The merit of the first is that it is a conceptual description of how the fundamental group of the knot is gleaned from the groups of its cross-sections. The second, however, is easier and quicker for practical use. Think of Algorithm 6.3 as a lemma to Algorithm 6.5.

Suppose we are given a standard description of a knot Σ by cross-sections as described in Section 4. It is arranged such that its critical points occur at $t = s_1, s_2, \ldots, s_n$ with $s_1 < s_2 < \cdots < s_n$. We will assume that Σ has at most one non-locally flat point. If it has one, we will assume that it is at s_n. We have seen in Section 5.1 that there is no loss in generality in assuming that Σ has no more than one non-locally flat point.

Now, for all $i \in \{1, 2, \ldots, n-1\}$, let C_{s_i} be the cross-section of S^4 containing the critical point s_i and let R_i denote the region $\{(x, y, z, t) : s_i < t < s_{i+1}\} - \Sigma$. Let $R_0 = \{(x, y, z, t) : t < s_1\}$ and $R_n = \{(x, y, z, t) : t > s_n\}$. To visualize this setup, see Figure 15. To simplify the discussion, let X' denote $X - \Sigma$ for any region X.

There is a strong deformation retraction of R_i' to one of its cross-sections, denoted C_i', so the fundamental group of R_i' is isomorphic to the fundamental group of C_i'. So, by Theorem 6.2 we can calculate the Wirtinger presentation W_i of $\pi(C_i')$ to get a presentation of the fundamental group of R_i'. (Note that W_0 and W_n represent the trivial group.)

We now have a finite set of spaces R_i', each of which has fundamental group W_i. The fundamental group of the space $S^4 - \Sigma$ is derived from these groups. We will describe how to derive the fundamental group of the union of two adjacent regions and the cross-section between them. This process can be repeated as many times as necessary to get the fundamental group of the entire space $S^4 - \Sigma$.

Given two adjacent regions R_{i-1}' and R_i', look at the cross-section between them, C_{s_i}'. This cross-section contains, in addition to ordinary arcs with possible crossing points, a critical point that we will assume is at $(0, 0, 0, s_i)$. In the cross-section this critical point will appear as

FIGURE 15
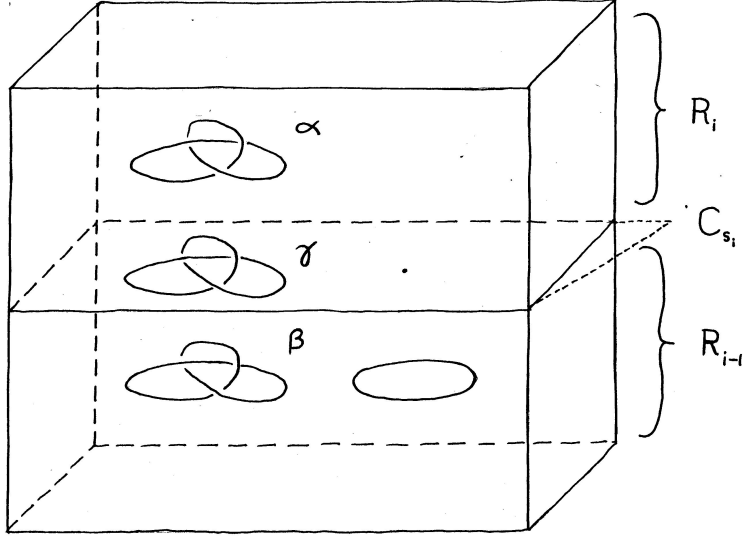
either an isolated point (if the critical point is a relative extremum) or as an intersection of arcs (if it is a saddle point).

Now, just as in the Wirtinger presentation, consider the set of arcs of $C_{s_i} \cap \Sigma$. These arcs will have endpoints not only at the undercrossings as before, but also at the intersection in the case of a saddle point. Denote these $\gamma_1, \ldots, \gamma_\nu$. For each arc γ_j there is a corresponding arc α_j in $R_{i-1} \cap \Sigma$ immediately below $C_{s_i} \cap \Sigma$; and a corresponding arc β_j in $R_i \cap \Sigma$ immediately above. Let a_j be the element of W_{i-1} corresponding to a loop around α_j in R'_{i-1} and similarly let b_j be the element of W_i corresponding to a loop around β_j in R'_i.

Algorithm 6.3. *With the notation as above, the Fox Presentation of the fundamental group of $R'_{i-1} \cup C'_{s_i} \cup R'_i$ is given by all the generators and relations of W_{i-1} and W_i, together with the relations $a_1 = b_1, a_2 = b_2, \ldots, a_\nu = b_\nu$.*

Note that $R'_{i-1} \cup C'_{s_i} \cup R'_i$ is the same as $(R_{i-1} \cup C_{s_i} \cup R_i) - \Sigma$.

In order to prove that this algorithm works we need the following theorem concerning the fundamental group of the union of two spaces. A discussion of this theorem, including a proof, can be found in [13, 17].

Theorem 6.4 (Seifert-VanKampen)**.** *Let U and V be path-connected open spaces such that $U \cap V$ is non-empty and path-connected, and $\pi(U \cap V)$ is finitely generated. Let $(g_1, \ldots, g_m : r_1, \ldots, r_n)$ and $(h_1, \ldots, h_p :$*

$s_1, \ldots, s_q)$ *be presentations of the fundamental groups of U and V*
respectively. Then the fundamental group of $U \cup V$ is presented by
$(g_1, \ldots, g_m, h_1, \ldots, h_p : r_1, \ldots, r_n, s_1, \ldots, s_q, u_1 = v_1, \ldots, u_t = v_t)$. *The*
u_i*'s above are expressions for the generators of $U \cap V$ in terms of $\pi(U)$;*
similarly, the v_i's are expression for the same generators of $U \cap V$ in
terms of $\pi(V)$.

This theorem tells us how to get a presentation of the fundamental
group of the union of two spaces given the presentations for the funda-
mental groups of the original two spaces. To paraphrase the theorem,
the group presentation for $\pi(U \cup V)$ contains all the generators and
relations of $\pi(U)$ and $\pi(V)$ plus some new relations that relate the
generators of $\pi(U)$ to the generators of $\pi(V)$. Using this theorem, we
are now ready to prove Algorithm 6.3.

Proof. [Algorithm 6.3] Let $W_{i-1} = (c_1, \ldots, c_l : u_1, \ldots, u_p)$ and $W_i =$
$(d_1, \ldots, d_m : v_1, \ldots, v_q)$. The sets R'_{i-1} and R'_i are open so we would like
to apply the VanKampen theorem to these two spaces. Unfortunately
they are disjoint so we can't do this. The strategy, then, will be to
create an intermediate space U that intersects both spaces. We will
calculate $\pi(U')$ and use the Van Kampen theorem twice: once to join
R'_{i-1} and U', and again to join that union to R'_i.

Let V be a closed epsilon ball about $(0, 0, 0, s_i)$, small enough that
$V \cap \Sigma$ is homeomorphic to a two dimensional disk. (It is a prop-
erty of any PL sphere that this can be done.) Then there is a strong
deformation retraction of V radially to its boundary. This may be ex-
tended to a strong deformation retraction of $(R_{i-1} \cup C_{s_i} \cup R_i) - \Sigma$ to
$(R_{i-1} \cup C_{s_i} \cup R_i) - \Sigma - V$. Because a strong deformation retraction
induces an isomorphism of the fundamental group, we may calculate
the fundamental group of $(R_{i-1} \cup C_{s_i} \cup R_i) - \Sigma - V$ to get the group
of $(R_{i-1} \cup C_{s_i} \cup R_i) - \Sigma$.

Let $U = \{(x, y, z, t) : s_i - \delta < t < s_i + \delta\}$. Choose δ small enough
so that the only single critical point of U is $(0, 0, 0, s_i)$. This insures
that there are no critical points in $U - V$. Additionally, choose δ
smaller than the radius of V to make sure all cross-sections of $U' - V$
are homeomorphic to each other. The fundamental group of $U' - V$
can then be calculated from the single cross-section $C'_{s_i} - V$. The
presentation of $\pi(U' - V)$ is given by $(e_1, \ldots, e_n : w_1, \ldots, w_r)$, where e_j
represents a loop around the arc γ_j and the relations are drawn from
the crossings of C'_{s_i}.

Now, to find $\pi(R'_{i-1} \cup U' - V)$ we must consider the intersection of
$U' - V$ and $R'_{i-1} - V$. In order to apply Van Kampen's theorem we

must find the generators of the group of this intersection. Take a representative cross-section C_τ of $R'_{i-1} \cap U'$ close enough to C_{s_i} that the Wirtinger presentation of $\pi(C'_\tau)$ is the same as the Wirtinger presentation of $\pi(C'_{s_i})$: $(e_1, \ldots, e_n : w_1, \ldots, w_r)$. For each e_j, there is a corresponding element a_j of W_{i-1} (expressed as a word in the c_j's). Using the Van Kampen theorem, $\pi(U' \cup R'_{i-1} - V) = (c_1, \ldots, c_l, e_1, \ldots, e_n : u_1, \ldots, u_p, w_1, \ldots, w_r, e_1 = a_1, \ldots, e_n = a_n)$.

We repeat this process to adjoin $R'_i - V$ to $U' \cup R'_{i-1} - V$. For each e_j, there is a corresponding element b_j of W_i (expressed as a word in the d_j's). Thus $\pi(R'_i \cup U' \cup R'_{i-1} - V) = (c_1, \ldots, c_l, d_1, \ldots d_m, e_1, \ldots, e_n : u_1, \ldots, u_p, v_1, \ldots v_q, w_1, \ldots, w_r, e_1 = a_1, \ldots, e_n = a_n, e_1 = b_1, \ldots e_n = b_n)$.

But notice that we can algebraically eliminated the e_i's since those generators can be expressed in terms of the a_i's or the b_i's. At the same time, each relation w_i can be obtained from the relations of the u_i's or the v_i's. Hence the fundamental group presentation can be reduced to $\pi(R'_i \cup U' \cup R'_{i-1} - V) = (c_1, \ldots, c_l, d_1, \ldots d_m : u_1, \ldots, u_p, v_1, \ldots v_q, a_1 = b_1, \ldots a_n = b_n)$. $\qquad\square$

Algorithm 6.3 is a good description of how the fundamental groups of the regions R'_i together form the the fundamental group of $S^4 - \Sigma$. However, in practice it is not a very efficient method. In general, group presentations are clumsy to work with. The method of Algorithm 6.3 involves taking two presentations together and then adding a number of relations on top of that. This could provide for some very large presentations, especially if Σ has many critical points. Fortunately, these presentations can be greatly simplified. We will develop an algorithm that incorporates the simplification directly in the calculation of the group. It will be based on Algorithm 6.3 but is much easier to use.

Assume we have a description of a knot Σ by cross-sections as before. Each region R_i has a fundamental group that is represented by a Wirtinger presentation W_i. The top region (that is, the region with the greatest value of t) R_n doesn't intersect Σ at all, so $\pi(R_n)$ is trivial. As we move down from R_n we first come across the critical point that is necessarily some sort of maximum and possibly a non-locally flat point. Because it could be a non-locally flat point, R_{n-1} could be a non-trivial knot with Wirtinger presentation W_{n-1}. Now by a rather trivial application of Algorithm 6.3, W_{n-1} is also a presentation for the fundamental group of the complement of $R_n \cup C_{s_n} \cup R_{n-1}$. This will be our base group.

From here we continue down the knot, and as we pass each critical point, s_i, we will add $C'_{s_i} \cup R'_{i-1}$ to our space. Each time we add a

region to our space, the fundamental group will change. It turns out that each change is rather small and manageable. We can do this until we have added the last space R_1'. In this way we have calculated the group of the entire space $S^4 - \Sigma$.

Now let's use Algorithm 6.3 to see exactly how the fundamental group changes when we add a region. The rest of the critical points will be locally flat and will be either relative maxima, relative minima, or saddle points. Assume that we have a critical point $P = (0, 0, 0, s_i)$ and that W is a presentation for the fundamental group of $\{(x, y, z, t) : t > s_i\} - \Sigma$. Then W_{i-1} is a presentation for the fundamental group of R_{i-1}. The algorithm's process of identifying arcs across the cross-section means that outside a neighborhood of P, the two groups simplify into one, since each generator and each relation of W_{i-1} can be reduced to and expressed in W. Then we only need to worry about what happens near P.

First we will consider the case that P is a relative maximum. Then W_{i-1} has a generator corresponding to the simple closed circle that was born at P. This generator will not be identified with anything in W. Hence, when we take the union of the two spaces the net result for the group will be an added generator.

In the case of a minimum, the group W_{i-1} has one less generator than W has. Then taking the union of the two spaces will have no effect on W.

Finally, consider the case when P is a saddle point. Refer to Figure 10 for this case. In the cross-section C_{s_i} there are four arcs near P, γ_1, γ_2, γ_3, γ_4. These correspond to two arcs in R_i: α_1 and α_2 and two arcs in R_{i-1}: β_1 and β_2. Let a_1 and a_2 be elements in W that represent loops around α_1 and α_2. Similarly let b_1 and b_2 be elements in W_{i-1} that represent loops around β_1 and β_2. By Algorithm 6.3 the group of the union includes the four relations: $a_1 = b_1$, $a_1 = b_2$, $a_2 = b_1$, and $a_2 = b_2$. In terms of W, this reduces to $a_1 = a_2$. Hence, the net effect on W is this one new relation.

In this process, our base group was taken from the region R_{n-1}. However, there is nothing special about this particular region. We may start from any region and then work out toward the top and the bottom, adding new generators and relations as we go. (Notice that as we go up, our concepts of maxima and minima switch.) It is usually convenient to start with the region with the most complicated fundamental group. In the case of a non-locally flat knot, this will most likely be R_{n-1}; but for a locally flat knot this isn't always the case.

We can summarize these results in the following algorithm:

Algorithm 6.5. *Suppose we are given a knot Σ embedded in S^4 such that its critical points occur at $t = s_1, s_2, \ldots, s_n$ with $s_1 < s_2 < \cdots < s_n$ and if Σ has a non-locally flat point, it is at s_n. Then the fundamental group $\pi(S^4 - \Sigma)$ can be calculated as follows.*

(1) *Calculate the Wirtinger presentation W of the knot of any cross-section C_k of R_k.*

(2) *Move down (in the negative direction of t) the knot until you reach a critical point P and make the following change to W:*
   (a) *If P is a maximum, add a generator to W. This generator represents a loop around the new arc that appears at the maximum.*

   (b) *If P is a minimum, there is no change.*

   (c) *If P is a saddle point, add the relation $\alpha_1 = \alpha_2$ to W; α_1 and α_2 are expressions in W for loops around the two elements a_1 and a_2 that come together at the saddle point.*

(3) *Repeat step 2 until you have reached the bottom of the knot.*

(4) *Move up (in the positive direction of t) the knot from R_k until you reach a critical point P and make the appropriate change to W:*
   (a) *If P is a maximum, there is no change.*

   (b) *If P is a minimum, add a generator to W. This generator represents a loop around the new arc that appears at the minimum.*

   (c) *If P is a saddle point, add a relation as in step 2c.*

(5) *Repeat step 4 until you have reached the top of the knot.*

Example 2: Consider the knot in Figure 8. We will use Algorithm 6.5 to calculate the fundamental group of this knot. The middle cross-section ($t = 0$) is exactly the square knot of Example 1. We calculated the fundamental group to be presented by $(a, b, f : aba = bab, afa = faf)$. This is our base group. Then as we move in the negative direction we come across a saddle point in which the arcs b and f come together. This identity simplifies our group to only two generators ad one relation: $(a, b : aba = bab)$.

As we continue in the negative direction, the two components disappear. These minima have no effect on the group, according to Algorithm 6.5.

Then we move in the positive direction from $t = 0$; again we come to a saddle point in which arcs b and f come together. The relation $b = f$ has already been added to our presentation; our group will not change. Finally, the two components disappear at the top of the knot. When we are moving in the positive direction of t, the maxima have no effect on the group. So the fundamental group of the knot in Figure 8 is $(a, b : aba = bab)$. (This group is the same group as the fundamental groups of the trefoil knot in 3 dimensions and of the trefoil suspension in 4 dimensions.)

We can show that this group is not isomorphic to $\mathbb{Z}$ by finding a homomorphism ϕ from this group to a group of permutations on three elements. Let ϕ be defined by $\phi(a) = (1\ 2)$ and $\phi(b) = (1\ 3)$. Notice that the image of the relation holds true. That is, $\phi(aba) = (1)(2\ 3) = \phi(bab)$. Therefore ϕ is a homomorphism. However, notice that $(1\ 2)(1\ 3) = (1\ 2\ 3) \neq (1\ 3\ 2) = (1\ 3)(1\ 2)$. Since $\phi(a)$ does not commute with $\phi(b)$, it must be that a does not commute with b. Hence, the fundamental group of this knot is not isomorphic to $\mathbb{Z}$. According to Theorem 6.1, this proves the existence of a knotted 2-sphere in 4-space!
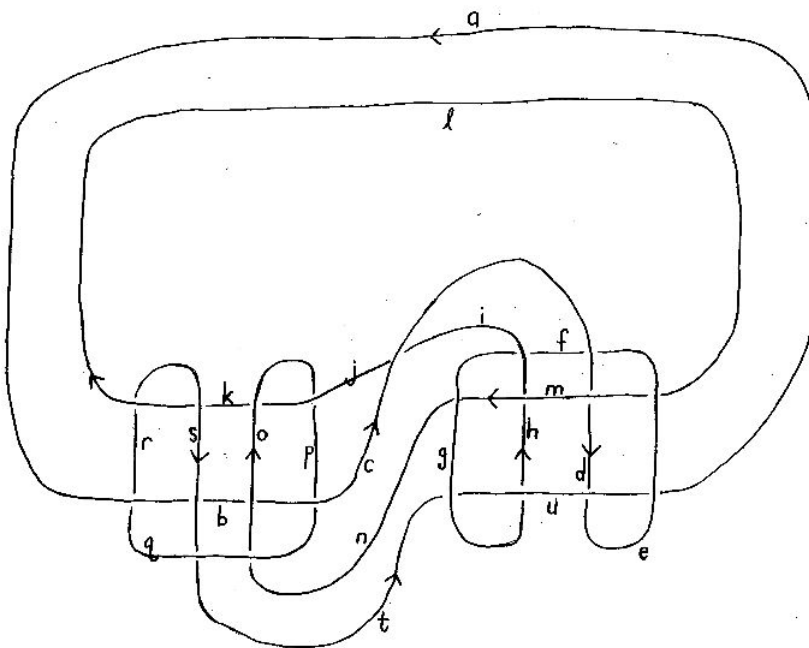
FIGURE 16
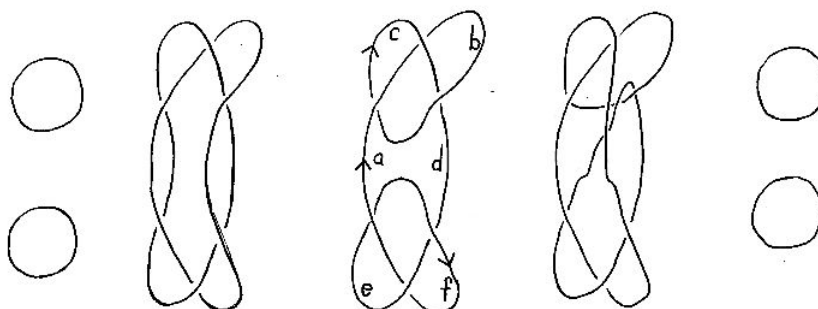
Example 3: Figure 16 shows a slight variation of the knot in Figure 8. The middle cross-section and the saddle point in the negative direction are the same as that of Figure 8. However, in the positive direction the saddle point is different than that of the previous example.

The calculation of the fundamental group will be similar to that of Example 2. We start with the base group of the square knot: $(a, b, f : aba = bab, afa = faf)$. In the negative direction we get one new

relation, as before, reducing the group to $(a, b : aba = bab)$. Now observe that the saddle point on the positive side is between the arcs b and e. We expressed e in terms of the other generators: $e = a^{-1}fa$. But now, we know that $b = f$, so we have $e = a^{-1}ba$. Then, the relation that we get when we join arc b to arc e is $b = a^{-1}ba$. This means that a and b commute, so our relation $aba = bab$ becomes $a^2b = ab^2$ or $a = b$. Therefore, the group of the complement of the knot in Figure 16 is $(a : -) \cong \mathbb{Z}$. In this case Theorem 6.1 tells us that this embedding is unknotted! This is an interesting example of an embedding of S^2 into S^4 because it is not knotted but has a cross-section that *is* knotted in S^3. This example was first presented by Stallings.

Example 4: This is an example of a non-locally flat (hence non-trivial) knot whose fundamental group of the complement is isomorphic to the group of the integers. This shows that Freedman's Theorem is not true for all piecewise linear knots; it requires that they be locally flat. First we will describe the knot, and then we will calculate the fundamental group.

FIGURE 17

This knot has a non-locally flat point at the top and is, therefore, a nontrivial knot. Below that point is the slice knot shown in Figure 17. As we move down, the next critical point is a saddle point in which

arc a and arc l come together. Observe that after this modification, what we have left is a pair of unlinked, unknotted components. Finally, these unknots disappear at minima.

To calculate the Fox presentation of this knot, we first calculate the Wirtinger presentation of the knot in Figure 17 and then add the relation $a = l$. The Wirtinger presentation of a 21-crossing knot is rather messy. It has 21 generators: $a, b, \ldots, u$. The 21 relations are:

i) $ar^{-1}a^{-1}q$
ii) $asb^{-1}s^{-1}$
iii) $bo^{-1}c^{-1}o$
iv) $cpc^{-1}q^{-1}$
v) $cj^{-1}c^{-1}i$
vi) $cec^{-1}f^{-1}$
vii) $cmd^{-1}m^{-1}$
viii) $du^{-1}e^{-1}u$
ix) $eue^{-1}a^{-1}$
x) $em^{-1}e^{-1}l$
xi) $fig^{-1}i^{-1}$
xii) $gmg^{-1}n^{-1}$
xiii) $gu^{-1}g^{-1}t$
xiv) $guh^{-1}u^{-1}$
xv) $hm^{-1}i^{-1}m$
xvi) $jp^{-1}j^{-1}o$
xvii) $jok^{-1}o^{-1}$
xviii) $ks^{-1}l^{-1}s$
xix) $lrl^{-1}s^{-1}$
xx) $nq^{-1}o^{-1}q$
xxi) $qt^{-1}q^{-1}s$

(While only 20 of these relations are necessary, for convenience we can consider all 21.)

Using these relations it is possible to reduce the presentation to only six generators and five relations. However, it is more efficient to consider the unreduced presentation.

When we add the relation $a = l$, the fundamental group becomes $\mathbb{Z}$. To show this it is sufficient to show that all generators become equal to each other.

Consider relations i and xix. They can be expressed as $q = ara^{-1}$ and $s = lrl^{-1}$ respectively. But if $a = l$ then $q = s$. Now that we know $q = s$, we can cancel the q^{-1} and s in xxi and we have $qt^{-1} = 1$, or $q = t$. We leave it to the reader to verify that, by this process of manipulation of relations and identification of generators, all generators

can be shown to be equal. Once we are down to one generator, each of the relations becomes trivial.

Therefore, the fundamental group of the complement of this knot is isomorphic to $\mathbb{Z}$. So Theorem 6.1 does not hold for all piecewise linear knots, only locally flat knots.

7. Minimizing Critical Points

We do not have tables of knotted 2-spheres as we did for classical knots. It is difficult to come up with a notion of complexity to use as an index that would be analogous to the crossing number, which is used in the classical knot case. One possibility is the number of critical points that occur in the embedding. An embedding that is non-locally flat can be knotted with only two critical points, such as a suspension with only a maximum and minimum. In what follows we will determine the minimum number of critical points necessary for a locally flat embedding to be globally knotted.

Theorem 7.1. *A locally flat, PL, knotted embedding of a 2-sphere in S^4 must have at least two maxima, two minima, and two saddle points.*

Proof. First, recall from Section 3 that any embedding of a locally flat PL 2-sphere is ambient isotopic to a 2-sphere in normal form. Thus, the only critical points that need occur are maxima, saddle points, and minima, with all the saddle points between the maxima and all the minima. Furthermore, each component is unknotted and unlinked when it first appears. This has been shown in [7].

There is another restriction on these critical points that arises from what is called the *Euler Characteristic* of the 2-sphere. Euler's formula is commonly stated in terms of planar graphs, as $vertices + faces - edges = 2$. It turns out that a strong deformation retraction can be made between a distorted 2-sphere that is the domain of the embedding and a planar graph; we can associate minima with vertices, maxima with faces, and saddle points with edges. Thus we have $minima + maxima - saddle\,points = 2$.

Intuitively, a standard 2-sphere will have one maximum, one minimum, and no saddle points, thus satisfying the formula. A distortion which adds a maximum or a minimum will also add a saddle point, thus preserving the total. The above result is stated explicitly in [7]; a proof of a more general case can be found in [14].

With these restrictions on our critical points, only certain combinations are possible involving fewer than two maxima, two minima, or two saddle points. We will use Algorithm 6.5 to show that fundamental group in each of these cases is isomorphic to $\mathbb{Z}$.

Case 1: An embedding with one minimum, no saddle points, and one maximum. Because the component that first appears at the maximum must be unknotted, any cross-section between the maximum and the minimum will have the fundamental group of the complement isomorphic to $\mathbb{Z}$. By Algorithm 6.5, this is the fundamental group of the whole sphere.

Case 2: An embedding with n minima, $n-1$ saddle points, and one maximum. Again, because of our result from [7], we know that a cross-section taken just below the maximum must be unknotted. Thus its fundamental group will have only one generator, say a, and will be isomorphic to $\mathbb{Z}$. This is our initial cross-section when applying Algorithm 6.5.

As we move down the knot, we will find only minima, which will not affect the fundamental group, and saddle points. Because we begin with one component and end with n components, each of the $n-1$ saddle points must add a component. Each of the components must be initially unknotted and unlinked. Thus the relations we add at the saddle points each relate two unlinked, unknotted components to the one component immediately above them. In this way we see that all the generators are equal to a, and we have no other relations. Therefore, the fundamental group of the whole sphere will be isomorphic to $\mathbb{Z}$.

Case 3: An embedding with one minimum, $n-1$ saddle points, and n maxima. This is identical to the preceding case; we simply take our initial cross-section just above the minimum and proceed up, according to Algorithm 6.5.

We now see that, with no maxima, one maximum, or n maxima and one minimum, the embedding must have a fundamental group isomorphic to $\mathbb{Z}$. By Theorem 6.1, these are all equivalent to the unknot. Therefore, a locally flat embedding must have at least two minima, two saddle points, and two maxima to be knotted.   □

We have already seen in Fibure 8 an example of a locally flat embedding with exactly this many critical points, and that has a fundamental group that is not isomorphic to $\mathbb{Z}$. Therefore, this is the fewest number of critical points possible in a locally flat knot.

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