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Algebra and Matrix Normed Spaces

Seth M. Hain
University of Nebraska, shain2@bigred.unl.edu

ALGEBRA AND MATRIX NORMED SPACES

SETH M. HAIN

Abstract. We begin by looking at why operator spaces are necessary in the study of operator algebras and many examples of and ways to construct operator algebras. Then we examine how certain algebraic relationships, for example the well known relationship $M_n(A) \cong Hom_A(A^{(n)})$, break down when norms are placed on them. This leads to ways to correct these ideas using matrix norms.

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1. Introduction and notation

This paper is really an extended investigation of some notions from pure algebra concerning rings, in the setting where the rings are operator algebras, i.e. closed subalgebras of the bounded operators on a Hilbert space (which we will define momentarily). In fact, for the purposes of our paper, the reader can take an 'operator algebra' to be a subalgebra of the set of $n \times n$ matrices, M_n. That is, an 'operator algebra' may be taken to be a vector subspace $A \subset M_n$ such that $xy \in A$ whenever $x, y \in A$. We also assume usually that the $n \times n$ identity matrix I_n is in A. In this case we say that A is a 'unital algebra'. We will first be involved with looking at why operator spaces may be useful in the study of operator algebras and some examples of operator algebras. This leads to a natural question as to how some algebraic notions transfer from the realm of algebra to the realm of analysis, in particular the realm of analysis concerned with Banach space theory. We will concentrate our efforts on primary algebraic relations that have problems when transferred to this area of analysis, but when viewed in the realm of matrix normed spaces hold up nicely.

The primary focus of our paper has its roots in the idea that the ring of linear transformations on $\mathbb{R}^n$ is isomorphic to the ring of $n \times n$ matrices, i.e. $M_n \cong Lin(\mathbb{R}^n)$. We will use $Lin(\mathbb{R}^n)$ to denote the set of linear transformations taking $\mathbb{R}^n \to \mathbb{R}^n$. This idea is central to much of mathematics including undergraduate courses in calculus and linear algebra. There is a more general algebraic result that states that, if A is a unital ring or algebra then the ring A-module homomorphisms from $A^{(n)}$ to $A^{(n)}$ is isomorphic to the $n \times n$ matrices with entries in A. Here $A^{(n)}$ is the direct sum of n copies of A. That is: $M_n(A) \cong Hom_A(A^{(n)})$ isomorphically. We will define $M_n(A)$ momentarily. The proofs of both of these results are quite straightforward. One simply checks that the canonical map $\theta : M_n(A) \to Hom_A(A^{(n)})$ defined as $\theta(B)(\vec{x}) = B\vec{x}$ where $B \in M_n(A)$ and $\vec{x} \in A^{(n)}$, is an isomorphism. Throughout this paper when we refer to a canonical map it will be defined in a similar manner with elements from the correct corresponding spaces. We will concern ourselves with how the algebraic relationship $M_n(A) \cong Hom_A(A^{(n)})$ transfers over to the analytic world. We will see that some problems occur when we replace the A-module homomorphisms on $A^{(n)}$ with the bounded A-module maps on $A^{(n)}$. We will define the terms 'bounded' and 'completely bounded' later.

We first need to define some of the notation we will be using. Throughout this paper we will assume that our field ($\mathbb{F}$) is either the real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers. A *Hilbert space* is an inner product space for which the associated norm is complete, i.e. Cauchy sequences converge. Here and throughout the paper H and K will be Hilbert spaces. We will also denote the set of $n \times n$ matrices with entries in A as $M_n(A)$ where A is some space. We will write M_n for the matrices with entries from the field, i.e. $M_n(\mathbb{F})$. Then $B(H, K)$ will denote the set of bounded linear maps taking $H \to K$. This means that for every $T \in B(H, K)$, $\|T\| = \sup\{\|T(x)\| : x \in H, \|x\| \leq 1\}$ is finite. Also $B(H) = B(H, H)$. When we discuss matrix norms on a space, X, we are referring to a set of norms, $\{\| \cdot \|_n\}_{n=1}^{\infty}$, where the nth norm is defined on $M_n(X)$, the space of $n \times n$ matrices with entries in X. It is well known to experts that $B(H)$ and $B(H, K)$ have natural matrix norms. However for us, we will only use a simple version of this fact. We will see momentarily that M_m has natural matrix norms, i.e. a natural norm on $M_n(M_m)$. It is a well known fact that any $B(H)$ may be viewed as some space of matrices M_m, so we can concentrate on M_m and forget about general $B(H)$'s for the most part.

An *operator space* is a vector subspace of $B(H)$, or for us, a vector subspace X of M_m, together with the natural matrix norms one gets on $M_n(X)$ from $M_n(M_m)$ for $n \in \mathbb{N}$. We can then define an operator T on a space X to be a *complete isometry* if $\|[T(x_{ij})]\|_n = \|[x_{ij}]\|_n$ for every $x_{ij} \in X$ and $n \in \mathbb{N}$. This simply states that the norm of an $n \times n$ matrix with entries $x_{ij} \in X$ equals the n norm of the matrix with entries $T(x_{ij})$. Recall that if this is true for $n = 1$ then T is *isometric.*

Throughout the paper we will be finding the norms of matrices. Recall, if $A \in M_n$ then the norm of A, $\|A\|$, can be calculated either by finding the square root of the largest eigenvalue of A^*A or $\|A\| = \sup\{\frac{\|A\vec{x}\|}{\|\vec{x}\|} : \vec{x} \in \mathbb{R}^n\}$. In more general terms, for any linear $T : X \to Y$, $\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$.

We now want to take note that the $M_n(A)$ matrices have a norm already induced on them. We will look at an example where $A = M_2(\mathbb{F})$. For this example we will examine the norm on $M_2(A)$. Then

$$\begin{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ 6 & -1 \end{bmatrix} \\ \begin{bmatrix} 7 & -5 \\ 2 & 0 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix} \end{bmatrix} \in M_2(M_2) \cong M_4.$$

Note that $M_2(M_2) \cong B(\mathbb{F}^2 \bigoplus_2 \mathbb{F}^2)$. Therefore $M_2(M_2)$ has a canonical norm on it, the norm that would be associated with the corresponding element in M_4. So we simply must find the norm of $\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 6 & -1 \\ 7 & -5 & 1 & -1 \\ 2 & 0 & -3 & 6 \end{bmatrix}$ and transfer it back to the element in $M_2(M_2)$.

There is another way to calculate the norm. It involves using the *C^*-identity* : $\|x^*x\| = \|x\|^2$ for any matrix x (or operator on a Hilbert space). Here the $*$ refers to the conjugate transpose of a matrix. That is $[a_{ij}]^* = [\bar{a}_{ji}]$. If the entries in the matrix are real, then $*$ is just the transpose. It is well known that $\|x^*\| = \|x\|$ for any matrix x. Thus using the C^*-identity twice we get:

$$\|x^*x\| = \|x\|^2 = \|x^*\|^2 = \|xx^*\|$$

The C^*-identity can be used in the following manner. First though we need some definitions which will be used throughout. Let

$$C_n = \left\{ \begin{bmatrix} a_1 & 0 & ... & 0 \\ a_1 & 0 & ... & 0 \\ ... & ... & ... & ... \\ a_n & 0 & ... & 0 \end{bmatrix} : a_i \in \mathbb{F} \right\}$$

$$R_n = \left\{ \begin{bmatrix} a_1 & a_1 & ... & a_n \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} : a_i \in \mathbb{F} \right\}.$$

Often times these are referred to as *Column n-space* and *Row n-space*, respectively. Applying the C^*-identity to an element $x \in C_n$ we see

$$\|x\| = \left\| \begin{bmatrix} a_1 & 0 & ... & 0 \\ a_1 & 0 & ... & 0 \\ ... & ... & ... & ... \\ a_n & 0 & ... & 0 \end{bmatrix} \right\| = \|x^*x\|^{\frac{1}{2}} = \left\| \begin{bmatrix} \bar{a_1} & \bar{a_2} & ... & \bar{a_n} \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 & ... & 0 \\ a_1 & 0 & ... & 0 \\ ... & ... & ... & ... \\ a_n & 0 & ... & 0 \end{bmatrix} \right\|^{\frac{1}{2}}$$

$$= \left\| \begin{bmatrix} |a_1|^2 + |a_2|^2 + ... + |a_n|^2 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} \right\|^{\frac{1}{2}} = \sqrt{|a_1|^2 + |a_2|^2 + ... + |a_n|^2}.$$

We can use the same idea to calculate the norm for any element of $y \in R_n$.

$$\|y\| = \left\| \begin{bmatrix} a_1 & a_2 & ... & a_n \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} \right\| = \|yy^*\|^{\frac{1}{2}} = \left\| \begin{bmatrix} a_1 & a_2 & ... & a_n \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} \bar{a_1} & 0 & ... & 0 \\ \bar{a_2} & 0 & ... & 0 \\ ... & ... & ... & ... \\ \bar{a_n} & 0 & ... & 0 \end{bmatrix} \right\|^{\frac{1}{2}}$$

$$= \left\| \begin{bmatrix} |a_1|^2 + |a_2|^2 + ... + |a_n|^2 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} \right\|^{\frac{1}{2}} = \sqrt{|a_1|^2 + |a_2|^2 + ... + |a_n|^2}.$$

This shows that the obvious linear isomorphism, the transpose map, between C_n and R_n is an isometry. In fact we have shown that both C_n and R_n are basically $\mathbb{F}^n$ with the Euclidean norm. So someone studying *Banach spaces*, i.e. vector spaces where the associated norm is complete, would consider C_n and R_n to be the same. Later we will see some differences that do arise though when viewed in a different way.

There are many useful tricks for calculating norms that we will use throughout this paper. The first of these is that switching rows and columns does not change the norm. This can be useful because it is also well known that the norm of a diagonal matrix is the absolute value of the largest entry on the diagonal. So often times we will switch rows and columns to make a matrix into a diagonal matrix. Another useful trick for computing norms is realizing that rows and columns entirely composed of zeros do not change the norm. So they can be inserted or deleted without any effect.

For more details on matrix norms, if needed, consult some of the references in our bibliography. We write $CB_A(A^{(n)})$ for the completely bounded A-module maps on $A^{(n)}$, denoted as $CB_A(A^{(n)})$, which are the completely bounded A-module maps taking $A^{(n)} \to A^{(n)}$. An operator T is *completely bounded* if $\sup\{\|[T(x_{ij})]\|_n : \|[x_{ij}]\|_n \leq 1\} < \infty$ for every $n \in \mathbb{N}$.

2. Matrix Norms and Operator Algebras

One of the first questions that arise when introduced to operator algebras is, "Why are operator spaces necessary?".

A first answer to this question is fairly basic in nature. Note that $C_n, R_n \subseteq M_n$. We checked above that $C_n \cong R_n$ isometrically isomorphically. Let's look at what happens, though, when we look at elements of $M_3(C_3)$ and $M_3(R_3)$. First we will look at the norm of a simple element in $M_3(C_3)$:

$$\left\| \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\| = \sqrt{3}$$

by the simple ideas illustrated in the introduction concerning calculating norms. It would seem that the norm of the corresponding element in $M_3(R_3)$, corresponding via the mapping

$$\begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from C_3 to R_3, would have the same norm. But instead

$$\left\| \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\| = 1.$$

It appears that something went wrong when we started looking at matrix norms. From this simple example we can note that since the norms are different, then C_3 and R_3 seem to have some underlying structure that is different and not perceived when viewed simply as normed spaces. This difference is not recognized by someone studying Banach spaces, because they will simply look at the first norm which are equal and not see this difference.

From this example it already seems evident that operator algebras seem to have a stronger set of conditions placed on them. The calculations above motivate the following theorem.

Theorem 2.1. *C_n is not completely isometrically isomorphic to R_n.*

Proof. Assume that $C_n \cong R_n$ completely isometrically isomorphically. Begin by looking at the matrices of size n with entries from R_n, i.e. $\left\{ \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} : x_i \in R_n \right\}$. Then

$\left\|\begin{bmatrix} x_1 & x_2 & ... & x_n \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}\right\|_n = \sqrt{\sum_{i=1}^n \|x_i\|^2}$ by using the C^*-identity to compute the norm in a similar manner as before. By our assumption there must exist a $T : C_n \to R_n$ which is a complete isometry since $C_n \cong R_n$ via some mapping. Let $\{e_1, e_2, ..., e_n\}$ be the standard orthonormal basis for C_n. Then $T(e_i) = x_i$ for some $x_i \in R_n$. Then we can calculate the norm of $\begin{bmatrix} e_1 & e_2 & ... & e_n \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix} \in M_n(C_n)$ by deleting rows of zeros and rearranging the columns to form the identity, thus $\left\|\begin{bmatrix} e_1 & e_2 & ... & e_n \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}\right\|_n = 1$. Then we have

$$1 = \left\|\begin{bmatrix} e_1 & e_2 & ... & e_n \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} T(e_1) & T(e_2) & ... & T(e_n) \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} x_1 & x_2 & ... & x_n \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}\right\|$$

because T is a complete isometry. Thus since $1 = \|e_i\| = \|x_i\|$ we have that

$$\left\|\begin{bmatrix} x_1 & x_2 & ... & x_n \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}\right\| = \sqrt{\sum_{i=1}^{n} \|x_i\|^2} = \sqrt{n}.$$

Thus we have shown that $1 = \sqrt{n}$ which is a contradiction. Therefore C_n is not completely isometrically isomorphic to R_n.

There is a more general result that can be proven concerning C_n. We will need the following lemma first though. Note that this lemma will be used at different times throughout the paper.

Lemma 2.2. $\|[a_{ij}]\|_n \geq \|a_{kl}\|$ *where* $a_{ij} \in A$ *and* A *is a subspace of* $B(H)$ *or* $B(H, K)$.

Proof. Let $[a_{ij}]$ be a p, q matrix with $a_{ij} \in A$. Then denote $\vec{e_k}^*$ to be a row of length p with all entries zero, except the kth entry, which would be 1. Similarly define $\vec{e_l}$ to be a vector of length q with all entries zero, except the lth entry which would be 1. Then note that $\vec{e_k}^*[a_{ij}]\vec{e_l} = a_{kl}$ and that $\|\vec{e_k}^*\| = \|\vec{e_l}\| = 1$ for any k and l. So then $\|a_{kl}\| = \|\vec{e_k}^*[a_{ij}]\vec{e_l}\| \leq \|\vec{e_k}^*\|\|[a_{ij}]\|\|\vec{e_l}\| = \|[a_{ij}]\|$.

So far we have seen an instance of some underlying structure in C_n and R_n that is not immediately visible. The ideas behind the structure of C_n motivate the following theorem which help to better describe the structure of C_n. The theorem states that an if an operator T taking elements of C_n into M_n is a linear complete isometry then up to a change of orthonormal basis, T must be a function that takes a vector of C_n (in this case we are viewing elements of C_n as vectors) and places it in the first column of M_n and fills the rest of the matrix with zeros. We first need to formalize what we mean by a change of orthonormal basis. A *unitary matrix* (often simply called a unitary) is a matrix A with

$AA^* = I$ and $A^*A = I$ where I is the identity. It is common to think of a unitary as a 'change of orthonormal basis' when it acts as an operator.

Theorem 2.3. *If $T : C_n \to M_n$ is a linear complete isometry then there exists unitary $n \times n$ matrices U and V, such that for any $\vec{x} \in C_n$, $UT(\vec{x})V = [\vec{x} : \vec{0} : ... : \vec{0}]$.*

Proof. An equivalent statement is if $T(\vec{e_i}) = A_i$ then $UA_iV = [\vec{e_i} : \vec{0} : ... : \vec{0}]$ where $\vec{e_i}$ are standard canonical basis elements, i.e. A_i are square matrices with $\vec{e_i}$ in the first column and the remaining entries 0. So we need to show that there exists two orthonormal bases $D = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ and $C = \{\vec{w_1}, \vec{w_2}, ..., \vec{w_n}\}$ such that the matrix of A_i with respect to D and C is $E_{i,1}$. That is the matrix with l, k entry equal to $\langle A_i\vec{w_k}, \vec{v_l}\rangle$, i.e. $[\vec{e_i} : \vec{0} : ... : \vec{0}]$. This is the statement we will prove. First let $T : C_n \to M_n$ be a linear complete isometry. Then let $\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ be the standard orthonormal basis for C_n. Then assume $T(\vec{e_i}) = A_i$, which by the complete isometry implies

$$\|e_i\| = \|T(e_i)\| = \|A_i\| = 1.$$

Thus $\|A_i^*A_i\| = 1$ We also have

$$\|[A_1A_2...A_n]\| = \|\sum_{i=1}^{n} A_iA_i^*\|^{\frac{1}{2}}$$

by the C^*-identity. Then

$$\|\sum_{i=1}^{n} A_iA_i^*\|^{\frac{1}{2}} = \|[\vec{e_1} : \vec{e_2} : ... : \vec{e_n}]\| = 1$$

by what was shown above and the fact that the matrix composed of basis elements can be arranged into the identity. Also note that

$$\left\| \begin{bmatrix} A_1 \\ ... \\ A_n \end{bmatrix} \right\| = \|\sum_{i=1}^{n} A_i^*A_i\|^{\frac{1}{2}} = \left\| \begin{bmatrix} \vec{e_1} \\ ... \\ \vec{e_n} \end{bmatrix} \right\| = \sqrt{n}.$$

Thus

$$\|\sum_{i=1}^{n} A_i^*A_i\| = n \ \ (\dagger).$$

Define $B_i = A_i^*A_i$ and $B = \sum_{i=1}^{n} B_i$. Note that B_i are positive matrices and therefore B is a positive matrix. Therefore B has eigenvalues ≥ 0. Also there exist unitary U such that $B = U^*RU$ where R is a diagonal matrix with the eigenvalues of B down the diagonal. Also $\|B\| \leq \|U^*\|\|R\|\|U\| = \|R\|$ and $\|R\| \leq \|U\|\|B\|\|U^*\| = \|B\|$ so $\|B\| = \|R\|$. Since $1 = \|A_i^*A_i\| = \|B_i\|$ then by what was marked ($\dagger$) earlier $\|B\| = n$. Note that the norm of R is the largest eigenvalue of B and $n = \|B\| = \|R\|$. Thus n is the largest eigenvalue of B. Let $\vec{x}$ be a normalized eigenvector for B for eigenvalue n, i.e., $\|\vec{x}\| = 1$ and $B\vec{x} = n\vec{x}$. This implies

$$\langle B\vec{x}, \vec{x}\rangle = \langle n\vec{x}, \vec{x}\rangle = n \Rightarrow \sum_{i=1}^{n} \langle B_i\vec{x}, \vec{x}\rangle = n.$$

By the Cauchy-Schwarz inequality

$$\langle B_i\vec{x}, \vec{x}\rangle \leq \|B_i\vec{x}\|\|\vec{x}\| \leq \|B_i\| = 1.$$

Therefore combining the last two ideas we have that $\langle B_i\vec{x}, \vec{x}\rangle = 1 \geq \|B_i\vec{x}\|\|\vec{x}\|$ for every $i \in 1, 2, ..., n$. By the converse of the Cauchy-Schwarz theorem $B_i\vec{x} = \vec{x}$ which implies that $\vec{x}$ is an eigenvector for every B_i for eigenvalue 1. Let $A_i\vec{x} = \vec{v_i}$. Let $D = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$. We need to show that D is an orthonormal basis. First $\|\vec{v_i}\|^2 = \|A_i\vec{x}\|^2 = \langle A_i\vec{x}, A_i\vec{x}\rangle = \langle A_i^*A_i\vec{x}, \vec{x}\rangle = \langle B_i\vec{x}, \vec{x}\rangle = 1$ by previous ideas. So now we need to show $\langle \vec{v_i}, \vec{v_j}\rangle = 0$ for $i \neq j$ and $\langle \vec{v_i}, \vec{v_j}\rangle = 1$ for $i = j$. We will assume that $\vec{v_i}$ is a column vector, then $\langle \vec{v_i}, \vec{v_j}\rangle = \vec{v_j^*}\vec{v_i} = \vec{v_j^*}A_i\vec{x}$. We have shown that $\|[A_1A_2...A_n]\| \leq 1$ and we also know that the diagonal matrix $[\vec{x}I_n]$ with $\vec{x}$ down the diagonal has norm equal to 1. Then we have

$$\|[A_1A_2...A_n][\vec{x}I_n]\| \leq \|[A_1A_2...A_n]\|\|[\vec{x}I_n]\| \leq 1 \Rightarrow \|[A_1\vec{x} : A_2\vec{x} : ... : A_n\vec{x}]\| \leq 1.$$

We also have that

$$\vec{v_j^*}[A_1\vec{x} : A_2\vec{x} : ... : A_n\vec{x}] = [\vec{v_j^*}A_1\vec{x} : \vec{v_j^*}A_2\vec{x} : ... : 1 : ... : \vec{v_j^*}A_n\vec{x}] \in R_n$$

with 1 in the jth entry. Therefore

$$\|[\vec{v_j^*}A_1\vec{x} : \vec{v_j^*}A_2\vec{x} : ... : 1 : ... : \vec{v_j^*}A_n\vec{x}]\| \leq \|\vec{v_j^*}\|\|[A_1\vec{x} : A_2\vec{x} : ... : A_n\vec{x}]\| \leq 1.$$

It follows then that $\vec{v_j^*}A_i\vec{x} = 0$ for $i \neq j$. Therefore D is an orthonormal basis. Now let $\vec{w_1} = \vec{x}$ and let $C = \{\vec{w_1}, \vec{w_2}, ..., \vec{w_n}\}$ be any orthonormal basis with $\vec{w_1}$ as stated. A corollary of the Gram-Schmidt Theorem guarantees that C exists as prescribed. Now we'll look at the matrix $[\langle A_i\vec{w_l}, \vec{v_k}\rangle]$. If $l = 1$ then $\langle A_i\vec{w_1}, \vec{v_k}\rangle = \langle A_i\vec{x}, \vec{v_k}\rangle = \langle \vec{v_i}, \vec{v_k}\rangle$. So by what was shown previously $\langle A_i\vec{w_1}, \vec{v_k}\rangle = 1$ if $k = i$ and 0 otherwise. Now we will look at the case of $l \geq 2$. Then since C is an orthonormal basis $\|[\vec{w_1} : \vec{w_2} : ... : \vec{w_n}]\| \leq 1$. Since by our lemma then $\|A_i\| \leq 1$, then $\|A_i[\vec{w_1} : \vec{w_2} : ... : \vec{w_n}]\| \leq 1 \Rightarrow \|[A_i\vec{w_1} : A_i\vec{w_2} : ... : A_i\vec{w_n}]\| \leq 1$ and we have already shown that $\|A_i\vec{w_1}\| = 1$. Now notice that

$$\|[A_1\vec{w_1} : ... : A_1\vec{w_n} : A_2\vec{w_1} : ... : A_2\vec{w_n} : ... : A_n\vec{w_1} : ... : A_n\vec{w_n}]\| \leq$$

$$\|[A_1A_2...A_n]\| \left\| \begin{bmatrix} [\vec{w_1} : \vec{w_2} : ... : \vec{w_n}] & 0 & ... & 0 \\ 0 & [\vec{w_1} : \vec{w_2} : ... : \vec{w_n}] & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & [\vec{w_1} : \vec{w_2} : ... : \vec{w_n}] \end{bmatrix} \right\| \leq 1$$

by previous things stated in the paper and this proof. By rearranging columns and using the fact that $\vec{w_1} = x$ we have

$$1 \geq \|[A_1\vec{w_1} : ... : A_1\vec{w_n} : A_2\vec{w_1} : ... : A_2\vec{w_n} : ... : A_n\vec{w_1} : ... : A_n\vec{w_n}]\| =$$

$$\|[A_1\vec{w_1} : A_2\vec{w_1} : ... : A_n\vec{w_1} : J]\| = \|[\vec{v_1} : \vec{v_2} : ... : \vec{v_n} : J]\|$$

where $J = [A_1\vec{w_2} : ... : A_1\vec{w_n} : A_2\vec{w_2} : ... : A_2\vec{w_n} : ... : A_n\vec{w_2} : ... : A_n\vec{w_n}]$. Then once again by rearranging the columns composed of $\vec{v_i}$s we can form a unitary matrix W. So we have that $1 \geq \|[W : J]\|$. Using the C^*-identity we can easily deduce from the last result that $\|J\| = 0$. Therefore $A_i\vec{w_l} = 0$ for $l \geq 2$. Then $\langle A_iw_l, v_k\rangle = 0$ for all $l \geq 2$. We therefore have shown that the matrix $[\langle A_iw_l, v_k\rangle] = E_{i,1}$.

Now we will assume that $x \in C_n$. Then $x = \sum_{i=1}^n \vec{e_i}a_i$ where the a_i's are unique scalars. Then using the properties of a linear function $UT(\vec{x})V = \sum_{i=1}^n UT(\vec{e_i})Va_i =$

$\sum_{i=1}^{n}[\vec{e_i}:\vec{0}:...:\vec{0}]a_i$ by the previous theorem. Therefore $UT(x)V=\sum_{i=1}^{n}[\vec{e_i}a_i:\vec{0}:...:\vec{0}]=[\vec{x}:\vec{0}:...:\vec{0}]$.

After having shown these last two ideas it is now possible to use them to prove the following type of corollary. Note that C_n is naturally a subalgebra of M_n with a right identity of norm 1.

Corollary 2.4. *The only multiplication on C_n that could make C_n an operator algebra with a right identity of norm 1 is the usual matrix multiplication up to a change of unitary.*

Proof. Assume that C_n is an operator algebra with some multiplication that we will write as xy. Define $\phi: C_n \to B(C_n)$ as $\phi(x)(y)=xy$ for $x,y\in C_n$. It is easy to show that this map is a linear homomorphism. Let e be the identity of C_n then $\|\phi(x)\|\geq\|\phi(x)(e)\|=\|xe\|=\|x\|$ and $\|\phi(x)\|=\sup\{\|xy\|:\|y\|\leq 1\}\leq\sup\{\|x\|\|y\|:\|y\|\leq 1\}=\|x\|$. Thus ϕ is isometric and one can similarly check that it is a complete isometry. Noting that $B(C_n)\cong M_n$ there then exists some $\theta: C_n\to M_n$ which is a homomorphic complete isometry. Then let $x,y\in C_n$. Since C_n is an operator algebra then $xy\in C_n$ also. Therefore we can write x,y and xy as follows $x=\begin{bmatrix} x_1 & 0 & ... & 0\\ x_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ x_n & 0 & ... & 0\end{bmatrix}$, $y=\begin{bmatrix} y_1 & 0 & ... & 0\\ y_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ y_n & 0 & ... & 0\end{bmatrix}$, and $xy=\begin{bmatrix} m_1 & 0 & ... & 0\\ m_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ m_n & 0 & ... & 0\end{bmatrix}$. Then by our last corollary $\theta(x)\theta(y)=U\begin{bmatrix} x_1 & 0 & ... & 0\\ x_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ x_n & 0 & ... & 0\end{bmatrix}VU\begin{bmatrix} y_1 & 0 & ... & 0\\ y_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ y_n & 0 & ... & 0\end{bmatrix}V=\theta(xy)=$ $U\begin{bmatrix} m_1 & 0 & ... & 0\\ m_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ m_n & 0 & ... & 0\end{bmatrix}V$ for some unitaries U and V. Then let $W=VU$. So multiplying the above equality on the left by U^* and on the right by V^* we have

$$\begin{bmatrix} x_1 & 0 & ... & 0\\ x_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ x_n & 0 & ... & 0\end{bmatrix}W\begin{bmatrix} y_1 & 0 & ... & 0\\ y_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ y_n & 0 & ... & 0\end{bmatrix}=\begin{bmatrix} m_1 & 0 & ... & 0\\ m_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ m_n & 0 & ... & 0\end{bmatrix}.$$

Then define $W\begin{bmatrix} y_1 & 0 & ... & 0\\ y_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ y_n & 0 & ... & 0\end{bmatrix}=\begin{bmatrix} \beta_1 & 0 & ... & 0\\ \beta_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ \beta_n & 0 & ... & 0\end{bmatrix}$. Thus we have that

$$\begin{bmatrix} x_1\beta_1 & 0 & ... & 0\\ x_2\beta_1 & 0 & ... & 0\\ ... & ... & ... & ...\\ x_n\beta_1 & 0 & ... & 0\end{bmatrix}=\begin{bmatrix} m_1 & 0 & ... & 0\\ m_2 & 0 & ... & 0\\ ... & ... & ... & ...\\ m_n & 0 & ... & 0\end{bmatrix}.$$

Thus multiplication must then be defined in this manner, which is normal matrix multiplication up to a change of unitary.

3. Examples of operator algebras

It is quite advantageous to have a number of examples of operator algebras at hand. This section is concerned with providing numerous examples of operator algebras and ways to construct them from others. We begin this section by noting that the class of operator algebras contains all finite dimensional algebras. To show this we will begin with a lemma:

Lemma 3.1. *Every finite dimensional vector space U is a Hilbert space.*

Proof. Let U be a finite dimensional vector space. We then can fix a finite basis for U, denote it as $\{u_1, u_2, ..., u_n\}$. Then for every $v \in U$ there exists unique $a_i \in \mathbb{F}$ such that $v = \sum_{i=1}^n a_i u_i$. Then let $\langle \cdot, \cdot \rangle : U \times U \to \mathbb{F}$ be the function defined as $\langle u, v \rangle = \sum_{i=1}^n a_i \bar{b_i}$ where a_i 's and b_i 's are determined uniquely by expressing u and v as the sum of the basis described previously. It can easily be verified that this is an inner product on U and will therefore induce a norm on U, i.e. $\|x\| = \sqrt{\langle x, x \rangle}$ for every $x \in U$. Thus we have created a finite dimensional normed space and it is well know that all finite dimensional normed spaces are complete. Therefore U is a Hilbert space.

By use of this lemma we are able to prove the following result. While this result is considered known, we are not aware of any straightforward proof which has been published. A similar, but not identical, result appears in [4]. The following result simply shows that every finite dimensional operator algebra is isomorphic to a subalgebra of the bounded operators on a Hilbert space.

Theorem 3.2. *Let A be an algebra which is finite dimensional as a vector space. Then there exists a Hilbert space H, and a 1-1 homomorphism $\pi : A \to B(H)$.*

Proof. First we will assume that A is a unital algebra. Using the previous lemma we can associate A with its own Hilbert space. Then for every $a \in A$ let $L_a(b) = ab$ for $b \in A$, thus L_a is a linear transformation from $A \to A$. Now let $\phi : A \to B(H)$ be the function $\phi(a) = L_a$. It can be checked that this is a homomorphism. If $\phi(a) = 0$ then $L_a = 0$. Therefore $0 = L_a(1) = a \cdot 1 = a$. Thus ϕ is injective and the theorem is proven in this case.

Now assume that A is not a unital algebra. In this case we will consider the Cartesian product of $A \times \mathbb{F}$ with multiplication defined as $(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta)$ for $a, b \in A$ and $\alpha, \beta \in \mathbb{F}$. This is a unital algebra with multiplicative identity $(0, 1)$. Denote this algebra as $\acute{A}$. Note that A is a subalgebra of $\acute{A}$. We can create an inner product on $\acute{A}$ defined as $\langle (u, a), (v, b) \rangle = \langle u, v \rangle + a\bar{b}$ for $u, v \in A$ and $a, b \in \mathbb{F}$, where the inner product on A is the one discussed in the previous lemma. It is easy to check that this is an inner product space. $\acute{A}$ is still a finite dimensional normed algebra and thus is complete. Therefore $\acute{A}$ is a Hilbert space. Then the map $\rho : A \to \acute{A}$ defined as $\rho(a) = (a, 0)$ for every $a \in A$ is an injective homomorphism. Let $\theta : \acute{A} \to B(\acute{A})$ be the same injective homomorphism discussed earlier. Then $\psi : A \to B(\acute{A})$ can be defined as $\psi = \theta \circ \rho$, which is an injective homomorphism by composition.

A corollary follows directly from that theorem:

Corollary 3.3. *Any finite dimensional algebra is an operator algebra.*

Proof. This follows directly from the definitions of an operator algebra and the previous theorem.

In a similar manner we can show that a finite dimensional $A - B$ bimodule is isomorphic to a subalgebra of the bounded operators on a Hilbert space. We will define bimodules and operator modules in the same manner as Blecher in [2].

Theorem 3.4. *For X an $A - B$ bimodule with X, A, and B having finite dimension, there exists a Hilbert space, H, and a 1-1 bimodule map $\pi : X \to B(H)$.*

Proof. Let X be an $A - B$ bimodule. Then we can create a space

$$U = \left\{ \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \ : \ a \in A, b \in B, x \in X \right\}.$$

Define the addition as the normal matrix addition and define multiplication as normal matrix multiplication. We need to check that U is closed under multiplication. Let $\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \in U$, then $\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_2x_2 + x_1b_2 \\ 0 & b_1b_2 \end{bmatrix}$. Since X is an $A - B$ module, $a_2x_2 + x_1b_2 \in X$ and therefore U is closed under multiplication. It is easy to check then that U is an algebra. Let $\theta : X \to U$ be defined as $x \mapsto \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ for every $x \in X$. It can be shown that θ is a 1-1 map. Since X, A, and B are all finite dimensional, then U must be also. Therefore we can apply the previous theorem and so there exists a $\rho : U \to B(H)$ which is a 1-1 homomorphism as described. Therefore define $\pi = \rho \circ \theta$. Then $\pi : X \to B(H)$ is a 1-1 map. Similarly define $\psi : A \to U$ as $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ for every $a \in A$. So once again by the previous theorem there exists $\varphi : U \to B(H)$ which is a 1-1 homomorphism. Then define $\varrho = \varphi \circ \psi$. Then $\varrho : A \to B(H)$ is a 1-1 map. Finally define $\alpha : B \to U$ as $b \mapsto \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ for every $b \in B$. So once again by the previous theorem there exists $\gamma : U \to B(H)$ which is a 1-1 homomorphism. Then define $\omega = \gamma \circ \alpha$. Then $\omega : B \to B(H)$ is a 1-1 map. Thus we can identify A and B as two subalgebras of $B(H)$. Then one can easily check that the map $\pi : X \to B(H)$ is a 1-1 $A - B$ bimodule map.

Corollary 3.5. *Any finite dimensional $A - B$ module X is an operator module.*

Proof. This also follows directly from the definitions of an operator module and the above theorem.

We will now concern ourselves with ways to create operator algebras from other operator algebras.

Define the following construction. Let V be a linear subspace of $B(H)$ then define

$$B_V = \left\{ \begin{bmatrix} \lambda I & u \\ 0 & \lambda I \end{bmatrix} \ : \ \lambda \in \mathbb{F} \ \& \ u \in V \right\}.$$

Note that B_V is an operator algebra, a subalgebra of $M_2(B(H))$.

Lemma 3.6. *Let V and W be subspaces of $B(H)$ and $B(K)$, respectively. Then $\theta : B_V \to B_W$ is a 1-1, onto, homomorphism if and only if*

$$\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right) = \begin{bmatrix} \lambda I & \rho(v) \\ 0 & \lambda I \end{bmatrix}$$

where $\rho : V \to W$ is a 1-1, onto, linear map.

Proof. We will begin by looking at possible idempotents. Define $x = \begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix} \in B_W$ then $x^2 = \begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix} = \begin{bmatrix} \lambda^2 I & 2\lambda v \\ 0 & \lambda^2 I \end{bmatrix}$. Since $x = x^2$ then λ is either 0 or 1 and $v = 2\lambda v \Rightarrow 0 = (2\lambda - 1)v$. Therefore either $2\lambda - 1 = 0$ or $v = 0$, but by possible values of λ, $2\lambda - 1 = -1$ or 1. Thus $v = 0$ for all idempotents. Thus the only idempotents are the identity matrix and the zero matrix. Since $\theta([I])^2 = \theta([I])$ and θ is linear and 1-1, thus $\ker\theta = \{0\}$, we then have that $\theta([I]) = [I]$. Let $C = \theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right)$. Then $C^2 = \theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right)^2 = \theta(0) = 0$. Now we will examine possible nilpotents. Assume $x^2 = 0$ for $x \in C_W$ then $x^2 = \begin{bmatrix} \lambda^2 I & 2\lambda w \\ 0 & \lambda^2 I \end{bmatrix} = [0]$ for some $w \in W$. Therefore $\lambda = 0$ for every nilpotent. So since $C \in C_W$ then $\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$ for some $z \in W$. Then define $\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \rho(v) \\ 0 & 0 \end{bmatrix}$ where $\rho : V \to W$. So combining these ideas we have $\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right) = \theta\left(\begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I \end{bmatrix} + \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \lambda\theta(I) + C = \lambda[I] + \begin{bmatrix} 0 & \rho(v) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda I & \rho(v) \\ 0 & \lambda I \end{bmatrix}$. Thus this is the only way to define θ. The proof that ρ is 1-1, onto, and linear follows directly from what we know about θ.

($\Leftarrow$) Let $x \in B_W$. Then there exists $v \in V$ such that $x = \begin{bmatrix} \lambda I & \rho(v) \\ 0 & \lambda I \end{bmatrix} = \theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right)$. Thus θ is onto. Suppose $\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right) = \theta\left(\begin{bmatrix} \gamma I & \acute{v} \\ 0 & \gamma I \end{bmatrix}\right)$. Then $\begin{bmatrix} \lambda I & \rho(v) \\ 0 & \lambda I \end{bmatrix} = \begin{bmatrix} \gamma I & \rho(\acute{v}) \\ 0 & \gamma I \end{bmatrix}$. So that $\lambda = \gamma$, and $\rho(v) = \rho(\acute{v})$. Since ρ is 1-1 then $v = \acute{v}$, thus θ is 1-1. Now to show it is a homomorphism, let $x, y \in B_V$. Then there exists $u, v \in V$ such that

$$\theta(xy) = \theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\begin{bmatrix} \gamma I & u \\ 0 & \gamma I \end{bmatrix}\right) = \theta\left(\begin{bmatrix} \lambda\gamma I & \lambda u + \gamma v \\ 0 & \lambda\gamma I \end{bmatrix}\right) =$$

$$\begin{bmatrix} \lambda\gamma I & \rho(\lambda u+\gamma v) \\ 0 & \lambda\gamma I \end{bmatrix} = \begin{bmatrix} \lambda\gamma I & \lambda\rho(u)+\gamma\rho(v) \\ 0 & \lambda\gamma I \end{bmatrix} = \begin{bmatrix} \lambda I & \rho(v) \\ 0 & \lambda I \end{bmatrix} \begin{bmatrix} \gamma I & \rho(u) \\ 0 & \gamma I \end{bmatrix} =$$

$$\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right) \cdot \theta\left(\begin{bmatrix} \gamma I & u \\ 0 & \gamma I \end{bmatrix}\right) = \theta(x)\theta(y).$$

Thus it is a homomorphism.

The next two lemmas answer questions concerning when θ and ρ are isometric and completely isometric. It will be assumed that θ and ρ are defined as previously.

Lemma 3.7. *θ is isometric if and only if ρ is isometric.*

Proof. ($\Rightarrow$) Let $v \in V$. Then $\|v\| = \left\|\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\|$. Since θ is an isometry then $\left\|\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\| = \left\|\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right)\right\| = \left\|\begin{bmatrix} 0 & \rho(v) \\ 0 & 0 \end{bmatrix}\right\| = \|\rho(v)\|$. So by the string of equalities ρ is an isometry.

($\Leftarrow$) It is an easy consequence of a result by Foias and Frazho in chapter IV of [5] that $\left\|\begin{bmatrix} \lambda I_H & S \\ 0 & \mu I_H \end{bmatrix}\right\| = f(|\lambda|, |\mu|, \|S\|)$ for some function f. Therefore $\left\|\begin{bmatrix} \lambda I_H & S \\ 0 & \mu I_H \end{bmatrix}\right\| = \left\|\begin{bmatrix} \lambda I_K & Q \\ 0 & \mu I_K \end{bmatrix}\right\|$ if and only if $\|S\| = \|Q\|$. So if $\rho : V \to W$ is an isometry then $\left\|\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right\| = \left\|\begin{bmatrix} \lambda I & \rho(v) \\ 0 & \lambda I \end{bmatrix}\right\| = \left\|\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix}\right)\right\|$ Therefore θ is an isometry.

Lemma 3.8. *θ is completely isometric if and only if ρ is completely isometric.*

Proof. ($\Rightarrow$) Let $v_{ij} \in V$. Then $\|[v_{ij}]\|_n = \left\|\begin{bmatrix} 0 & v_{ij} \\ 0 & 0 \end{bmatrix}\right\|_n$. Since θ is a complete isometry then $\left\|\begin{bmatrix} 0 & v_{ij} \\ 0 & 0 \end{bmatrix}\right\|_n = \left\|\theta_n\left(\begin{bmatrix} 0 & v_{ij} \\ 0 & 0 \end{bmatrix}\right)\right\|_n = \left\|\begin{bmatrix} 0 & \rho(v_{ij}) \\ 0 & 0 \end{bmatrix}\right\|_n = \|[\rho(v_{ij})]\|_n$. So by the string of equalities ρ is a complete isometry.

($\Leftarrow$) Paulsen has proven that if ρ is a complete isometry then the function $\theta : \begin{bmatrix} \lambda I & x \\ y^* & \mu I \end{bmatrix} \to \begin{bmatrix} \lambda I & \rho(x) \\ \rho(y)^* & \mu I \end{bmatrix}$ for $x, y \in V$ is a complete isometry. Setting $y = 0$ and $\mu = \lambda$ we have our necessary result.

From these results we are now able to create many different operator algebras from ones we are already aware of. The last two lemmas we proved provide details on exactly which are operator algebras and which are not. Following directly from this example we can prove similar results about another type of example.

We will take any subspace $U \subset B(H)$ and look at the subalgebra A_U of $M_2(B(H)) \cong B(H^{(2)})$ consisting of the following matrices:

$$A_U = \left\{\begin{bmatrix} \lambda I & u \\ 0 & \mu I \end{bmatrix} \;:\; \lambda, \mu \in \mathbb{F} \;\&\; u \in U\right\}.$$

The following three lemmas we will combine to create a new set of examples of operator algebras.

Lemma 3.9. *Let V and W be subspaces of $B(H)$ and $B(K)$, respectively. Then $\theta : A_V \to A_W$ is a 1-1, onto, isometric, homomorphism if and only if*

$$\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \mu I \end{bmatrix}\right) = \begin{bmatrix} \lambda I & \rho(v) \\ 0 & \mu I \end{bmatrix}$$

where $\rho : V \to W$ is a 1-1, onto, isometric, linear map.

Proof. ($\Rightarrow$) We will begin by noting that

$$\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right)^2 = \theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \theta(0) = 0.$$

So $\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right)$ is a nilpotent element. Now we will examine possible nilpotents in A_W. Define $x = \begin{bmatrix} \lambda I & v \\ 0 & \mu I \end{bmatrix} \in A_W$ to be a nilpotent with $x^2 = 0$, then $x^2 = \begin{bmatrix} \lambda^2 I & (\lambda+\mu)v \\ 0 & \mu^2 I \end{bmatrix} = 0$. This implies that $\lambda = \mu = 0$. Therefore $\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}$ for some $w \in W$. Define $\theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \rho(v) \\ 0 & 0 \end{bmatrix}$ where $\rho : V \to W$. Now define $B = \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right)$ and $C = \theta\left(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right)$. Then note that

$$B^2 = \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^2\right) = \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = B.$$

Similarly $C^2 = C$. Therefore B and C are idempotents. Also

$$B \cdot C = \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right)\theta\left(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right) = \theta\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right) = \theta\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$

and similarly $C \cdot B = 0$, so $B \cdot C = C \cdot B = 0$. Now we will examine the possible forms that idempotents can be. Define $x = \begin{bmatrix} \lambda I & v \\ 0 & \mu I \end{bmatrix} \in A_V$, then $x^2 = \begin{bmatrix} \lambda^2 I & \lambda v + \mu v \\ 0 & \mu^2 I \end{bmatrix}$. Therefore if $x = x^2$, the possible values for λ and μ are 0 and 1 and $v = \lambda v + \mu v$ implies $(\lambda+\mu-1)v = 0$. Thus either $\lambda + \mu = 1$ or $v = 0$. Therefore B and C are each one of these three forms

$$\begin{bmatrix} I & v \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & v \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

for some $v \in V$ Since $B \cdot C = C \cdot B = 0$ then neither can be of the last form. Note that $\|B\| = \|C\| = 1$. Since θ is an isometry then

$$\left\|\begin{bmatrix} I & v \\ 0 & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 0 & v \\ 0 & I \end{bmatrix}\right\| = 1.$$

Using the C^* identity we can see that the norm of either of these elements is $\sqrt{1+|v|^2}$. Thus $v = 0$ in both cases. Assume that $C = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ then $C \cdot \theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = 0 \Rightarrow \theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ where $a, b \in W$. This is a contradiction. Thus $C = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ and $B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Combining all of this we then have

$$\theta\left(\begin{bmatrix} \lambda I & v \\ 0 & \mu I \end{bmatrix}\right) = \theta\left(\begin{bmatrix} \lambda I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mu I \end{bmatrix}\right) =$$

$$\lambda B + \theta\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) + \mu C = \begin{bmatrix} \lambda I & \rho(v) \\ 0 & \mu I \end{bmatrix}.$$

We have shown that this is the only way to define θ. It can be shown that ρ is 1-1, onto, and linear quite easily from what is known about θ.

($\Leftarrow$) The proof this direction is also very similar to the proof provided in 3.6.

The result concerning complete isometries that we proved for the previous example carry over nicely to this example. Allow θ and ρ to be defined as in the previous example.

Corollary 3.10. *θ is a complete isometry if and only if ρ is a complete isometry.*

Proof. This also follows from the proof from the last example.

These previous results enable us to look at examples of matrices which are isometrically isomorphic but not completely isometrically isomorphic. Our next example is concerned with another such set of matrices.

First we will define a few different algebras. Let RI_n be an algebra with elements of the form $\begin{bmatrix} a & b_1 & b_2 & \dots & b_{n-1} \\ 0 & c & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c \end{bmatrix}$, where $a, b_i, c \in \mathbb{F}$. Similarly we will define CI_n to be an algebra with elements of the form $\begin{bmatrix} a & 0 & 0 & \dots & 0 \\ b_1 & c & 0 & \dots & 0 \\ b_2 & 0 & c & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-1} & 0 & 0 & \dots & c \end{bmatrix}$, where $a, b_i, c \in \mathbb{F}$. Note that both RI_n and CI_n are subalgebras of M_n.

Example 3.11. *RI_n and CI_n are isometric but not completely isometric.*

Proof. We will first begin by showing that RI_n and CI_n are isometric. This is quite easy to see by applying the C^*-identity to both of them. Let $\rho : CI_n \to RI_n$ be the mapping defined as

$$\begin{bmatrix} a & 0 & 0 & ... & 0 \\ b_1 & c & 0 & ... & 0 \\ b_2 & 0 & c & ... & 0 \\ ... & ... & ... & ... & ... \\ b_{n-1} & 0 & 0 & ... & c \end{bmatrix} \mapsto \begin{bmatrix} a & b_1 & b_2 & ... & b_{n-1} \\ 0 & c & 0 & ... & 0 \\ 0 & 0 & c & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & c \end{bmatrix}.$$

It is well known that the transpose map is isometric and ρ is the transpose map. Thus ρ is isometric. It is quite easy to show that ρ is a 1-1, onto, homomorphism. Therefore we have that $RI_n \cong CI_n$ isometrically isomorphically. To show that they are not completely isometric we will need the following constructions. Note that the constructions are isometric. Let $r \in RI_n$. Then construct the following matrix $\begin{bmatrix} aI_{n-2} & 0 \\ 0 & r \end{bmatrix}$. Note that $\left\| \begin{bmatrix} aI_{n-2} & 0 \\ 0 & r \end{bmatrix} \right\| = \|r\|$. By switching rows and columns this matrix can be made into the form $\begin{bmatrix} aI_{n-1} & r' \\ 0 & cI_{n-1} \end{bmatrix}$ where $r' \in R_{n-1}$. So then $\left\| \begin{bmatrix} aI_{n-1} & r' \\ 0 & cI_{n-1} \end{bmatrix} \right\| = \|r\|$. Similarly by switching rows and columns the matrix $\begin{bmatrix} s & 0 \\ 0 & cI_{n-1} \end{bmatrix}$ with $s \in CI_n$ can be transformed into the matrix $\begin{bmatrix} cI_{n-1} & s' \\ 0 & aI_{n-1} \end{bmatrix}$ for $s' \in C_{n-1}$. With $\left\| \begin{bmatrix} cI_{n-1} & s' \\ 0 & aI_{n-1} \end{bmatrix} \right\| = \|s\|$. Therefore we have shown that $RI_n \cong A_{R_{n-1}}$ completely isometrically and $CI_n \cong A_{C_{n-1}}$ completely isometrically. So $CI_n \cong RI_n$ completely isometrically if and only if $A_{R_n} \cong A_{C_n}$ completely isometrically. Lemma 3.10 showed that $A_{R_n} \cong A_{C_n}$ completely isometrically if and only if $C_n \cong R_n$ completely isometrically. We showed in Theorem 2.1 that C_n is not completely isometric to R_n therefore CI_n is not completely isometric to RI_n. So we have another example of two subalgebras of M_n which are isometric to one another but not completely isometric.

4. Algebra and operator algebras

In this section we will examine a few ideas from algebra and consider the corresponding ideas in the context of operator algebras. In particular, we will generalize the idea that the linear transformations on an n-dimensional vector space are isomorphic to the $n \times n$ matrices.

We will examine the algebraic idea that the for any ring or unital algebra A the A-module homomorphisms from $A^{(n)}$ to itself are isomorphic to the $n \times n$ matrices with entries in A, i.e. $Hom_A(A^{(n)}) \cong M_n(A)$ isomorphically. Note that in the case that $A = \mathbb{F}$ this becomes $M_n \cong Lin(\mathbb{F})$ isomorphically, which as discussed in the introduction, is central to much of mathematics. To investigate if there is any good isometric version of this relation, it is necessary to assign a norm to the different elements and make the isomorphism an isometric isomorphism. For this portion of the paper let A be a unital operator algebra, whose identity has norm 1. Note then by the discussion in section 1, $M_n(A)$ has a canonical norm on it. Throughout the rest of the paper $C_n(A)$ will be defined in the same manner as C_n except entries will be in A rather then in $\mathbb{F}$. We often think of $A^{(n)}$ as being a vector of length n with entries in A. Following this line of reasoning it makes sense to view $A^{(n)}$ as the first column of an $M_n(A)$, i.e. $A^{(n)} = C_n(A)$. Since $C_n(A) \subset M_n(A)$ it therefore has a norm on it also. So it seems very natural to place a norm on $A^{(n)}$ in this manner. The following lemma shows that one direction of the relationship $M_n(A) \cong B_A(C_n(A))$ isometrically isomorphically holds up.

Lemma 4.1. *The canonical map $\theta : M_n(A) \to B_A(C_n(A))$ is a contraction.*

Proof. For any $x \in M_n(A)$ $\|\theta(x)\| = \sup\{\|x\vec{z}\| : \|\vec{z}\| \leq 1\} = \sup\{\|[x\vec{z} : \vec{0} : ... : \vec{0}]\| : \|\vec{z}\| \leq 1\}$ where $[x\vec{z} : \vec{0} : ... : \vec{0}]$ is a square matrix. We then have that $\|\theta(x)\| = \sup\{\|x[\vec{z} : \vec{0} : ... : \vec{0}]\| : \|\vec{z}\| \leq 1\} \leq \sup\{\|x\|\|[\vec{z} : \vec{0} : ... : \vec{0}]\| : \|\vec{z}\| \leq 1\} = \|x\| \sup\{\|[\vec{z} : \vec{0} : ... : \vec{0}]\| : \|\vec{z}\| \leq 1\} = \|x\|$. Thus θ is a contraction.

The example below shows that this relationship does not hold up so well in the other direction.

Example 4.2. *$M_n(A)$ is not necessarily isometrically isomorphic to $B_A(C_n(A))$ via the canonical map.*

Proof. Let A be the algebra with elements of the form $\begin{bmatrix} a & c & d & e \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$ with $a, b, c, d, e \in \mathbb{F}$. It is easy to check that A is an operator algebra. Let

$$r = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \in M_2(A).$$

It is easy to calculate using the C^*-identity that $\|r\| = \sqrt{2}$. Let $\theta : M_2(A) \to B_A(C_2(A))$ be the canonical map. Then $\|\theta(r)\| =$

$$\sup\left\{ \left\| \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a & b & c & d \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \\ \begin{bmatrix} f & g & h & j \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix} \end{bmatrix} \right\| : \left\| \begin{bmatrix} \begin{bmatrix} a & b & c & d \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \\ \begin{bmatrix} f & g & h & j \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix} \end{bmatrix} \right\| \leq 1 \right\}$$

$$= \sup\left\{ \left\| \begin{bmatrix} \begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} & \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a&b&c&d\\0&e&0&0\\0&0&e&0\\0&0&0&e\\f&g&h&j\\0&k&0&0\\0&0&k&0\\0&0&0&k \end{bmatrix} \end{bmatrix} \right\| : \left\| \begin{bmatrix} a&b&c&d\\f&g&h&j\\0&e&0&0\\0&k&0&0\\0&0&e&0\\0&0&k&0\\0&0&0&e\\0&0&0&k \end{bmatrix} \right\| \le 1 \right\}.$$

By Lemma 2.2 $\left\| \begin{bmatrix} a&b&c&d\\f&g&h&j\\0&e&0&0\\0&k&0&0\\0&0&e&0\\0&0&k&0\\0&0&0&e\\0&0&0&k \end{bmatrix} \right\| \ge \left\| \begin{bmatrix} e\\k \end{bmatrix} \right\|$. So then

$$\|\theta(r)\| = \sup\{\|[e\ \ k]\| : \left\| \begin{bmatrix} e\\k \end{bmatrix} \right\| \le 1\} = 1.$$

Thus $\|\theta(r)\|$ is not equal to $\|r\|$.

The next logical question is whether or not there is a different type of norm that could be assigned that would make the relationship $M_n(A) \cong B_A(A^{(n)})$ isometrically isomorphically true. There only seems to be one logical way to place a norm on $M_n(A)$ as explained in the introduction. Thus the only option we would have is to place a different norm on $A^{(n)}$. The obvious possibilities are the following, where $\vec{a}$ is a vector of size n with entries from A,

$$\|\vec{a}\|_p = \sqrt[p]{\sum_{i=1}^{n} \|a_i\|^p} \ : \ p \ge 1$$

$$\|\vec{a}\|_\infty = \max\{\|a_i\|\}$$

$$\|\vec{a}\|_{R_n(A)} = \| \sum_{i=1}^{n} a_i a_i^* \|^{\frac{1}{2}}$$

$$\|\vec{a}\|_{C_n(A)} = \| \sum_{i=1}^{n} a_i^* a_i \|^{\frac{1}{2}}.$$

Previously we showed that the last norm listed would not work. The following lemma and corollary show that none of these nor any other sensible norm will work for the relationship.

First define a *sensible norm*, $|||\cdot|||$, as one that satisfies $\left|\left|\left| \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \right|\right|\right| \geq \|a_i\|$ for $a_i \in A$ and $\left|\left|\left| \begin{bmatrix} a_1 \\ 0 \\ ... \\ 0 \end{bmatrix} \right|\right|\right| = \|a_1\|$.

Lemma 4.3. *There does not exist any sensible norm on $A^{(n)}$ for which the canonical map $\theta : M_n(A) \to B_A(A^{(n)})$ is an isometric isomorphism and which makes $A^{(n)}$ a Banach A-module.*

Proof. Let $\theta : M_n(A) \to B_A(A^{(n)})$ be the canonical map. Suppose that there exists a norm, $|||\cdot|||$, on $A^{(n)}$, with the property that $\left|\left|\left| \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \right|\right|\right| \geq |||a_i|||$ for $a_i \in A$, which makes θ an isometric isomorphism and makes $A^{(n)}$ a Banach A-module. Then consider the map $\phi : A^{(n)} \to A^{(n)}$ defined as $\begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \mapsto \begin{bmatrix} b_1a_1 \\ b_2a_1 \\ ... \\ b_na_1 \end{bmatrix}$ for some fixed $b_1, b_2, ..., b_n \in A$. Note that ϕ is a linear right A-module map. Indeed $\phi(\vec{a}) = \begin{bmatrix} b_1 & 0 & ... & 0 \\ b_2 & 0 & ... & 0 \\ ... & ... & ... & ... \\ b_n & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix}$. That is $\theta(B) = \phi$ where $B = \begin{bmatrix} b_1 & 0 & ... & 0 \\ b_2 & 0 & ... & 0 \\ ... & ... & ... & ... \\ b_n & 0 & ... & 0 \end{bmatrix}$. Since θ is isometric by assumption then $\|\phi\| = \|B\| = \|\sum_{i=1}^n b_i^* b_i\|^{\frac{1}{2}}$. By definition

$$\|\phi\| = \sup\left\{ \left|\left|\left| B \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \right|\right|\right| : \left|\left|\left| \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \right|\right|\right| \leq 1 \right\} = \sup\left\{ \left|\left|\left| \begin{bmatrix} b_1a_1 \\ b_2a_1 \\ ... \\ b_na_1 \end{bmatrix} \right|\right|\right| : \left|\left|\left| \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \right|\right|\right| \leq 1 \right\}.$$

By our assumptions then we have that

$$\|\phi\| = \sup\left\{ \left|\left|\left| \begin{bmatrix} b_1a_1 \\ b_2a_1 \\ ... \\ b_na_1 \end{bmatrix} \right|\right|\right| : \|a_1\| \leq 1 \right\} \geq \left|\left|\left| \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} \right|\right|\right|.$$

This uses the assumption that the norm is sensible. Following from the fact that $A^{(n)}$ is a Banach A-module, we have that

$$\|\phi\| = \sup\left\{\left|\left|\left|\begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} a_1\right|\right|\right| : \|a_1\| \leq 1\right\} \leq \sup\left\{\left|\left|\left|\begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}\right|\right|\right| \|a_1\| : \|a_1\| \leq 1\right\} = \left|\left|\left|\begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}\right|\right|\right|.$$

Therefore $\left|\left|\left|\begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}\right|\right|\right| = \|\phi\| = \|\sum_{i=1}^n b_i^* b_i\|^{\frac{1}{2}}$ by what we showed earlier in the proof. So with these assumptions the norm

$$||| \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} ||| = \| \sum_{i=1}^n b_i^* b_i \|^{\frac{1}{2}} = \| \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} \|_{C_n(A)}.$$

Previously in the paper we showed that $M_n(A)$ is not isometrically isomorphic to $B_A(A^{(n)})$ when the norm on $A^{(n)}$ is the $C_n(A)$ norm. Thus we have a contradiction. and there does not exist a norm on $A^{(n)}$ that satisfies our conditions.

So we have shown that in this manner we can not make the relationship isometric. Some would argue that the relationship between $M_n(A)$ and $B_A(C_n(A))$ does not need to be isometric, rather it would just need to be bicontinuous. A function $\theta : X \to Y$ is *bicontinuous* if there exists $k_1, k_2 > 0$ such that $k_1\|x\| \leq \|\theta(x)\| \leq k_2\|x\|$ for every $x \in X$. In some areas of analysis they are correct, these types of relationships need only be bicontinuous. In some instances though bicontinuity is not enough. In fact the lack of an isometry in this simple relation actually causes the complete breakdown of many more sophisticated ideas. To illustrate this we will look at the following simple relationship, $M_\infty^{cf}(A) \cong Hom_A(A^{(\infty)})$ isomorphically. By $M_\infty^{cf}(A)$ we mean the infinite matrices, $\begin{bmatrix} a_{11} & a_{12} & ... \\ a_{21} & a_{22} & ... \\ ... & ... & ... \end{bmatrix}$, with $a_i \in A$ which are column finite, that is finitely many entries of each column are not the zero element. We will also define $A^{(\infty)}$ as the algebraic direct sum of a countably infinite number of copies of A. That is, $A^{(n)}$ is the set of $(a_i)_{i=1}^\infty$ where all but finitely many a_i are the zero element. Before anything else we will verify this algebraic relationship.

Proposition 4.4. *$M_\infty^{cf}(A) \cong Hom_A(A^{(\infty)})$ isomorphically.*

Proof. Let $\theta : M_\infty^{cf}(A) \to Hom_A(A^{(\infty)})$ be the canonical mapping, i.e. $\theta(B)(a) = Ba$ for $B \in M_\infty^{cf}(A)$ and $a \in A^{(\infty)}$. First we will show θ is linear. Let $B_1, B_2 \in M_\infty^{cf}(A)$. Then $\theta(B_1 + B_2)(x) = B_1x + B_2x = \theta(B_1)(x) + \theta(B_2)(x)$ for $x \in A^{(\infty)}$. Similarly let $\lambda \in \mathbb{F}$ and $B \in M_\infty^{cf}(A)$ then $\theta(\lambda B)(x) = \lambda Bx = \lambda\theta(B)(x)$for $x \in A^{(\infty)}$. Therefore θ is linear. Let $\theta(A) = 0$ then $\theta(A)(\vec{x}) = A\vec{x} = 0$ for every $\vec{x} \in A^{(\infty)}$. Define $\vec{e_i}$ as the vector with all zeros except 1 in the ith entry. Then $A\vec{e_i}$ is the ith column of A. We then have that for any i, $A\vec{e_i} = 0$. Therefore every column of A is 0. Thus $A = 0$ and θ is 1-1. Given $T \in Hom_A(A^{(\infty)})$ define $\vec{v_i} = T(\vec{e_i})$ where $\vec{e_i}$ is the vector of infinite size with zero entries except the identity in the ith position. Then let $B = [\vec{v_1} : \vec{v_2} : ...]$. Since $T(\vec{e_i}) \in A^{(\infty)}$ then

$B \in M_\infty^{cf}(A)$. We now need to check that $\theta(B) = T$. Let $\vec{v} \in A^{(\infty)}$ then express $\vec{v}$ as a finite sum $\vec{v} = \sum_{i=1}^n \vec{e_i}a_i$ where the a_i are uniquely determined by the elements $\vec{e_i}$. Then $T(\vec{v}) = T(\sum_{i=1}^n \vec{e_i}a_i) = \sum_{i=1}^n T(\vec{e_i})a_i = \sum_{i=1}^n \vec{v_i}a_i$ using the fact that T is linear. Similarly $\theta(B)(\vec{v}) = B\vec{v} = B\sum_{i=1}^n \vec{e_i}a_i = \sum_{i=1}^n B\vec{e_i}a_i = \sum_{i=1}^n \vec{v_i}a_i$. Therefore we have the equality we were looking for. Let $x, y \in M_\infty^{cf}(A)$. Denote $x = \begin{bmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}$ and $y = \begin{bmatrix} y_{11} & y_{12} & \dots \\ y_{21} & y_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}$. Note that it can be shown that $M_\infty^{cf}(A)$ is an algebra and thus $xy \in M_\infty^{cf}(A)$. Let $\vec{z} \in A^{(\infty)}$, then $\theta(x)\theta(y)(\vec{z}) = \theta(x)\left(\begin{bmatrix} y_{11} & y_{12} & \dots \\ y_{21} & y_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}\vec{z}\right) = \begin{bmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}\begin{bmatrix} y_{11} & y_{12} & \dots \\ y_{21} & y_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix}\vec{z} = \theta(xy)(\vec{z})$. Thus θ is a homomorphism. Thus $M_\infty^{cf}(A) \cong Hom_A(A^{(\infty)})$ isomorphically.

We will now try to move this idea over to the realm of analysis we have been discussing. Once again it makes sense to replace $Hom_A(A^{(\infty)})$ with $B_A(A^{(\infty)})$ by the same reasoning as before. We will denote $M_\infty^*(A) = \theta^{-1}(B_A(A^{(\infty)})$ where θ is the canonical 1-1 mapping described in the previous theorem. Let $\theta' : M_\infty^*(A) \to B_A(A^{(\infty)})$ be the associated restriction of θ. We will also define $M_\infty^f(A)$ to be the set of infinite matrices with a finite number of nonzero entries. We now can look at the following example.

Example 4.5. *The canonical $\theta' : M_\infty^*(A) \to B_A(A^{(\infty)})$ is not necessarily bicontinuous.*

Proof. We will check that $M_\infty^f(A) \subseteq M_\infty^*(A)$. Note that for any element of $B \in M_\infty^f(A)$,

$$B = \begin{bmatrix} a_{11} & \dots & a_{1n} & 0 & \dots \\ \dots & \dots & \dots & 0 & \dots \\ a_{n1} & \dots & a_{nn} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} = \sum x_{ij},$$

where x_{ij} is the matrix with all zero entries except a_{ij} in the ijth entry with $i, j \leq n$. This is a finite sum because the number of x_{ij} are finite. Then $\theta'(x_{ij})(\vec{z})$ is the vector with all zero entries except $a_{ij}z_j$ in the ith entry where z_j is the jth entry of $\vec{z}$, denote this as $[a_{ij}z_j]_i$. Then $\|\theta'(x_{ij})(\vec{z})\| = \|[a_{ij}z_j]_i\| = \|a_{ij}z_j\| \leq \|a_{ij}\|\|z_{ij}\| \leq \|a_{ij}\|\|\vec{z}\|$ for every $\vec{z} \in A^{(\infty)}$. Thus $\theta'(x_{ij})$ is bounded. Since θ' is linear then $\theta'(B)$ is a sum of bounded operators and thus is bounded. So $M_\infty^f(A) \subseteq M_\infty^*(A)$. So it will suffice to show that for every $n \in \mathbb{N}$ there exists an $x_n \in M_\infty^f(A)$ such that $\frac{\|\theta(x_n)\|}{\|x_n\|} \leq \frac{1}{n}$. This implies that there does not exist any constant k_1 such that $k_1\|x\| \leq \|\theta(x)\|$ for every $x \in M_\infty^f(A)$. Therefore θ would not be bicontinuous.

Let A be a subalgebra of $M_\infty(\mathbb{F})$ with elements of the form $\begin{bmatrix} a & c_1 & c_2 & c_3 & \dots \\ 0 & b & 0 & \dots & \dots \\ 0 & 0 & b & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$ that have a finite norm. Then let $n \in \mathbb{N}$ and write $x_n = [a_{ij}]$ where $a_{ij} \in A$ with a_{ij} the zero element if $i \geq 2$, $j > n^2 + 1$ or $i = j = 1$, otherwise let $a_{1,j}$ be a matrix with elements in A

with a 1_A in the $1,j$ entry and zeros everywhere else, i.e.

$$x_n = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & ... \\ 0 & 0 & 0 & ... \\ ... & ... & ... & ... \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & ... \\ 0 & 0 & 0 & ... \\ ... & ... & ... & ... \end{bmatrix} & ... & \begin{bmatrix} 0 & 0 & ... & 1 & ... \\ 0 & 0 & ... & 0 & ... \\ ... & ... & ... & ... & ... \end{bmatrix} & [0] & ... \\ [0] & [0] & [0] & [0] & [0] & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}.$$

Then $\|x_n\| = \sqrt{n^2} = n$ and $\|\theta(x_n)\| = \sup\{\|x_n[y_j]_{\infty,1}\| : \|[y_j]_{\infty,1}\| \leq 1\}$ where

$$y_j = \begin{bmatrix} a_j & c_{j,1} & c_{j,2} & c_{j,3} & ... \\ 0 & b_j & 0 & ... & \\ 0 & 0 & b_j & ... & \\ ... & ... & ... & ... & \end{bmatrix}.$$

Using the lemma proven before in this paper, computation, and the fact that the transpose of rows and columns does not change the norm we have

$$\|\theta(x_n)\| \leq \sup \left\{ [b_2 : b_3 : ... : b_{n^2+1}] : \left\| \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_{n^2+1} \end{bmatrix} \right\| \leq 1 \right\} =$$

$$\sup\{\sqrt{|b_2|^2 + |b_3|^2 + ... + |b_{n^2+1}|^2} : \sqrt{|b_1|^2 + |b_2|^2 + ... + |b_{n^2+1}|^2} \leq 1\} = 1.$$

Thus $\frac{\|\theta(x_n)\|}{\|x_n\|} \leq \frac{1}{n}$ for every $n \in \mathbb{N}$.

This last example shows that even bicontinuity is not assured when things are transferred from the algebraic world to the realm of analysis. With things falling apart in this fairly simple example it would seem there is little hope when looking at the bigger picture.

Our first attempts at transferring these ideas into the realm of analysis has not been very successful. This may be because we have been working in the wrong framework. It is possible that if we looked at something with more structure maybe these ideas would hold together better. We will try restricting ourselves to the $CB_A(C_n(A))$, i.e. the completely bounded right A-module maps from $C_n(A) \to C_n(A)$, instead of just $B_A(C_n(A))$ as before.

Theorem 4.6. *$CB_A(C_n(A)) \cong M_n(A)$ isometrically isomorphically via the canonical map.*

Proof. Let $\theta : M_n(A) \to CB_A(C_n(A))$ be the canonical map. First we will check that this map is isomorphic. For any $x, y \in M_n(A)$, $\theta(x+y)(\vec{z}) = x(\vec{z}) + y(\vec{z}) = \theta(x)(\vec{z}) + \theta(y)(\vec{z})$ for $\vec{z} \in C_n(A)$. Similarly for $x \in M_n(A)$ and scalar λ, $\theta(\lambda x)(\vec{z}) = \lambda x(\vec{z}) = \lambda\theta(x)(\vec{z})$ for $\vec{z} \in C_n(A)$. So θ is linear. Then if $\theta(A) = 0$ then $Ax = 0$ for every $x \in C_n(A)$. Let $E_{i,1} \in C_n(A)$ denote the matrix with all entries zeros except 1 in the $i,1$ position. Then $AE_{i,1}$ is the matrix with the ith column of A in the first column and the rest zeros. Thus since $AE_{i,1} = 0$ for every i, $A = 0$. So θ is 1-1. Let $T \in CB_A(C_n(A))$ then define $\vec{v_i}$ to be the first column of $T(E_{i,1})$ and let $B = [\vec{v_1} : \vec{v_2} : ... : \vec{v_n}]$. B is obviously an $n \times n$ matrix with entries in A and thus is in $M_n(A)$. Then let $\{e_1, e_2, ..., e_n\}$ be the standard orthonormal basis for C_n. Then for any $x \in C_n(A)$ there exists unique $a_i \in M_n(A)$ such that $x = \sum_{i=1}^n e_i a_i$. Thus $T(x) = T(\sum_{i=1}^n e_i a_i) = \sum_{i=1}^n T(e_i)a_i = \sum_{i=1}^n v_i a_i$. Similarly

$\theta(B)(x) = B\sum_{i=1}^{n} e_i a_i = \sum_{i=1}^{n} Be_i a_i = \sum_{i=1}^{n} v_i a_i$. Therefore $\theta(B) = T$. In a similar manner to Proposition 4.4 it can be shown that θ is a homomorphism. So we have shown that θ is isomorphic. Now we just need to show that it is isometric. First we will check that θ is a contraction. $\|\theta(x)\|_{cb} = \sup\{\|[x\vec{z_{ij}}]\|_m : \|[\vec{z_{ij}}]\|_m \leq 1\} = \sup\{\|[x\vec{z_{ij}} : \vec{0} : \vec{0} : ... : \vec{0}]\|_m : \|[\vec{z_{ij}}]\|_m \leq 1\}$ such that $[x\vec{z_{ij}} : \vec{0} : \vec{0} : ... : \vec{0}]$ is a square matrix. Then following with the equalities

$$\sup\left\{\left\|[x\vec{z_{ij}} : \vec{0} : \vec{0} : ... : \vec{0}]\right\|_m : \|[\vec{z_{ij}}]\|_m \leq 1\right\} =$$

$$\sup\{\| \begin{bmatrix} x & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & x \end{bmatrix} [\vec{z_{ij}} : \vec{0} : \vec{0} : ... : \vec{0}]\|_m : \|[\vec{z_{ij}}]\|_m \leq 1\} \leq$$

$$\left\| \begin{bmatrix} x & ... & 0 \\ 0 & ... & 0 \\ 0 & ... & x \end{bmatrix} \right\| \sup\{\|[\vec{z_{ij}} : \vec{0} : \vec{0} : ... : \vec{0}]\|_m : \|[\vec{z_{ij}}]\|_m \leq 1\} = \|x\|.$$

This shows that the θ is a contraction. Let $[\vec{y_{ij}}]$ be the matrix with $\vec{y_{ij}} = \vec{0}$ for all $i \geq 2$ and $\vec{y_{1j}}$ be the vector with all zero entries except the identity in the jth row. Then $\|[\vec{y_{ij}}]\|_m = 1$ and also $[x\vec{y_{ij}}] = x$ for every x. Therefore $\|\theta(x)\|_{cb} = \sup\{\|[x\vec{z_{ij}}]\|_m : \|[\vec{z_{ij}}]\|_m \leq 1\} \geq \|[x\vec{y_{ij}}]\|_m = \|x\|_m$. Therefore we have shown that $\|\theta(x)\|_{cb} = \|x\|$.

This result may also be given a longer indirect proof by deducing it from some general theorems in [3], for example.

We have now shown for the finite cases that this relationship holds exactly as we wanted. Before we saw though that things went drastically wrong when we started looking at matrices of infinite size. The following theorem shows though that things hold up quite well even for the infinite matrices when looking at the completely bounded maps. We will first need another definition. Define $M_\infty(A)$ to be the set of all countably infinite matrices $\begin{bmatrix} a_{11} & a_{12} & ... \\ a_{21} & a_{22} & ... \\ ... & ... & ... \end{bmatrix}$ with $a_{ij} \in A$ for which there exists $k \geq 0$ such that $k \geq \left\| \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \right\|$ for every $n \in \mathbb{N}$.

Theorem 4.7. *$M_\infty^{cf}(A) \cap M_\infty(A) \cong CB_A(A^{(\infty)})$ isometrically isomorphically via the canonical map if A is a unital operator algebra.*

Proof. Let $\theta : M_\infty^{cf}(A) \cap M_\infty(A) \to CB_A(A^{(\infty)})$ be the canonical map. It is easy to show that θ is linear and 1-1. To check that it is onto, let $T \in CB_A(A^{(\infty)})$. Then let $\vec{v_i} = T(\vec{e_i})$ where $\vec{e_i}$ is a basis element. Then let $B = [\vec{v_1} : \vec{v_2} : \vec{v_3} : ...]$. Note that $\vec{v_i} \in A^{(\infty)}$ and thus have a finite number of entries nonzero. Therefore $B \in M_\infty^{cf}(A)$. Let $n \in \mathbb{N}$ then let B_n denote the matrix that is the first n rows and columns of B. Then B_n is a finite square submatrix of $[\vec{v_1} : \vec{v_2} : ... : \vec{v_n}]$. Thus by Lemma 2.2 $\|B_n\| \leq \|[\vec{v_1} : \vec{v_2} : ... : \vec{v_n}]\| = \|[T(\vec{e_1}) : T(\vec{e_2}) : ... : T(\vec{e_n})]\| \leq \|T\|_{cb} = k$ for some constant k because T is completely bounded.

Since this is true for any $n \in \mathbb{N}$ then $B \in M_\infty(A)$. Therefore $B \in M_\infty^{cf}(A) \cap M_\infty(A)$. In a similar manner to Proposition 4.4 you can show that $T = \theta(B)$. That proposition also shows in a similar manner that θ is a homomorphism. We will now concentrate on showing that it is isometric also. Let $x \in M_\infty^{cf}(A)$. Then $\|\theta(x)\|_{cb} = \sup\{\|[x\vec{z_{ij}} : 0 : ... : 0]\|_n : \|[\vec{z_{ij}} : 0 : .. : 0]\|_n \leq 1\} \leq \sup\{\|x\|\|[\vec{z_{ij}} : 0 : ... : 0]\|_n : \|[\vec{z_{ij}} : 0 : ... : 0]\|_n \leq 1\} \leq \|x\| \sup\{\|[\vec{z_{ij}} : \vec{0} : ... : \vec{0}]_n\| : \|[\vec{z_{ij}} : \vec{0} : ... : \vec{0}]\|_n \leq 1\} = \|x\|$ for $n \in \mathbb{N}$. Thus θ is a contraction. Now we will define $[\vec{y_{ij}}]$ as an $m \times m$ matrix with $\vec{y_{ij}} = \vec{0}$ for $i \geq 2$ and $j > m$ and $\vec{y_{1j}}$ for $j \leq m$ to be the vector of infinite length with zero entries except the identity in the jth position. Note then that $\|[x\vec{y_{ij}}]\| = \|x_k\|$ where x_k is a block of x. Also note that $\|[\vec{y_{ij}}]\| = 1$. We then have that $\|\theta(x)\|_{cb} = \sup\{\|[x\vec{z_{ij}}]\|_n : \|[\vec{z_{ij}}]\|_n \leq 1\} \geq \|[x\vec{y_{ij}}]\| = \|x_k\|$ for $n \in \mathbb{N}$. Then the supremum of $\|x_k\|$ as the size of the matrix $[\vec{y_{ij}}]$ goes to infinite is $\|x\|$. Thus we have that $\|\theta(x)\| \geq \|x\|$ Thus we have shown that θ is isometric in this case.

In a very similar manner you can show that the following relationship is also true.

Theorem 4.8. *$M_\infty(A) \cong CB_A(C_\infty(A))$ isometrically isomorphically via the canonical map if A is a unital subalgebra of M_n.*

Throughout this paper we have been looking at different relationships in algebra. In many cases our first attempt at transferring them over to the realm of matrix spaces failed. We saw things that appeared to be the same turn out to be quite different when looked at with Banach space norms and then we saw algebraic relationships that did not transfer over as we thought they would have. But after closer examination we saw some ways around these problems.

Now that we have identified these problems and a context in which some of them seem to be taken care of the next step is to transfer over some of the ideas from algebra and see how they hold up in this environment. In a few cases this has already been done. One example is the Memoir [3] in which a Morita theory for operator algebras is developed. The next step will be to transfer other common ideas.

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Department of Mathematics and Statistics, University of Nebraska - Lincoln
E-mail address: shain2@bigred.unl.edu