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Paul Gallagher
University of Pennsylvania

David Hu
Georgetown University

Zane Martin
Willams College

Maggie Miller
University of Texas at Austin

Byron Perpetua
Williams College

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Paul Gallagher[a] David Hu[b] Zane Martin[c]
Maggie Miller[d] Byron Perpetua[e]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]University of Pennsylvania
[b]Georgetown University
[c]Williams College
[d]University of Texas at Austin
[e]Williams College

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Abstract. We study the isoperimetric problem in the plane with weighting or density $e^{-1/r}$. The isoperimetric problem seeks to enclose prescribed weighted area with minimum weighted perimeter. For density $e^{-1/r}$, isoperimetric curves are conjectured to pass through the origin. We provide numerical and theoretical evidence that such curves have an angle at the origin approaching 1 radian from above as area approaches zero and provide further estimates.

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1 Introduction

A *density* on a manifold is a positive function that weights volume equally in all dimensions. Manifolds with density have long been studied on an *ad hoc* basis in mathematics, and were instrumental in Perelman's proof of the Poincaré conjecture [M2, 18.11]. The isoperimetric problem in manifolds with density has been well studied, but few cases have been solved. In Gauss space, $\mathbb{R}^n$ with radially symmetric density Ce^{-br^2}, it has been proved that isoperimetric curves are hyperplanes [B], [S]. In the plane with density r^p, $p > 0$, Dahlberg *et al.* [D] proved that isoperimetric curves are circles that pass through the origin. Recently the Log-Convex Density Conjecture, which states that any radial log-convex density on $\mathbb{R}^n$ has spheres around the origin as unique isoperimetric surfaces, has been proven by Gregory Chambers [Cha].

In this paper we study the plane with radial density $e^{-1/r}$, where isoperimetric regions are conjectured to be curves passing through the origin. Intuition for this conjecture might come from a result of Dahlberg *et al.* [D, Thm. 3.16], which states that isoperimetric curves in the plane with density $r^p, p > 0$ are circles through the origin. As in the case of density r^p, the plane with density $e^{-1/r}$ has density 0 at the origin and density strictly increasing with distance from the origin. Although an isoperimetric curve must be smooth away from the origin, we provide numerical and theoretical evidence that it passes through the origin at a sharp angle, decreasing to 1 radian as the enclosed area approaches 0 as in Figure 1. Our computations show that other candidate (constant generalized curvature) curves either do not close up or have unstable oscillation as in Figure 2. In Remark 2.3, we give a lower bound on the weighted perimeter required to enclose a given weighted area. Propositions 2.5 and 2.6 establish the existence of isoperimetric curves enclosing any area and describe their behavior away from the origin. Throughout the remainder of the paper, we discuss behavior at the origin. By Proposition 2.10, an isoperimetric curve through the origin must be continuous differentiable there on one side. Propositions 2.11 and 2.12 prove that a certain class of isoperimetric curves cannot have a cusp at the origin, while Proposition 2.13 demonstrates the existence along an isoperimetric curve of exactly two extrema of distance from the origin. Finally, in Proposition 2.15, we show that perimeter-minimizing circular sectors at the origin, which approximate our conjectured isoperimetric curves, approach an angle of 1 radian for sufficiently small area.

2 Isoperimetric problem in the plane with density $e^{-1/r}$

We study the plane with density $e^{-1/r}$. Kolesnikov and Zhdanov [K, Sect. 5] have found interesting numerical results on isoperimetric curves in the plane with density Ce^{-r^α} for $\alpha > 1$. The $\alpha = 2$ case corresponds to Gaussian density, for which isoperimetric curves have been proven by Borell [B] and Sudakov and Tsirelson [S] to be straight lines. As α decreases from 2 to 1, they discover a transition in isoperimetric curves from the straight line to curved lines to large closed curves and circles around (though not centered at) the origin. As α increases from 2, they see the same curving of the straight line in the other

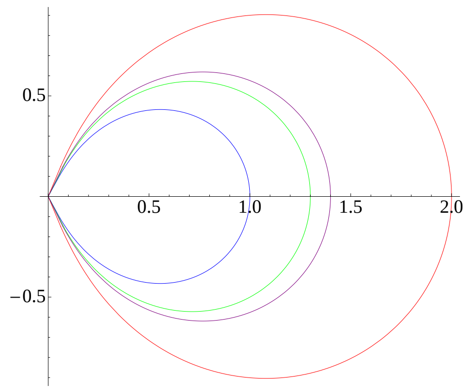

Figure 1: The conjectured isoperimetric curves with area 0.1132, 0.2783, 0.3540, 1.0519 in the plane with density $e^{-1/r}$ have angles at the origin: $2.1438, 2.1956, 2.2368, 2.3560$, respectively. These curves were found with Mathematica using the constraint of constant generalized curvature (see Def. 2.1 and Prop. 2.5.).

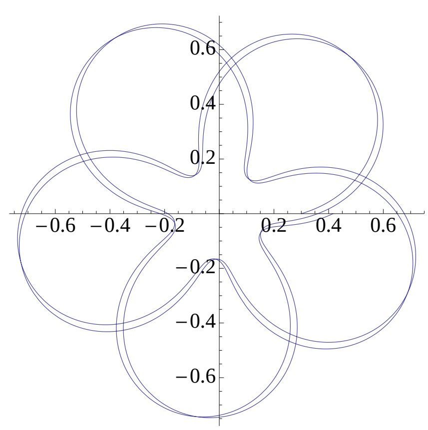

Figure 2: Computed curves of constant generalized curvature exhibit periodicity and instability. Curvature: 5; initial conditions: $r(0) = .3$, $r'(0) = 1$.

direction. This paper considers the case of $\alpha < 0$, in particular $\alpha = -1$, for which density vanishes at the origin but approaches 1 at infinity.

Definition 2.1. A *density* is a positive continuous function that weights perimeter and area equally. Given a Riemannian manifold with density e^{ϕ} and underlying Riemannian volume element dV_0 and hypersurface area element dA_0, the new weighted volume and area forms are given by

$$dV_{\phi} = e^{\phi} dV_0$$
$$dA_{\phi} = e^{\phi} dA_0.$$

On a smooth two-dimensional Riemannian manifold with density e^{ϕ}, the *generalized curvature* of a smooth curve with unit normal $\mathbf{n}$ and Riemannian curvature κ is

$$\kappa_{\phi} = \kappa - \frac{d\phi}{d\mathbf{n}}. \tag{1}$$

To justify this, note that the first variation $\delta^1(v) = dL_{\phi}/dt$ of the length of a smooth curve in a two-dimensional Riemannian manifold with density e^{ϕ} under a smooth variation with initial velocity v satisfies

$$\frac{dL_{\phi}}{dt} = \delta^1(v) = -\int \kappa_{\phi} v ds_{\phi}$$
$$\frac{dA_{\phi}}{dt} = -\int v ds_{\phi}$$

(see e.g. [Co, Prop. 3.2]), where A_{ϕ} is the weighted area of the curve on the side of the normal. Thus $\kappa_{\phi} = dL_{\phi}/dA_{\phi}$. Therefore, a smooth isoperimetric curve must have constant generalized curvature.

Figures 2 and 3 show some constant-curvature curves.

Conjecture 2.2. *Isoperimetric curves in the plane with density* $e^{-1/r}$ *pass through the origin, at an acute angle.*

Remark 2.3. Grigor'yan *et al.* [G, Thm. 5.3] show that for some constants $C > 0, \tau \in (0,1)$, for given volume V, a lower bound on minimum weighted surface area A necessary to enclose weighted area V in the plane with density $e^{-1/r^{\alpha}}$ is

$$A \geq C \begin{cases} V(\log\frac{1}{V})^{1+\frac{1}{\alpha}} & 0 \leq V \leq \tau \\ V^{\frac{n-1}{n}} & V > \tau \end{cases}.$$

In the plane with density $e^{-1/r}$, this yields

$$P \geq C \begin{cases} A(\log\frac{1}{A})^{2} & 0 \leq A \leq \tau \\ \sqrt{A} & A > \tau \end{cases},$$

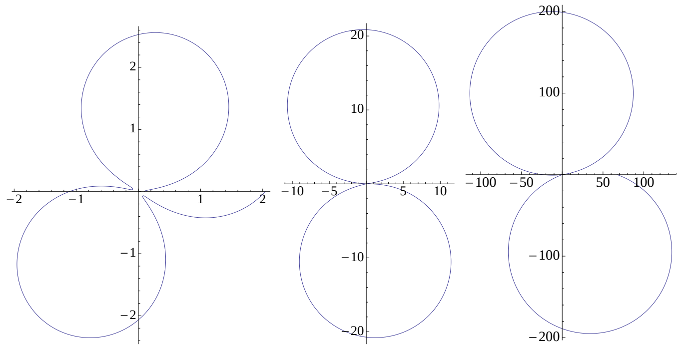

Figure 3: By carefully choosing the curvature of the constant-generalized-curvature curve, we may approach a period that divides 2π. Curvature (left to right): 1, .1, .01; initial conditions: $r(0) = .1$, $r'(0) = 0$.

where P is the least weighted perimeter needed to enclose weighted area A. The lower bound is easy for large area, since $(1-\epsilon)A$ is in a ball complement where the density is greater than $1-\epsilon$, requiring perimeter order of A to enclose it. Small volume is much more delicate. As for large A, disks about the origin show that the bound is asymptotically sharp for small A. For small A,

$$A = 2\pi \int_0^r r\mathrm{e}^{-1/R} dR \sim 2\pi^2 r^3 \mathrm{e}^{-1/r},$$
$$P = 2\pi r \mathrm{e}^{-1/r},$$

where r is the radius of the disk. It follows that

$$P \geq CA(\log \frac{1}{A})^2$$

for small A and some C, as desired.

The following proposition gives an easy proof of a weaker estimate.

Proposition 2.4. *The least perimeter P to enclose small area A in the plane with density $\mathrm{e}^{-1/r}$ satisfies $P \geq cA^{3/2}(\log 1/A)^3$.*

Proof. Given A, let r_0 be the radius of a disk of area $A_0 = A/2$. Then the unweighted area outside the disk of a region of area A is greater than $A/2$. Hence (see [Cho]), the unweighted perimeter of the region outside the disk is at least $c_1 A^{1/2}$, so the weighted perimeter of the region outside the disk is at least

$$c_1 \mathrm{e}^{-1/r_0} A^{1/2} \sim c_2 (A_0/r_0^3) A^{1/2} \sim c_3 A_0 (\log(1/A_0))^3 A^{1/2} \sim c_4 A^{3/2} (\log(1/A))^3,$$

as desired. As A approaches zero, this yields the estimate given by circles about the origin up to a constant. Therefore, this lower bound is asymptotically sharp up to a constant. □

Proposition 2.5. *Isoperimetric curves exist for all areas in the plane with density $e^{-1/r}$, and such curves are smooth constant-generalized-curvature curves except possibly at the origin.*

Proof. To show that isoperimetric regions exist for all areas in the plane with density $e^{-1/r}$, by Morgan and Pratelli [MP, Thm. 7.11] it suffices to show that for all $c > 0$ and $\rho > 0$, there exists an $R > \rho$ such that

$$e^{-1/R} \leq 1 - e^{-cR}.$$

(Morgan and Pratelli actually define density to be positive, but do not make use of this property in proving Theorem 7.11.) But this is true, as for R large enough, $e^{-1/R} \leq 1 - 1/(2R)$, which is clearly less than $1 - e^{-cR}$ for R large enough.

For any smooth density e^{ϕ}, an isoperimetric curve is smooth [M3, 3.10]. For equilibrium, the generalized curvature must be constant (see Def. 2.1). □

Proposition 2.6. *A smooth curve $r(\theta)$ in the plane with density $e^{-1/r}$ has generalized curvature*

$$\kappa_{-1/r} = \frac{r + 1 - r''}{r\sqrt{r^2 + r'^2}} + \frac{r'^2(r + r'')}{r(r^2 + r'^2)^{3/2}} = \frac{r^3 + r^2 + 2r(r')^2 + (r')^2 - r''r^2}{r(r^2 + (r')^2)^{3/2}}, \tag{2}$$

or equivalently in Cartesian coordinates:

$$\kappa_{-1/r} = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}} - \frac{x \cdot f'(x) - f(x)}{(1 + f'(x)^2)^{1/2}(x^2 + f(x)^2)^{3/2}}. \tag{3}$$

Proof. Setting $\phi = -1/r$ in the equation derived in Corwin *et al.* [Co, Prop. 3.6], which describes curves of constant generalized curvature for density e^{ϕ}, gives the desired result. □

Remark 2.7. Equation (2) is singular at the origin. While an isoperimetric curve in the plane with density $e^{-1/r}$ must be smooth away from the origin, its behavior at the origin is not well understood. A conjectured isoperimetric curve, computationally derived from Equation 2.6(2), can be seen in Figure 1.

The following propositions begin to describe possible behaviors of isoperimetric curves through the origin.

Proposition 2.8. *Suppose that C is an isoperimetric curve that passes through the origin. Then C can be written as a radial function of θ, the angle with the $x-$axis.*

Proof. See Figure 4. Suppose C is a curve enclosing a region R, and C is not a function of θ. Then there exists a ray L emanating from the origin that intersects C transversely in at least two points neither of which is the origin. Pick some segment s of this ray that lies outside the interior of R. Then s must split C into two parts, C_1 and C_2, and $s \cup C_1$ and $s \cup C_2$ must each be the boundary of a region in the plane. One of these regions will contain R. Say that $s \cup C_1$ contains R. Then we have constructed a curve that has strictly more area than the original region, but, since straight lines are the shortest paths from a point to the origin, $s \cup C_1$ has less perimeter than C, giving a contradiction. □

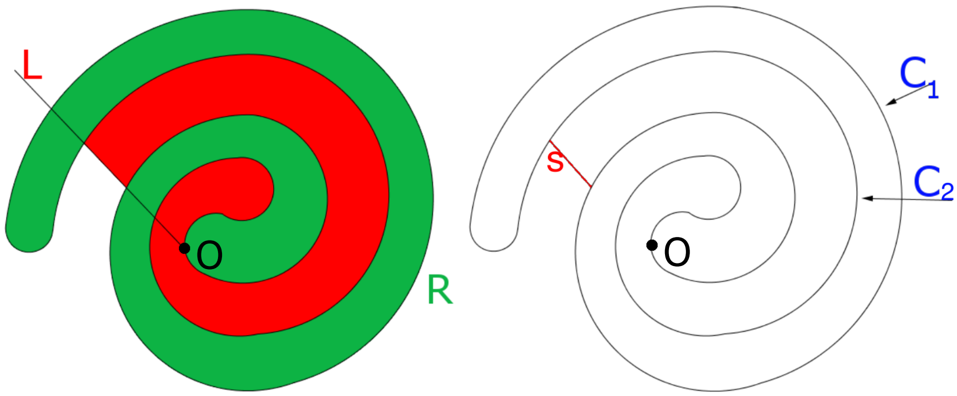

Figure 4: A curve that passes through the origin and is not a radial function of angle with respect to the x-axis is not isoperimetric.

Corollary 2.9. *All isoperimetric curves through the origin are one-sided differentiable at the origin.*

Proof. Let C be defined as the radial function $r = r(\theta)$. Suppose $r(\theta_0) = 0$. Then as $\theta \to \theta_0$ from one side, the secant lines are just the rays from the origin, and they approach the tangent line $\theta = \theta_0$. Thus, the secant lines have a limit, and so the curve is one-sided differentiable at the origin. $\square$

By using the Cartesian differential equation for constant generalized curvature, we can get more:

Proposition 2.10. *All isoperimetric curves through the origin must be one-sided C^1 there.*

Proof. Write the curve as the graph of a function $f(x) = y$, and rotate such that $f'(0) = 0$. Then being a radial graph implies that $f(x)/x < f'(x)$, and so $f'(x) \cdot x - f(x) > 0$. But then by Equation (3), $\partial\phi/\partial\mathbf{n} > 0$. So, in order for $\kappa_\phi > 0$, we must have $f''(x) > 0$. But this implies that f is C^1. $\square$

Proposition 2.11. *Suppose that $\{y = f(x)\}$ is a curve defined for $a > x \geq 0$ ($a \leq \infty$) with constant generalized curvature $\kappa_{-1/r}$, $f(0) = 0$, $f'(0) = 0$, and f'' is continuous at the origin. Then $f''(0) = 0$, and $f(x) = ax^3 + h_3(x)x^3$ for some h_3 where $\lim_{x\to 0} h_3(x) = 0$.*

Proof. Since f is C^2, use Taylor's Theorem to write

$$f(x) = \frac{f''(0)}{2}x^2 + h_2(x)x^2$$
$$f'(x) = f''(0)x + h_2'(x)x^2 + 2h_2(x)x,$$

where $\lim_{x\to 0} h_2(x) = 0$. By Equation (3), the only way that anything could be singular at zero is in the second term, which is the $\partial\phi/\partial\mathbf{n}$ summand. Now consider the following:

$$\begin{aligned}\frac{\partial\phi}{\partial\mathbf{n}} &= \frac{x(f''(0)x + h_2'(x)x^2 + 2h_2(x)x) - f''(0)x^2/2 + h_2(x)x^2}{(1+f'(x)^2)^{1/2}\left(x^2 + x^4\left(\frac{f''(0)^2}{4} + f''(0)h_2(x) + h_2(x)^2\right)\right)^{3/2}} \\ &= \frac{x^2(f''(0)/2 + h_2(x)) + x^3 h_2'(x)}{(1+f'(x)^2)x^3(1 + x^2(f''(0)^2/4 + f''(0)h_2(x) + h_2(x)^2))^{3/2}}.\end{aligned}$$

From this equality, it is clear that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x < \delta$, the following inequality holds, letting $f''(0)/2 = C$:

$$\frac{C + h_2(x)}{x} + h_2'(x) \geq \frac{\partial\phi}{\partial\mathbf{n}} \geq \frac{1}{\epsilon}\frac{C + h_2(x)}{x} + h_2'(x).$$

Since $\lim_{x\to 0} \partial\phi/\partial\mathbf{n}$ has a finite limit, it follows that

$$\lim_{x\to 0} \frac{C + h_2(x)}{x} + h_2'(x) = K < \infty.$$

It then follows that for all $\epsilon' > 0$, there exists $\delta' > 0$ such that if $x < \delta'$ then

$$a - \epsilon' < \frac{C + h_2(x)}{x} + h_2'(x) < a + \epsilon'.$$

Therefore,

$$x(a - \epsilon') < C + (x \cdot h_2(x))' < x(a + \epsilon').$$

Integrating yields

$$\frac{1}{2}x^2(a - \epsilon') < Cx + xh_2(x) < \frac{1}{2}x^2(a + \epsilon'),$$

and we divide by x to get

$$\frac{1}{2}x(a - \epsilon') < C + h_2(x) < \frac{1}{2}x(a + \epsilon').$$

Both the left and the right sides of this inequality approach zero, and h_2 approaches zero as $x \to 0$; therefore, $C = f''(0)/2 = 0$, and

$$\frac{1}{2}(a - \epsilon') < \frac{h_2(x)}{x} < \frac{1}{2}(a + \epsilon).$$

Thus $\lim_{x\to 0} h_2(x)/x = \lim_{x\to 0} h_2'(x) = a/2$. Applying Taylor's theorem to $h_2(x)$ gives

$$h_2(x) = ax + h_3(x)x,$$

where $\lim_{x\to 0} h_3(x) = 0$. Substituting this back into the original equation for $f(x)$ yields that $f(x) = ax^3 + h_3(x)x^3$. $\square$

Proposition 2.12. *An isoperimetric curve that satisfies the hypotheses of Proposition* 2.11 *cannot have a cusp (0-degree angle) at the origin.*

Proof. Let γ be a constant generalized curvature curve with a cusp at the origin. Orient it so that the common tangent line is the x-axis. Then by Proposition 2.11, the top part of the curve must be ax^3 to third order and the bottom half is $-bx^3$, where $a, b > 0$, as in Figure 5. Then the Euclidean distance between A and B is equal to $(a+b)x^3$ (again, up to third order).

If the region bounded by this curve contains the points in the small curvilinear triangle, then lines OA and OB will be shorter than the curves and will contain more area, so the region cannot be isoperimetric. Therefore, assume that the region is above the upper line and below the lower line. If the line AB is shorter than the sum of OA and OB, then adding the line AB and removing the curves AO and BO will decrease perimeter while adding area.

Note that if $p \in AB$, then $\phi(p)$, the density at p, must be less than the density at the endpoints because the density is increasing. Thus, letting $c = a+b$, the total weighted length of AB, $h(x)$, must satisfy

$$h(x) \leq (a+b)x^3 e^{-(x^2+c^2x^6)^{-1/2}}$$

up to third order. On the other hand,

$$OA + OB \geq 2\int_0^x e^{-1/r}dr = 2e^{-1/x}x - 2\Gamma(0, 1/x),$$

again up to third order, where Γ is the incomplete gamma function $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$. Taking the third-order Taylor expansion of this near $x = 0$ yields

$$OA + OB \geq e^{-1/x}(2x^2 - 4x^3).$$

Taking this and dividing it by our estimate for $h(x)$ gives the following:

$$\exp(-\frac{1}{x} + \frac{1}{(x^2+c^2x^6)^{1/2}})\frac{2x^2-4x^3}{x^3}.$$

As $x \to 0$, the exponential part goes to 1, while the rational part goes to infinity. Furthermore, since higher order terms are negligible, this estimate holds for x near zero. Thus the length AB must be shorter than the sum of OA and OB, and the original curve could not have been isoperimetric. □

Dahlberg *et al.* [D, Prop. 3.12] show that an isoperimetric curve in the plane with density r^p has one maximum and one minimum of radius. Following Dahlberg *et al.*, the following proposition proves the same result for the plane with density $e^{-1/r}$.

Proposition 2.13. *In an isoperimetric curve around the origin that is not a circle around the origin, there must be exactly two extrema of distance from the origin. Furthermore, the point at minimum distance from the origin must be a global minimum of curvature.*

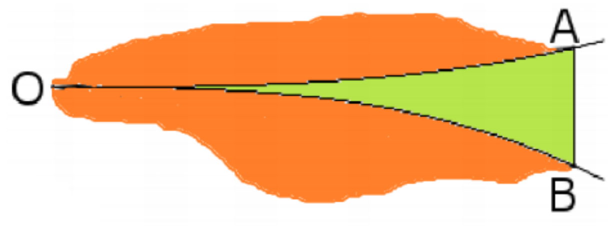

Figure 5: An isoperimetric curve in the plane with density $e^{-1/r}$ cannot have a zero-degree angle at the origin. Here A and B are sufficiently close to O so that AB has less weighted perimeter than does the cusped curve, as shown in Proposition 2.12.

Proof. We follow the proof of Dahlberg *et al.* [D, Prop. 3.12]. By Dalhberg *et al.* [D, Cor. 2.6, Prop. 2.5], any constant generalized curvature curve γ that is not a circle has finitely many critical points of r, and furthermore, all of these critical points are strict extrema. Without loss of generality, suppose that there are two maxima of distance from the origin. Suppose additionally that the maximum distance is r_0. Then take a circle around the origin of radius $r_0 - \epsilon$. This will intersect γ in at least four points. These four points generate four sectors of the region. Rearrange the sectors so that the two regions containing the maxima are next to each other, and note that the boundary curve remains continuous. This new region has the same area but now has a non-differentiable boundary. Therefore, it cannot be stationary, and the original region could not have been isoperimetric.

Let p_1 be the point on γ closest to the origin. At this point, $-\partial\phi/\partial\mathbf{n} = 1/|p_1|^2$. Let p be any point on γ. Then $-\partial\phi/\partial\mathbf{n}(p) \leq 1/|p_1|^2$ with equality if and only if $p = p_1$, by the previous part. Thus, in order to keep generalized curvature the same, $\kappa(p) > \kappa(p_1)$ if $p \neq p_1$. □

We note that Chung *et al.* [Chu, Prop. 2.22] similarly show that a component of an isoperimetric curve in the plane with density e^r that is not a circle about the origin has one maximum and one minimum of radius, although of course the Log-Convex Density Conjecture implies that isoperimetric curves in the plane with density e^r are in fact circles about the origin.

Remark 2.14. The question remains as to what angle an isoperimetric curve makes at the origin. Consider two points located at a distance r from the origin, separated by angle φ. Figure 6 gives a graphical representation, computed with Mathematica, of when the line segment between the two points has greater perimeter than two line segments connecting each point to the origin. Unfortunately, for small angles and radii, Mathematica cannot compute the curve's behavior at the origin. Figure 7 shows angle at the origin vs. area for constant generalized curvatures such as half of those pictured in Figure 3.

To further investigate the angle at the origin, we examine circular sectors at the origin that are perimeter minimizing among other sectors with the same area. We thank Bill Huber for help with Mathematica in the following proposition.

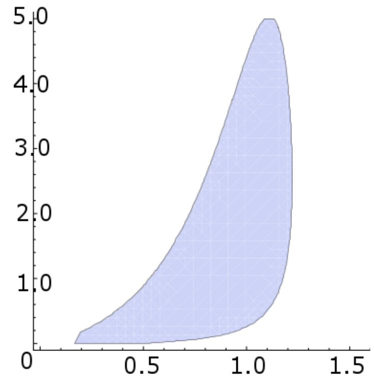

Figure 6: The shaded region indicates where a line segment between two points has less perimeter than the line segments connecting the two points to the origin. The horizontal axis represents the angle in radians between the points, and the vertical axis represents distance from the origin.

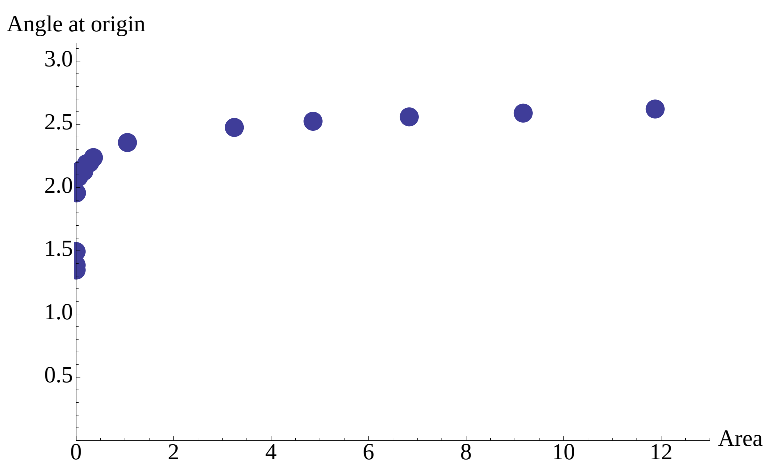

Figure 7: Angle at the origin vs. area for some constant-generalized-curvature curves, as pictured in Figure 3.

Proposition 2.15. *As area approaches zero, the angle of the perimeter-minimizing circular sector at the origin of the given area approaches one radian, as indicated by Figure 7.*

Proof. The following two formulas for area and perimeter in terms of angle φ and ρ are easily derivable:

$$A(\rho,\varphi) = \varphi \int_0^\rho re^{-1/r}dr$$
$$P(\rho,\varphi) = 2\int_0^\rho e^{-1/r}dr + \varphi\rho e^{-1/\rho}.$$

We use Lagrange multipliers to compute which values of ρ and φ minimize perimeter for given area. To do this, we differentiate:

$$\frac{\partial A}{\partial \rho} = \varphi\rho e^{-1/\rho} \qquad \frac{\partial A}{\partial \varphi} = \int_0^\rho r^{-1/r}dr$$
$$\frac{\partial P}{\partial \rho} = e^{-1/\rho}\left(2+\varphi+\frac{\varphi}{\rho}\right) \qquad \frac{\partial P}{\partial \varphi} = \rho e^{-1/\rho}.$$

Applying the method of Lagrange multipliers gives the following equation:

$$\frac{\partial A}{\partial \rho}\frac{\partial P}{\partial \varphi} = \frac{\partial A}{\partial \varphi}\frac{\partial P}{\partial \rho}$$
$$\int_0^\rho re^{-1/r}dr\left(2+\varphi+\frac{\varphi}{\rho}\right) = \rho^2\varphi e^{-1/\rho},$$

where the common $e^{-1/\rho}$ has already been canceled. Mathematica exactly integrates the integral on the left hand side:

$$\int_0^\rho re^{-1/r}dr = \frac{1}{2}\left[e^{-1/\rho}(\rho-1)\rho - \mathrm{Ei}(-1/\rho)\right],$$

where $\mathrm{Ei}(z) = -\int_{-z}^\infty e^{-t}/tdt$. The term $e^{-1/\rho}$ now cancels from both sides, leaving

$$\frac{1}{2}\left[\rho(\rho-1) - e^{1/\rho}\mathrm{Ei}(-1/\rho)\right]\left(2+\varphi+\frac{\varphi}{\rho}\right) = \rho^2\varphi.$$

The only problematic term remaining is $e^{1/\rho}\mathrm{Ei}(-1/\rho)$. Since we are only concerned with its behavior near the origin, approximate it by its fourth-order Maclaurin series computed by Mathematica:

$$e^{1/\rho}\mathrm{Ei}(-1/\rho) \approx -\rho + \rho^2 - 2\rho^3 + 6\rho^4.$$

Substituting this above and simplifying yields

$$2(\varphi-1) + 3\rho(2+\varphi) = 0.$$

As $\rho \to 0$, $\varphi \to 1$. (See Figure 7). □

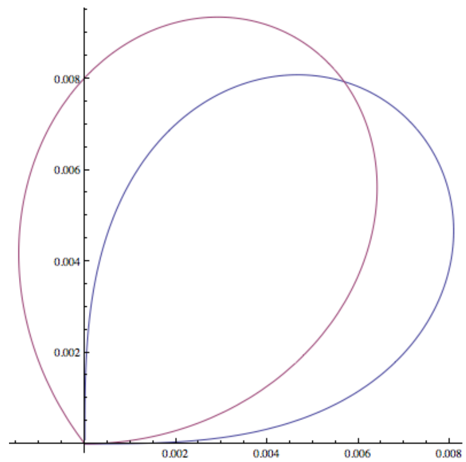

Figure 8: While not isoperimetric, curves of the form $a\sin^{1/2}(2\theta)$ and $a\sin(\sqrt{2}\cdot\theta)$ are the best explicitly known candidates for very small areas.

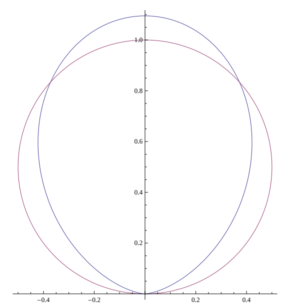

Figure 9: The curve $r = \sin^2(\theta)$, scaled to have the same area as the unit circle through the origin, has less perimeter.

Remark 2.16. For a very small given area, the best explicitly known curves of the form $a\sin^b(c\theta)$ are $b = 1/2, c = 2$ and $b = 1, c = \sqrt{2}$, with a chosen to enclose the given area. For example, to enclose area equal to that of the circle of diameter .01 passing through the origin, the curves $a\sin^{1/2}(2\theta)$ and $a\sin(\sqrt{2}\cdot\theta)$ (see Fig. 8) both have perimeter $9.25114\cdot 10^{-47}$, while the circle has perimeter $9.26748\cdot 10^{-47}$ (although finding a curve that encloses area more efficiently than the circle is not difficult, see Fig. 9). However, these curves do not have constant generalized curvature, so they are not isoperimetric.

The curves described above are not optimal for larger areas. For example, neither beats the circle of diameter 10 through the origin. When b is fixed at 1, the optimal value of c is conjectured to approach 1 as the area enclosed approaches infinity. Similarly, when c is held constant at 1 and area goes to infinity, the best value of b appears to approach 1. This is expected, as the plane with density $e^{-1/r}$ increasingly resembles the Euclidean plane at greater distances from the origin.

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