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Continuously diagonalizing the shape operator

Matthew M. Lukac

Abstract. In this paper, we investigate the behavior of the curvature of non-developable surfaces around an umbilic point at the origin. The surfaces are of the form $z = f(x, y)$ where f is a nonhomogeneous bivariate polynomial with cubic and quartic terms. We do this by looking at the continuity of the principal directions around the origin as well as the rate that the principal curvatures converge to zero as they approach the origin. This is done by considering the eigenvectors and eigenvalues of the shape operator. In our main result, we prove that a continuously diagonalizable shape operator implies the existence of a path through the origin with noncomparable principal curvatures.

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1 Introduction

Let M be a smooth surface immersed in $\mathbb{R}^3$. At each point p on M, the unit vectors pointing in directions of maximum and minimum curvature are called the *principal directions* at p. The maximum and minimum curvatures are called the *principal curvatures*. Points in which every direction is a principal direction are called *umbilics*. When the principal curvatures converge at the same rate along some path towards an umbilic, they are said to be *comparable*. The bivalued vector field on the surface created with the principal directions is called the *principal distribution* on M. One way to understand these definitions intuitively is to consider an eggshell. The points on the poles of the eggshell are umbilics while every other point has unique directions of maximum and minimum curvature. Another surface to examine is that of a cylinder. There are no umbilics on this surface, and every point has a minimum principal direction pointing up (or down) the shaft.

Ando has done much work with investigating the behavior of the principal distribution around isolated umbilics [1] [2]. Building on the work of Darboux [6], Sotomayor and Gutierrez have explored the stability of the principal distribution around isolated umbilics [20]. However, the surfaces used by Ando are defined by homogeneous polynomials in two real variables. Sotomayor and Gutierrez study nonhomogeneous polynomials of two real variables using quadratic and cubic terms with isolated umbilics. We will be focusing on surfaces defined by nonhomogeneous quartic bivariate polynomials in two real variables in which the umbilics need not be isolated.

We will be working with the perspective of the *shape operator* of our surface M. The shape operator is defined as the negative directional derivative of the unit normal vector field in the direction of a tangent vector v_p at a point $p \in M$. Since the shape operator is symmetric and linear, we may ask if we are able to continuously diagonalize it in some neighborhood N_u of an umbilic point u. If we cannot, we say the shape operator is nondiagonalizable in N_u. For example, near the origin, the shape operator for the surface $z = x^3/6 - xy^2/2$ can be written as $S = \begin{bmatrix} x & -y \\ -y & -x \end{bmatrix}$. The principal curvatures, which happen to be the eigenvalues of S, are $\lambda = \pm\sqrt{x^2+y^2}$. As we approach the umbilic at the origin along any path, the principal curvatures vanish at a linear rate and are therefore comparable. Moreover, the principal directions, which happen to be the eigenvectors of S, are discontinuous at the origin since approaching the origin from the x-axis will yield $[1, 0]$ and $[0, 1]$ as our principal directions while approaching along the y-axis yields $[1, -1]$ and $[1, 1]$. This gives us a nondiagonalizable shape operator in a neighborhood of the origin. For more details on this example, see Example 3.1 in Section 3.

The complex analog for the shape operator is called the Levi form. The research that inspired the work in this paper was performed by Derridj, in which he showed that if the eigenvalues of the Levi form are positive and comparable near a point where they both tend to zero, then the Levi form is nondiagonalizable near said point [8]. Derridj has also done some work with block decomposable Levi forms for hypersurfaces in $\mathbb{C}^n$ [9] [10].

The structure of this paper is as follows: Section 2.1 will provide several definitions from differential geometry that are used frequently for the rest of this paper. Section 2.2 provides a rotation of our coordinate axes in order to simplify some later computations. Section 2.3 classifies exactly what type of surfaces we will be working with for the rest of the paper. Section 3 provides several examples showcasing how to work with the definitions from Section 2.1. Section 4.1 guarantees the existence of comparable paths. Section 4.2 provides a coordinate rotation similar to that in Section 2.2 as well as necessary conditions for a continuously diagonalizable shape operator. Section 4.3 provides a short proof of the existence of a noncomparable path whenever the shape operator is continuously diagonalizable. Section 5 ties the motivation for this work to the theory of PDEs for functions of several complex variables and we discuss the possible steps to continue this investigation. Finally, Section 6 is an appendix which provides the Mathematica code used to generate the figures in the paper.

2 Preliminaries

2.1 Definitions

Let M be a twice differentiable surface immersed in $\mathbb{R}^3$ parametrized by the *Monge patch*:

$$\mathbf{r}(x,y) = [x, y, f(x,y)].$$

Taking the partial derivatives with respect to x and y, denoted $\mathbf{r}_x$ and $\mathbf{r}_y$, respectively, yields

$$\mathbf{r}_x = [1, 0, f_x] \qquad \text{and} \qquad \mathbf{r}_y = [0, 1, f_y].$$

Now, $\mathbf{r}_x$ and $\mathbf{r}_y$ are in the tangent plane of any point $(x, y, f(x,y))$ on M. Since $\mathbf{r}_x$ and $\mathbf{r}_y$ are linearly independent, they span the tangent plane and hence we can compute the field of unit normal vectors $\hat{n}$ by

$$\hat{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{[-f_x, -f_y, 1]}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Now we are ready to define a 2×2 matrix of functions that completely describes the curvature of M. The process of computing this matrix can be found in Oprea's text on elementary differential geometry [19].

Definition 2.1. *The* ***shape operator****, denoted with S, of a twice differentiable surface in $\mathbb{R}^3$ parametrized by a Monge patch $\mathbf{r}$ is the 2×2 symmetric matrix*

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix},$$

where

$$L = \mathbf{r}_{xx} \cdot \hat{n}, \qquad M = \mathbf{r}_{xy} \cdot \hat{n}, \qquad N = \mathbf{r}_{yy} \cdot \hat{n}.$$

Since the shape operator describes the curvature of M, a natural question to ask would be about the significance of the solutions to the eigenvalue equation $Sk = \lambda k$.

Definition 2.2. *The eigenvectors of the shape operator, denoted with k_1 and k_2, are called the **principal directions**.*

Note that since the shape operator is a symmetric matrix, the principal directions are necessarily orthogonal. At each point on M, these vectors point in the directions of maximum and minimum curvature, where the curvature is considered positive if the surface bends toward the unit normal, and negative if it bends away from the unit normal.

Definition 2.3. *The eigenvalues of the shape operator, denoted with λ_1 and λ_2, are called the **principal curvatures**.*

The principal curvatures give the curvature in the directions that the principal directions point. It is also worth defining another type of curvature on a surface.

Definition 2.4. *The product of the principal curvatures, $\lambda_1\lambda_2$, is called the **Gaussian curvature** at a point on M.*

A notable feature of Gaussian curvature is that it is an intrinsic property of a surface. One way to compute the Gaussian curvature of a surface is by finding the determinant of the shape operator.

Definition 2.5. *An **umbilic point** is a point on a surface in which the principal curvatures are equal.*

At an umbilic point, every direction is a principal direction and so the surface can be approximated by a sphere or a plane up to second order at these points. We say an umbilic point is isolated if it is the only umbilic point in a neighborhood of said point.

Now, using our patch, we get

$$\mathbf{r}_{xx} = [0, 0, f_{xx}], \qquad \mathbf{r}_{xy} = [0, 0, f_{xy}], \qquad \mathbf{r}_{yy} = [0, 0, f_{yy}]$$

which immediately gives us

$$L = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \qquad M = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \qquad N = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}.$$

This paper will be solely focused on the class of surfaces defined by

$$z = f(x, y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^4 + Fx^3y + Gx^2y^2 + Hxy^3 + Iy^4, \tag{2.1}$$

where f is not identically zero. Observe that $\mathbf{r}_x$ and $\mathbf{r}_y$ are not orthogonal, but are close to orthogonal for lower order terms. We will also only be working in a neighborhood of the umbilic point at the origin. The shape operator for these surfaces is

$$S = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

Centered at $t = 0$, we can use the Taylor expansion $\sqrt{1+t} = 1 + t/2 + \mathcal{O}(t^2)$ and the fact that the lowest order terms in f_x and f_y are quadratic, we can write

$$\sqrt{1 + f_x^2 + f_y^2} = 1 + \mathcal{O}(x^4 + y^4).$$

We make use of another Taylor expansion for $\frac{1}{1+t} = 1 - t + \mathcal{O}(t^2)$ centered at $t = 0$, so that we can say

$$\frac{1}{1 + \mathcal{O}(x^4 + y^4)} = 1 + \mathcal{O}(x^4 + y^4).$$

Hence,

$$S = \frac{1}{1 + \mathcal{O}(x^4 + y^4)} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \left(1 + \mathcal{O}(x^4 + y^4)\right).$$

It follows that in a neighborhood of the origin, the shape operator is approximated with the Hessian of f. The study of the principal distribution fields on surfaces with isolated umbilics has been performed by Bruce and Fidal [3]. These surfaces were defined such that the trace of S is zero. We will not be restricting ourselves to this case. At this point, it should be said that when computing the shape operator, we will sometimes be computing the second partial derivatives in Cartesian coordinates and then converting to polar coordinates with the change of variable $x = r\cos\theta$ and $y = r\sin\theta$.

We will now define two properties that the eigenvalues and eigenvectors of the shape operator can have. Understanding the relationship between the following two definitions is the focus of this paper.

Definition 2.6. *The eigenvalues λ_1 and λ_2 of a 2×2 matrix of functions are said to be* ***comparable*** *along some path γ ending at an umbilic if there exists a nonzero scalar c such that*

$$\frac{\lambda_1 - \lambda}{c} \leq \lambda_2 - \lambda \leq c(\lambda_1 - \lambda)$$

on γ, where λ is the limiting value of the eigenvalues at the umbilic.

Essentially, this means that two comparable eigenvalues have the same rate of convergence as they approach the umbilic along some path. If the shape operator has comparable eigenvalues along some direction and noncomparable eigenvalues along a different direction, we will say that it has *mixed* eigenvalues.

Definition 2.7. *A 2×2 matrix of functions is said to be* ***continuously diagonalizable*** *around a point p if it has continuous eigenvectors in a neighborhood of p.*

Since the shape operator is symmetric it can always be diagonalized point-wise, but not always continuously. Moreover, there will always be an umbilic at the origin due to our choice of surfaces defined by (2.1).

The following remark is a well known result from linear algebra, but we include the proof because it gives us a nice formula for computing the principal curvatures.

Remark 2.8. *The eigenvalues of a real symmetric matrix are real.*

Proof. Let A be real and symmetric so that $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. The characteristic equation is

$$\lambda^2 - (a+b)\lambda + ac - b^2 = 0$$

which gives

$$\lambda = \frac{1}{2}\left(a + c \pm \sqrt{(a-c)^2 + 4b^2}\right). \tag{2.2}$$

Since a sum of two squares is nonnegative, λ must be real. ■

As we move further, we will use (2.2) to compute the principal curvatures. In other words, for f defined in (2.1), our principal curvatures become

$$\lambda = \frac{1}{2}\left(f_{xx} + f_{yy} \pm \sqrt{(f_{xx} - f_{yy})^2 + 4f_{xy}^2}\right). \tag{2.3}$$

2.2 A Convenient Rotation

We will now be considering the bivariate polynomial from (2.1). Note the absence of constant, linear, and quadratic terms. We can get away with this since the constant and linear terms will have no influence on the curvature of the surface, which is what we are focusing our attention on. Furthermore, we can neglect the quadratic terms because, if we did include them, we would simply be adding a constant multiple of the identity matrix to the shape operator. This is due to our requirement of having an umbilic at the origin. This constant addition will change what the principal curvatures and principal directions converge to at the origin, but not whether or not they are comparable or continuous, respectively, while approaching the origin. Therefore, we may effectively neglect any quadratic terms in (2.1).

Now, the first order of business is to rotate our coordinate axes to slightly depress our polynomial. To see this rotation, we start with the change of coordinates

$$\begin{aligned} x &= u\cos\varphi - v\sin\varphi, \text{ and} \\ y &= u\sin\varphi + v\cos\varphi. \end{aligned}$$

Taking inspiration from Sotomayor and Gutierrez's coordinate rotation in [20], we will rotate our xy plane so that B, the coefficient of the x^2y term in (2.1), is zero. Assuming $B \neq 0$, we substitute the new coordinates into (2.1) and set the coefficient of the u^2v term equal to zero to obtain

$$B\cos^3\varphi + (2C - 3A)\cos^2\varphi\sin\varphi + (3D - 2B)\cos\varphi\sin^2\varphi - C\sin^3\varphi = 0$$

which is equivalent to

$$1 + \frac{2C - 3A}{B}\tan\varphi + \frac{3D - 2B}{B}\tan^2\varphi - \frac{C}{B}\tan^3\varphi = 0 \tag{2.4}$$

whenever $\cos\varphi \neq 0$. Since cubic polynomials always have at least one real solution, we can conclude that there exists some $\varphi \in (-\pi/2, \pi/2)$ that solves (2.4). Hence, we may always rotate our coordinates to make $B = 0$ in (2.1).

2.3 Minding's Theorem and Plane Isometries

Before moving on, we must first admit that it is a bit deceiving to say that we will be looking at any surface defined by (2.1). We will not be considering a small class of surfaces in this paper: surfaces that are isometric to a plane. These types of surfaces are called *developable surfaces*. We will be making use of the following theorem.

Theorem 2.9 (Minding's Theorem)**.** *All surfaces of the same constant Gaussian curvature are isometric*

The proof of Minding's theorem can be found in Struik's text on differential geometry [22]. An immediate result of Minding's theorem is that a surface is isometric to a plane if and only if it has Gaussian curvature that is identically zero. With this in hand, we can classify exactly which cases (2.1) will describe a developable surface.

Theorem 2.10. *A surface M defined by* (2.1) *is developable if and only if one of the following cases holds*

$$f(x,y) = \begin{cases} Ax^3 + Ex^4 & \text{if } A \neq 0 \\ Dy^3 + Iy^4 & \text{if } A = 0 \text{ and } D \neq 0 \\ Iy^4 & \text{if } A = D = E = 0 \\ E\left(x + \frac{F}{4E}y\right)^4 & \text{if } A = D = 0 \text{ and } E \neq 0. \end{cases} \tag{2.5}$$

Proof. We assume that M is developable, so $\det S$ is identically zero. Suppose we have already used the above rotation to depress (2.1) so that $B = 0$. Now, carefully computing $\det S$, one will obtain

$$\begin{aligned} \det S = 12ACx^2 &+ 36ADxy - 4C^2y^2 + 12(AG + 2CE)x^3 \\ &+ 36(AH + 2DE)x^2y + 12(6AI + 3DF - CG)xy^2 \\ &+ 12(DG - CH)y^3 + 3(8EG - 3F^2)x^4 \\ &+ 12(6EH - FG)x^3y + 6(24EI + 3FH - 2G^2)x^2y^2 \\ &+ 12(6FI - GH)xy^3 + 3(8GI - 3H^2)y^4. \end{aligned} \tag{2.6}$$

Setting each coefficient equal to zero, the y^2 term immediately implies $C = 0$. What remains of the coefficients is the following system of equations.

$$\begin{aligned} AD &= 0 & 8EG &= 3F^2 \\ AG &= 0 & 6EH &= FG \\ AH + 2DE &= 0 & 6FI &= GH \\ 6AI + 3DF &= 0 & 8GI &= 3H^2 \\ DG &= 0 & 24EI + 3FH &= 2G^2 \end{aligned}$$

Now, if $A \neq 0$, the five equations on the left imply $D = G = H = I = 0$. The first equation on the right then implies $F = 0$ and we have satisfied the system. This leaves us with

$$f(x,y) = Ax^3 + Ex^4$$

whenever $A \neq 0$.

Now assume $A = 0$ and $D \neq 0$. Then $E = F = G = 0$ by the third, fourth, and fifth equations on the left. The right column of equations are satisfied only if $H = 0$ as well. This leaves us with

$$f(x, y) = Dy^3 + Iy^4$$

whenever $A = 0$ and $D \neq 0$.

Next, assume $A = D = 0$ as well as $E = 0$. The leftmost equations are immediately satisfied and the rightmost imply $F = G = H = 0$, leaving us with

$$f(x, y) = Iy^4.$$

Finally, assume $A = D = 0$ and $E \neq 0$. Again, the left column of equations are satisfied immediately. We can now use the first, second, and third equations on the right to solve for G, H, and I, respectively, in terms of E and F. Explicitly, these are

$$G = \frac{3F^2}{8E}, H = \frac{F^3}{16E^2}, \text{ and } I = \frac{F^4}{256E^3}.$$

Note that these also satisfy the last two equations. Now, by substitution, we see that

$$f(x, y) = E\left(x + \frac{F}{4E}y\right)^4.$$

Conversely, assume (2.1) is defined as in (2.5). Then we simply see that, in each case,

$$\det S = f_{xx}f_{yy} - (f_{xy})^2 = 0$$

and so M is developable. ■

Since we are not considering any developable surfaces for the rest of the paper, we are safe in assuming that there always exists some path γ through the origin such that $\det S \neq 0$. So along γ, neither of the principal curvatures are zero. It follows that if $\operatorname{tr} S = 0$ along γ, then $\lambda_1 = -\lambda_2 \neq 0$, yielding comparable principal curvatures along γ. This fact can be useful while doing calculations to find paths of comparable principal curvatures.

3 Examples

Before we head into the following examples, it should be noted that the figures that follow will attempt to allow the reader to visualize comparable principal curvatures as well as discontinuous (or continuous) principal directions. For the figures that show information about comparable eigenvalues, red signifies noncomparable eigenvalues while any other color signifies comparable eigenvalues. For the figures that show information about whether the shape operator is continuously diagonalizable, the colors will show the continuity (or lack thereof) of the principal directions around the origin. In other words, if there is a color that we may assign to the origin, then shape operator is continuously diagonalizable. Otherwise, the shape operator is not continuously diagonalizabile. The Mathematica code used to produce these figures can be found in the appendix.

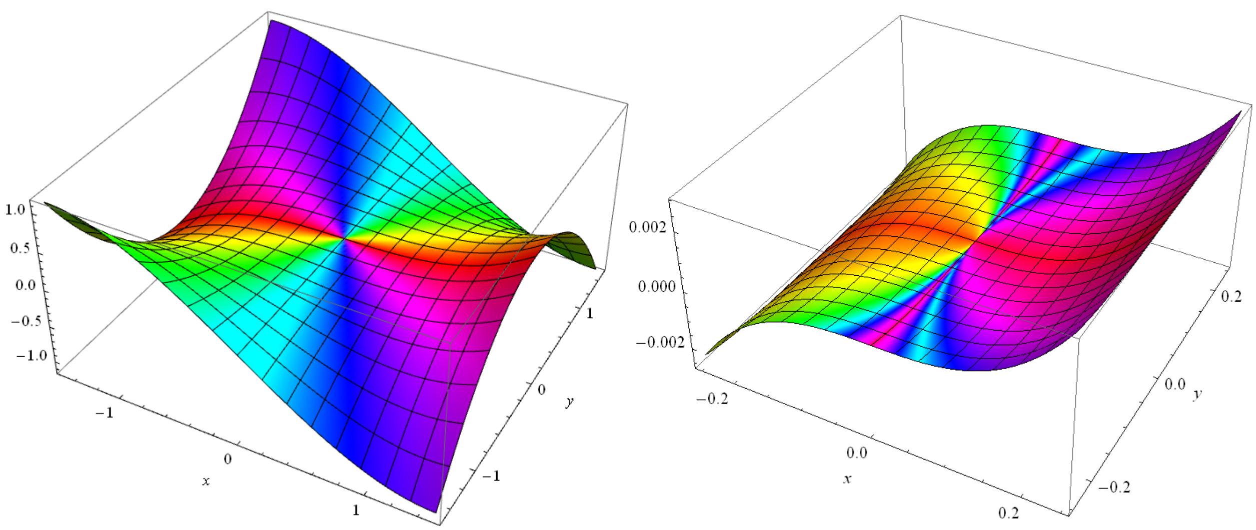
Example 3.1. Consider the monkey saddle defined by $f(x, y) = x^3/6 - xy^2/2$. Then, in polar coordinates

$$S = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ -r\sin\theta & -r\cos\theta \end{bmatrix} \left(1 + \mathcal{O}(r^4)\right).$$

Now, the eigenvalues are

$$\lambda_1 = r + \mathcal{O}(r^5) \qquad \text{and} \qquad \lambda_2 = -r + \mathcal{O}(r^5)$$

which are clearly comparable for all θ. Notice that when $\theta = 0$, $k_1 = [1, 0]$ and $k_2 = [0, 1]$. When $\theta = \pi/2$, $k_1 = [1, -1]$ and $k_2 = [1, 1]$. The eigenvectors are discontinuous and so S is not continuously diagonalizable. Thus, it is possible for the shape operator to not be continuously diagonalizable with comparable eigenvalues along any path through the origin. The discontinuous principal directions can be visualized in Figure 1a. Notice that the origin appears to be approaching different colors as we come in from different directions.



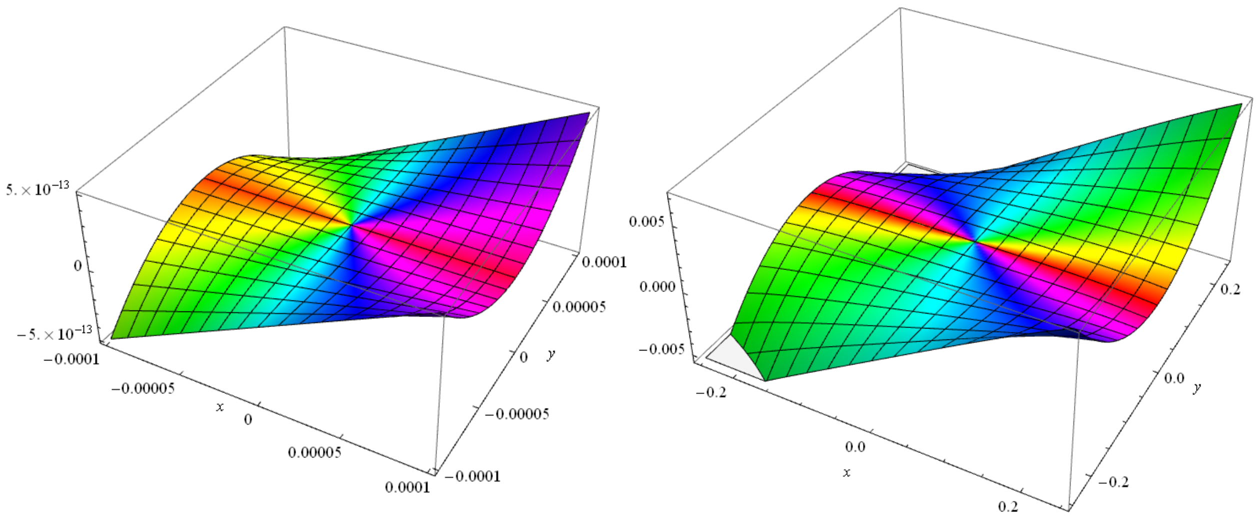
(a) not continuously diagonalizable (b) mixed principal curvatures

Figure 1: The surfaces from (a) Example 3.1 showing discontinuous principal directions and (b) Example 3.2 showing a mix of comparable and noncomparable principal curvatures

Example 3.2. Now consider $f(x, y) = x^3/6 + y^4/12$. Then

$$S = \begin{bmatrix} x & 0 \\ 0 & y^2 \end{bmatrix} \left(1 + \mathcal{O}(r^4)\right)$$

which is diagonal and hence continuously diagonalizable. We can also read the eigenvalues right from the diagonal of S. Since one is linear and the other is quadratic, they are never comparable along any radial direction. However, if we approach along the path $y = \sqrt{x}$, the eigenvalues will be comparable. Thus, there exists a surface with continuous principal



(a) mixed principal curvatures

(b) not continuously diagonalizable

Figure 2: The surface from Example 3.3

directions as well as a mix of comparable and noncomparable principal curvatures, depending on our choice of path to the origin. Notice that in Figure 1b there is a mix of red and other colors. After seeing this, one may ask if we could construct an example in which the shape operator is continuously diagonalizable and the principal curvatures are never comparable. We answer this question in Theorem 4.2.

Example 3.3. Now consider $f(x,y) = xy^2/2 - x^4/12 - y^4/12$. When $\theta = 0$,

$$S = \begin{bmatrix} -r^2 & 0 \\ 0 & r \end{bmatrix} \left(1 + \mathcal{O}(r^4)\right), \qquad \lambda_1 = -r^2 + \mathcal{O}(r^5), \qquad \lambda_2 = r + \mathcal{O}(r^5),$$

$$\hat{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \hat{k}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

When $\theta = \pi/2$,

$$S = \begin{bmatrix} 0 & r \\ r & -r^2 \end{bmatrix} + \mathcal{O}(r^5)$$

$$\lambda = \pm r + \mathcal{O}(r^2) \qquad k_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mathcal{O}(r) \qquad k_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mathcal{O}(r).$$

Thus, it is possible for the shape operator to not be continuously diagonalizable with mixed principal curvatures. Once again, we can visualize these results in Figure 2. Figure 2a contains a mix of both red and other colors, showing mixed principal curvatures. Figure 2b has different colors radiating from the origin, showing discontinuous principal directions.

Finally, we look at one last interesting example.

Example 3.4. Consider $f(x,y) = y^3/6 + x^2y^2/2$. When $y = 0$, we get

$$S \approx \begin{bmatrix} 0 & 0 \\ 0 & x^2 \end{bmatrix} \qquad \lambda_1 = 0 \qquad \lambda_2 = x^2 \qquad \hat{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{k}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now let $x = y$ so we get

$$S \approx \begin{bmatrix} x^2 & 2x^2 \\ 2x^2 & x + x^2 \end{bmatrix}$$

and

$$S \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now since $[1, 0]$ does not satisfy the eigenvalue equation when $x = y$, S has discontinuous eigenvectors. Let θ be fixed such that $\sin\theta \neq 0$. Making use of $\sqrt{1+t} = 1 + t/2 + \mathcal{O}(t^2)$, we have the noncomparable principal curvatures

$$\lambda = \frac{1}{2}\left(\sin\theta \pm |\sin\theta|\right) r + \mathcal{O}(r^2).$$

So there is seemingly no direction in which the shape operator has comparable eigenvalues. However, let us consider approaching the origin along the curve $y = -x^2$. Then

$$S \approx \begin{bmatrix} x^4 & -2x^3 \\ -2x^3 & 0 \end{bmatrix}$$

and

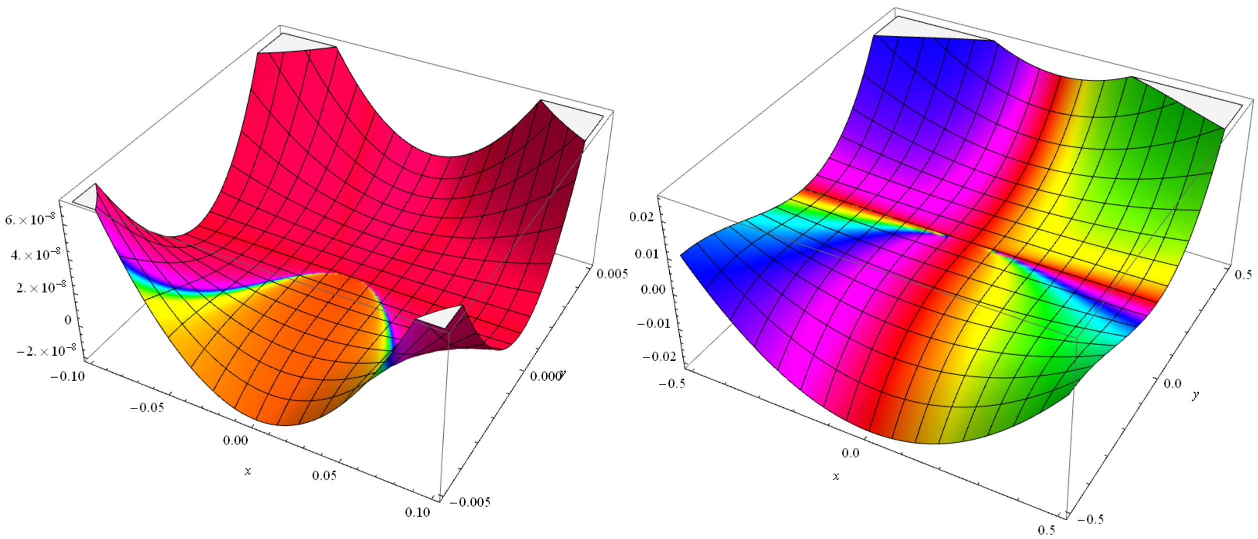
$$\begin{aligned} \lambda &= \frac{1}{2}\left[x^4 \pm \sqrt{x^8 + 16x^6}\right] \\ &= \pm 2x^3 + \mathcal{O}(x^4). \end{aligned}$$

This example gives us some insightful information. Not only do we need to consider approaching along parabolic paths when looking for comparable principal curvatures, but we may also encounter cubic principal curvatures upon doing so. This knowledge will be useful in the proof of Theorem 4.2. Notice that we can see a red (noncomparable eigenvalues) parabolic curve passing through the origin in Figure 3a and we have different colors approaching the origin in Figure 3b displaying discontinuous eigenvectors.

So far, we have shown that the shape operator may be continuously diagonalizable with mixed principal curvatures, or have discontinuous eigenvectors with mixed or always comparable eigenvalues. In summary,

	always comparable	always noncomparable	mixed
diagonalizable	X	X	✓
nondiagonalizable	✓	X	✓

where the X's come from the results in the next section.



(a) mixed principal curvatures

(b) not continuously diagonalizable

Figure 3: The surface from Example 3.4

4 Results

4.1 Comparable Paths

We start this section with a lemma we will need later. Note the use of polar coordinates.

Lemma 4.1. *Let S be a 2×2 symmetric matrix of continuous functions defined in a neighborhood of the origin. Let γ be a smooth curve through the origin. If $\det S$ is of order $2m$ on γ, m a natural number, and $\operatorname{tr} S$ is of order at least m on γ, then both of the eigenvalues of S are of order m on γ.*

Proof. While approaching the origin on a smooth curve γ, let

$$\det S = \psi r^{2m} + \mathcal{O}(r^{2m+1})$$

and

$$\operatorname{tr} S = \phi r^m + \mathcal{O}(r^{m+1})$$

where $\psi \in \mathbb{R}\backslash\{0\}$ and $\phi \in \mathbb{R}$. Then the eigenvalues are

$$\begin{aligned}
\lambda &= \frac{1}{2}\left(\operatorname{tr} S \pm \sqrt{(\operatorname{tr} S)^2 - 4\det S}\right) \\
&= \frac{1}{2}\left(\phi r^m + \mathcal{O}(r^{m+1}) \pm \sqrt{(\phi^2 - 4\psi) r^{2m} + \mathcal{O}(r^{2m+1})}\right) \\
&= \frac{1}{2}\left(\phi \pm \sqrt{\phi^2 - 4\psi}\right) r^m + \mathcal{O}(r^{m+1}).
\end{aligned}$$

It does not matter whether $\phi = 0$ or not, λ will always have order m. ■

Recall that if $\operatorname{tr} S = 0$ along γ, we still have comparable principal curvatures since $\lambda_1 = -\lambda_2 \neq 0$. So when we apply Lemma 4.1, we need not worry about $\operatorname{tr} S = 0$. We are now ready to prove our first main result, which shows the generality of paths of comparable principal curvatures.

Theorem 4.2. *Let M be a non-developable surface defined by* (2.1) *under the rotation to make $B = 0$. Then there always exists some path through the origin in which the principal curvatures are comparable.*

Proof. To prove this theorem, we will consider several different cases for the coefficients from (2.1) and using Lemma 4.1 heavily, provide a path with comparable principal curvatures. To this end, we let $B = 0$ by our previous depressing rotation, and start with the assumption that $C \neq 0$. Then the second partial derivatives of (2.1) are

$$\begin{aligned} f_{xx} &= 6Ax + 12Ex^2 + 6Fxy + 2Gy^2, \\ f_{xy} &= 2Cy + 3Fx^2 + 4Gxy + 3Hy^2, \text{ and} \\ f_{yy} &= 2Cx + 6Dy + 2Gx^2 + 6Hxy + 12Iy^2. \end{aligned}$$

Referring to $\det S$ given by (2.6) and

$$\operatorname{tr} S = 2(3A + C)x + 6Dy + 2(6E + G)x^2 + 6(F + H)xy + 2(G + 6I)y^2, \tag{4.1}$$

we can see that when $x = 0$, $\det S$ is of order 2 and $\operatorname{tr} S$ is of order at least 1. By Lemma 4.1, $x = 0$ is a comparable path.

Next, assume $C = 0$ and both $A \neq 0$ and $D \neq 0$. We can again refer to (2.6) and (4.1) as well as apply Lemma 4.1 to see that $y = x$ is a comparable path.

Next, assume that $A = C = 0$, $D \neq 0$, $E \neq 0$, and consider the parabola $y = kx^2$ where

$$k \neq 0 \text{ and } k \neq \frac{3F^2 - 8EG}{24DE}$$

for reasons which will soon be clear. Notice that along this parabola, $\operatorname{tr} S$ is of order at least 2 in x and that

$$\det S = 3(24DEk + 8EG - 3F^2)x^4 + \mathcal{O}(x^5).$$

Because $k \neq (3F^2 - 8EG)/24DE$, $\det S$ is guaranteed to be of order 4 and we may apply Lemma 4.1 to show that this parabola is a comparable path.

Now, we let $A = C = E = 0$ but require $D \neq 0$ and $F \neq 0$. If we consider the parabola $y = x^2$, then we have

$$\det S = -9F^2x^4 + \mathcal{O}(x^5).$$

Since $\operatorname{tr} S$ is of order at least 2, Lemma 4.1 tells us that $y = x^2$ is a comparable path.

Now, we let $A = C = E = F = 0$ but require $D \neq 0$ and $G \neq 0$. We will also consider the parabola $y = kx^2$ where $k = \frac{-G}{3D}$. With our choice of k, notice that $\operatorname{tr} S$ is of order at least 3. Moreover,

$$\det S = -\frac{16G^4}{9D^2}x^6 + \mathcal{O}(x^7)$$

which gives us cubic, and thus comparable, principal curvatures along this parabola by Lemma 4.1.

Coming to the last case of this leg of the journey, we consider the case with $A = C = E = F = G = 0$. Then the x-axis is a path with trivial principal curvatures since all terms in the shape operator share a factor of y.

Now, we move back up to the case where $C = 0$, $A \neq 0$, and $D \neq 0$. Let us see now what happens when we let $D = 0$, $A \neq 0$, and $I \neq 0$. If we look at the previous set of cases, but replace x with y, A with D, E with I, and F with H, then the logic follows exactly the same as before. Taking advantage of this symmetry, we are effectively done with this case.

Finally, when $A = C = D = 0$, (2.1) is a homogeneous degree 4 bivariate polynomial. In this case, $\det S$ is of order 4 along some path γ through the origin. To see this, let γ be the line $y = kx$. Then choose k so that $\det S \neq 0$. Moreover, trS is of order 2 along γ. Yet again, Lemma 4.1 makes short work of this case, showing that γ is a comparable path. ■

By this theorem, we are justified in placing X's in the always noncomparable column in the table at the end of section 3.

4.2 Conditions for Diagonalizability

Lemma 4.3. *Let M be a non-developable surface defined by* (2.1)*. If the shape operator is continuously diagonalizable, then there exist coordinates such that the principal directions converge to* $[1, 0]$ *and* $[0, 1]$.

Proof. Since M is not isometric to a plane, there exists some line γ through the origin that is not a line of umbilics. Let the eigenpairs of S in a neighborhood of the origin converge to (λ_1, κ_1) and (λ_2, κ_2) as we approach the origin along γ. Let e_1 and e_2 denote the standard basis vectors. Let R be the counterclockwise rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

such that

$$R\kappa_1 = e_1 \text{ and } R\kappa_2 = e_2.$$

We define a new set of coordinates, x' and y' by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

Let the shape operator in these primed coordinates be denoted by S'. Now, the chain rule gives us

$$\begin{aligned} f_{xx} &= f_{x'x'}\cos^2\theta + 2f_{x'y'}\cos\theta\sin\theta + f_{y'y'}\sin^2\theta, \\ f_{xy} &= -f_{x'x'}\cos\theta\sin\theta + f_{x'y'}(\cos^2\theta - \sin^2\theta) + f_{y'y'}\cos\theta\sin\theta, \text{ and} \\ f_{yy} &= f_{x'x'}\sin^2\theta - 2f_{x'y'}\cos\theta\sin\theta + f_{y'y'}\cos^2\theta \end{aligned}$$

which is equivalent to $S = R^T S' R$, or

$$S' = RSR^T.$$

Now, since $R^T = R^{-1}$,

$$\kappa_1 = R^T e_1 \text{ and } \kappa_2 = R^T e_2.$$

Recall that these are the limiting values of the principal directions as we approach the origin. Consider the vectors k_1' and k_2' such that

$$k_1' = Rk_1 \text{ and } k_2' = Rk_2.$$

Then

$$k_1' = Rk_1 \to R\kappa_1 = e_1$$

and similarly, $k_2' \to e_2$. What is left is to show that k_1' and k_2' are the principal directions of S' along γ. Now,

$$\begin{aligned} S'k_1' &= RSR^T k_1' \\ &= RSk_1 \\ &= R(\lambda_1 k_1) \\ &= \lambda_1 k_1'. \end{aligned}$$

Similarly,

$$S'k_2' = \lambda_2 k_2'.$$

■

For the rest of the paper, we will be working in polar coordinates. With this lemma in hand, we will now be taking the primed coordinate system to be our canonical coordinates. This means that whenever the shape operator is continuously diagonalizable, we can assume that we are in the coordinate system with principal directions

$$k_1 = \begin{bmatrix} 1 + ar \\ br \end{bmatrix} \text{ and } k_2 = \begin{bmatrix} cr \\ 1 + dr \end{bmatrix}.$$

Using $1/(1-t) = \sum_{n=0}^{\infty} t^n$, we can write

$$k_1 = (1 + ar) \begin{bmatrix} 1 \\ \frac{br}{1+ar} \end{bmatrix} = (1 + ar) \begin{bmatrix} 1 \\ br + \mathcal{O}(r^2) \end{bmatrix}$$

and since $1 + ar$ is a scalar multiple of k_1, we can instead use the eigenvectors

$$k_1 = \begin{bmatrix} 1 \\ br + \mathcal{O}(r^2) \end{bmatrix}.$$

Similarly,

$$k_2 = \begin{bmatrix} cr + \mathcal{O}(r^2) \\ 1 \end{bmatrix}.$$

Now, since the principal directions are always orthogonal, it follows from $k_2^T k_1 = 0$ that $c = -b$. Along the x-axis, the shape operator is

$$S = \begin{bmatrix} 6A & 2B \\ 2B & 2C \end{bmatrix} r + \begin{bmatrix} 12E & 3F \\ 3F & 2G \end{bmatrix} r^2$$

and we have

$$k_2^T S k_1 = 2Br + [3F + 2b(C - 3A)]r^2 + \mathcal{O}(r^3).$$

So, up to lower order, to satisfy $k_2^T S k_1 = 0$, we can find b if

$$B = 0 \text{ and } (3A \neq C \text{ or } F = 0)$$

When $3A = C$, we can use higher order terms to solve for b. These cases, which follow from our choice of coordinates, are what we will be calling our basic assumptions, as they will be assumed to be true whenever the shape operator for (2.1) is continuously diagonalizable.

BASIC ASSUMPTIONS
$B = 0$ and either $3A \neq C$ or $F = 0$

Recall our convenient rotation earlier that depressed our polynomial so that $B = 0$. It turns out that we get this depression for free with our rotation to these coordinates. With all of this in our tool belt, we can now prove a very useful result.

Lemma 4.4. *Let M be a non-developable surface defined by* (2.1)*. If the shape operator is continuously diagonalizable, then there exist coordinates such that one of the following two cases hold:*

I. $C = F = G = H = 0$

II. $C = 0$, *either* $A \neq 0$ *or* $D \neq 0$, $3FD^2 + 4GAD + 3HA^2 = 0$, *and* $(6E - G)D^2 + 3(F - H)AD + (G - 6I)A^2 = 0$.

Proof. We start by assuming that $k_1 = [1, b]$ and $k_2 = [-b, 1]$, where b is a continuous function of r and θ such that $b \to 0$ as $r \to 0^+$. We will be using the fact that

$$k_2^T S k_1 = f_{xy} + b(f_{yy} - f_{xx}) - b^2 f_{xy} = 0 \tag{4.2}$$

to determine what the coefficients of (2.1) must be in order to ensure that the principal directions are the same in all other directions. To this end, if $f_{xy} \neq 0$, then we can solve for b to get

$$b = \frac{f_{xx} - f_{yy} \pm \sqrt{(f_{yy} - f_{xx})^2 + 4f_{xy}^2}}{-2f_{xy}} = \frac{2f_{xy}}{f_{xx} - f_{yy} \mp \sqrt{(f_{yy} - f_{xx})^2 + 4f_{xy}^2}}.$$

where

$$f_{xy} = 2Cr\sin\theta + (3F\cos^2\theta + 4G\sin\theta\cos\theta + 3H\sin^2\theta)r^2$$

and

$$\begin{aligned} f_{yy} - f_{xx} = {} & [(2C - 6A)\cos\theta + 6D\sin\theta]r \\ & + [(2G - 12E)\cos^2\theta + 6(H - F)\sin\theta\cos\theta + (12I - 2G)\sin^2\theta]r^2. \end{aligned}$$

Now, for (4.2) to hold for lower order terms for any θ, we must have $C = 0$. More explicitly, if $f_{xy} \neq 0$,

$$b = \frac{(3F\cos^2\theta + 4G\sin\theta\cos\theta + 3H\sin^2\theta)r}{g(\theta, r) \mp \sqrt{[g(\theta, r)]^2 + (3F\cos^2\theta + 4G\sin\theta\cos\theta + 3H\sin^2\theta)^2 r^2}} \tag{4.3}$$

where

$$\begin{aligned} g(\theta, r) = {} & 3A\cos\theta - 3D\sin\theta \\ & + [(6E - G)\cos^2\theta + 3(F - H)\sin\theta\cos\theta + (G - 6I)\sin^2\theta]r. \end{aligned} \tag{4.4}$$

Note that g is simply $(f_{xx} - f_{yy})/2r$. When $f_{xy} = 0$ for all values of θ, we have $F = G = H = 0$ and (4.2) implies $b = 0$. This is exactly case **I** in the statement of the lemma. Now suppose $f_{xy} \neq 0$ for some value of θ. Note that if $A = D = 0$, then (4.3) is independent of r and hence $b \nrightarrow 0$ as $r \to 0^+$ for some θ.

Now we consider the case that $A \neq 0$ or $D \neq 0$ such that the constant (with respect to r) term in (4.4) is not identically zero. When $A\cos\theta - D\sin\theta \neq 0$ there will be a term in the denominator that does not depend on r, allowing $b \to 0$ as $r \to 0^+$. Next we ask what happens in the direction such that $A\cos\theta - D\sin\theta = 0$. This is the direction, call it $\theta = \theta_0$, in which

$$\cos\theta_0 = \frac{\pm D}{\sqrt{A^2 + D^2}} \qquad \text{and} \qquad \sin\theta_0 = \frac{\pm A}{\sqrt{A^2 + D^2}}.$$

Notice that if $f_{xy} \neq 0$ when $\theta = \theta_0$,

$$b\big|_{\theta=\theta_0} = \frac{3FD^2 + 4GAD + 3HA^2}{g(\theta_0, 1) \mp \sqrt{[g(\theta_0, 1)]^2 + (3FD^2 + 4GAD + 3HA^2)^2}}. \tag{4.5}$$

Now, if the numerator is nonzero, then the denominator is certainly nonzero. However, (4.5) is independent of r and so to enforce our requirement that $b \to 0$ as $r \to 0^+$, we must have

$$3FD^2 + 4GAD + 3HA^2 = 0 \tag{4.6}$$

Adopting the convention that $s = \sin\theta$, $c = \cos\theta$, and $\sigma = \sqrt{A^2 + D^2}$, we consider approaching the origin along the curve

$$r = k|\sin(\theta - \theta_0)| = \frac{k}{\sigma}|Ds - Ac| \tag{4.7}$$

for some positive real number k. Under the assumption that $g(\theta_0, 1) \neq 0$, we claim that there exists some k such that $b \nrightarrow 0$ as $\theta \to \theta_0$. Showing this claim will imply discontinuous principal directions, a contradiction. Now, plugging in our substitutions and our prescribed path to the origin, for $\theta \neq \theta_0$ we obtain

$$b = \frac{g_2(\theta) \pm \sqrt{[g_2(\theta)]^2 + [k(3Fc^2 + 4Gsc + 3Hs^2)]^2}}{-k(3Fc^2 + 4Gsc + 3Hs^2)} \tag{4.8}$$

where

$$\begin{aligned} g_2(\theta) &= \sigma \frac{g(\theta, \frac{k}{\sigma}|Ds - Ac|)}{|Ds - Ac|} \\ &= 3\sigma \operatorname{sgn}(Ac - Ds) + [(6E - G)c^2 + 3(F - H)sc + (G - 6I)s^2]k. \end{aligned}$$

In case it is not immediately clear to the reader that the sign of $Ac - Ds$ will be different depending on how θ approaches θ_0, observe that $\sin(\theta - \theta_0) = \pm(Ds - Ac)/\sigma$. Then for θ near θ_0,

$$\operatorname{sgn}(Ac - Ds) = \mp \operatorname{sgn}(\theta - \theta_0).$$

Let us denote the numerator of (4.8) by η as well as the substitution

$$\omega = g(\theta_0, 1) = \frac{(6E - G)D^2 + 3(F - H)AD + (G - 6I)A^2}{\sigma^2} \neq 0.$$

Now let

$$0 < k < \frac{3\sigma}{|\omega|}.$$

Let us look at the two cases in which $\omega < 0$ or $\omega > 0$. Whenever $\omega < 0$, we have

$$0 < k < -3\sigma/\omega \qquad \Longrightarrow \qquad 3\sigma + \omega k > 0.$$

Now, as we approach along the direction in which $Ac - Ds > 0$, we find that

$$\eta \to 3\sigma + \omega k \pm |3\sigma + \omega k|. \tag{4.9}$$

Recall that as $\theta \to \theta_0$, the denominator of (4.8) approaches zero by (4.6). If there is any hope for (4.8) to go to zero as $\theta \to \theta_0$, we must be in the indeterminate form 0/0. So we must choose the minus sign in (4.9). However, when approaching along the direction in which $Ac - Ds < 0$, then for b to be continuous, we must keep the minus sign and so

$$\eta \to -3\sigma + \omega k - |-3\sigma + \omega k|. \tag{4.10}$$

Notice that since $-3\sigma < 0$ and $\omega k < 0$, then $-3\sigma + \omega k < 0$ and hence η does not approach zero in (4.10). Now, we assume $\omega > 0$. This argument will follow similar to the last one. So we have

$$0 < k < \frac{3\sigma}{\omega} \qquad \Longrightarrow \qquad -3\sigma + \omega k < 0.$$

Approaching along the direction in which $Ac - Ds < 0$, we find that

$$\eta \to -3\sigma + \omega k \pm |-3\sigma + \omega k| \tag{4.11}$$

which will equal zero only if we use the plus sign. Again, we want b to be continuous, so we keep the plus sign when approaching along the direction in which $Ac - Ds > 0$. Along this direction, we find that

$$\eta \to 3\sigma + \omega k + |3\sigma + \omega k|. \tag{4.12}$$

Notice that $3\sigma + \omega k > 0$ since $3\sigma > 0$ and $\omega k > 0$. This implies that as $\theta \to \theta_0$, $\eta \nrightarrow 0$, and thus $b \nrightarrow 0$, in either of the two cases. It follows by this contradiction that for b to approach zero along this path, we need $\omega = 0$, or

$$(6E - G)D^2 + 3(F - H)AD + (G - 6I)A^2 = 0.$$

■

4.3 Main Theorem

Now that we have the classifications for a diagonalizable shape operator, we are able to prove (quite quickly) what was predicted in the table at the end of section 3.

Theorem 4.5. *Let M be a non-developable surface given by f from* (2.1) *that satisfies our basic assumptions. If the shape operator is continuously diagonalizable, then there exists some curve through the origin with noncomparable principal curvatures.*

Proof. Under the assumption that the shape operator is continuously diagonalizable, we need to find a curve passing through the origin in which the principal curvatures are non-comparable for both cases in Lemma 4.4. We start with case **I**, in which S is diagonal. Then we can read the principal curvatures off the diagonal of S. In Cartesian coordinates, the eigenvalues are

$$\lambda_1 = (6Ax + 12Ex^2)(1 + \mathcal{O}(x^4 + y^4))$$
$$\lambda_2 = (6Dy + 12Iy^2)(1 + \mathcal{O}(x^4 + y^4)).$$

Here, we can find a noncomparable path since approaching the origin along one of the coordinate axes (the choice of which axis depends on the values of A, E, D, and I) will produce a nonzero principal curvature as well as an identically zero principal curvature. Note that $A = D = E = I = 0$ is forbidden since that would make 2.1 identically zero and hence developable.

Next, we assume case **II** from Lemma 4.4. Let us first assume that $A \neq 0$. In polar coordinates, when $\theta = 0$,

$$S = \begin{bmatrix} 6Ar + 12Er^2 & 3Fr^2 \\ 3Fr^2 & 2Gr^2 \end{bmatrix}$$

and by (2.3) the principal curvatures are

$$\lambda = 3Ar + (6E + G)r^2 \pm \sqrt{[3Ar + (6E - G)r^2]^2 + 9F^2r^4}. \tag{4.13}$$

Applying the Taylor approximation trick outlined in Theorem 4.2, (4.13) becomes

$$\lambda = 3(A \pm |A|)r + \mathcal{O}(r^2)$$

which gives us principal curvatures that vanish at different rates. Now, when $A = 0$, this implies $D \neq 0$ and we proceed in a similar fashion as above, but with $\theta = \pi/2$, to obtain

$$\lambda = 3(D \pm |D|)r + \mathcal{O}(r^2).$$

Again, these principal curvatures are noncomparable. ■

5 Looking Ahead

The motivation for this paper comes from the theory of partial differential equations of functions of several complex variables. These functions are usually defined over some domain in $\mathbb{C}^n$ and it is the geometry of this domain that influences if and how we can solve the PDE. It is therefore beneficial to study the curvature of the manifold defined by the boundary of the domain. This is where the Levi form (the complex analogue of the shape operator) and its eigenvalues come into play. In general, the domains in question will yield a solvable PDE, but it is impossible to fully estimate all partial derivatives of our function. However, having comparable eigenvalues of the Levi form for the domain enables us to fully estimate all but one of the partial derivatives. Achieving this estimate of our function is the best case we can hope for. This should stress the importance of being able to establish some criteria for comparable eigenvalues. In this paper, it turned out that there are always comparable eigenvalues.

To this end, let u be a function and f a known 1-form, both defined over some domain $\Omega \subset \mathbb{C}^n$, and suppose we want to solve the $\bar{\partial}$-problem $\bar{\partial}u = f$ where $\bar{\partial}$ is the Cauchy-Riemann operator. The solvability of this PDE is not in question since Kohn showed in 1964 that $\bar{\partial}u = f$ is solvable whenever every eigenvalue of the domain's Levi form is positive [17] [18]. Then in 1965, Hörmander weakened Kohn's hypothesis by only requiring nonnegative eigenvalues [13]. In the same publication, Hörmander also showed that the upper bound of the norm of u depends on the norm of f. As stated in the previous paragraph, we cannot estimate all partial derivatives of u in general. So we would like to find a function that is the weak derivative of u that estimates as many of u's partial derivatives as possible. Our best case scenario would be to find a solution that estimates the partial derivatives of u in every direction of $\mathbb{C}^n$ except one. If this can be done, then we have achieved what is called the maximal estimate. Derridj has shown that the maximal estimate can be achieved if and only if the eigenvalues of the Levi form for the manifold $\partial\Omega$ are comparable [8]. For Derridj's original discussion on maximal estimates, see [7]. For a lower dimensional discussion of the maximal estimate, refer to Straube's lecture notes [21]. For more analysis on this topic, refer to [11] and [14].

Charpentier and Dupain also have some nice results involving both the Levi form and the Bergman projection. Let $L^2(\Omega)$ denote the function space of square integrable functions defined over $\Omega \subset \mathbb{C}^n$, $\mathscr{O}(\Omega)$ denote the function space of holomorphic functions defined over Ω, and $\mathscr{H}^2(\Omega) = L^2(\Omega) \cap \mathscr{O}(\Omega)$. Since $\mathscr{H}^2(\Omega)$ is a subspace of $L^2(\Omega)$, we can orthogonally project functions in $L^2(\Omega)$ to $\mathscr{H}^2(\Omega)$. This is the Bergman projection. Charpentier and Dupain showed in [5] that we can estimate all derivatives of the Bergman projection of a function u whenever the Levi form is diagonalizable and the domain is of finite type (a technical term that is beyond the scope of this paper). Related results can be found in [15] and [16].

It would be nice if we could have the previous two results simultaneously. Unfortunately, having comparable eigenvalues and having a diagonalizable Levi form are mutually exclusive events. It is therefore important for the vigilant mathematician to know which situation they might find themselves in. Although this paper works solely in the case where $\partial\Omega$ is a two dimensional manifold embedded in $\mathbb{R}^3$, it is a good step forward in determining whether or not the Levi form has comparable eigenvalues.

At this point, one may wonder what the next possible step in this research would be. We could move forward by increasing the size of our surface sample space by assuming that (2.1) is a lower order Taylor approximation for some arbitrary surface. Under our current assumptions, having lines of umbilics was a possibility. However, there are some points that appear umbilic with lower order terms, but are not umbilic after bringing in the higher order terms. So with a Taylor approximation, we would be working under the assumption of having an isolated umbilic at the origin. This was the initial ambition of this work before moving to the case with non-isolated umbilics.

Finally, we could move even further with this work by attempting to extend our results to manifolds in $\mathbb{R}^n$. Proving this would likely require considerably more finesse than the exhaustive methods used in this work. And of course, the work of mathematicians such as Charpentier, Dupain, Derridj, Harrington, Raich, (see [4], [8], and [12]) and many others provide the motivation for the ultimate goal of this research: to move up to manifolds in $\mathbb{C}^n$ and push the understanding of the $\bar{\partial}$ problem forward.

6 Appendix

This appendix is dedicated to showing the reader the Mathematica 9 code used to generate the images in the examples. Since $\det S = \lambda_1\lambda_2$, it is easy to see that

$$\lambda_1/\lambda_2 = \lambda_1^2/\det S.$$

Whenever $\det S = 0$, we replace the eigenvalue ratio with its limiting value at the origin. The choice of approaching along the line $y = x$ (as apposed to the coordinate axes) is because this was the direction that, in practice, yielded the most accurate results.

```
(*Define function, calculate second derivatives*)

f[x_, y_] := x^3/6 + y^3/6 + x^4/12 - y^4/12;
fxx = D[f[x, y], {x, 2}];
fxy = D[f[x, y], x, y];
fyy = D[f[x, y], {y, 2}];

(*Define trace and determinant of S as well as other useful
parameters*)

tr = fxx + fyy;
det = fxx*fyy - fxy^2;
A = fxx - fyy;
B = fxy;

(*Compute the eigenvalues of S*)

eval1 = (tr + Sqrt[A^2 + 4 B^2])/2;
eval2 = (tr - Sqrt[A^2 + 4 B^2])/2;

(*Compute eval1/eval2*)

evalratio = eval1^2/(det);

(*Compute limit of evalratio as the line y = x approaches the
origin from the right*)

y = x;
evallim = Limit[evalratio, x -> 0, Direction -> 1];

(*Reset y*)

y =.;

(*Rescale evalratio and evallim to be in the interval [0,1] for the
Hue function. Piecewise function takes care of any singularities*)

evalhue :=
```

```
  Piecewise[{{(ArcTan[Log[Abs[evalratio]]] + Pi/2)/Pi,
     det != 0 }, {(ArcTan[Log[Abs[evallim]]] + Pi/2)/Pi, det == 0}}];

(*Plot surface in neighborhood of the origin to visualize comparable
eigenvalues. Red points have noncomparable eigenvalues while points
with any other color have comparable eigenvalues*)

Plot3D[f[x, y], {x, -.25, .25}, {y, -.25, .25},
  AxesLabel -> Automatic,
  ColorFunction -> Function[{x, y, z}, Hue[evalhue]],
  ColorFunctionScaling -> False, BoxRatios -> {1, 1, .5},
  PlotPoints -> 50, BaseStyle -> {FontSize -> 10}]

(*Compute the ratio of the components of an eigenvector of S*)

evec = A/(2 B) + (Sqrt[(A/B)^2 + 4])/2;

(*Normalize evec to be in the interval [0,1] for the Hue function.
Piecewise takes care of any singularities*)

evechue := Piecewise[{{2 (ArcTan[evec] + Pi/2)/Pi, B != 0}, {1,
     B == 0}}];

(*Plot surface to visualized diagonalizability. If a neighborhood of
the origin is a single color, the shape operator is diagonalizable.
Otherwise, the shape operator is nondiagonalizable.*)

Plot3D[f[x, y], {x, -.5, .5}, {y, -.5, .5}, AxesLabel -> Automatic,
  ColorFunction -> Function[{x, y, z}, Hue[evechue]],
  ColorFunctionScaling -> False, BoxRatios -> {1, 1, .5},
  PlotPoints -> 50]
```

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