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ROSE-HULMAN UNDERGRADUATE MATHEMATICS JOURNAL
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SURFACE-AREA-MINIMIZING n-HEDRAL TILES

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Abstract. We provide a list of conjectured surface-area-minimizing n-hedral tiles of space for n from 4 to 14, previously known only for n equal to 5 and 6. We find the optimal "orientation-preserving" tetrahedral tile ($n = 4$), and we give a nice new proof for the optimal 5-hedron (a triangular prism).

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1 Introduction

Certain polygons have the property that they can tile the plane without overlapping or gaps between them. The most common polygons that tile the plane are squares, triangles, and regular hexagons. Suppose we scale all polygonal tiles in the universe to unit area. Then which tile has the smallest perimeter? It is well known that the optimal polygon with four sides is a square, and the optimal triangle is the equilateral triangle. Recently, Chang et al [CFS] proved that the optimal convex 5-gon is a tie between the so-called Prismatic and Cairo pentagons (the convexity assumption has yet to be removed). In 2001, Thomas Hales [Hal] proved that the regular hexagon is the optimal tile among all possible unit-area tiles. Therefore, for $n > 6$, the best n-gon is a degenerate n-gon, namely, the regular hexagon with extra vertices.

In this paper, we study the same perimeter minimization problem in three dimensional space: we seek the n-hedral tile of unit volume that minimizes surface area. Past research, including studies by Goldberg, Minkowski, and Fejes Tóth, has been in finding overall surface-area-minimizing polyhedra, which need not tile. For example, Minkowski [M] showed that for any n, there exists as surface-area-minimizing n-hedron. Some of the optimal polyhedra identified such as the regular tetrahedron ($n = 4$) and the regular dodecahedron ($n = 12$) are, in fact, not tiles. Therefore, it is natural to ask what are the optimal polyhedral tiles, or even, for any n, whether we can always find a surface-area-minimizing n-hedral tile.

In section 2 of this paper, Conjecture 2.1 provides candidates for the optimal n-hedral tiles from $n = 4$, a certain irregular tetrahedron, to $n = 14$, Kelvin's truncated octahedron, and beyond (see Figs. 1-9). The conjecture is known for $n = 6$ and $n = 5$. That a cube is the best 6-hedron, tile or not, is well known [FT1] (see Thm. 2). In section 3, we show that among convex polyhedra, for fixed n, there exists a surface-area-minimizing n-hedral tile of space. Section 4 gives some properties of prisms and a proof that a certain hexagonal prism is the surface-area-minimizing prism that tiles space. In section 5, we find the optimal 5-hedron and "orientation-preserving" 4-hedral tile. Theorem 5.10 gives a nice new proof that a certain triangular prism is the surface-area-minimizing 5-hedron. Because the triangular prism tiles space, it is also the optimal 5-hedral tile. Additionally, Theorem 5.2 proves that a third of a particular triangular prism is the surface-area-minimizing "orientation-preserving" 4-hedral tile, based on a classification of tetrahedral tiles by Sommerville [So].

2 Tiling of Space

We assume that a polyhedron tiles $\mathbb{R}^3$ with congruent copies of itself and the copies are face-to-face, i.e., that they meet only along entire faces, entire edges, or at vertices. Moreover, we do not assume that a tile is convex. We have the following conjecture:

Conjecture 2.1. *For fixed n and unit volume, the following provide the surface-area-minimizing n-hedral tiles of* $\mathbb{R}^3$ *(see Figs. 1-9 and Table 1):*

1. $n = 4$*: a tetrahedron formed by four isosceles triangles with two sides of* $\sqrt{3}$ *and one side of*

2 (scaled to unit volume). It is also formed by cutting a triangular prism into three congruent tetrahedra;

2. $n = 5$*: a right equilateral-triangular prism;*

3. $n = 6$*: a cube;*

4. $n = 7$*: a right Cairo or Prismatic pentagonal prism;*

5. $n = 8,9$*: a gabled rhombohedron, described by Goldberg [G7] as having four pentagonal and four quadrilateral sides, and a regular hexagonal prism;*

6. $n = 10$*: a half elongated rhombic dodecahedron, found in Fejes Tóth [FT2] and described by Goldberg [G6] as a "bee cell;"*

7. $n = 11$*: a half of a deformed, elongated truncated octahedron, described by Fejes Tóth [FT2] as a more optimal shape than the 10-hedral "bee cell;"*

8. $n = 12$*: a rhombic dodecahedron and a trapezo-rhombic dodecahedron of Types 12-IV and 12-VI as described by Goldberg [G3];*

9. $n = 13$ *: half of the bee cell and the Fejes Tóth cell, combined at their hexagonal bases (described by Goldberg as the 13-hedron of Type 13-II [G1]);*

10. $n = 14$*: Kelvin's truncated octahedron ([Ke], see [Mo1, pp. 157-171]);*

11. $n > 14$*: a polyhedral approximation of the curvilinear Kelvin Cell ([Ke], see Morgan [Mo1, pp. 157-171] and Weaire [W, p. 74]).*

Remark 2.2. Goldberg ([G2, p.231], see Florian [F, p. 213]), conjectured that a surface-area-minimizing n-hedron has vertices of degree three only, but it may well not tile. All of our conjectured polyhedra, except for $n = 10, 12$ and 13, only contain vertices of degree three.

Unfortunately, a regular tetrahedron, which is the surface-area-minimizing tetrahedron, does not tile because the dihedral angles of 70.53° cannot add up to 360° (Fig. 10). We provide the best orientation-preserving tetrahedral tile in Theorem 5.2, but have not been able to remove the orientation-preserving assumption.

In the known cases $n = 5$ or $n = 6$, the candidates are surface-area-minimizing unit-volume n-hedra and hence, of course, the optimal n-hedral tiles. Minkowski [M] proved that such a n-hedron exists, as does Steinitz [Ste]. (We do not think that their arguments imply the existence of a surface-area-minimizing unit-volume n-hedral *tile*.)

The case $n = 6$ follows immediately from a theorem of Goldberg [G2, p. 230] and also given by Fejes Tóth.

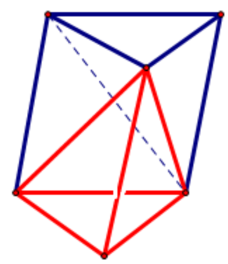


Figure 1: A tetrahedron formed by cutting a triangular prism into three congruent tetrahedra is the conjectured surface-area-minimizing tetrahedral tile.

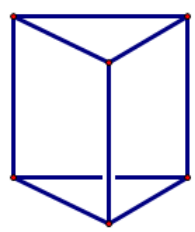


Figure 2: A right equilateral-triangular prism is the surface-area-minimizing 5-hedron.

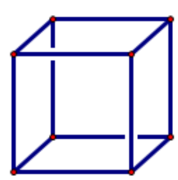


Figure 3: A cube is the surface-area-minimizing 6-hedron.

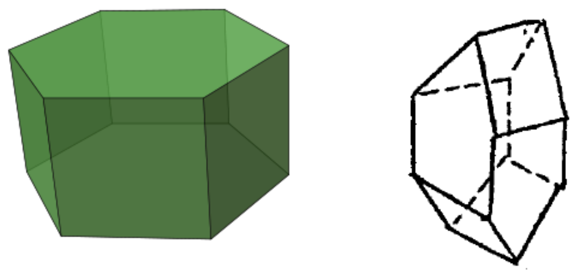


Figure 4: A hexagonal prism [Wi] and Goldberg's [G7, Fig. 8-VI] gabled rhombohedron are the conjectured surface-area-minimizing 8-hedral tiles. They have the same surface area.

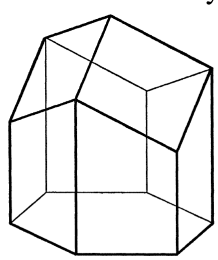


Figure 5: A half elongated rhombic dodecahedron, described by Fejes Tóth [FT2], is the conjectured surface-area-minimizing 10-hedral tile. Image from [FT2].

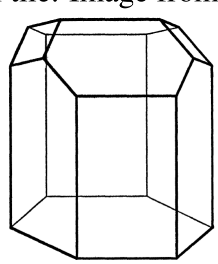


Figure 6: A half elongated, deformed truncated octahedron, described by Fejes Tóth [FT2], is the conjectured surface-area-minimizing 11-hedral tile. Image from [FT2].

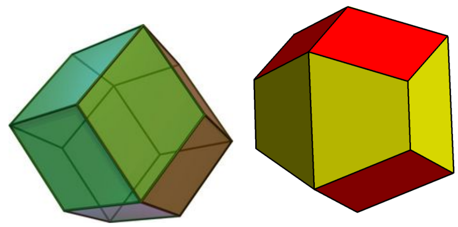


Figure 7: A rhombic dodecahedron and a trapezo-rhombic dodecahedron are the conjectured surface-area-minimizing 12-hedral tiles. They have the same surface area. Images from [Wi].

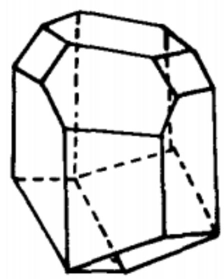


Figure 8: Combining half of the bee cell and half of a Fejes Tóth Cell yields our conjectured surface-area-minimizing 13-hedral tile. Image from [G1].

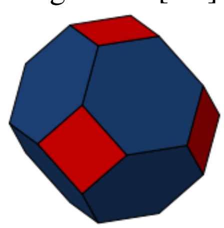


Figure 9: Kelvin's truncated octahedron is the conjectured surface-area-minimizing convex polyhedral tile. Image from [Wi].

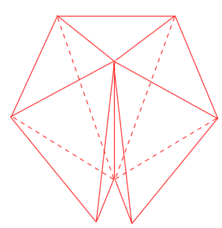


Figure 10: Because the dihedral angles (70.53°) of a regular tetrahedron cannot add up to 360°, the regular tetrahedron does not tile; there is a small gap. Image from [U].

Theorem. *([FT1, pp. 174 - 180],* see *[F, pp. 212 - 213]). If F denotes the surface area and V the volume of a three-dimensional convex polyhedron with f faces, then*

$$\frac{F^3}{V^2} \geq 54(f-2)\tan\omega_f(4\sin^2\omega_f - 1)$$

where $\omega_f = \pi f/6(f-2)$*. Equality holds only for regular tetrahedra, cubes, and regular dodecahedra.*

Regarding $n = 7$, Goldberg [G2] claims that a right regular pentagonal prism is the surface-area-minimizing 7-hedron. However, the proof, which was given by Lindelöf, is – in Lindelöf's words – "only tentative". Furthermore, regular pentagons cannot tile the plane. Therefore, we cannot tile $\mathbb{R}^3$ with the right regular pentagonal prism. Cairo and Prismatic pentagons (Fig. 11) have recently been proved by Chung et al. [CFS, Thm. 3.5] as the best pentagonal planar tiles. They are circumscribed about a circle, with three angles of $2\pi/3$ and two angles of $\pi/2$, adjacent in the Prismatic pentagon and non-adjacent in the Cairo pentagon. We conjecture a right Cairo or Prismatic prism is the surface-area-minimizing 7-hedral tile.

For $n = 8$ and 9, Goldberg [G2] shows that the regular octahedron does not minimize surface area, supporting his conjecture that the surface area minimizer cannot have vertices of degree greater than three. We found that a gabled rhombodecahedron has the same surface area as a regular octahedron (which does not tile) and hexagonal prism (which tiles). Moreover, it has less surface area than the gyrobifastigium suggested by Li et al. [LMPW, p. 30] as the optimal 8-hedral tile. Note that an n_0-hedron may be considered a (degenerate) n-hedron for any $n > n_0$ by subdividing its faces, as in 2.1(5). The gabled rhombodecahedron is distinguished among Goldberg's [G7] octahedral tiles by having all vertices of degree three. We examined another 8-hedron as a possible minimizing tile – the so-called Schmitt-Conway biprism (see [Wei] and [Haf2], pictured in Fig. 12). The optimal biprism is worse than the cube, because rotating the lower triangular prism 90 degrees yields a polyhedron with 6 faces.

For $n = 9$, we have also considered halves of various so-called "pencil cubes" described by Goldberg [G5, Types 9-I and 9-II] as well as a quarter truncated octahedron [G5, Type 9-IV].

For $n = 10$ and 11, we initially conjectured that a decahedral "barrel" with congruent square bases and eight congruent pentagonal sides was optimal. The decahedron was inspired by a non-trivial geodesic nets on the sphere meeting in threes at $2\pi/3$, classified by Heppes (see [T] and [Mo1]), although these polyhedra inscribed in spheres are not circumscribed about spheres as surface area minimizers would be. However this decahedron does not tile. Our current conjectured optimal 10-hedral tile is a half elongated rhombic dodecahedron having a regular hexagonal cross-section. It is described by Fejes Tóth [FT2] and Goldberg [G4], and it is inspired by the shape of honeycombs used by bees. Our conjectured 11-hedron is similarly described by Fejes Tóth [FT2] and Goldberg [G4] as forming a slightly more efficient "honeycomb" than that derived from a half rhombic dodecahedron. It is formed by truncating an elongated, irregular truncated octahedron. We have compared it to truncating the regular truncated octahedron (with equal edge lengths) in half in such a way that the resulting polyhedron has 11 faces.

For $n = 12$, we conjecture that a rhombic dodecahedron and a trapezo-rhombic dodecahedron are surface area-minimizing tiles. These are described as dodecahedra of Types 12-IV and 12-VI

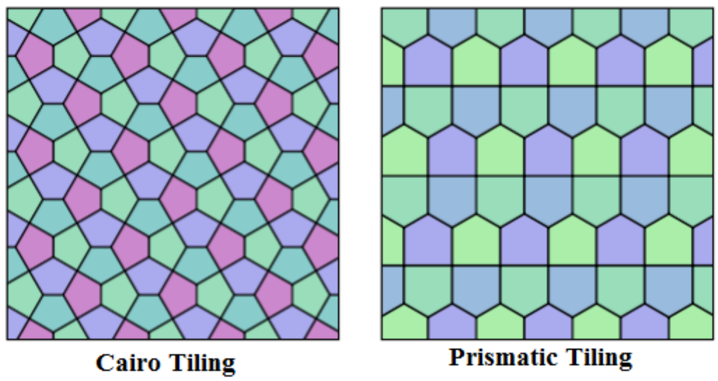


Figure 11: Cairo and Prismatic pentagons have recently been proved as the best pentagonal planar tiles (Chung et al. [CFS, Thm. 3.5]). Image from [Wi].

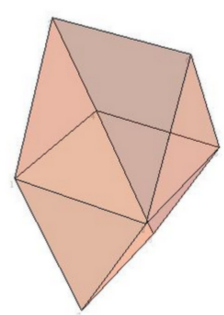


Figure 12: The optimal Schmitt-Conway biprism [Wei] is an 8-hedron with surface area greater than that of the cube. Image from [Haf2].

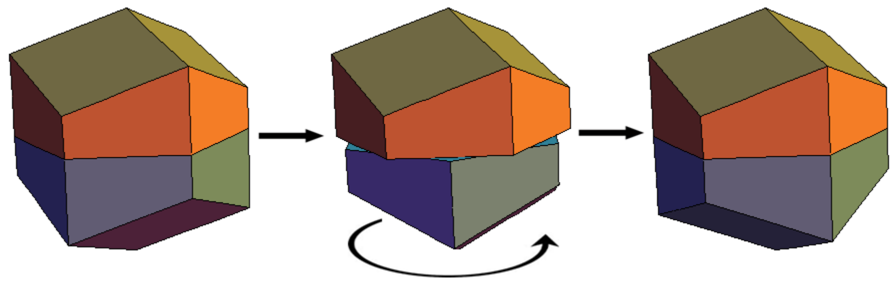


Figure 13: Rotating half of a rhombic dodecahedron yields the trapezo-rhombic dodecahedron. Images from [Haf1].

by Goldberg. Note that if one were to take half of either dodecahedron (getting the conjectured 10-hedron), such that both halves contained exactly three rhombi, rotating one of the halves by $\pi/3$ and rejoining them yields the other shape (Figure 13). Therefore, they clearly have the same surface area. We have also considered a 12-hedron of Type 12-VIII described by Goldberg [G3] with 20 vertices of degree three and none of degree four (one half the truncated octahedron). This has higher surface area than our conjectured candidates.

For $n = 13$, we take our conjectured optimal 10- and 11-hedra and glue them together at their bases. This idea came from the observation that the two polyhedra compared by Fejes Tóth both have regular hexagonal bases [FT2]. It is Goldberg's Type 13-II [G1].

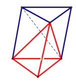
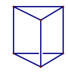
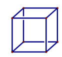
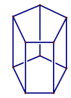
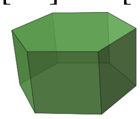
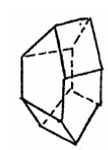
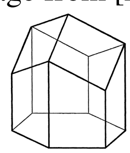
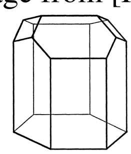
For $n = 14$, we follow the famous Kelvin Conjecture $-$ that a truncated octahedron is the surface-area-minimizing convex n-hedron that tiles space. Table 1 gives the surface areas of the conjectured minimizers, computed using Proposition 4.5, known formulae for surface area and volume, or via direct calculation, using Mathematica to optimize. In the case of the 11-hedra, Fejes Tóth [FT2] provides volume and surface area formulae which were used in our calculations. We initially used the Quickhull algorithm [BDH] to help with initial calculations, but didn't use the algorithm in our final results.

As Yoav Kallus, Aládar Heppes, and Endre Makai pointed out to us, since the ideal Kelvin cell is not polyhedral, there are better and better n-hedral approximations for large n.

Proposition 2.3. *There exists an $n > 14$ for which the truncated octahedron does not provide a surface-area-minimizing n-hedral tile.*

Proof. If the truncated octahedron is allowed to relax while retaining unit volume it produces the curvilinear Kelvin cell ([Ke], see Morgan [Mo1, pp. 157-171], and Weaire [W, p. 74]), which has less surface area and tiles 3-space. A single Kelvin cell tiles a 3-torus and therefore approximating the curvilinear Kelvin cell in the torus results in a polyhedron that tiles space but has strictly less surface area than the truncated octahedron. □

The following proposition shows that you can always reduce surface area by a small truncation and rescaling; however, the resulting polyhedron may not tile. We think the truncated octahedron is as far as you can go with this method and still tile.

$n = 4$ One third of a triangular prism	**7.4126** 
$n = 5$ A triangular prism	**6.5467** 
$n = 6$ A cube	**6.0000** 
$n = 7$ A Cairo or Prismatic pentagonal prism	**5.8629** 
$n = 8, 9$ A hexagonal prism or a gabled rhombohedron	**5.7191** (Images from [Wi] and [G7, Fig. 8-VI])  
$n = 10$ A half elongated rhombic dodecahedron	**5.5386** (Image from [FT2]) 
$n = 11$ A half deformed truncated octahedron	**5.5351** (Image from [FT2]) 

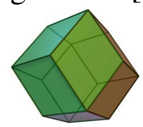
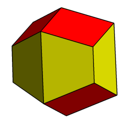
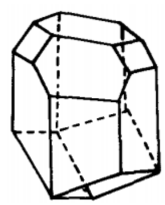
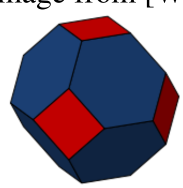
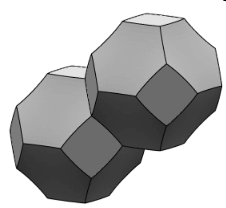
$n = 12$ A rhombic dodecahedron or a trapezo-rhombic dodecahedron	**5.3454** (Images from [Wi])  
$n = 13$ Goldberg's [G1] Type 13-II	**5.3416** (Image from [G1]) 
$n = 14$ Kelvin's truncated octahedron	**5.3147** (Image from [Wi]) 
$n = \infty$ The Kelvin cell	**5.3051** (From Princen and Levinson [P], image from Brakke [B]) 

Table 1: Conjectured surface-area-minimizing unit-volume n-hedral tiles.

Proposition 2.4. *A slight truncation at any strictly convex vertex and rescaling to the original volume reduces the surface area of a polyhedron.*

Proof. Instead of rescaling, we show the decrease of the scale invariant area-volume ratio A^3/V^2. Under truncation by a distance t, the logarithmic derivative

$$\frac{3A'}{A} - \frac{2V'}{V}$$

is negative for all sufficiently small t because A' is proportional to $-t$, while V' is proportional to $-t^2$. $\square$

Heppes drew our attention to Wolfram Online's [Wei] discussion of convex polyhedral tiles. It notes the extensive categorization of polyhedral tiles by Goldberg [G1-G7]. Grünbaum and Shephard [Gr] and Wells [Wel] discuss the known polyhedral tiles pre-1980, when the maximal n for n-hedral tiles was believed to be 26. In 1980, P. Engel [Wel, pp. 234-235] found 172 additional polyhedral tiles with 17 to 38 faces, and Wolfram Online [Wei] says more polyhedral tiles have been found subsequently.

3 Existence of a surface-area-minimizing tile

For fixed n, Minkowski [M] proved that among convex polyhedra, there exists a surface-area-minimizing n-hedron. (For this problem, the convexity hypothesis is unnecessary, because for fixed orientations of the n faces, the minimizer is convex; see [Mo2]). We show that if we assume the polyhedra tile space, then there exists a surface-area-minimizing polyhedral tile.

Definition 3.1. A polyhedron is *nondegenerate* if it does not have any unnecessary edges, i.e. if there are no dihedral angles equal to π.

The furthest distance between two vertices is the *diameter* of a polyhedron.

We call two polyhedra P and Q combinatorially equivalent if there exists a bijection f between the set of the vertices of P and Q such that:

1. *v_1v_2 is an edge of P if and only if $f(v_1)f(v_2)$ is an edge Q.*

2. *$v_1,\ldots,v_k$ is a face of P if and only if $f(v_1),\ldots,f(v_k)$ is a face Q.*

Proposition 3.2. *For any n, there are a finite number of combinatorial types of n-hedra.*

Proof. Fix n. First, a n-hedron's face can have at most $(n-1)$-edges. Assume, on the contrary, that a n-hedron contains an n-gon. Then since each edge is shared by two faces and two faces share at most one edge, there are at least $n+1$ faces in the n-hedron, which is a contradiction. This means that the biggest face can have $n-1$ edges and the smallest is a triangle (3 edges).

Therefore, we have $n-3$ choices for each face. Hence, the number of possible combinations of $n-3$ faces is equivalent to the number of solutions to the equation

$$x_3 + x_4 + \ldots + x_{n-1} = n$$

where x_i corresponds to the number of faces with i edges. The number of solutions to the equation is $\binom{2n-4}{n}$. It follows that for each combination, we can arrange the faces in a finite number of ways. Therefore, there are a finite number of combinatorial types. $\square$

Remark 3.3. Not all possible combinations of faces can make a polyhedron. For example for $n = 5$, it is possible to have 6 combinations of different faces, but in Proposition 5.4, we will prove that the only combinatorial types are either triangular prisms or quadrilateral pyramids.

Theorem 3.4. *For a fixed n, there exists a surface-area-minimizing unit-volume convex n-hedral tile.*

The minimizer could be a degenerate n-hedron (with fewer than n faces), as we conjecture occurs for $n > 14$ (Conj. 2.1 (10)).

Proof. Take a sequence of unit-volume convex n-hedral tiles with surface areas approaching the infimum. We may assume that the surface areas are bounded above by P_0. To obtain a surface-area-minimizing limit, by compactness of the set of possible vertices inside a large ball, it suffices to show that the diameters are bounded.

Consider a unit-volume convex polyhedron. Take the slice of largest area a_0 perpendicular to the diameter D. Consider a pyramid with based a_0 and the apex at the most distant end of the diameter. By convexity, the pyramid lies inside the polyhedron. Therefore,

$$1 \geq \left(\frac{1}{3}\right) a_0 \frac{D}{2}$$

and

$$a_0 \leq \frac{6}{D}.$$

For every slice perpendicular to the diameter, by the isoperimetric inequality, the perimeter p and area a satisfy

$$p \geq \sqrt{4\pi a}.$$

Since $\sqrt{a} \geq a/\sqrt{a_0}$, we have

$$\sqrt{4\pi a} \geq \frac{a\sqrt{4\pi}}{\sqrt{6/D}} = a\sqrt{\frac{2\pi D}{3}}.$$

Integrating over all slices, the area becomes the volume which equals 1 and the perimeter-area P_0 satisfies

$$P_0 \geq \sqrt{\frac{2\pi D}{3}}.$$

Therefore,

$$D \leq \frac{3P_0^2}{2\pi}, \tag{1}$$

as desired. That the limit tiles the plane follows from a compactness argument. $\square$

Remark 3.5. As Makai pointed out to us, Wrase, Gritzman, and Wills [GWW, Eqn.(4) p. 23] show that in fact diameter in Proposition 3.4(1) is bounded above by $P_0^2/4\pi$, though our bound is sufficient for our purposes.

Remark 3.6. In general an area-minimizing n-hedral tile need not be unique. For $n = 8$, the conjectured gabled rhombohedron and hexagonal prism have the same surface area, and for $n = 12$, the conjectured rhombic dodecahedron and trapezo-rhombic dodecahedron have the same surface area.

4 Properties of Prisms

In this section, we give some properties of prisms, which are useful in the next section. We begin by giving a definition of prisms. Then we characterize prisms by showing that if a polyhedron has two n-gonal bases and n quadrilateral faces, then it must be a prism (Prop. 4.3 and 4.4). Moreover, we show that a prism with a regular polygonal base uniquely minimizes surface area among all prisms of fixed volume and number of faces and provide a method to calculate the surface area and optimal height (Prop. 4.5). Lastly, in Proposition 4.6, we relate tiling of the plane with tiling of space in order to prove that a certain hexagonal prism is the surface-area-minimizing prism (Prop. 4.8).

Definition 4.1. A *prism* is a polyhedron consisting of a polygonal planar base, a translation of that base to another plane, and edges between corresponding vertices. We say an n-hedron is a *combinatorial prism* if it is combinatorially equivalent to the n-hedron made when an $n-2$-gon is translated to another plane.

Remark 4.2. Bernd Sturmfels [Stu] asked us the following question: given a specific combinatorial type for some n-hedron, can we determine whether there exists a tile of that type. We conjecture that the pentagonal pyramid is the combinatorial polyhedron with the fewest faces which does not tile. Wolfram Online [Wei] remarks that there are no known pentagonal pyramids which tile.

The next two propositions characterize when we know that a n-hedron must be a combinatorial prism.

Proposition 4.3. *Let P be a nondegenerate polyhedron with three quadrilateral faces and two triangular faces. Then P is a combinatorial triangular prism.*

Proof. Since each edge lies on two faces, the total number of edges is 9. By Euler's formula, the number of vertices is 6. Since the sum over the faces of the number of vertices is 18, each vertex must have degree 3. (By the nondegeneracy hypothesis, no vertex can have degree 2.)

Suppose that the triangular faces $\triangle ABC$ and $\triangle ABY$ meet. Because each vertex has degree 3, they must share an edge, as in Figure 14. The other faces at edges AC and BC must be quadrilaterals. Quadrilateral $ACXY$ has vertices X and Y, distinct because the polyhedron has degree 3. It follows that the vertex B is not of degree 3, a contradiction. Therefore, the triangular faces are disjoint and the polyhedron is a combinatorial triangular prism, as desired. $\square$

Proposition 4.4 shows that more generally a nondegenerate polyhedron with n quadrilateral faces and two n-gonal faces is a combinatorial n-gonal prism. The proof is similar to the proof of Proposition 4.3.

Proposition 4.4. *Let P be a nondegenerate polyhedron with n quadrilateral faces and two n-gonal faces. Then P is a combinatorial n-gonal prism.*

Proof. By the same argument in Proposition 4.3, we can show that every vertex has degree 3 and that $V = 2n$ and $E = 3n$.

(Case 1): $n = 4$.

Since no vertex can have degree greater than three, it must be the case that two of the faces do not share a vertex. Since the six faces of this polyhedra will be quadrilaterals, we can identify any two faces as bases.

(Case 2): $n \geq 5$

Suppose that the two n-gonal faces meet. If they only share one vertex, then the degree of this vertex is at least four, a contradiction. So they should meet at an edge. Let us call this edge cd and the two n-gonal faces $a_1a_2\ldots a_{n-2}cd$ and $b_1b_2\ldots b_{n-2}cd$. c is contained in the edges ca_{n-2}, cb_{n-2}, and cd. Therefore, there exists a quadrilateral face containing the edges ca_{n-2} and cb_{n-2}, namely $ca_{n-2}xb_{n-2}$. Similarly, there exists a vertex y such that db_1ya_1 is a face of P. If $x = y$, then the degree of x is at least four, a contradiction. So x and y are distinct.

Now note that since b_1 is contained in the three edges b_1d, b_1y, and b_1b_2, there exists a face containing the edges b_1b_2 and b_1y. This face must be a quadrilateral, so there exists a vertex z such that b_2b_1yz is a face of P. Since there are $2n$ vertices of P, $z \in \{a_1,\ldots,a_{n-2},b_1,\ldots,c_{n-2},c,d,x,y\}$. Moreover, since two faces meet at most at two vertices, $z \in \{b_3,\ldots,b_{n-2},x\}$. It follows that $\deg z$ is at least four, a contradiction. Therefore, the two n-gonal faces do not share an edge, and it follows that they cannot meet.

We now show that P is a combinatorial n-gonal prism. Let $a_1a_2\ldots a_n$ be an n-gonal face described above. Let the other n-gonal face have vertices $b_1,b_2,\ldots,b_n$. By permuting the vertices $b_1,b_2,\ldots,b_n$, we may assume that a_ib_i is an edge of P for each $i = 1,2,\ldots,n$. It follows that P is a combinatorial n-gonal prism. $\square$

The following proposition gives the optimal height for any right regular prism:

Proposition 4.5. *The optimal unit-volume prism with a base similar to a region R of area A_0 and perimeter P_0 is a right prism of height $h = (4\sqrt{A_0}/P_0)^{2/3}$ and surface area $S = 3(P_0^2/2A_0)^{1/3}$. If the base is a regular polygon, it uniquely minimizes surface area among all prisms of fixed volume and number of faces.*

Proof. Since the top is a translation of the bottom, we may assume that both are horizontal. Since shearing a right prism preserves volume but increases surface area, we may assume that our prism is a right prism. A simple calculus computation minimizing surface area as a function of base and height for fixed volume shows that the optimal right prism has height and surface area as

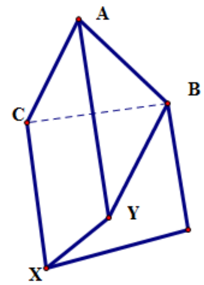


Figure 14: Two triangular faces cannot meet in a nondegenerate polyhedron with three quadrilateral faces and two triangular faces.

asserted. Since a regular n-gon uniquely minimizes perimeter for given area, the right n-gonal prism of optimal dimensions uniquely minimizes surface area among all prisms of fixed volume and number of faces. □

The next proposition gives an example of how we can relate tiling of the plane with tiling of space. We use Proposition 4.6 and Hales' honeycomb theorem [Hal, Thm. 1-A] to prove that a hexagonal prism is the surface-area-minimizing prism.

Proposition 4.6. *Given $n \geq 5$, a monohedral tiling of space by a unit-volume right prism with n faces is surface-area-minimizing among prisms if and only if the bases are perimeter-minimizing tilings of parallel planes by fixed-area $(n-2)$-gons and the height is optimal as in Proposition* 4.5.

Proof. We claim that bases must match up with bases and sides with sides. For $n \neq 6$, this is trivial. For $n = 6$, the prism is a cube and the claim is even more trivial. Therefore, the bases tile parallel planes. Furthermore, the bases minimize perimeter for fixed area if and only if the prism minimizes surface area for fixed volume. □

Remark 4.7. Proposition 4.6 assures that a surface-area-minimizing tile which is combinatorial prism of seven faces is a Cairo or Prismatic prism.

Proposition 4.8. *A right regular hexagonal prism of base length $(2/9)^{1/3}$ and height $2^{1/3}3^{-1/6}$ provides the least-surface area tiling of space by unit-volume prisms. Its surface area is $2^{2/3}3^{7/6}$.*

Proof. Hales' honeycomb theorem [Hal, Thm. 1-A] says that a regular hexagon provides the least-perimeter way to tile the plane into equal parts. By Proposition 4.6, a regular hexagonal prism is the least-surface-area way to tile space by equal volume prisms. The best right regular hexagonal prism has height given by Proposition 4.5. Since the base length of a unit-volume right regular hexagonal prism is determined by its height, we have the desired result. □

5 The surface-area-minimizing tetrahedron and 5-hedron tiles

The regular tetrahedron is the surface-area-minimizing tetrahedron by Theorem 2, but, unfortunately, does not tile space (Fig. 10). While the problem of tetrahedral tilings has been considered in the literature, there does not seem to be a discussion of *surface-area-minimizing* tetrahedral tiles. In this section, we use Sommerville's classification of space-filing tetrahedra to find the surface-area-minimizing tetrahedron. However, we are unable to remove the orientation-preserving assumption. We first define an orientation-preserving tiling as follows:

Definition 5.1. A tiling is *orientation preserving* if any two tiles are equivalent under an orientation preserving isometry of $\mathbb{R}^3$.

Sommerville [So, p.57] describes four types of tetrahedral tiles and claims that, "in addition to these four, no tetrahedral tiles exist in euclidean space". Edmonds [E] addresses some concerns about Sommerville's proof and proves that Sommerville's four candidates are indeed the only four face-to-face, orientation-preserving tiles. The No. 1 tetrahedron is given by cutting a triangular prism into three (see Fig. 15). The No. 2 tetrahedron is given by cutting No. 1 or cutting No. 3 in half (Fig. 16). The No. 3 tetrahedron is given by cutting a square pyramid in half across the diagonal of the base (Fig. 17). This means No. 3 is 1/12 a cube. Note that No. 3 was incorrectly suggested by Li et al. [LMPW] as a surface-area-minimizing tetrahedral tile. Lastly, the No. 4 tetrahedron is given by cutting No. 1 into 4 (Fig. 18).

Goldberg [G8] considered general tetrahedral tilings and found infinitely many families of them, including some which are not face-to-face and some which are face-to-face but not orientation preserving. Edmonds does not consider tilings which are not orientation-preserving. Further investigation is needed regarding what is known about nonorientation-preserving tilings, and whether the orientation-preserving hypothesis can be removed from the Theorem 5.2.

Marjorie Senechal [Se] provides an excellent survey on tetrahedral tiles. Senechal explains that Sommerville's initial consideration of this question goes back to an error made by a student. The student stated that three tetrahedra which divide a triangular prism are congruent, though he meant equal volume. This prompted Sommerville's initial study of congruent tetrahedra which tile space. Senechal points out that Sommerville seems to consider only orientation-preserving, face-to-face tetrahedral tilings, and she stresses the need for more consideration of the problem.

We now proceed to show that the No. 1 tetrahedron provides the optimal orientation-preserving tetrahedral tiling of space.

Theorem 5.2. *Let T be the No. 1 tetrahedron formed by four isosceles triangles with two sides of $\sqrt{3}$ and one side of 2 (Fig.* 15*). Then T provides the least-surface-area unit-volume orientation-preserving tetrahedral tiling.*

Proof. Since Sommerville provides edge lengths and dihedral angles for each of the four types, we scaled the various tetrahedra to unit volume and calculated the surface area of each. The four types had surface areas of $7.4126, 7.9635, 8.1802$, and 10.3646 (to four decimal places), respectively. Thus, T is the surface-area-minimizing orientation-preserving tetrahedral tile. □

Remark 5.3. For all prisms, the sum of all dihedral angles is a multiple of 2π. This does not hold for every polyhedron that tiles $\mathbb{R}^3$, as shown by Sommerville's tetrahedra (as seen in 5.2).

Although Conjecture 2.1(2) for $n = 5$ is well known, there seems to be no nice proof in the literature. The more specific problem of tiling space with prisms was put forth by Steiner ([St];

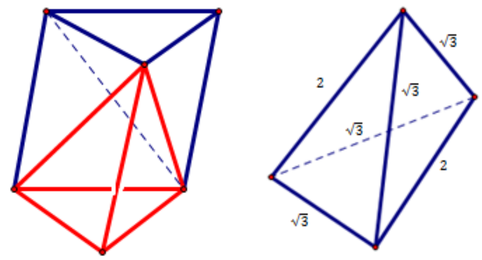


Figure 15: The tetrahedron (Sommerville No. 1) formed by four isosceles triangles with two sides of $\sqrt{3}$ and one side of 2 minimizes surface area among all orientation-preserving tetrahedral tiles [So, Fig. 7].

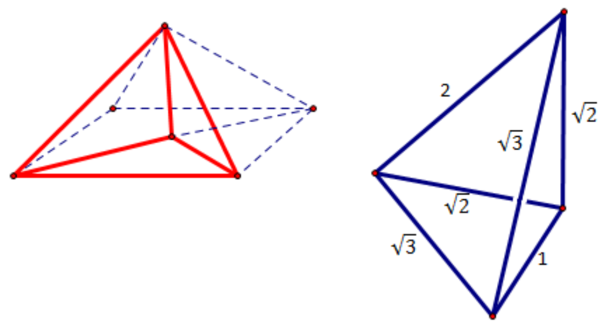


Figure 16: No. 2 tetrahedron is given by cutting No. 3 in half. [So, Fig. 8].

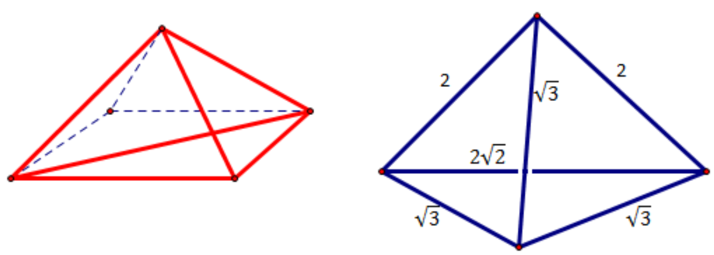


Figure 17: No. 3 tetrahedron is given by cutting a square pyramid into two. [So, Fig. 9].

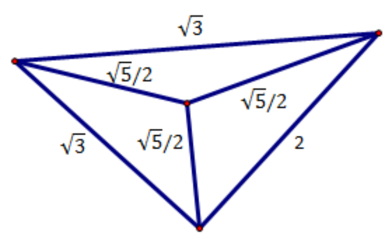


Figure 18: No. 4 tetrahedron is given by cutting No. 1 into 4. [So, Fig. 10].

see [F, p. 209]) who conjectured that a right prism with a regular polygonal base was surface area minimizing among all combinatorial prisms. Steinitz apparently proved the conjecture for triangular prisms but the result was never published (see [F, p. 209]). Brass, Moser, and Pach [BMP] assert that the optimal n-hedron is known for $n \leq 7$ but do not provide candidates, though they do reference Goldberg [G2]. Goldberg says that the optimal candidate among 5-hedra is known, but offers no proof or specific reference in his paper. We are happy to add our proof and Corollary 5.11 to the literature.

Earlier, Sucksdorff [Suc] gave a proof which Florian [F, p. 211] calls "very troublesome". Sucksdorff first eliminates other combinatorial types by noting that the well-known best representative, a square pyramid, has more surface area than the optimal triangular prism. Then follow eighteen pages of algebraic and trigonometric inequalities to show that the right equilateral triangular prism of optimal height minimizes surface area in its combinatorial type. The editor, M. Catalan, appends a note that Sucksdorff's conclusion agrees with the theorem published by Lindelöf [Li] twelve years later, of which Sucksdorff was apparently unaware. The editor had heard of the result somewhere, from "Mr. Steiner, I believe." We thank Bill Dunbar for help reading the original French.

Our proof of the least surface area 5-hedron begins by first showing that the faces characterize a combinatorial triangular prism (Prop. 4.3). Then we show that a polyhedron with five faces is combinatorially equivalent to a square pyramid or a triangular prism (Prop. 5.4). Furthermore, we prove that a certain square pyramid is the least-surface-area combinatorial pyramid (Prop. 5.6) and find a triangular prism that has less surface area than that square pyramid (Prop. 5.7). Therefore, the best 5-hedron must be a combinatorial triangular prism. By computation, we eliminated nonconvex 5-hedra. Therefore, the most efficient must be convex. Finally, using Lindelöf's Theorem (Thm. 5.9), we show that the 5-hedron with the least surface area is a right equilateral triangular prism (Thm. 5.10).

In section 4, we gave the following proposition, which shows that faces characterize a combinatorial triangular prism.

Proposition 4.3. *Let P be a nondegenerate polyhedron with three quadrilateral faces and two triangular faces. Then P is a combinatorial triangular prism.*

We now show a nondegenerate polyhedron with five faces is combinatorially equivalent to a square pyramid or a triangular prism by using Euler's formula to limit the number of possible combinations of quadrilateral and triangular faces to three. Then we show one case is impossible and apply Proposition 4.3 to complete the proof.

Proposition 5.4. *A nondegenerate polyhedron with five faces is combinatorially equivalent to a square pyramid or a triangular prism.*

Proof. Because P has five faces and is nondegenerate, each face is either a triangle or a quadrilateral. Let a be the number of triangular faces and b be the number of quadrilateral faces. Since P has five faces, we have $a+b=5$. Let V be the number of vertices of P and E be the number of edges of P. By Euler's formula, we have $V-E+5=2$. By calculating the sum of the number of edges of each face of P, we have $2E=3a+4b$. Therefore, a is even.

(Case 1): $a = 0$ and $b = 5$.

From the above formulas, we have $V = 7$ and $E = 10$. By counting the number of edges from each vertex, we have that the sum of degrees of vertices of P is $2E = 20$. By the Pigeonhole principle, there exists a vertex which has degree less than or equal to $20/7$. Since every degree is at least three, we get a contradiction.

(Case 2): $a = 2$ and $b = 3$.

By Proposition 4.3, P is a combinatorial triangular prism.

(Case 3): $a = 4$ and $b = 1$.

From the above formulas, we have $V = 5$ and it easily follows that P is a quadrilateral pyramid. Therefore, we have shown that P is either a combinatorial triangular prism or quadrilateral pyramid. □

Next, we give a lower bound on the surface area of a given pyramid and use it to show that a certain quadrilateral pyramid with a square base has the least surface area of among quadrilateral pyramids.

Lemma 5.5. *Let P be a pyramid with apex V, base $A_1A_2...A_n$ and height h. Suppose that the base has area S and perimeter p, then the sum of the areas of side faces of P is greater than or equal to $(1/2)\sqrt{(2S)^2 + p^2h^2}$. Equality holds if and only if the base is circumscribed about a circle and the foot of the perpendicular line from V to the base is the center of the circumscribing sphere.*

Proof. Let B be the foot of the perpendicular line from V to the base. Let $a_1, a_2, ..., a_n$ be the lengths of the sides of the base. Let $x_1, x_2, ..., x_n$ be the distances from B to the sides of the base. Then we have $\sum_i \pm a_i x_i = 2S$. This implies that $\sum_i a_i x_i \geq 2S$. Equality holds when B lies in the interior of the base. The sum of areas of side faces of P is given by

$$\frac{1}{2}\sum_i a_i\sqrt{x_i^2 + h^2} = \frac{1}{2}\sum_i \sqrt{(a_i x_i)^2 + (a_i h)^2}.$$

By the triangle inequality,

$$\sum_i \sqrt{(a_i x_i)^2 + (a_i h)^2} \geq \sqrt{\left(\sum_i a_i x_i\right)^2 + \left(\sum_i a_i h\right)^2}.$$

Together with the inequality $\sum_i a_i x_i \geq 2S$, we get the desired inequality. It is easy to verify the equality condition. □

Proposition 5.6. *Let P be a unit-volume quadrilateral pyramid. Then the surface area of P is greater than or equal to $2^{5/3}3^{2/3}$. Equality holds if and only if it is a right regular pyramid with base-length $2^{-1/3}3^{2/3}$ and height $2^{2/3}3^{-1/3}$.*

Proof. Let S be the area and p be the perimeter of the base of P. Let h be the height of P. Since P has unit volume, we have $Sh = 3$. Moreover, for given perimeter, the square is the area maximizer among quadrilaterals. Therefore, $p \geq 4\sqrt{S}$. From Lemma 5.5, the surface area of P is greater than or equal to

$$S + \frac{1}{2}\sqrt{(2S)^2 + p^2h^2} = S + \frac{1}{2}\sqrt{(2S)^2 + \frac{9p^2}{S^2}}.$$

Furthermore, we have the following inequalities:

$$S + \frac{1}{2}\sqrt{(2S)^2 + \frac{9p^2}{S^2}} \geq S + \frac{1}{2}\sqrt{(2S)^2 + \frac{9(16S)}{S^2}} = S + \sqrt{S^2 + \frac{36}{S}}.$$

Therefore, it suffices to show that

$$S + \sqrt{S^2 + \frac{36}{S}} \geq 2^{5/3}3^{2/3}$$

or equivalently that

$$S^2 + \frac{36}{S} \geq \left(2^{5/3}3^{2/3} - S\right)^2.$$

By direct calculation, this is equivalent to $2^{8/3}3^{2/3}S + 36/S \geq 2^{10/3}3^{4/3}$. This follows directly from arithmetic-geometric mean (AM-GM) inequality. It is easy to check the equality condition from the equality condition of AM-GM inequality and Lemma 5.5. □

Proposition 5.7 shows that a triangular prism has less surface area than a square pyramid. Therefore, it has less surface area than any unit-volume quadrilateral pyramid. It follows that the optimal 5-hedral tile must be a combinatorial triangular prism.

Proposition 5.7. *Let P be the unit-volume right equilateral-triangular prism circumscribed about a sphere and Q be a unit-volume quadrilateral pyramid. Then P has less surface area than Q.*

Proof. By direct computation, we have that P has base-length $4^{1/3}$ and height $4^{1/3}3^{-1/2}$. P has surface area $2^{1/3}3^{3/2}$. Therefore, by Proposition 5.6, the triangular prism has less surface area than any unit-volume quadrilateral pyramid. □

Before, we proceed to the main theorem, we use a linear algebra argument to show that the edges of the sides of a triangular prism are either parallel or concur at a point. We then use this lemma in a our main theorem.

Lemma 5.8. *Let $ABC - DEF$ be a combinatorial triangular prism such that ABC and DEF are triangular faces. Then the lines AD, BE, and CF are either parallel to each other or concur at a point (Fig. 19).*

Proof. Imagine the prism $ABC - DEF$ is placed in an Euclidean space such that ABC lies in the plane $z = 0$. Pick vectors v_1, v_2 and v_3 such that they are parallel to $\overrightarrow{AD}$, $\overrightarrow{BE}$ and $\overrightarrow{CF}$, respectively and they all have z coordinate 1. Consider the vector space V spanned by the vectors v_1, v_2 and v_3.

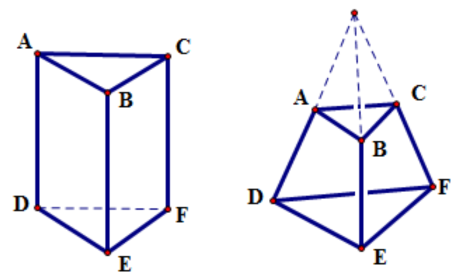


Figure 19: In a combinatorial triangular prism, the lines AD, BE, and CF are either parallel to each other or concur at a point.

(Case 1): $\dim(V) = 1$.

v_1, v_2 and v_3 are the same. Therefore AD, BE and CF are parallel to each other, as desired.

(Case 2): $\dim(V) = 2$.

Since the vectors v_1, v_2 and v_3 are not all the same, there exists a vector among them that is different from the others. Without loss of generality, suppose v_3 is different from v_1 and v_2. Then, v_3 and v_1 span the plane $ACFD$. Hence, V contains the vector $\overrightarrow{AC}$. Similarly, we can show that the vector $\overrightarrow{BC}$ is contained in V. Because $\overrightarrow{AC}$, $\overrightarrow{BC}$, and v_3 are linearly independent, $\dim(V) = 3$, a contradiction.

(Case 3): $\dim(V) = 3$.

It follows that v_1, v_2 and v_3 are distinct. Since v_2 and v_3 span the plane $BCFE$, there exists a real number α_1 such that the vector $\overrightarrow{BC} = \alpha_1(v_2 - v_3)$. Similarly, there exist real numbers α_2 and α_3 such that the vector $\overrightarrow{CA} = \alpha_2(v_3 - v_1)$ and the vector $\overrightarrow{AB} = \alpha_3(v_1 - v_2)$. Take the sum of these equations. We have

$$(\alpha_3 - \alpha_2)v_1 + (\alpha_1 - \alpha_2)v_2 + (\alpha_2 - \alpha_3)v_3 = 0.$$

Since v_1, v_2 and v_3 are linearly independent, $\alpha_1 = \alpha_2 = \alpha_3 (:= \alpha)$. It follows that

$$A + \alpha v_1 = B + \alpha v_2 = C + \alpha v_3.$$

Therefore, the lines AD, BE and CF meet at a point. $\square$

Lorenz Lindelöf [Li] proved that a surface-area-minimizing n-hedron is circumscribed about a sphere, with each face tangent at its centroid. See the beautiful survey by Florian [F, pp. 174-180] and [CFS, Prop. 3.1] from before we knew about Lindelöf. For a given combinatorial type, in order to find the surface-area-minimizing polyhedron of that type, it is usually enough to make sure it satisfies Lindelöf's condition. We prove that a certain right equilateral-triangular prism minimizes surface area among unit-volume 5-hedra, by showing that if a 5-hedron must satisfy Lindelöf's conditions, then the only possibility is that it is a right equilateral-triangular prism.

Theorem 5.9 (Lindelöf Theorem [Li].). *A necessary condition for a polyhedron P to be the surface-area-minimizing polyhedron is that P circumscribes a sphere, and the inscribed sphere is tangent to all the faces of P at their respective centroids.*

Theorem 5.10. *The right equilateral-triangular prism circumscribed about a sphere minimizes surface area among unit-volume 5-hedra.*

Proof. A surface-area-minimizing 5-hedron X exists [M]. By Proposition 2.4, we may assume that it is nondegenerate. By Lindelöf's Theorem [Li], X is circumscribed about a sphere tangent to each face of X at its centroid. By Proposition 5.7, X cannot be a square pyramid; therefore by Proposition 5.4, X is a combinatorial triangular prism. Define ABC and DEF as the triangular bases of X and AD, BE, and CF as the edges. To simplify notation, we refer to the bases ABC and DEF as B_1 and B_2, respectively and the three quadrilateral faces - $ABED$, $BCFE$, and $CADF$ - as Q_3, Q_4 and Q_5, respectively (Fig. 20).

Let O be the center of a sphere inscribed in X. Let T_1, T_2, T_3, T_4 and T_5 be the touching points between the sphere and faces B_1, B_2, Q_3, Q_4 and Q_5, respectively. Finally, let M_1, M_2 and M_3 be midpoints of AD, BE and CF, respectively. Imagine we place X in Euclidean space such that O is at the origin (Fig. 21).

(Step 1) The midpoint of T_1T_2 is the centroid of $T_3T_4T_5$.

This follows from the observation that both of them are the centroid of X.

(Step 2) The quadrilaterals $M_1T_3T_4T_5$, $M_2T_3T_5T_4$, and $M_3T_4T_3T_5$ are parallelograms.

Since T_3 is the centroid of Q_3, we have that $M_1+M_2 = 2T_3$. Similarly, we have $M_2+M_3 = 2T_4$ and $M_3+M_1 = 2T_5$. By solving this linear equation for M_1, M_2, and M_3, we have $M_1 = T_5+T_3-T_4$, $M_2 = T_3+T_4-T_5$, and $M_3 = T_4+T_5-T_3$, as desired.

(Step 3) $T_3T_4T_5$ is an equilateral triangle.

Observe that the face $BEFC$ is perpendicular to the line OT_4. Therefore, $\overrightarrow{OT_4}\cdot\overrightarrow{M_2M_3} = 0$. Additionally, from **(Step 2)**, we have $\overrightarrow{M_2M_3} = 2\overrightarrow{T_3T_5}$. Hence $\overrightarrow{OT_4}\cdot\overrightarrow{T_3T_5} = 0$. This is equivalent to $\overrightarrow{OT_4}\cdot\overrightarrow{OT_5} = \overrightarrow{OT_4}\cdot\overrightarrow{OT_3}$. Together with the fact that $|OT_5| = |OT_3|$, we have that $|T_4T_5| = |T_3T_4|$. Similarly, we can show that $|T_4T_5| = |T_3T_5|$. Therefore, $T_3T_4T_5$ is an equilateral triangle.

(Step 4) X is a right equilateral-triangular prism circumscribed about a sphere.

By Lemma 5.8, AD, BE, and CF are parallel to each other or they concur at a point.

(Case 1): AD, BE, and CF are parallel to each other.

We orient X such that AD, BE, and CF are parallel to the z-axis and O is at the origin. Define

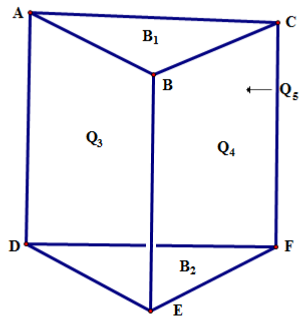


Figure 20: By Proposition 5.7, the surface-area-minimizing 5-hedron X cannot be a square pyramid; therefore by Proposition 5.4, X is a combinatorial triangular prism.

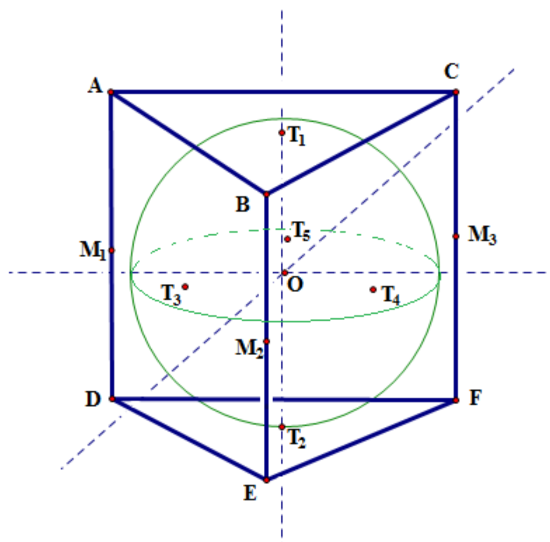


Figure 21: A right equilateral-triangular prism circumscribed about a sphere tangent to each face at its centroid minimizes surface area among unit-volume 5-hedra.

$\pi : R^3 \to R^2$ be the projection map from the whole Euclidean space to xy-plane. Let $z(X)$ denote the z-component of any point X in Euclidean space.

First, observe that the tangent planes of the sphere at points T_3, T_4 and T_5 are parallel to the z-axis. It follows that $z(T_3) = z(T_4) = z(T_5) = 0$, so T_3, T_4 and T_5 lie on the xy-plane. Then, by **(Step 3)**, we have that the centroid of $T_3T_4T_5$ is the origin O. It follows, by **(Step 2)**, that the centroid of $M_1M_2M_3$ is also the origin.

Because projection maps are linear, they preserve centroids. Since the triangle $\pi(A)\pi(B)\pi(C)$ is equivalent to $M_1M_2M_3$, $\pi(T_1)$ is the origin O. Similarly, $\pi(T_2)$ is O.

Therefore, the B_1 and B_2 are perpendicular to the lines AD, BE, and CF. This implies that B_1, B_2, and $M_1M_2M_3$ are congruent to each other. From **(Step 2)** and **(Step 3)**, the triangle $M_1M_2M_3$ is equilateral. Then B_1 and B_2 are also equilateral. Hence, X is the unit-volume equilateral-triangular prism circumscribed about a sphere.

(Case 2): AD, BE, and CF concur at a point.

We now orient X such that $T_3T_4T_5$ is parallel to the xy-plane and O is at the origin. Since $T_3T_4T_5$ is an equilateral triangle, the projection of T_1 to the xy-plane is the origin O. By **(Step 1)**, the midpoint of T_1T_2 also projects to the origin in xy-plane.

From the assumption of this case, AD, BE, and CF are not parallel to the z-axis. Therefore, the plane containing $T_3T_4T_5$ does not contain the origin. Hence, the distances from the plane containing $T_3T_4T_5$ to T_1 and to T_2 are different. Therefore, we deduce that $OT_1 \neq OT_2$, a contradiction. It follows that this case is impossible. □

Corollary 5.11. *A right equilateral-triangular prism circumscribed about a sphere, having base-length* $4^{1/3}$ *and height* $4^{1/3}3^{-1/2}$*, is the surface-area-minimizing 5-hedral tile.*

Proof. Since the prism is surface-area-minimizing by Theorem 5.10 and is a tile, it gives the surface-area-minimizing tiling. □

Remark 5.12. Since equilateral triangles are the perimeter-minimizing polygons of 3 sides, Corollary 5.11 also follows directly from Proposition 4.6.

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