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Michael A. Bush
The College of Wooster

Katelyn R. French
The College of Wooster

Joseph R. H. Smith
The College of Wooster, jsmith15@wooster.edu

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Michael A. Bush[a] Katelyn R. French[b]
Joseph R. H. Smith[c]

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[a]The College of Wooster
[b]The College of Wooster
[c]The College of Wooster

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Total Linking Numbers of Torus Links and Klein Links

Michael A. Bush Katelyn R. French Joseph R. H. Smith

Abstract. We investigate characteristics of two classes of links in knot theory: torus links and Klein links. Formulas are developed and confirmed to determine the total linking numbers of links in these classes. We find these relations by examining the general braid representations of torus links and Klein links.

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1 Introduction

Knot theory is the study of mathematical knots and links. We remind the reader of some standard definitions found in [1, 8]. A mathematical **knot** is a knotted loop of string in $\mathbb{R}^3$ that has no thickness, and a **link** is a set of disjoint knotted loops. Each of the individual knots in a link is called a **component**, so a knot is the special case of a one-component link. Links are represented in two-dimensions using projections which preserve which string passes over another. A projection of the Hopf link is shown in Figure 1. The places where a link crosses over itself in a projection are called **crossings**. A link is **oriented** when a direction is assigned to each component in the link. The direction of the orientation is shown with arrows on a projection (Figure 1).

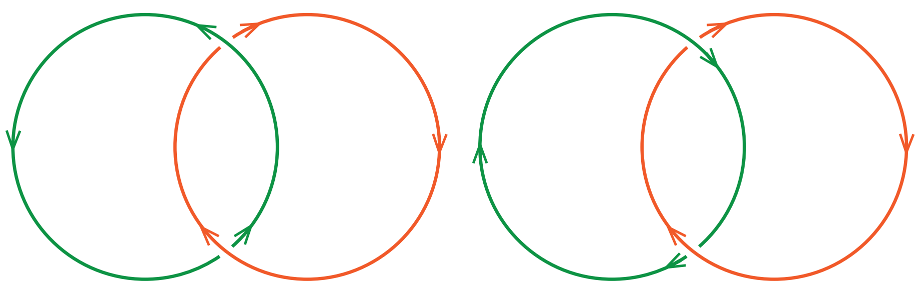

Figure 1: Two orientations of the Hopf link.

Links are classified based on characteristics, called **invariants**, that are the same for projections of equivalent links. Invariants are used to distinguish between different types of links. We will be examining the link invariant called total linking number.

Definition 1.1 ([1, 8]). *The* **total linking number** *of a link l, independent of l's orientation, is*

$$L(l) = \frac{1}{2}\left|\sum_{i=1}^{n} c_i\right| \tag{1}$$

where c_i is the value for each crossing between different components of the link. The value for crossings is $+1$ or -1 as determined by the relationship in Figure 2.

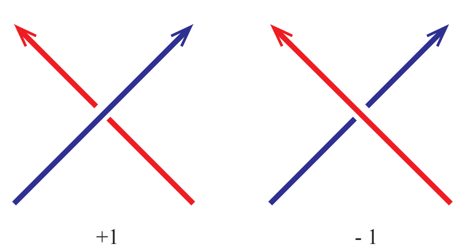

Figure 2: A right handed crossing (+1) and a left handed crossing (-1).

As in [5], we will refer to the total linking number as simply **linking number**. The linking number of the Hopf link (Figure 1) is 1. We will explore the linking numbers for two classes of links: torus links and Klein links.

In Section 2 we discuss the construction of torus links and Klein links. Section 3 introduces the concept of braids, and provides the general braid representation of torus links and Klein links. We then derive the linking number of torus links in Section 4. In Section 5 we partition the general braid word of a Klein link into three pieces and in Section 6 we derive the contribution that each of these pieces gives to the linking number of Klein links. In Section 7 we combine the results from Sections 5 and 6 to form expressions that may be used to calculate the linking numbers of Klein links.

2 Torus Links and Klein Links

Torus links are links that can be placed on the surface of a torus so that they do not cross over themselves [1, 8]. These links are classified by the number of times they wrap around the longitude of a torus (m) and the meridian (n), labeled $T(m,n)$. Figure 3 shows $T(3,2)$ on a rectangular diagram and the corresponding three-dimensional torus. For a $T(m,n)$ torus link, formulas for invariants including crossing number and the number of components have already been determined [1, 8]. It is known that a $T(m,n)$ torus link has $gcd(m,n)$ components [8]. This will be useful information for determining their linking numbers. When m and n are relatively prime, the torus link is a knot (a one-component link), so the linking number is 0. For example, $T(3,2)$ (also known as the trefoil knot) has linking number 0.

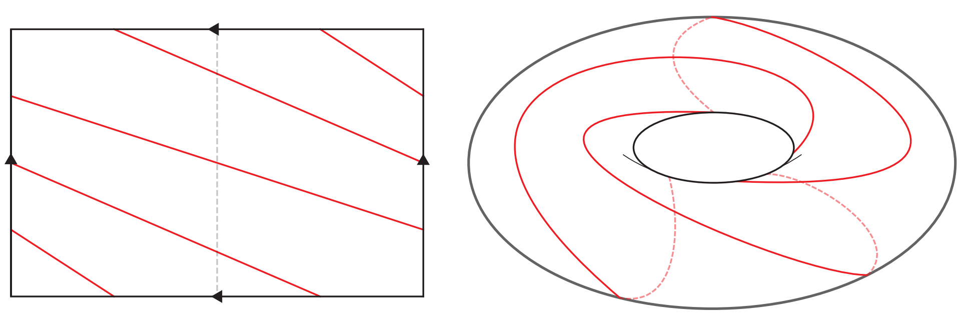

Figure 3: The torus link $T(3,2)$ on the rectangular representation and the equivalent torus.

Klein links are links that are placed on the surface of a once-punctured Klein bottle represented in three-dimensions [2, 3, 4]. In a similar manner to torus links, Klein links can be represented with a rectangular diagram, although there are several key differences between the representations in Figures 3 and 4. In the diagram of a Klein link, there is a hole in the upper left corner to represent the puncture of the three-dimensional representation of the Klein bottle. Similar to torus links, Klein links are denoted $K(m,n)$, where the m strands connecting to the left edge of the diagram are below the hole and the n strands connecting to the top edge of the diagram are above the hole. As shown in Figure 4, the top edge of the diagram has a reverse orientation from the bottom edge, due to the non-orientable nature of Klein bottles [10]. The Klein links discussed here will use this construction. Altering this specific hole placement and construction can change the resulting links that are formed. It can be shown relatively easily that hole placement will categorize knots differently. Distinct hole placement produces another predictable Klein link in our catalogue.

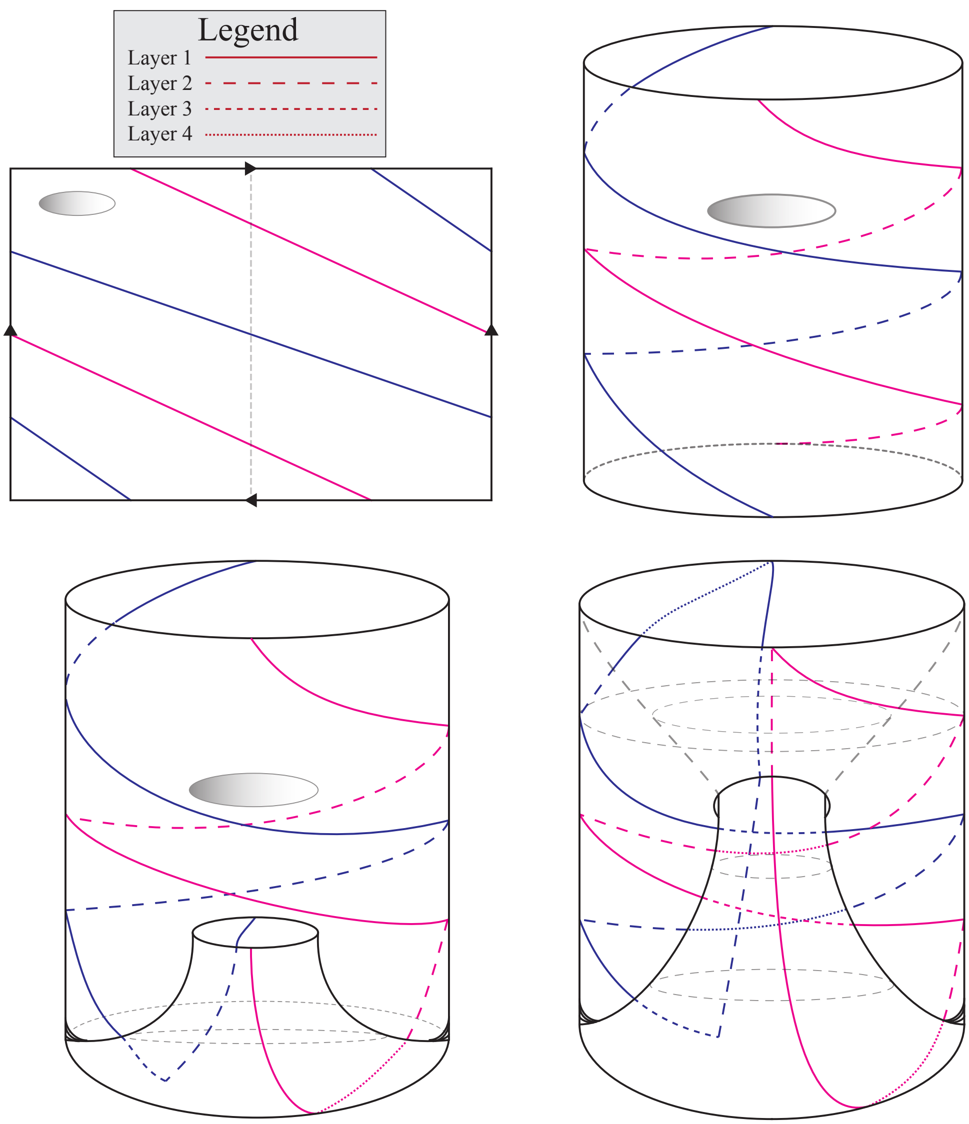

Figure 4: $K(3,2)$ on the rectangular representation and the equivalent once-punctured Klein bottle where dashed lines represent hidden layers of the projection.

3 Braids

One useful way of representing links is with **braids**. Braids are made up of n strings traveling down vertically, weaving around each other. At the top and bottom of the braid, the strings are attached to horizontal bars. An example of a braid can be seen in Figure 8. In a **closed braid**, the strings on the top bar attach to the corresponding strings in the same position on the bottom bar. Braids are an important tool for knot theory because all knots and links can be represented in a closed braid form [1]. Braids are commonly described using what are called *braid words*, which sequentially organize the crossings and determine how the strings interact. Braid words are made up of **braid generators** σ_i^ϵ (Figure 5), which are used to describe each crossing in the braid. The i represents the position of the leftmost string in each crossing, with $\epsilon = 1$ if the i^{th} string crosses over the $(i+1)^{st}$ and $\epsilon = -1$ if it crosses under the $(i+1)^{st}$ string [1].

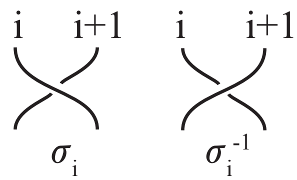

Figure 5: Braid generators [2].

Braid multiplication forms a new braid by stacking the two braids on top of each other and connecting corresponding strings [1]. In addition, **multiplying** two n-string braids results in the product of their braid words [1]. In Figure 6, $\sigma_1^{-1}\sigma_2^{-1}\sigma_3$ and $\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1$ are braid words multiplied to form the braid $\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1$. Now we will define some braid terminology that will be necessary for the development of our results.

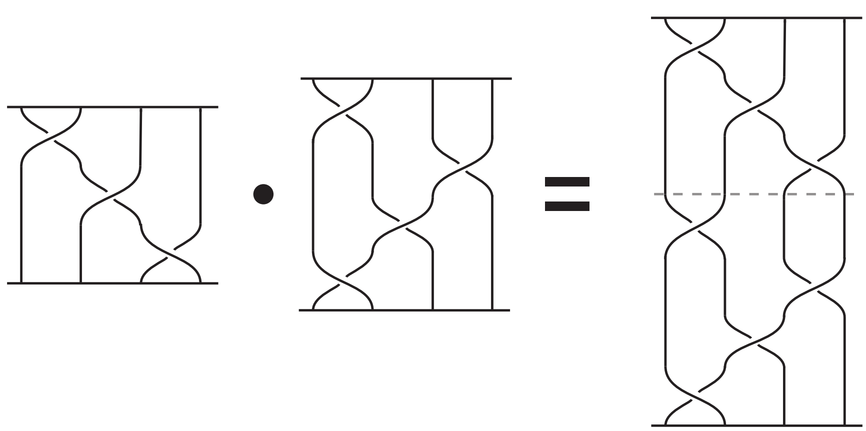

Figure 6: The multiplication of $\sigma_1^{-1}\sigma_2^{-1}\sigma_3$ and $\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1$.

Definition 3.1. *A* **sweep** *in an n-string braid is the braid word $\sigma_1\sigma_2\ldots\sigma_{n-1}$.*

A sweep has $(n-1)$ crossings since the string on the far left of the braid crosses over each other string once, as illustrated in Figure 7.

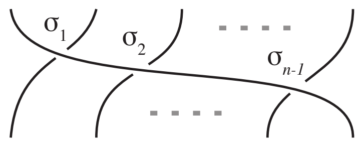

Figure 7: A sweep in an n-string braid.

Definition 3.2 ([6])**.** *A* **full twist** *on an n-string braid word, denoted Δ_n^2, is $(\sigma_1^\epsilon\sigma_2^\epsilon\ldots\sigma_{n-2}^\epsilon\sigma_{n-1}^\epsilon)^n$, and a* **half twist** *on an n-string braid word denoted Δ_n is $\prod_{i=1}^{n-1}(\sigma_1^\epsilon\sigma_2^\epsilon\ldots\sigma_{n-i}^\epsilon)$, where all $\epsilon = 1$, or all $\epsilon = -1$.*

In a full twist (Δ_n^2), the initial position of each string is equivalent to its final position, while in a half twist Δ_n the initial positions of the strings and their final positions are reversed, as illustrated in Figure 8 with $\epsilon = 1$ on a 5-string braid. This can be visualized by keeping the top of the braid fixed and twisting the bottom by π for a half twist and 2π for a full twist.

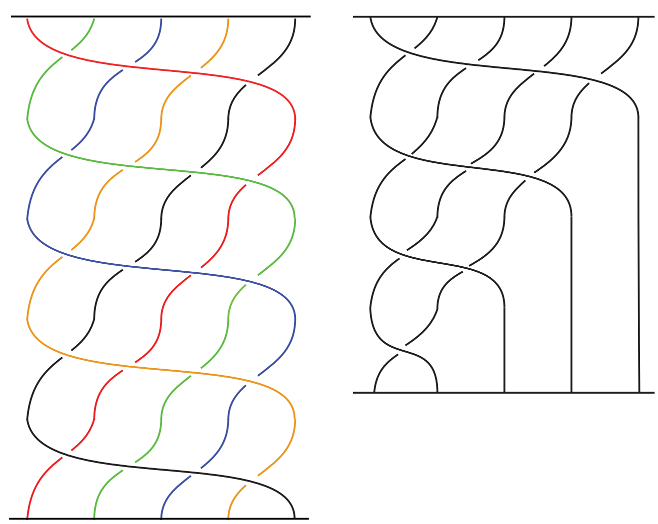

Figure 8: A full twist ($\Delta_5^2 = (\sigma_1\sigma_2\sigma_3\sigma_4)^5$) on a 5-string braid colored by components (left) and a half twist on a 5-string braid ($\Delta_5 = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$) shown on the right.

Definition 3.3 ([6]). *A braid is* **positive** *if all crossings in the braid have the same sign (ϵ).*

Note that both full twists and half twists are positive braids according to Definition 3.3. To calculate the linking number of positive braids, the number of crossings between different components may simply be summed and divided by two since all crossings have the same sign and will not cancel each other out when calculating linking number using Equation 1. This fact is useful because all of the links examined here can be represented in positive braid forms. General braid words for torus links and Klein links are known, and their braid representation can be used to determine linking number when the string positions of each component are known.

Definition 3.4 ([7]). *A general braid word for a $T(m,n)$ torus link is $(\sigma_1\sigma_2\ldots\sigma_{n-1})^m$.*

This braid word represents a positive braid with m sweeps on an n-string braid and when $m \geq n$, there is at least one full twist in the braid, since there is exactly one full twist for each n sweeps.

Proposition 3.1 ([3]). *A general braid word w for a Klein link, $K(m,n)$, where $m \geq n$ is*

$$w = (\sigma_1\sigma_2\ldots\sigma_{n-1})^{m-n+1}(\sigma_1\sigma_2\ldots\sigma_{n-2})(\sigma_1\sigma_2\ldots\sigma_{n-3})\ldots\sigma_1.$$

□

Both of these general braid words for torus links and Klein links are positive braids with $\epsilon = 1$ for all crossings (σ_i^ϵ). As previously mentioned, this means the linking numbers of these links are equal to half the number of crossings between different components of the link.

4 Linking Number of Torus Links

Torus knot invariants have been extensively studied and the linking numbers of torus links have already been computed [5]. We will develop different techniques for calculating linking numbers using braid representations of torus links, and then extend them to determine linking numbers of Klein links. The number of components in torus links and the positions of their strings in the braid will be examined to see how many crossings between different components occur in each sweep of the braid. For example, the braid of the torus link $T(6,4)$ is shown in Figure 9. It has two components and linking number 6. The first and third string make up one component and the second and fourth strings make up the other component.

Lemma 4.1 ([2]). *Let i be the initial position of a string in the general torus link braid, and i' be the string's final position, then*

$$i' \equiv (i-m) \mod n.$$

□

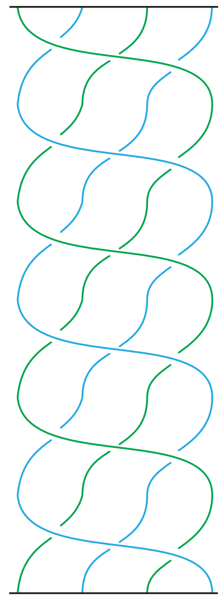

Figure 9: T(6,4) with two components and linking number 6.

Lemma 4.1 is true because each sweep in the torus braid shifts the string positions to the left by one, with the string in the far left wrapping around to rightmost position. This results in the relationship $i' \equiv (i - m) \mod n$, shown with an inductive argument in [2].

Theorem 4.2. *The linking number of $T(m, n)$ is*

$$L(T(m, n)) = m\left(n - \frac{n}{gcd(m, n)}\right).$$

Proof. A torus link has $gcd(m, n)$ components (let $x = gcd(m, n)$). We will now determine the number of strings between a string's initial position i and final position i' in the general torus braid representation. From the relationship in Lemma 4.1,

$$i' \equiv (i - m) \mod n,$$

which means

$$i' - (i - m) = kn$$

for some $k \in \mathbb{Z}$, thus

$$i' - i = m + kn.$$

Since m and n are both divisible by x,

$$i' \equiv i \mod x.$$

This shows that the strings of each component have the same number of strings between them, since there is a constant difference between the initial and final position of the string.

Also this implies that there are $\frac{n}{x}$ strings per component. There are $n-1$ crossings per sweep for m sweeps and $\frac{n}{x}-1$ crossings between a component and itself for each sweep. This gives

$$L(T(m,n)) = m\left(n-1-\left(\frac{n}{x}-1\right)\right) = m\left(n-\frac{n}{gcd(m,n)}\right).$$

$\square$

5 Klein Links as Braids

Now we will examine linking numbers for Klein links. By examining the general braid word w for $K(m,n)$ from Proposition 3.1, we can see that there will always be a half twist Δ_n at the end of the braid word, so w can be written

$$\begin{aligned} w &= (\sigma_1\sigma_2\ldots\sigma_{n-1})^{m-n+1}(\sigma_1\sigma_2\ldots\sigma_{n-2})(\sigma_1\sigma_2\ldots\sigma_{n-3})\ldots\sigma_1 \\ &= (\sigma_1\sigma_2\ldots\sigma_{n-1})^{m-n}\Delta_n. \end{aligned}$$

Also, when $m-n \geq n$, there is at least one full twist at the beginning of the braid word, if the sweep in Δ_n is not included. After the maximum number of full twists at the beginning of the braid word and the half twist at the end of the braid are considered, there will be between 0 and $(n-1)$ sweeps in the middle that have not been accounted for.

Definition 5.1. *A general braid word w for $K(m,n)$, where $m \geq n$, can be partitioned into three pieces: k full twists $(\Delta_n^2)^k, B$, and a half twist Δ_n. Here B is the set of remaining sweeps between the k full twists and the final half twist.*

So the general braid word w representing $K(m,n)$, with $m \geq n$, can be partitioned into three pieces ($(\Delta_n^2)^k, B$, and Δ_n), as shown with $K(12,5)$ and $K(15,4)$ in Figure 10, such that

$$w = (\Delta_n^2)^k B\Delta_n,$$

where $k \geq 0$ and $k \in \mathbb{Z}$. There are $(m-n)$ sweeps before the half twist in w, and for each n consecutive sweeps there is a full twist, so k (the number of full twists before the half twist) is equal to

$$k = \left\lfloor \frac{m-n}{n} \right\rfloor.$$

As shown in Figure 10, $K(12,5)$ has one full twist ($k=1$) and two sweeps in B. In addition, $K(15,4)$ has two full twists ($k=2$) and three sweeps in B.

6 Linking Number and Braid Multiplication

By dividing w into the three pieces $(\Delta_n^2)^k, B$, and Δ_n, we can find general results regarding the linking number of each piece given information about Klein link component number and

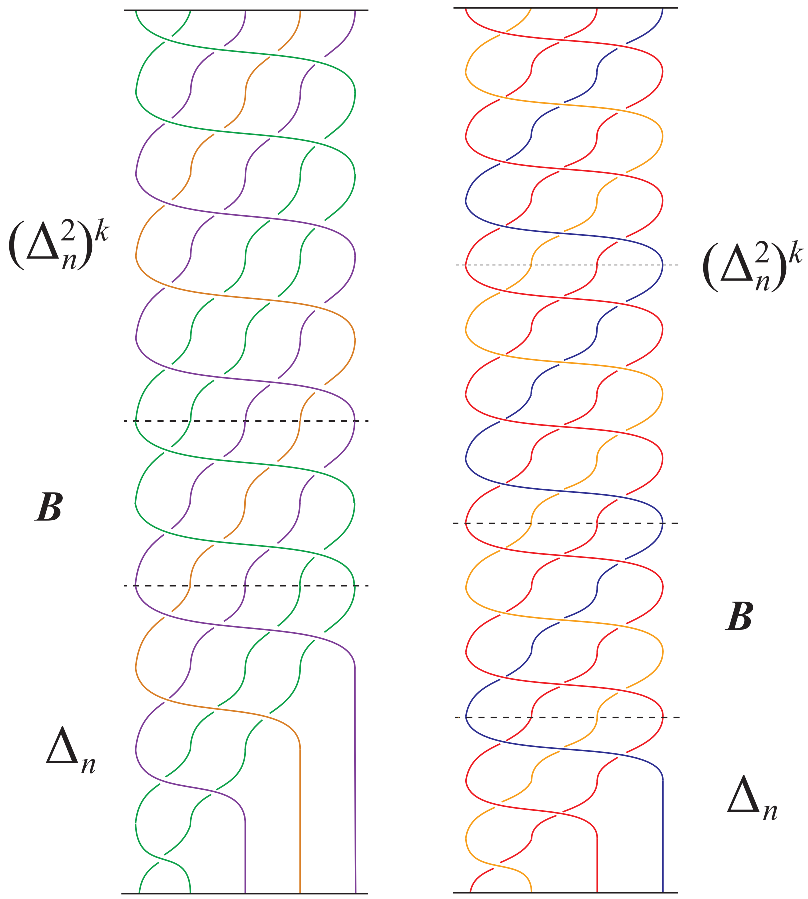

Figure 10: $K(12,5)$ (left) and $K(15,4)$ (right) with string color corresponding to the different components.

string positions in the braid. Then the pieces can be recombined with braid multiplication to provide results about Klein link linking number. Braids with two types of components will be considered. The types will be determined based on how many strings are used to represent a component in the general braid.

Definition 6.1. *In an n-string braid, α is the number of components represented with two strings, where the final position of each string is the initial position of the other string, and β is the number of components with strings that have the same initial and final positions.*

Figure 11 shows a section of a braid with $\alpha = 2$ and $\beta = 0$. In Figure 10, the $K(12,5)$ has $\alpha = 2,\ \beta = 1$, and the $K(15,4)$ has $\alpha = 1,\ \beta = 2$. Since linking number is determined from crossings between different components, the number of strings used to represent each component will be considered.

Proposition 6.1 ([2])**.** *The maximum number strings for each component in the Klein link general braid word is two.* □

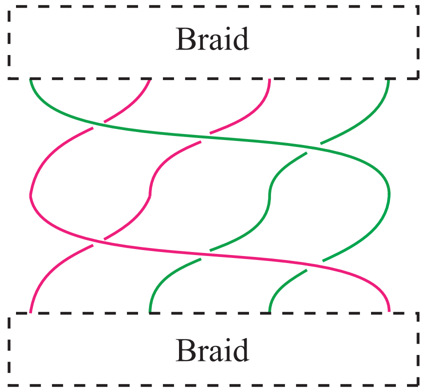

Figure 11: A section of a braid with $\alpha = 2$ and $\beta = 0$.

Following Proposition 6.1, we need not consider cases when a component is represented by more than two strings. For $K(m,n)$ represented by w,

$$2\alpha + \beta = n, \tag{2}$$

since w has n strings and only components represented by either two (α) or one (β) strings in the braid. From this relationship, the values of α and β can be determined if the number of components for a Klein link is known.

Proposition 6.2 ([2]). *If m is even, then $K(m,n)$ has $\lceil \frac{n}{2} \rceil$ components and if m is odd, $K(m,n)$ has $\lceil \frac{n+1}{2} \rceil$ components.* □

Proposition 6.3. *For a Klein link $K(m,n)$,*

1. *if n and m are both even then $\beta = 0$ and $\alpha = \frac{n}{2}$;*
2. *if n is even and m is odd then $\beta = 2$ and $\alpha = \frac{n-2}{2}$;*
3. *if n is odd then $\beta = 1$ and $\alpha = \frac{n-1}{2}$.*

Proof. There are $\frac{n}{2}$ components for m even and n even from Proposition 6.2. Since the sum of α and β is equal to the number of components, we have $\frac{n}{2} = \alpha + \beta \implies n = 2\alpha + 2\beta$. By Equation 2, we also know that $n = 2\alpha + \beta$. By computing we can see

$$2\alpha + 2\beta = 2\alpha + \beta$$

$$\beta = 0 \implies \alpha = \frac{n}{2}.$$

Similarly, for m even and n odd, the Klein link $K(m,n)$ has $\frac{n+2}{2}$ components from Proposition 6.2, so $\frac{n+2}{2} = \alpha + \beta$, which implies $n = 2\alpha + 2\beta - 2$. Since $n = 2\alpha + \beta$, we can solve this system to get $\alpha = \frac{n-2}{2}$ and $\beta = 2$. Likewise, for n odd, the Klein link $K(m,n)$ will have $\frac{n+1}{2}$ components, which gives us $n = 2\alpha + 2\beta - 1$. By solving the system, we get $\alpha = \frac{n-1}{2}$ and $\beta = 1$. □

We will examine the full twists in w represented by the piece $(\Delta_n^2)^k$. Since multiplying a braid by a full twist does not alter the string positions in the original braid, we must determine how many crossings between distinct components occur in the full twist.

Lemma 6.4. *The product of a link with a positive braid representation b (on n-strings following the relationship $2\alpha + \beta = n$) and a full twist Δ_n^2, with crossings of the same sign of b, has linking number*

$$L(b\Delta_n^2) = L(b) + \frac{2\alpha(n-2) + \beta(n-1)}{2}. \tag{3}$$

Proof. The multiplication of any braid and a full twist does not alter the initial and final positions of each string in the original braid. This fixes the number of components of the link the closed braid represents, so α and β remain the same. From Definition 3.2, each string has exactly one sweep in Δ_n^2. Since all crossings in Δ_n^2 and b have the same sign,

$$L(b\Delta_n^2) = L(b) + L(\Delta_n^2)$$

from Equation 1.

We will now find $L(\Delta_n^2)$ in terms of α, β, and n. For each sweep of a one-string component, $\frac{(n-1)}{2}$ is added to $L(\Delta_n^2)$ since each of the $(n-1)$ crossings in the sweep must be with a different component. Similarly, for each sweep of a two-string component, $\frac{(n-2)}{2}$ is added to the linking number because one of the crossings in the sweep will be a crossing between two strings of the same component.

Each two-string component will have two sweeps in Δ_n^2, and each one-string component will have just one sweep in Δ_n^2. Therefore

$$L(\Delta_n^2) = 2 \cdot \frac{\alpha(n-2)}{2} + \frac{\beta(n-1)}{2}.$$

Thus the linking number of the resultant braid is

$$L(b\Delta_n^2) = L(b) + \frac{2\alpha(n-2) + \beta(n-1)}{2}.$$

□

We will also develop a similar linking number result regarding the product of a braid and a half twist (Δ_n). This result will be useful in determining the effect of the half twist piece of w on linking number for Klein links.

Lemma 6.5. *The product of a link with a positive braid representation b (on n-strings) and a half twist Δ_n with crossings of the same sign as b will have linking number*

$$L(b\Delta_n) = L(b) + \frac{1}{2}\left(\frac{n^2 - n}{2} - \alpha\right), \tag{4}$$

where α is the number of components represented by two strings in b.

Proof. Since all of the crossings in Δ_n and b have the same sign we know that

$$L(b\Delta_n) = L(b) + L(\Delta_n).$$

In the half twist Δ_n, there are $\frac{n^2-n}{2}$ crossings, where each string has one crossing with each other string (see an example in Figure 12). Thus each two-string component will cross itself one time, eliminating α crossings from consideration in the calculation of the half twist's linking number. This gives the half twist a linking number of

$$L(\Delta_n) = \frac{1}{2}\left(\frac{n^2-n}{2} - \alpha\right).$$

Therefore, we can conclude

$$L(b\Delta_n) = L(b) + \frac{1}{2}\left(\frac{n^2-n}{2} - \alpha\right).$$

□

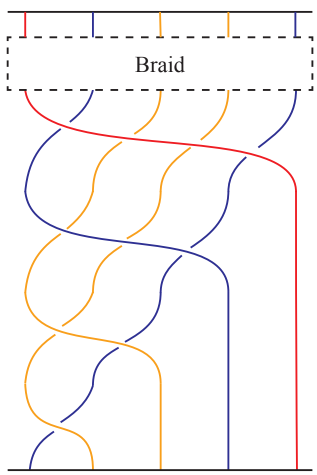

Figure 12: A half twist on a 5-string braid, showing how each string shares exactly one crossing with each other string.

Now that we have computed the contribution of the full twists and half twist to the linking number of these braids, we will determine how many sweeps occur in B given any m and n.

Lemma 6.6. *The number of sweeps, S, in B for a Klein link $K(m, n)$ represented by the general braid word w, is $S \equiv m \mod n$, where S is an integer and $0 \leq S \leq n-1$.*

Proof. As stated in Section 5, there are $(m-n)$ sweeps in the braid appearing before the half twist Δ_n in $(\Delta_n^2)^k$ and B. Since there are n sweeps in each full twist, for $k \in \mathbb{Z}$, there are $(n \cdot k)$ sweeps in $(\Delta_n^2)^k$. Thus S, the number of sweeps in B, is

$$S = (m-n) - nk = m - n \cdot (k+1).$$

In modular form we have

$$S \equiv m \mod n.$$

From this partitioning of w,

$$k = \left\lfloor \frac{m-n}{n} \right\rfloor = \left\lfloor \frac{m}{n} - 1 \right\rfloor = \left\lfloor \frac{m}{n} \right\rfloor - 1.$$

This gives

$$k + 1 = \left\lfloor \frac{m}{n} \right\rfloor$$

and

$$S = m - n \left\lfloor \frac{m}{n} \right\rfloor,$$

which implies S is the representative of the modular class of m in $\mathbb{Z}_n$. By the definition of a modular class representative,

$$S \equiv m \mod n,$$

where S is an integer and $0 \leq S \leq n-1$. □

Since all components in w are represented by either one or two strings, it is important to know, for a given m and n, if w has any one-string components, and if it does, where they occur in the braid. Since one-string components have the same initial and final string positions, each one-string component corresponds to a string in the braid being fixed.

Proposition 6.7 ([2]). *Any string of w is fixed, and thus a one-string component, if*

$$2i \equiv (m+1) \mod n, \tag{5}$$

where i is the initial position of a string in w, and i' is the string's final position. □

The only possible initial string positions for these components are given by the following lemma. Then we will see whether these initial positions cause the braid w to have a sweep in B, so it can be accounted for in our calculation of linking numbers.

Lemma 6.8. *The possible initial positions i of one-string components for $K(m,n)$, where $m \geq n$ for the general braid word w, are*

$$i = \frac{S+1}{2} \quad \text{or} \quad i = \frac{S+n+1}{2},$$

where $i \in \mathbb{N}$ such that $1 \leq i \leq n$.

Proof. Following Proposition 6.7, any string of w is fixed, and represents a one-string component if

$$2i \equiv (m+1) \mod n, \tag{6}$$

and since $(\Delta_n^2)^k$ fixes all string positions, we may just consider the sweeps $S \equiv m \mod n$ in B from Lemma 6.6, hence

$$2i \equiv (S+1) \mod n.$$

Since S is in the range $0, 1, \ldots n-1$,

$$2i = (S+1) + dn,$$

for some $d \in \mathbb{Z}$. Since $d < 0$ would result in $i < 1$, d must not be negative. For $d = 0$,

$$i = \frac{S+1}{2} < \frac{n+1}{2} < n$$

so $d = 0$ results in $i < n$. Similarly for $d = 1$,

$$i = \frac{S+n+1}{2} \leq \frac{(n-1)+n+1}{2} = n,$$

so $d = 1$ results in $1 \leq i \leq n$. For the cases where $d \geq 2$,

$$i = \frac{S+dn+1}{2} \leq \frac{(n-1)+dn+1}{2} = \frac{(d+1)n}{2} \not\leq n$$

so any $d \geq 2$ results in $i > n$, which is not a possibility for an n-string braid.

Thus the only possible initial positions of one string components for $K(m, n)$ are

$$i = \frac{S+1}{2}, \ \textit{or} \ \ i = \frac{S+n+1}{2}.$$

□

Corollary 6.9. *If w, a general braid word representing the Klein link $K(m, n)$, has a one-string component $i = \frac{(S+1)}{2}$, it will contain a sweep within the portion B of w, and if w has a one-string component $i = \frac{(S+n+1)}{2}$, it will not contain a sweep within the portion B of w.*

Proof. To contain a sweep in B, $i \leq S$ (where S is the number of sweeps in B).

$$i = \frac{S+1}{2} = \frac{S}{2} + \frac{1}{2},$$

since $S \geq 1$ (0 would result in a rational number less than 1, and thus an extraneous solution for i),

$$\frac{S}{2} + \frac{1}{2} \leq S.$$

For $i = (S+n+1)/2$,

$$i = \frac{S+n+1}{2} = \frac{S}{2} + \frac{n}{2} + \frac{1}{2} > S,$$

since $\frac{n}{2} > \frac{S}{2}$ (S has a maximum of $(n-1)$ from Lemma 6.6). This implies that $i > S$ and there will not be a sweep of the one-string component in B. □

7 Klein Link Linking Number

We will use modular arithmetic $(\bmod n)$ to classify $K(m,n)$ Klein links using our construction and partitioning with the general braid word $w = (\Delta_n^2)^k B\Delta_n$. Klein links with the same B are members of the same modular class. In modular arithmetic, an even modulus preserves parity within a modular class. This gives us our first two cases where n is even, and m is either even or odd. However, in the case of an odd modulus, parity is not preserved. So when n is odd, two cases will be combined into a single theorem covering both odd and even values of S. Recall from Lemma 6.6 that $S \equiv m \mod n$, where S is an integer and $0 \leq S \leq n-1$.

Theorem 7.1. *For the Klein link $K(m,n)$, with n even and m even, $m \geq n$, the linking number is*

$$L(K(m,n)) = \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{n(n-2)}{2} + \frac{S(n-2)}{2} + \frac{n(n-2)}{4}.$$

Proof. For m even, n even, $K(m,n)$ has $\alpha = \frac{n}{2}$ and $\beta = 0$ from Proposition 6.3. From the second term in Equation 3 from Lemma 6.4,

$$\frac{2 \cdot \frac{n}{2}(n-2) + 0 \cdot (n-1)}{2} = \frac{n(n-2)}{2}$$

is added to the linking number for each full twist. There are k full twists $\left(k = \left\lfloor \frac{m-n}{n} \right\rfloor\right)$, which adds

$$\left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{n(n-2)}{2}$$

to the linking number.

Next, we must consider the portion of w denoted by B. Since $\beta = 0$, every sweep in B will be a sweep of a two-string component. Therefore, in each sweep there will be one crossing between two strings of the same component. So for each sweep in B, there will be $(n-2)$ crossings with other components, which contribute $\frac{n-2}{2}$ to the linking number of w. By Lemma 6.6, we know that there are S sweeps in B, so the total that B contributes to the linking number is $\frac{n-2}{2} \cdot S$.

Now to calculate the linking number added from the half twist, we will use Lemma 6.5;

$$\frac{1}{2}\left(\frac{n^2-n}{2} - \alpha\right) = \frac{1}{2}\left(\frac{n^2-n}{2} - \frac{n}{2}\right) = \frac{n(n-2)}{4}$$

is added to the linking number from Δ_n portion of w.

In order to obtain the linking number for w, we must sum the linking numbers of the three pieces of w,

$$\begin{aligned} L(K(m,n)) &= L((\Delta_n^2)^k) + L(B) + L(\Delta_n) \\ &= \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{n(n-2)}{2} + \frac{S(n-2)}{2} + \frac{n(n-2)}{4}. \end{aligned}$$

$\square$

Theorem 7.2. *In the Klein link $K(m,n)$, for n even and m odd, $m \geq n$, the linking number is*

$$L(K(m,n)) = \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{n^2-2n+2}{2} + \frac{(n-1)+(n-2)(S-1)}{2} + \frac{n^2-2n+2}{4}.$$

Proof. For m odd and n even, $K(m,n)$ has $\alpha = \frac{n-2}{2}$ and $\beta = 2$ from Proposition 6.3. From Lemma 6.4,

$$\frac{2 \cdot \frac{n-2}{2}(n-2) + 2 \cdot (n-1)}{2} = \frac{n^2-4n+4+2n-2}{2} = \frac{n^2-2n+2}{2}$$

is added to the linking number of w for each full twist. There are $\left\lfloor \frac{m-n}{n} \right\rfloor$ full twists in w so the total contribution to the linking number by all of the full twists in w is

$$\left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{n^2-2n+2}{2}.$$

In this case, the braid word w has two one-string components, but Corollary 6.9 tells us that only one of these one-string components will have a sweep in B. Thus, one of the sweeps in B will contribute $\frac{n-1}{2}$ to the linking number. From Lemma 6.6, we know that there are S sweeps in B, and since only one of these are from a one-string component, the rest of the sweeps, $S-1$, will be sweeps of strings of two-string components. As previously stated, sweeps of strings of two-string components will contribute $\frac{n-2}{2}$ to the linking number. Thus the total linking number contributed by the B portion is

$$\frac{n-1}{2} + (S-1)\frac{n-2}{2} = \frac{(n-1)+(n-2)(S-1)}{2}. \tag{7}$$

Now we will examine the contribution that the half twist makes to the linking number of w. From Lemma 6.5, we know that the linking number of the half twist is

$$\frac{1}{2}\left(\frac{n^2-n}{2} - \alpha\right) = \frac{1}{2}\left(\frac{n^2-n}{2} - \frac{n-2}{2}\right) = \frac{n^2-2n+2}{4}.$$

By taking the sum of the linking numbers of the three portions of w, we obtain the total linking number for $K(m,n)$:

$$\begin{aligned} L(K(m,n)) &= L((\Delta_n^2)^k) + L(B) + L(\Delta_n) \\ &= \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{n^2-2n+2}{2} + \frac{(n-1)+(n-2)(S-1)}{2} + \frac{n^2-2n+2}{4}. \end{aligned}$$

□

Now the case where n is odd will be considered. The same techniques as used for n even will be utilized, although the number of sweeps in B must be considered. Since an odd modulus does not preserve parity, the cases where S is odd and S is even will be combined to give a formula for the linking number when n is odd.

Theorem 7.3. *In the Klein link $K(m,n)$, for n odd, $m \geq n$, the linking number is*

$$L(K(m,n)) = \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{(n-1)^2}{2} + ((S+1) \mod 2) \cdot \frac{S(n-2)}{2}$$
$$+ (S \mod 2) \cdot \frac{(n-1)+(n-2)(S-1)}{2} + \frac{(n-1)^2}{4}.$$

Proof. For n odd, $K(m,n)$ has $\alpha = \frac{n-1}{2}$ and $\beta = 1$ from Proposition 6.3. From Lemma 6.4, we know that

$$\frac{2 \cdot \frac{n-1}{2}(n-2) + 1 \cdot (n-1)}{2} = \frac{n^2 - 3n + 2 + n - 1}{2} = \frac{(n-1)^2}{2}$$

is added to the linking number for each full twist in w. Given that the number of full twists in w is $\left\lfloor \frac{m-n}{n} \right\rfloor$, the total contribution of the full twists to the linking number is

$$\left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{(n-1)^2}{2}.$$

From Lemma 6.5, the linking number from adding the half twist of $K(m,n)$ is given by

$$\frac{1}{2}\left(\frac{n^2-n}{2} - \alpha\right) = \frac{1}{2}\left(\frac{n^2-n}{2} - \frac{n-1}{2}\right) = \frac{(n-1)^2}{4}.$$

Case 1. (even number of sweeps, S):

In order to determine the contribution of B to the linking number, we must examine the one-string component that could sweep in B, the string with initial position $i = \frac{S+1}{2}$. If S is even, i will be a non-integer, and therefore B contains no sweeps of single string components. Thus all of the sweeps in B will add $\frac{n-2}{2}$ to the linking number, so the total linking number contributed by B is

$$\frac{S(n-2)}{2}.$$

By summing our three expressions, we have:

$$\begin{aligned} L(K(m,n)) &= L(B) + L(\Delta_n) + L((\Delta_n^2)^k) \\ &= \frac{S(n-2)}{2} + \frac{(n-1)^2}{4} + \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{(n-1)^2}{2}. \end{aligned}$$

Case 2. (odd number of sweeps, S):

Since S is odd, the value of $i = \frac{S+1}{2}$ will be an integer. Thus B will contain one sweep that will contribute $\frac{n-1}{2}$ to the linking number. As in Equation 7 from Theorem 7.2, the total linking number of B will be

$$\frac{n-1}{2} + (S-1)\frac{n-2}{2} = \frac{(n-1)+(n-2)(S-1)}{2}.$$

Thus the total linking number for $K(m,n)$ in this case is

$$\begin{aligned} L(K(m,n)) &= L(B) + L(\Delta_n) + L((\Delta_n^2)^k) \\ &= \frac{(n-1)+(n-2)(S-1)}{2} + \frac{(n-1)^2}{4} + \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{(n-1)^2}{2}. \end{aligned}$$

These cases can be combined to conclude

$$\begin{aligned} L(K(m,n)) = \left\lfloor \frac{m-n}{n} \right\rfloor \cdot \frac{(n-1)^2}{2} + ((S+1) \mod 2) \cdot \frac{S(n-2)}{2} \\ + (S \mod 2) \cdot \frac{(n-1)+(n-2)(S-1)}{2} + \frac{(n-1)^2}{4}. \end{aligned}$$

□

8 Conclusions

These results can be extended to cases where $m < n$ since the Klein link $K(m,n)$ is the disjoint union of $K(m,m)$ and the mirror image of $K(n-m, n-m)$ for $m < n$, as proven in [9], where the mirror image flips each crossing so the part of the link that originally crossed under the other part now crosses over it [1]. If we examine these links with the natural orientation given to braids (all strings with orientation downward or all upward), the linking number [8, 9] of these Klein links is $L(K(m,n)) = |L(K(m,m) - L(K(n-m, n-m))|$.

This work has increased our knowledge of Klein links and torus links through their braid representations. Partitioning the general Klein link braid allowed us to find new connections between Klein links. This let us develop classes of Klein links that differ only by the number of full twists in their braid representations. These classes have related linking numbers and we suspect that similar properties will hold for other invariants.

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