Rose-Hulman Undergraduate Mathematics Journal

Volume 15
Issue 1

Article 1

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Rose-
Hulman
Undergraduate
Mathematics
Journal

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Volume 15, No. 1, Spring 2014

Sponsored by

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Abstract. In this paper we geometrically provide a necessary and sufficient condition for points on a cubic to be associated with an infinite family of other cubics who have nine-pointic contact at that point. We then provide a parameterization of the family of cubics with nine-pointic contact at that point, based on the osculating quadratic.

Acknowledgements: This research took place at the University of Wisconsin – Stout REU 2013, which was sponsored by NSF grant DMS 1062403. The authors wish to acknowledge the support of the University of Wisconsin – Stout. In particular, Dr. Seth Dutter, for all of his guidance and support throughout this project.

1 Introduction

Algebraic geometry studies geometry of sets defined by the common solutions to polynomial equations. One of the classic topics of study is the intersections between algebraic curves in the plane. Algebraic geometers quantify intersection between two curves by computing the intersection multiplicity of the curves at that point. Curves that intersect each other to a high multiplicity at a point will be good local approximations for each other. In the simplest sense, osculating curves are the best local approximations to a curve at a point and correspondingly, extactic points are the points on a curve where even an osculating curve intersects to a greater multiplicity than expected. The product of the degree of two curves provides an upper bound for the intersection multiplicity between them at any given point so investigating points on a curve at which that bound is able to be acheived is of particular interest.

The study of osculating curves was of interest to Cayley and Salmon in the 1800's. Cayley was interested in finding the sextactic points of a curve, the points on a curve such that the curve intersects a conic with multiplicity six at that point. Salmon was interested in something slightly different: cubic curves that intersect other cubic curves to high degree. This is the phenomena we are focused on in this paper. What Salmon did was count the points on a cubic at which it is possible to have another cubic intersect it with multiplicity 9, or equivalently, intersect at exactly one point. He called these the points of nine-pointic contact. He determined that there are 81 of these points on any cubic, including the 9 inflection points. He also supplied a simple equation for finding these points. These points and their defining equation were further examined by A.S. Hart in 1875.

Other research, including that of Halphen in his 1876 thesis, focuses on coincidence points. The concept of these points is not quite the same as that of the points of nine-pointic contact. However, both of these sets of points are related to the construction of infinite families of cubic curves that intersect to degree nine at a point.

Though this topic possesses a rich history, research on it continues. While Cayley focused on finding the sextactic points of a plane curve of arbitrary degree, Kamel and Farahat, in 2012, looked closely at the total sextactic points of quartic plane curves in [4]. Their investigation centered around investigating the points on a quartic curve at which there exists a quadratic curve that intersects it completely at that point, which is to say that they intersect with multiplicity 8 at that point and nowhere else.

The points of nine-pointic contact investigated by Salmon give rise to infinite families of cubic curves that intersect to degree nine at that point. Examining which cubic curves fit into which of these families is an interesting and complicated question. We provide insight into the answer by demonstrating the geometric conditions required for a point to be a point of nine-pointic contact and describing how these conditions relate to osculating curves. The purpose of this paper is to build the infinite family of cubics that intersect a smooth cubic curve to degree nine at a non-flex point directly from the theory of osculating curves. Furthermore, we show that these cubic curves are in bijection with the points on the osculating conic, thus showing that this family is parametrizable.

2 Preliminaries

We will assume a basic familiarity of Algebraic Geometry. For the reader who is new to the subject the authors recommend "Algebraic Curves" by Fulton [3]. However, as our proofs depend upon them, we will give a brief overview of the necessary definitions and theorems. By convention, for a polynomial f, the notation $\mathrm{V}(f)$ will denote the vanishing of f, or equivalently the variety defined by f.

2.1 Intersection Multiplicity

Definition 2.1. Let $C, D \in \mathbb{C}[x,y]$ and p be a maximal ideal. We define the *intersection multiplicity* of C and D at p to be

$$\mathrm{I}_p(C,D) = \dim_{\mathbb{C}} \mathbb{C}[x,y]_p / \langle C, D \rangle,$$

where the dimension is taken to be the dimension as a $\mathbb{C}$ vector space.

We can extend this definition to the intersection multiplicity of two projective curves at a point p by selecting an affine chart containing p and then applying the above definition to the defining equations of the given curves at the maximal ideal associated to p. This is independent of the choice of affine chart and defining equations. By abuse of language we will speak of the intersection multiplicity of two curves. Additionally we recall the following properties of intersection multiplicity.

Corollary 2.2. *Let $C, D, E \in \mathbb{C}[x,y]$, and $\alpha \in \mathbb{C}^*$, then intersection multiplicity satisfies the following properties:*

i) $\mathrm{I}_p(C,D) = \mathrm{I}_p(D,C)$

ii) $\mathrm{I}_p(C,C) = \infty$

iii) $\mathrm{I}_p(CD,E) = \mathrm{I}_p(C,E) + \mathrm{I}_p(D,E)$

iv) $\mathrm{I}_p(\alpha C,D) = \mathrm{I}_p(C,D)$

v) $\mathrm{I}_p(C,D) = 0 \iff p \neq \mathrm{V}(C) \cap \mathrm{V}(D)$

vi) $\mathrm{I}_p(C,D) = \mathrm{I}_p(C, D + EC)$

vii) $I_{(0,0)}(x,y) = 1$

Proof. See Fulton [3], Chapter 3, Section 3, Theorem 3. □

Lemma 2.3. *Let $C, D, E \in \mathbb{C}[x, y]$ and $\mathrm{I}_p(C, D) = n$ and $\mathrm{I}_p(D, E) = m$. If $\mathrm{V}(D)$ is smooth at p, then $\mathrm{I}_p(C, E) \geq \min\{n, m\}$.*

Proof. We may assume that $p \in \mathrm{V}(D)$ because if p is not on the curve $\mathrm{V}(D)$ then $\mathrm{I}_p(C, D) = 0$ and $\mathrm{I}_p(D, E) = 0$ so the lemma follows trivially.

Since $\mathrm{V}(D)$ is smooth, by [3], Chapter 3, Section 2, Thorem 1, the local ring $\mathbb{C}[x, y]_p/\langle D\rangle$ is a discrete valuation ring. Let t be a local parameter. Next, we consider C and E as elements of this local ring. Since $\mathrm{I}_p(C, D) = n$ and $\mathrm{I}_p(D, E) = m$, we can write $C = t^n\alpha$ and $E = t^m\beta$ for some α and β which are rational functions that are not contained in the maximal ideal p. Without a loss of generality, we assume $n \leq m$, so

$$\mathbb{C}[x, y]_p/(C, D, E) = \mathbb{C}[x, y]_p/\langle C, D\rangle.$$

We also have a surjective morphism $\mathbb{C}[x, y]_p/\langle C, E\rangle \to \mathbb{C}[x, y]_p/\langle C, D, E\rangle$, so transitively there is a surjective morphism $\mathbb{C}[x, y]_p/\langle C, E\rangle \to \mathbb{C}[x, y]_p/\langle C, D\rangle$. Thus,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]_p/\langle C, E\rangle) \geq \dim_{\mathbb{C}}(\mathbb{C}[x, y]_p/\langle C, D\rangle)$$

which is the minimum of n and m by assumption. □

Lemma 2.4. *Intersection multiplicity is invariant under projective linear transformations.*

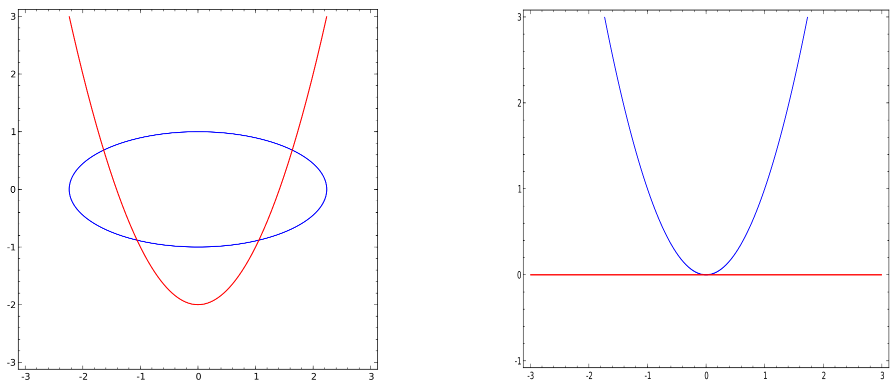
Proof. For details, see [3] page 37. □

In affine space, it can be difficult to determine the number of times two curves intersect. For example, a parabola may intersect a line at two distinct points or only at one distinct point. However, in projective space we have a definitive answer as to how many times two curves intersect, given by the following theorem.

Theorem 2.5 (Bézout's Theorem)**.** *Let $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ be projective curves of degree m and n. If C and D share no common components, then the number of points of intersection of C and D, counting multiplicity, is mn.*

Proof. For details concerning the proof of Bézout's Theorem, see [2] Chapter 8, Section 7, Theorem 10. □

Figure 1



(a) $\mathrm{V}(x^2 - y - 2)$,$\mathrm{V}(x^2 + 5y^2 - 5)$ (b) $\mathrm{V}(y - x^2)$,$\mathrm{V}(y)$

Example 2.6. Consider the affine curves $\mathrm{V}(x^2 - y - 2)$ and $\mathrm{V}(x^2 + 5y^2 - 5)$ as depicted in Figure 1 (a). Since the polynomials $x^2 - y - 2$ and $x^2 + 5y^2 - 5$ are both of second degree, Bézout's Theorem implies that there are a total of 4 intersection points including multiplicity. As shown in the plots of these curves, one can see that there are indeed four intersection points.

Example 2.7. Recall that Bézout's Theorem counts multiplicity. In Figure 1 (b). The line $\mathrm{V}(y)$ lies tangent to the parabola $\mathrm{V}(y - x^2)$. We will see in section 2.2 that this tangency implies an intersection multiplicity of 2 at the point, which must account for all of the points of intersection since $y - x^2$ is of second degree and y is of first degree.

Bézout's Theorem is essential to understanding intersection theory and it will play a key role in many of our proofs. For example, the following lemma follows nicely from Bézout's Theorem.

Lemma 2.8. *Any two pairs of distinct lines in $\mathbb{P}^2_{\mathbb{C}}$ are projectively equivalent.*

Proof. By Bézout's Theorem, we know that any two lines will intersect precisely once. Let p_0 be that point of intersection and let p_1 and p_2 be points distinct from p_0 on each line respectively. By Theorem 3.4 of [1], there is a projective change of coordinates that takes p_0, p_1, p_2 to $[0 : 0 : 1]$, $[1 : 0 : 1]$, and $[0 : 1 : 1]$ respectively. Thus, since every pair of distinct lines is projectively equivalent to the union of the lines $x = 0$ and $y = 0$, and because a projective change of coordinates is invertible by definition, we conclude that any two pairs of distinct lines are projectively equivalent. $\square$

2.2 Osculating Curves

Let $C \in \mathbb{C}[x, y]$. When considering a point p on $\mathrm{V}(C)$, one can construct the tangent line of $\mathrm{V}(C)$ at p. When p is a smooth point the tangent line has the same slope as $\mathrm{V}(C)$ at p and the intersection multiplicity of $\mathrm{V}(C)$ and the tangent line is at least 2 at the point p. In order to give a definition of tangent lines that extends to singular points we will first need a lemma on homogeneous polynomials in 2 variables.

Lemma 2.9. *If $C \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree n, then C factors into n linear homogeneous terms.*

Proof. Since C is homogeneous in x and y of degree n, we may write

$$C = \sum_{i=0}^{n} a_i x^i y^{n-i}.$$

Suppose that $y \nmid C$, then $a_n \neq 0$. We divide through by y^n, so that

$$\frac{C}{y^n} = \sum_{i=1}^{n} a_i \left(\frac{x}{y}\right)^i$$

is degree n in x/y. By the Fundamental Theorem of Algebra there exists an $a \in \mathbb{C}$ and some $\alpha_i \in \mathbb{C}$ for $i = 1, \ldots, n$ such that

$$\frac{C}{y^n} = a \prod_{i=1}^{n} \left(\frac{x}{y} - \alpha_i\right).$$

Multiplying through by y^n yields

$$C = a \prod_{i=1}^{n} (x - \alpha_i y).$$

In the case that $y \mid C$, we may write $C = y^k \cdot C'$ where k is the greatest positive integer such that $y^k \mid C$. Then by the previous reasoning, C' is a product of $n - k$ linear factors so C is a product of n linear factors where k of them are the polynomial y. $\square$

While tangent lines are commonly understood analytically, it will be useful for us to define a tangent line of a curve at a point formally instead.

Definition 2.10. Let C be a homogeneous polynomial defining a projective curve of degree n passing through the point $[0 : 0 : 1]$. We can write C as

$$C = L_m Z^{n-m} + L_{m+1} Z^{n-m-1} + \cdots + L_n$$

where each $L_i \in \mathbb{C}[X, Y]$ is a homogeneous polynomial of degree i and $m \geq 1$. By Lemma 2.9 we can factor L_m to get

$$L_m = \prod_{i=1}^{m} T_i.$$

The line defined by each T_i is said to be *tangent* to the curve defined by C at $[0 : 0 : 1]$.

Remark 2.11. When working in affine coordinates we would have a similar decomposition

$$C = L_m + L_{m-1} + \cdots + L_n,$$

where $L_i \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree i. The tangent lines at $[0 : 0 : 1]$ would be given by the factors of L_m.

We can extend our definition of tangent lines to any appoint p by applying a projective linear transformation that takes p to $[0 : 0 : 1]$, finding the tangent lines, and then applying the inverse linear transformation. The resulting lines will be independent of the choice of projective linear transformation.

Example 2.12. Consider the curve defined by the polynomial $C = x - 3xy - y + x^2$. We can group the terms by degree and see that $C = (x-y)+(x^2-3xy)$. Then by the definition of tangent line, the line defined by $\mathrm{V}(x - y)$ is a tangent line of $\mathrm{V}(C)$ at $(0, 0)$ in affine coordinates. This is the same result one would obtain using implicit differentiation.

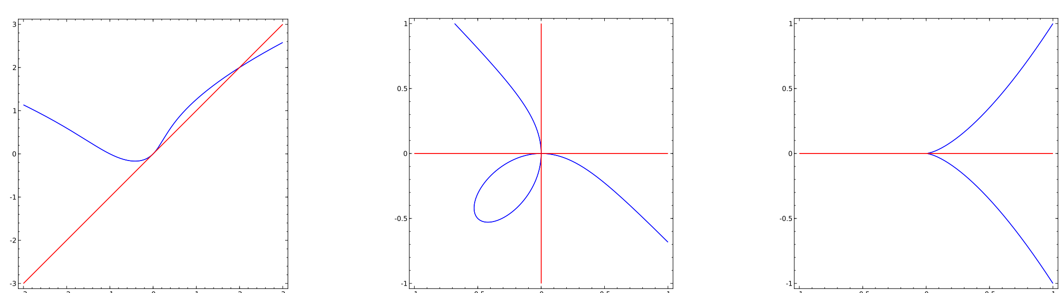
Definition 2.13. Let $C = L_2 + L_3 \in \mathbb{C}[x, y]$ define an irreducible cubic curve where each L_i is a homogeneous polynomial of degree i and $\mathrm{V}(C)$ singular at $(0, 0)$. If $L_2 = T_1T_2$ where $\mathrm{V}(T_1) \neq \mathrm{V}(T_2)$ then we say that $(0, 0)$ is a *node* of $\mathrm{V}(C)$. If $\mathrm{V}(T_1) = \mathrm{V}(T_2)$ then we say that $(0, 0)$ is a *cusp* of $\mathrm{V}(C)$.

Example 2.14. Let $D = x^3 - 2xy^2 + x^2 - y^2$ define a curve. Again, we may group the terms by degree to see that $D = (x^2 - y^2) + (x^3 - 2xy^2)$. Then the grouping of lowest degree is $(x^2 - y^2)$. Following the definition of a tangent line, since $x^2 - y^2 = (x + y)(x - y)$, we see that the lines $\mathrm{V}(x + y)$ and $\mathrm{V}(x - y)$ are the tangent lines of $\mathrm{V}(D)$ at $(0, 0)$. Furthermore, since there were no first degree terms of $g(x, y)$ and the second degree terms decomposed into distinct linear factors, we see that $\mathrm{V}(g)$ has a node at $(0, 0)$.

Remark 2.15. There are precisely two distinct tangent lines of a cubic at a node and only one tangent line of a cubic at a cusp.

Figure 2 shows the tangent lines on cubic curves at a smooth point, node, and a cusp.

Figure 2: Tangent Lines



(a) $\mathrm{V}(x - y + x^2 + xy - y^3)$, $\mathrm{V}(x - y)$

(b) $\mathrm{V}(xy + x^3 + y^3)$, $\mathrm{V}(x)$, $\mathrm{V}(y)$

(c) $\mathrm{V}(y^2 - x^3)$, $\mathrm{V}(y)$

At smooth points we can consider tangent lines to be the best linear approximation to the curve. We can similarly consider the best approximation by a quadratic curve at a given point. Such a quadratic curve is said to be osculating at the given point. Formally, we define the osculating quadratic as follows.

Definition 2.16. Let $C, Q \in \mathbb{C}[x, y]$ where $\deg(C) \geq 3$, $\deg(Q) = 2$, and $p \in \mathrm{V}(C)$. We say that $\mathrm{V}(Q)$ is an *osculating quadratic* of $\mathrm{V}(C)$ at p if $\mathrm{I}_p(Q, C) \geq 5$.

Example 2.17. Figure 3 depicts the osculating quadratic of a particular smooth curve at the point $(0, 0)$.

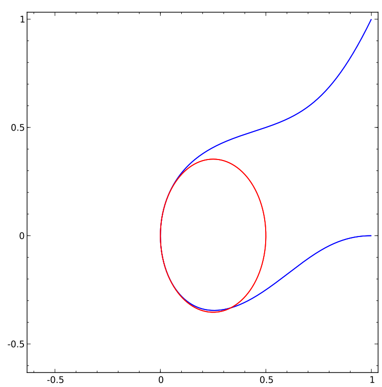


Figure 3: $\mathrm{V}(-y^2 + x^2y + x^3 - 2x^2 + x), \mathrm{V}(-y^2 - 2x^2 + x)$

Definition 2.18. Let $C \in \mathbb{C}[x, y]$ define an irreducible affine quadratic curve. We say that $\mathrm{V}(C)$ is a *parabola* if $\mathrm{V}(C)$ intersects the line at infinity at one distinct point with intersection multiplicity 2.

Definition 2.19. Let $C \in \mathbb{C}[x, y]$ define an irreducible curve that is smooth at p. We say p is a *flex*, or a *flex point*, of $\mathrm{V}(C)$ if there exists a line that intersects $\mathrm{V}(C)$ at p with degree 3 or more.

Example 2.20. Consider $C = y - x^3 \in \mathbb{C}[x, y]$. To see that $p = (0, 0)$ is a flex point, observe that $\mathrm{I}_p(y, y - x^3) = I_p(y, x^3) = 3$ so the line $\mathrm{V}(y)$ intersects $\mathrm{V}(C)$ at p with degree 3. This example is depicted in Figure 4 with $\mathrm{V}(y - x^3)$ shown in blue and the $\mathrm{V}(y)$ shown in red.

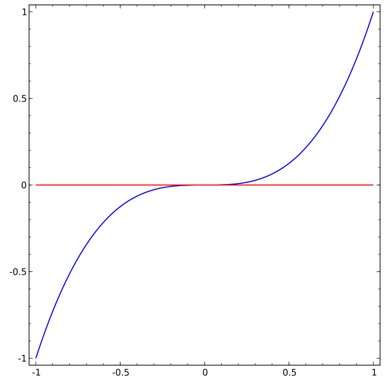


Figure 4: $\mathrm{V}(y-x^3), \mathrm{V}(y)$

Definition 2.21. Let $C \in \mathbb{C}[x,y]$ define an irreducible curve that is smooth at p. We say p is a *sextactic* point of $\mathrm{V}(C)$ if there exists a quadratic curve that intersects $\mathrm{V}(C)$ at p with degree 6 or more.

Lemma 2.22. *Let $C \in \mathbb{C}[X,Y,Z]$ be a homogeneous polynomial defining a cubic curve that is smooth at a point $p \in \mathrm{V}(C) \subset \mathbb{P}^2_{\mathbb{C}}$. If p is not a flex of $\mathrm{V}(C)$, then there exists a projective linear transformation taking p to $[0:0:1]$ and $\mathrm{V}(C)$ to a curve defined by*

$$C(X,Y,Z) = Z^2X + ZY^2 + G_3(X,Y)$$

where $G_3(X,Y)$ is a homogeneous polynomial of degree 3 depending only upon X and Y.

Proof. By Lemma 2.8, after a linear change of coordinates we may assume that $p = [0:0:1]$ and that the tangent line to the projective curve associated to C is given by $X = 0$. After which we can write C as

$$C = Z^2L_1(X,Y) + ZL_2(X,Y) + L_3(X,Y)$$

where each $L_i(X,Y)$ is homogeneous of degree i. Since our curve is smooth at $[0:0:1]$ with tangent line defined by $X = 0$ we know that $L_1(X,Y)$ is a non-zero multiple of X. After scaling by a nonzero constant we get a new equation

$$C' = Z^2X + Z(aX^2 + bXY + cY^2) + L'_3(X,Y).$$

If $c = 0$ then the intersection multiplicity with the line $X = 0$ would be 3 which would contradict our assumption that p is not a flex. Therefore we may assume that $c \neq 0$. Our next step is to perform the linear change of variables sending Z to $Z - \frac{a}{2}X - \frac{b}{2}Y$ and fixing X and Y. After making this substitution we are left with

$$Z^2X + Z(cY^2) - \frac{(aX+bY)(aX^2+bXY+2cY^2)}{4} + L'_3(X,Y).$$

Finally we scale the variable Y by $1/\sqrt{c}$ for some choice of $\sqrt{c}$ to get the desired form. Each step of this construction is an invertible linear change of coordinates, therefore their composition is a projective linear transformation. $\square$

Remark 2.23. Note that restricting curves to affine coordinates after projectively transforming them will not change the intersection multiplicity of the two curves at that point. Thus in the following lemma, we will homogenize the cubic polynomial given, projectively change the curve into the simpler form described in Lemma 2.22 and then restrict this curve to the affine plane.

Lemma 2.24. *Let $C \in \mathbb{C}[x, y]$ be the defining polynomial for the cubic curve $\mathrm{V}(C)$ and p a smooth non-flex point of $\mathrm{V}(C)$. An osculating conic of $\mathrm{V}(C)$ at p exists.*

Proof. Suppose that we have homogenized C, applied the projective linear transformations described in Lemma 2.22 and then restricted the resulting curve, $\mathrm{V}(C)$, to affine coordinates. Our defining equation can then be written as

$$C = x + y^2 + fx^3 + gx^2y + hxy^2 + iy^3$$

for some $f, g, h, i \in \mathbb{C}$. To prove our claim, we will explicitly show that the polynomial $Q = (i^2 - h)x^2 - ixy + y^2 + x$ defines an osculating conic of $\mathrm{V}(C)$ at $(0, 0)$.

Note that

$$C - (iy + (i^2 + h)x + 1)Q = x^2[(2i^3 + 2hi + g)y + (i^4 + 2hi^2 + h^2 + f)x].$$

Therefore we have

$$I_p(C, Q) = I_p(x^2, Q) + I_p((2i^3 + 2hi + g)y + (i^4 + 2hi^2 + h^2 + f)x, Q).$$

Since $\mathrm{V}(x)$ is tangent to $\mathrm{V}(Q)$ at p, $I_p(x, Q) = 2$ and $I_p(x^2, Q) = 4$. On the other hand $(2i^3 + 2hi + g)y + (i^4 + 2hi^2 + h^2 + f)x$ and Q both vanish at p, so

$$I_p((2i^3 + 2hi + g)y + (i^4 + 2hi^2 + h^2 + f)x, Q) \geq 1.$$

In particular $I_p(C, Q) \geq 5$. Note that Q must be irreducible, since otherwise it would decompose into two lines, at least of one which would have to intersect C to degree 3 at p which would contradict our supposition that p is not a flex point. □

Lemma 2.25. *Every irreducible nodal cubic has at least two irreducible osculating quadratics at the node.*

Proof. Let $N \in \mathbb{C}[x, y]$ define a nodal cubic at the point $p = (0, 0)$. By Lemma 2.8, we may projectively change the two tangent lines of $\mathrm{V}(N)$ to be the lines $\mathrm{V}(x)$ and $\mathrm{V}(y)$ so after scaling by a factor, we may suppose that

$$N = xy + fx^3 + gx^2y + hxy^2 + iy^3$$

for some $f, g, h, i \in \mathbb{C}$. Note that the constants f and i cannot be zero since otherwise N would be reducible. A brief calculation will show that $Q_1 = x + gx^2 + hxy + iy^2$ intersects

N with multiplicity 6 at p.

$$\begin{aligned}
I_p(N, Q_1) &= I_p(xy + fx^3 + gx^2y + hxy^2 + iy^3, x + gx^2 + hxy + iy^2) \\
&= I_p(fx^3, x + gx^2 + hxy + iy^2) \\
&= 3I_p(x, x + gx^2 + hxy + iy^2) \\
&= 3I_p(x, iy^2) \\
&= 6.
\end{aligned}$$

By symmetry, the quadratic $Q_2 = y + fx^2 + gxy + hy^2$ also intersects N with multiplicity 6 at p. To see that Q_1 and Q_2 must be irreducible, observe that for a reducible quadratic to intersect a cubic curve at a point with multiplicity 6, it must be the product of two lines that intersect the cubic curve at that point with multiplicity 3 each. The only lines that have that property are $\mathrm{V}(x)$ and $\mathrm{V}(y)$. Since f and i are not zero, we notice that neither Q_1 nor Q_2 have a factor of x or y. Hence, Q_1 and Q_2 cannot be the product of two linear terms and thus are irreducible. □

Lemma 2.26. *Let $C \in \mathbb{C}[x, y]$ be an irreducible polynomial. If $p \in \mathrm{V}(C)$ is smooth, then the tangent line of $\mathrm{V}(C)$ at p is unique and the osculating quadratic of $\mathrm{V}(C)$ at p is unique.*

Proof. By the definition of smooth, C can be written as $C = L_1(x, y) + L_2(x, y) + \cdots + L_n(x, y)$ where $L_i \in \mathbb{C}[x, y]$ is homogeneous of degree i and L_1 is not identically zero. By definition of tangent line, $\mathrm{V}(L_1(x, y))$ is the tangent line of $\mathrm{V}(C)$.

Similarly, suppose that the polynomials $Q, Q' \in \mathbb{C}[x, y]$ both define osculating quadratics of $\mathrm{V}(C)$ at p. It follows that $\mathrm{I}_p(C, Q) \geq 5$ and $\mathrm{I}_p(C, Q') \geq 5$ so by Lemma 2.3 we have that $\mathrm{I}_p(Q, Q') \geq 5$. However, by Bézout's Theorem, we know that $\mathrm{V}(Q)$ and $\mathrm{V}(Q')$ intersect at precisely 4 points including multiplicity, so they cannot intersect each other with degree 5 at p. □

Remark 2.27. Recall that a flex point of a curve is a smooth point such that there exists a line that intersects the curve at that point with multiplicity 3. Since osculating quadratics are unique at a smooth point, the only osculating quadratic at a flex point is given by an equation for the tangent line squared. This double line will in fact intersect the curve with multiplicity 6 at the flex point. In particular, this means that there is no irreducible osculating quadratic of a curve at a flex point.

3 Families of Cubics with Ninth Order Intersections at A Point

Theorem 3.1. *Let $D \in \mathbb{C}[x, y]$ define an irreducible cubic curve through the point p. If there exists an irreducible quadratic polynomial Q, such that $I_p(D, Q) \geq 5$, then we can write*

$$D = T^2 l_1 + Ql_2$$

where T defines the tangent line to $\mathrm{V}(Q)$ at p and l_1 and l_2 are linear polynomials.

Proof. Since intersection multiplicity is invariant under a projective change of coordinates, it will suffice for our purposes to consider only the case in which $p = (0, 0)$. After a projective change of coordinates followed by localization to affine coordinates, we may suppose that $Q(x, y) = y^2 - x$, from which it follows that $T(x, y) = x$ defines a tangent line to $Q(x, y)$ at p.

Since $I_p(D, Q) \geq 5$, it follows that

$$D(y^2, y) = ay^6 + by^5.$$

Constructing the polynomial $g(x, y) = D(x, y) - (ax^3 + bx^2 y)$, we observe that

$$g(y^2, y) = D(y^2, y) - ay^6 + by^5 = 0.$$

Since $g(x, y)$ is identically zero in the polynomial ring $\mathbb{C}[x, y]/\langle y^2 - x\rangle$, we conclude that $g(x, y) \in \langle y^2 - x\rangle$ and thus $y^2 - x$ divides the polynomial $g(x, y)$. Using this divisibility relation, we can write

$$\begin{aligned} D(x, y) &= ax^3 + bx^2 y + g(x, y) \\ &= x^2(ax + by) + (y^2 - x)l_2 \\ &= T^2 l_1 + Ql_2, \end{aligned}$$

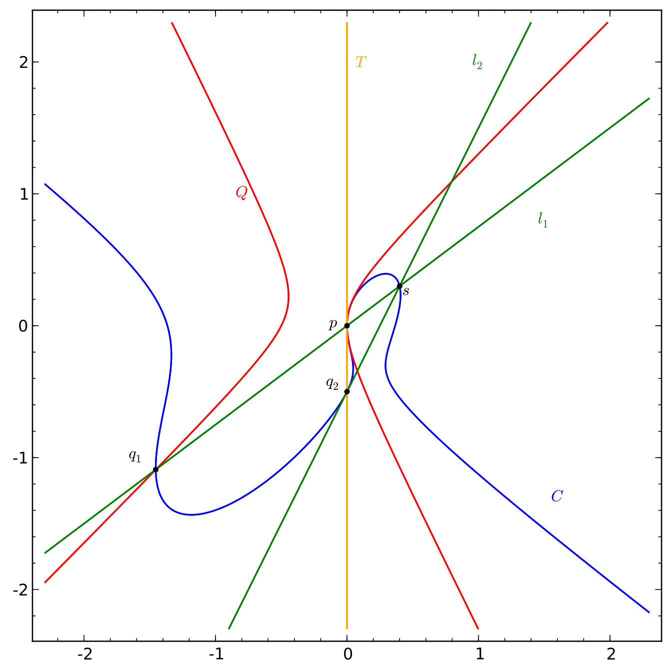
where l_1 and l_2 are linear polynomials in $\mathbb{C}[x, y]$. Note that $l_1 = ax+by$ vanishes at $p = (0, 0)$. This will be used in the next theorem. □

We have geometric interpretations for $\mathrm{V}(T)$ and $\mathrm{V}(Q)$. The following theorem describes the geometric properties of $\mathrm{V}(l_1)$ and $\mathrm{V}(l_2)$.

Theorem 3.2. *Let $D \in \mathbb{C}[x, y]$ define an irreducible cubic curve through the point p and let Q define an irreducible quadratic such that $I_p(D, Q) \geq 5$. In the form*

$$D = T^2 l_1 + Q l_2$$

afforded by the previous theorem, $\mathrm{V}(l_1)$ *contains both p and the sixth point of intersection of* $\mathrm{V}(Q) \cap \mathrm{V}(D)$*. Furthermore,* $\mathrm{V}(l_2)$ *is a tangent line to* $\mathrm{V}(D)$ *at the third point of intersection of* $\mathrm{V}(T) \cap \mathrm{V}(D)$.



Proof. Since $I_p(D, Q) \geq 5$, the notion of a sixth point of intersection of $\mathrm{V}(Q) \cap \mathrm{V}(D)$ is well defined and may or may not be the point p. Let q_1 denote this point. Recall that $l_1(p) = 0$, so if it is the case that $q_1 = p$, the result follows trivially. Suppose then, that q_1 is distinct from p and consider the following evaluation

$$\begin{aligned} D(q_1) &= T(q_1)^2 l_1(q_1) + Q(q_1) l_2(q_1) \\ 0 &= T(q_1)^2 l_1(q_1). \end{aligned}$$

We know that $I_p(T, Q) = 2$ since T defines the tangent line of Q at p and by Bézout, these curves can only intersect twice including multiplicity. It follows that since q_1 is distinct from p, that $T(q_1) \neq 0$ so l_1 must vanish at q_1.

Similarly, since $I_p(D, T) \geq 2$, it makes sense to talk about the "third" point of intersection of $\mathrm{V}(T) \cap \mathrm{V}(D)$. We will call this point q_2. An evaluation at q_2 gives us

$$\begin{aligned} D(q_2) &= T(q_2)^2 l_1(q_2) + Q(q_2) l_2(q_2) \\ 0 &= Q(q_2) l_2(q_2). \end{aligned}$$

However, if $Q(q_2) = 0$ then the curves $\mathrm{V}(T)$ and $\mathrm{V}(Q)$ would intersect at the point q_2 in addition to intersecting at p with a multiplicity of 2, totalling 3 intersection points. This would force $\mathrm{V}(Q)$ to contain the line $\mathrm{V}(T)$ by Bézout, a contradiction to the irreducibility of Q. Therefore $Q(q_2) \neq 0$ which means that $l_2(q_2) = 0$ and further implies that $q_2 \neq p$ since $Q(p) = 0$. It then follows that

$$\begin{aligned} I_{q_2}(D, l_2) = I_{q_2}(T^2 l_1 + Q l_2, l_2) &= I_{q_2}(T^2 l_1, l_2) \\ &= 2 I_{q_2}(T, l_2) + I_{q_2}(l_1, l_2) \\ &= 2 + I_{q_2}(l_1, l_2) \end{aligned}$$

From this, it follows that $I_{q_2}(D, l_2) \geq 2$. This implies that $\mathrm{V}(l_2)$ is tangent to $\mathrm{V}(D)$ at q_2 so long as q_2 is a smooth point of $\mathrm{V}(D)$. If q_2 is not smooth on $\mathrm{V}(D)$, then $I_{q_2}(T, D) \geq 2$. However, it is already the case that $I_p(T, D) \geq 2$ and since $q_1 \neq p$, this would lead to a contradiction since a line can only intersect a cubic at three points including multiplicity. Hence, q_1 must be a smooth point and thus $\mathrm{V}(l_2)$ is tangent to $\mathrm{V}(D)$ at q_2.

□

3.1 Case 1: Smooth, Non-Flex Points On Cubics

Throughout this section we will let $C \in \mathbb{C}[x, y]$ be an irreducible polynomial that defines a cubic curve that is smooth at the point $p = (0, 0)$ where p is not a flex point of $\mathrm{V}(C)$. As such, by Lemma 2.24, we know that there exists an osculating conic of $\mathrm{V}(C)$ at p. We will let Q denote a defining polynomial of this osculating conic and T denote a defining polynomial of the tangent line of $\mathrm{V}(C)$ at p. By Lemma 2.26, we know that both $\mathrm{V}(T)$ and $\mathrm{V}(Q)$ are unique. Of course there are infinitely many equations of these curves, all of which differ by multiplication of a constant. In each instance we simply fix one such equation.

Furthermore, by Theorem 3.1, we know we can write $C = T^2 l_1 + Q l_2$. From this point on we will use the notation l_1 and l_2 to refer precisely to these linear terms. We will also refer to q_1 and q_2 as defined in Theorem 3.2 as the third point of intersection of $\mathrm{V}(T, C)$ and the sixth point of intersection of $\mathrm{V}(Q, C)$ respectively.

Our goal now is to investigate whether we can find another cubic curve that intersects $\mathrm{V}(C)$ to degree 9 at p.

Lemma 3.3. *Let $D \in \mathbb{C}[x, y]$ define an irreducible curve. Then $\mathrm{I}_p(C, D) = 9$ if and only if D can be written as $D = T^2 l_3 + Q l_4$ where $l_3, l_4 \in \mathbb{C}[x, y]$ are linear polynomials and $\mathrm{V}(Q) = \mathrm{V}(l_1 l_4 - l_2 l_3)$. Furthermore, if $\mathrm{I}_p(C, D) = 9$, then p is not a sextactic point of $\mathrm{V}(C)$.*

Proof. We know that $\mathrm{I}_p(C, Q) \geq 5$, so if $\mathrm{I}_p(C, D) = 9$, then transitively by Lemma 2.3, $\mathrm{I}_p(D, Q) \geq 5$ which fulfills the necessary conditions in Theorem 3.1 for D to be written as $D = T^2 l_3 + Q l_4$.

Suppose that $\mathrm{V}(C)$ and $\mathrm{V}(D)$ intersect only at a point p with multiplicity 9. Then equivalently

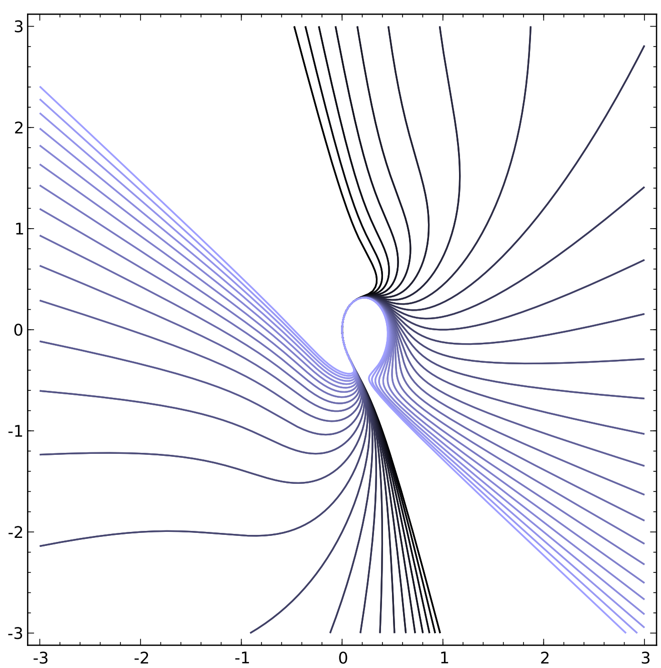
$$\begin{aligned}9 = \mathrm{I}_p(C,D) &= \mathrm{I}_p(T^2l_1 + Ql_2, T^2l_3 + Ql_4)\\ &= \mathrm{I}_p(T^2l_1 + Ql_2, T^2l_3l_2 + Ql_4l_2)\\ &= \mathrm{I}_p(T^2l_1 + Ql_2, T^2l_3l_2 - T^2l_4l_1)\\ &= \mathrm{I}_p(T^2l_1 + Ql_2, -T^2(l_1l_4 - l_2l_3))\\ &= \mathrm{I}_p(C, -T^2) + \mathrm{I}_p(C, (l_1l_4 - l_2l_3))\\ &= 4 + \mathrm{I}_p(C, (l_1l_4 - l_2l_3)).\end{aligned}$$

The above equality holds if and only if $\mathrm{I}_p(C, l_1l_4 - l_3l_2) \geq 5$. But $\mathrm{V}(l_1l_4 - l_3l_2)$ is a quadratic, so in order for it to intersect $\mathrm{V}(C)$ with degree 5, it must be the osculating quadratic of $\mathrm{V}(C)$, namely $\mathrm{V}(Q)$.

Furthermore, if p was a sextactic point of $\mathrm{V}(C)$, we would have that $\mathrm{I}_p(C,Q) = I_p(C, l_1l_4 - l_2l_3) = 6$ which would imply that $\mathrm{I}_p(C,D) = 10$. This is a contradiction by Bézout's Theorem since C and D are polynomials of degree 3 so therefore, p cannot be a sextactic point of $\mathrm{V}(C)$.

□

Lemma 3.4. *If there is one cubic curve defined by $D \in \mathbb{C}[x,y]$ such that $\mathrm{I}_p(C,D) = 9$, then there are infinitely many. Furthermore, every such cubic other than D itself is of the form $C + \alpha D$ for some $\alpha \in \mathbb{C}^*$.*

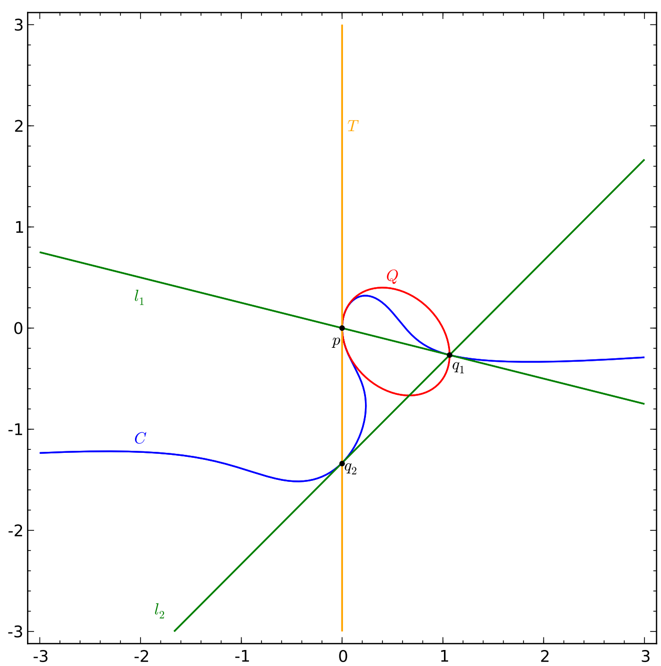


Proof. It follows directly from the properties of intersection multiplicity that $I_p(C,D) = 9$ implies $I_p(C, C + \alpha D) = 9$ for $\alpha \neq 0$.

We next show that every point in the plane is contained by one of the curves in this family. Let s be an arbitrary point in the plane. If $s \in \mathrm{V}(D)$ or $s \in \mathrm{V}(C)$ we are done. Suppose not, then letting $\alpha = -C(s)/D(s)$ yields a curve passing through s with $\alpha \neq 0$.

Let E be the defining polynomial of any curve intersecting C to degree 9 at p. Pick any point $s \in \mathrm{V}(E)$, $s \neq p$. By our previous observation there must exist some curve F in our infinite family passing through s. However, E must intersect F to degree 9 at p and to degree at least 1 at s which is a contradiction of Bézout's Theorem unless they describe the same curve. Therefore this family contains all such curves. □

Theorem 3.5. *There exists a cubic curve which intersects* $\mathrm{V}(C)$ *at* p *with degree* 9 *if and only if* $\mathrm{V}(l_1)$ *is tangent to* $\mathrm{V}(C)$ *at* q_1*.*



Proof. Recall that $p, q_1 \in \mathrm{V}(l_1)$ and $\mathrm{V}(l_2)$ is a tangent line of $\mathrm{V}(C)$ at q_2.

Suppose that $\mathrm{V}(l_1)$ is tangent to $\mathrm{V}(C)$ at q_1. Then it must be the case that $\mathrm{V}(l_1, C) = \{p, q_1\}$ since $\mathrm{V}(l_1)$ and $\mathrm{V}(C)$ intersect at 3 points including multiplicity, two of which are q_1 and the other p.

Let s denote the point of intersection of $\mathrm{V}(l_1)$ and $\mathrm{V}(l_2)$. An evaluation of the equation $C = T^2 l_1 + Q l_2$ at s shows that $C(s) = 0$ so $s \in \mathrm{V}(C)$.

Since $s \in \mathrm{V}(l_1, C)$ we must have that s is either p or q_1. Either way, $s \in \mathrm{V}(l_1, Q)$ so $s = \mathrm{V}(l_1, l_2) \subset \mathrm{V}(Q)$. According to the ideal-variety correspondence, it follows that

$$Q \in \langle l_1, l_2 \rangle,$$

so we can write

$$Q = l_1 A - l_2 B$$

for some linear polynomials A and B. Then it follows directly from Lemma 3.3 that the curve

$$D = T^2 B + QA,$$

intersects $\mathrm{V}(C)$ at p with degree 9.

Conversely, suppose that $\mathrm{V}(l_1)$ is not tangent to $\mathrm{V}(C)$ at q_1. Recall that if $I_p(C, D) = 9$, then p cannot be a sextactic point of C. This is precisely the same thing as saying that $p \neq q_1$. Since $p, q_1 \in \mathrm{V}(l_1, C)$, it makes sense to discuss the third point of $\mathrm{V}(l_1, C)$ which we will call r.

We know $r \neq q_1$ since otherwise, this would imply that $\mathrm{V}(l_1)$ is tangent to $\mathrm{V}(C)$ at q_1. We know $r \neq p$ since otherwise $\mathrm{V}(l_1) = \mathrm{V}(T)$ and it would follow that

$$I_p(C, Q) = I_p(T^3 + Ql_2, Q) = 3I_p(T, Q) = 6.$$

Which is equivalent to saying p is sextactic on $\mathrm{V}(C)$, a contradiction. Finally, we conclude that since $r \in \mathrm{V}(C)$ but $r \notin \{p, q_1\} = \mathrm{V}(C, Q)$ we must have that $r \notin \mathrm{V}(Q)$. Evaluating the usual form for C at r yields

$$\begin{aligned} C(r) &= T^2(r) l_1(r) + Q(r) l_2(r) \\ 0 &= Q(r) l_2(r) \end{aligned}$$

Since $r \notin \mathrm{V}(Q)$, it must be the case that $l_2(r) = 0$.

Suppose now toward a contradiction that $D = T^2 l_3 + Q l_4$ defines a cubic curve such that $I_p(C, D) = 9$. Then by Lemma 3.3 it follows that

$$Q = l_1 l_4 - l_2 l_3,$$

but since $l_1(r) = 0$ and $l_2(r) = 0$, this would imply that

$$Q(r) = 0,$$

which is contradiction since $r \notin \mathrm{V}(Q)$.

Therefore, if $\mathrm{V}(l_1)$ is not tangent to $\mathrm{V}(C)$ at p, we may conclude that there does not exist any cubic which intersects $\mathrm{V}(C)$ at p with degree 9. □

Remark 3.6. Note that if $D \in \mathbb{C}[x, y]$ defines a cubic curve that intersects $\mathrm{V}(C)$ at p with multiplicity 9, it must be irreducible. Otherwise, it would decompose into three lines, or a line and a conic. Either way, by properties of intersection multiplicity, the line would have to intersect $\mathrm{V}(C)$ with multiplicity 3 at p, contradicting the fact that p is not a flex point.

Lemma 3.7. *If there is an infinite family of cubics that intersect* $\mathrm{V}(C)$ *at* p *to degree 9, precisely one of those cubics has a singularity at* p*. In particular, the singularity is a node and the singular curve may be written as*

$$N = T^3 + QT',$$

where T' *defines the second tangent of* $\mathrm{V}(N)$ *at* p *and* $\mathrm{I}_p(N, Q) = 6$.

Proof. Let $D \in \mathbb{C}[x, y]$ define an irreducible cubic curve such that $I_p(C, D) = 9$. It follows that $I_p(T, D) \geq 2$ and thus $\mathrm{V}(T)$ is the tangent line of $\mathrm{V}(D)$ at p. Thus, the linear terms of C and D must differ merely by a constant. Since multiplying a polynomial by a constant does not change its solutions, we may assume that C and D have the same linear term.

Recall that if $I_p(C, D) = 9$, then $I_p(C, C+\alpha D) = 9$ for any $\alpha \in \mathbb{C}^*$. Let $N = C+(-1)D$. Since C and D have the same linear term, it follows that N has no linear term, and thus is singular at the point p. Since $I_p(C, Q) = 5$ and $I_p(C, N) = 9$ it follows that $I_p(N, Q) \geq 5$ so we may write $N = T^2 l_3 + Ql_4$ where l_4 vanishes at p since N is singular.

Next, we will show that l_3 and l_4 both define tangent lines to $\mathrm{V}(N)$ at p that are distinct. Consider the following

$$\begin{aligned} I_p(l_4, N) &= I_p(l_4, T^2 l_3 + Ql_4) \\ &= I_p(l_4, T^2 l_3) \\ &= 2I_p(l_4, T) + I_p(l_4, l_3) \\ &= 3 \end{aligned}$$

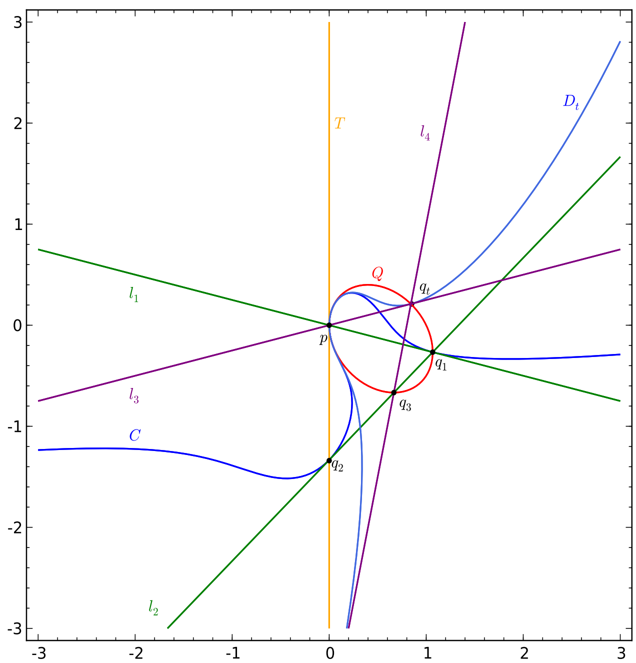
$$\begin{aligned} I_p(l_3, N) &= I_p(l_3, T^2 l_3 + Ql_4) \\ &= I_p(l_3, Ql_4) \\ &= I_p(l_3, (l_1 l_4 - l_2 l_3) l_4) \\ &= I_p(l_3, l_1 l_4^2 - l_2 l_3 l_4) \\ &= I_p(l_3, l_1 l_4^2) \\ &= I_p(l_3, l_1) + 2I_p(l_3, l_4) \\ &= 3 \end{aligned}$$

Note that $l_3 \neq l_4$ since otherwise $Q = l_1 l_4 - l_2 l_3$ would be reducible. Hence, l_3 and l_4 define distinct tangent lines to N at p which means N has a node at p. By the remark preceeding this lemma, $N = T^2 l_3 + Ql_4$ is irreducible so $\mathrm{V}(l_4) \neq \mathrm{V}(T)$. Therefore, since nodes only have two tangents, we conclude that $\mathrm{V}(l_3) = \mathrm{V}(T)$ and that l_4 must define the second tangent of $\mathrm{V}(N)$ at p. Thus, after multiplication by a constant, we may write $N = T^3 + QT'$ and consider $I_p(N, Q)$.

$$\begin{aligned} I_p(N, Q) &= I_p(T^3 + QT', Q) \\ &= I_p(T^3, Q) \\ &= 3I_p(T, Q) \\ &= 6. \end{aligned}$$

Finally, to see that this node is unique, notice that every cubic which intersects $\mathrm{V}(C)$ with multiplicity 9 at p is of the form $C + \alpha D$ and that the only choice of α that makes the linear term of $C + \alpha D$ vanish is $\alpha = -1$. $\square$

Theorem 3.8. *If there is an infinite family of cubics that intersect* $\mathrm{V}(C)$ *to ninth order at* p*, then the members of this family can be described by a one to one correspondence with points on* $\mathrm{V}(Q)$*.*



Proof. Define the point q_3 to be the point at which $\mathrm{V}(Q)$ and $\mathrm{V}(l_2)$ intersect other than q_1 and let q_t be any point on $\mathrm{V}(Q)$. Define A_t to be an equation of the line $\overline{pq_t}$ and B_t to be an equation of the line $\overline{q_3q_t}$. Let us examine the set $\mathrm{V}(A_tl_2, B_tl_1)$. We have by definition

$$\begin{aligned} \mathrm{V}(A_t, B_t) &= \{q_t\} \\ \mathrm{V}(A_t, l_1) &= \{p\} \\ \mathrm{V}(l_2, B_t) &= \{q_3\} \\ \mathrm{V}(l_2, l_1) &= \{q_1\}. \end{aligned}$$

So equivalently, we have that $\mathrm{V}(a_tl_2, B_tl_1) = \{p, q_1, q_3, q_t\} \subset \mathrm{V}(Q)$. By Max Noether's Fundamental Theorem, this implies that we can write

$$Q = \alpha l_1 B_t - \beta l_2 A_t,$$

where $\alpha, \beta \in \mathbb{C}$. But since multiplying by a scalar does not change the vanishing of a polynomial, we can define $A'_t = \beta A_t$ and $B'_t = \alpha B_t$ to be new equations for the lines defined by A_t and B_t.

Now since we have $Q = l_1 B_t' - l_2 A_t'$, by a previous theorem we have that the curve defined by

$$D_t = T^2 A_t' + Q B_t',$$

intersects the curve $\mathrm{V}(C)$ to degree nine at p.

Conversely, suppose that we have some D_t such that $\mathrm{V}(D_t)$ intersects $\mathrm{V}(C)$ to degree nine at p. We can write $D = T^2 A + QB$ where A and B are the analogous constructions of l_1 and l_2 in $C = T^2 l_1 + Q l_2$. We can then consider the point of intersection of $\mathrm{V}(A)$ and $\mathrm{V}(B)$ and name it s. Setting q_t in the above construction to be equal to s yields that $\mathrm{V}(D) = \mathrm{V}(D_t)$. So the points on $\mathrm{V}(Q)$ are in a one to one correspondence with the elements of the family of cubics that intersect C to degree 9 at p.

□

3.2 Constructing Families from an Irreducible Nodal Cubic

We know that in the family of cubic curves constructed in the previous section, there is one particular cubic curve that has a node at p. It is worthwhile to ask the inverse question: Does every nodal cubic give rise to such a family of smooth cubics? Are these families in a one to one correspondence with nodal cubics? Or can one nodal cubic be the unique singular curve of more than one family? We will explore questions such as this now.

Throughout this section, we will suppose that $N \in \mathbb{C}[x, y]$ defines an irreducible cubic curve with a node at $p = [0 : 0 : 1]$.

Remark 3.9. Recall that by Lemma 2.25, there exists two osculating conics to $\mathrm{V}(N)$ at p. Each conic corresponds to a different tangent line to $\mathrm{V}(N)$ at p so by Theorem 3.1 we may write N in either of the following forms.

$$N = T_1^3 + Q_1 T_2$$

$$N = T_2^3 + Q_2 T_1$$

Lemma 3.10. *Let $N = T_1^3 + Q_1 T_2 = T_2^3 + Q_2 T_1$. There exists two smooth cubic curves defined by $C_1, C_2 \in \mathbb{C}[x, y]$ such that $\mathrm{I}_p(N, C_1) = 9$ and $\mathrm{I}_p(N, C_2) = 9$ but $\mathrm{I}_p(C_1, C_2) \leq 9$. Furthermore, these cubic curves may be written as*

$$C_1 = T_1^2 l_1 + Q_1 l_2,$$

and

$$C_2 = T_2^2 l_3 + Q_1 l_4.$$

Proof. By Theorem 3.3, we only need to show that there exists l_1 and l_2 such that $\mathrm{V}(Q_1) = \mathrm{V}(T_1 l_2 - T_2 l_1)$ in order to prove that

$$C_1 = T_1^2 l_1 + Q_1 l_2$$

defines a smooth cubic curve with $\mathrm{I}_p(N, C_1) = 9$.

We will show that such l_1 and l_2 exists by construction.

Let q_3 be the second point of intersection of $\mathrm{V}(Q_1)$ and $\mathrm{V}(T_2)$ and fix some $q_4 \in \mathrm{V}(Q_1)$. Let l'_1 be an equation of the line connecting p and q_4 and let l'_2 be an equation of the line connecting q_3 and q_4.

Considering the variety $\mathrm{V}(T_1 l_2, T_2 l_1)$ we see that

$$\begin{aligned} \mathrm{V}(T_1, T_2) &= \{p\} \\ \mathrm{V}(T_1, l'_1) &= \{p\} \\ \mathrm{V}(l'_2, T_2) &= \{q_3\} \\ \mathrm{V}(l'_2, l'_1) &= \{q_4\}, \end{aligned}$$

so $\mathrm{V}(T_1 l'_2, T_2 l'_1) \subseteq \mathrm{V}(Q_1)$ so by the ideal variety correspondence we have that

$$Q_1 \in \langle T_1 l'_2, T_2 l'_1 \rangle.$$

It then follows that $Q = \alpha(T_1 l'_2 - T_2 l'_1) = T_1(\alpha l'_2) - T_2(\alpha l'_1)$. Setting $l_1 = \alpha l'_1$ and $l_2 = \alpha l'_2$ we see that l_1 and l_2 are defining equations of the same lines defined by l_1 and l_2. Thus, the linear polynomials l_1 and l_2 satisfy the conditions of 3.3 so setting $C_1 = T_1^2 l_1 + Q_1 l_2$ yields the result that $\mathrm{I}_p(C_1, N) = 9$.

We can repeat the same construction after fixing Q_2 instead of Q_1 to see that $C_2 = T_1^2 l_3 + Q_1 l_4$ yields the result of $\mathrm{I}_p(C_2, N) = 9$.

However, observe that if $\mathrm{I}_p(C_1, C_2) = 9$ then $\mathrm{I}_p(C_1, Q_2) = 5$ which would imply that Q_1 and Q_2 defined the same conic. This is a contradiction since we chose Q_1 and Q_2 to define the two unique conics that intersect $\mathrm{V}(N)$ to degree 6 at p. □

Lemma 3.11. *If there exists an irreducible cubic curve defined by $C \in \mathbb{C}[x, y]$ such that $\mathrm{I}_p(C, N) = 9$, then there is an infinite family of such cubic curves which all intersect each other at p with degree 9.*

Proof. Let $C \in \mathbb{C}[x, y]$ define an irreducible cubic curve such that $\mathrm{I}_p(C, N) = 9$. Consider the curve defined by $C_t = C + tN$. It directly follows that $\mathrm{I}_p(C_t, N) = 9$ and $\mathrm{I}_p(C_t, C) = 9$ for all $t \in \mathbb{C}$. □

Theorem 3.12. *For every nodal cubic N there are two families of cubic curves E_{Q_1} and E_{Q_2} such that $\mathrm{I}_p(A, B) = 9$ for every $A, B \in E_{Q_i}$ and if $A \in E_{Q_1}$ and $B \in E_{Q_2}$ then $\mathrm{I}_p(A, B) = 9$ if and only if $A = B = N$.*

Proof. By Lemma 3.10 there are at least two cubic curves defined by C and D such that $\mathrm{I}_p(C, N) = 9$, $\mathrm{I}_p(D, N) = 9$ and $\mathrm{I}_p(C, D) < 9$. However, by Lemma 3.11 we know that there are infinitely many C_t and D_t such that $\mathrm{I}_p(C, C_t) = 9$, and $\mathrm{I}_p(D, D_t) = 9$. However, since $\mathrm{I}_p(C, D) < 9$, we may conclude from Lemma 2.3 that $\mathrm{I}_p(C_i, D_j) < 9$ for all smooth C_i and D_j. Note that all C_i are smooth except N and all D_j are smooth except N so the only exception to $\mathrm{I}_p(C_i, D_j) < 9$ is when $C_i = C_j = N$. □

References

[1] Robert Bix. *Conics and Cubics: A Concrete Introduction to Algebraic Curves*. Springer-Verlag New York, Inc., New York, NY, USA, 1998.

[2] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, 3/e (Undergraduate Texts in Mathematics)*. Springer-Verlag New York, Inc., New York, NY, USA, 2007.

[3] William Fulton. *Algebraic Curves: An Introduction to Algebraic Geometry*. Addison-Wesley, 1989.

[4] Alwaleed Kamel and M. Farahat. On the moduli space of smooth plane quartic curves with a sextactic point. *Appl. Math. Inf. Sci.*, 7(2):509–513, 2013.