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KLEIN LINKS AND BRAIDS

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Abstract. We introduce the construction of Klein links through an alteration to the orientation on the rectangular representation of a torus knot. We relate the resulting Klein links to their corresponding braid representations, and use these representations to understand the relationship between Klein links and torus knots as well as to prove relationships between several different Klein links.

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1 Introduction to Klein Links

A commonly studied class of links in knot theory is the class of torus knots. A torus knot, such as the one in Figure 2, is created by wrapping a piece of string around a torus (Figure 1) without self-intersection and then gluing the two ends together. A general torus knot generalizes this process by wrapping multiple pieces of string around a torus. A torus knot that wraps m times around the longitude of the torus and n times around the meridian of the torus is called the (m, n) -torus knot [1].

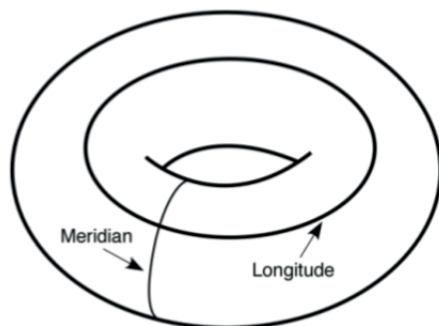


Figure 1: Torus surface [3].

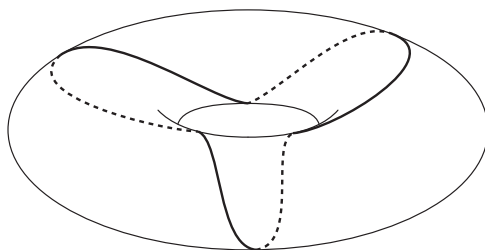


Figure 2: $(3, 1)$ -torus knot.

The algorithmic construction process for the torus knots give rise to numerous properties. For instance, the (m, n) -torus knot and (n, m) -torus knot are equivalent [1]. Another classic result is that the (m, n) -torus knot has $\gcd(m, n)$ components and so the (m, n) -torus knot is a knot only if $\gcd(m, n) = 1$ [1, 5].

Suppose we take a torus with any torus knot on it and cut the torus along the meridian curve. We can deform the resulting surface into a cylinder. Next, by cutting the cylinder along the longitude curve, we obtain an oriented rectangle. An example of this process is demonstrated in Figure 3. Notice that the arrows on the vertical edges of the rectangle are both pointing upward while the arrows on the horizontal edges are both pointing to the right, indicating that the corresponding edges have matching orientation. All torus knots have this basic form. The m strand connections on the left and right sides of the rectangle

correspond to m crossings of the longitude on a torus while the n strand connections on the top and bottom of the rectangle correspond to n crossings of the meridian on a torus.

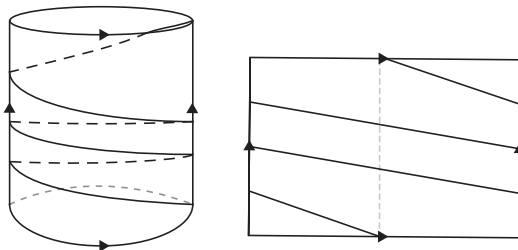


Figure 3: The cylindrical and rectangular representations of the $(3,1)$ -torus knot.

Let us alter the rectangular representation of the torus by reversing the orientation of the top horizontal edge while leaving everything else the same. When we identify the vertical edges of the rectangle, we create a cylinder (Figure 4). Since the top and bottom of our cylinder have opposite orientations (as shown in Figure 4) and we must identify the bottom circle with the top of the cylinder, we deform the bottom end of the cylinder and pass it through itself. In four dimensions, this self-intersection does not occur and it results in the formation of a surface known as the Klein bottle [7]. There are many different three-dimensional representations of the Klein bottle, such as the one in Figure 5.

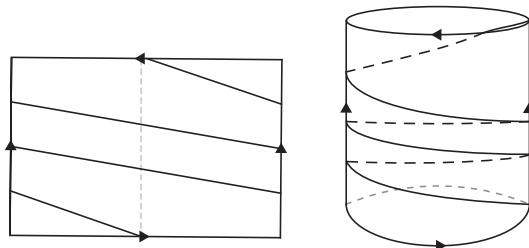


Figure 4: The rectangular and cylindrical representations of the $(3,1)$ -torus knot with opposite orientation.

For the remainder of this article, we will discuss our findings and results from reorienting the rectangular representation of the (m,n) -torus knot to form a related link on a Klein bottle. We have termed the resulting links **Klein links**. In the formation of these Klein links, there are several necessary standardizations to create a consistent link.

1. Three-Dimensional Representation: We have opted to use a three-dimensional representation of the Klein bottle for the Klein links because links in four dimensions are trivial; all such links reduce to the unlink [1, 6]. Hence, there are no nontrivial results that can be produced by examining these links on the four-dimensional Klein bottle.

2. Construction Standardizations: It is important to clarify the specific construction of the three-dimensional Klein bottle that we use to construct the Klein links. Starting with

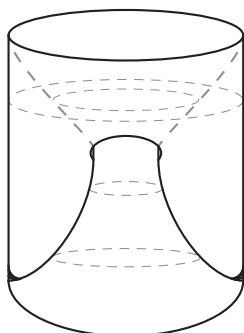


Figure 5: The three-dimensional representation of the Klein bottle.

the rectangle, we create a cylinder by folding the right side of the rectangle backwards, and attaching the left and right sides to create the left vertical side of the cylinder. Thus, the left side of the rectangle becomes the front of the cylinder while the right side of the rectangle becomes the back of the cylinder. Finally, the bottom and top circles of the cylinder are identified as described above. Figure 5 helps to illustrate this particular representation of the Klein bottle.

We chose to place the hole that results from the cylinder intersecting itself in the upper left hand corner of the rectangular representation, as shown in Figure 6. With this placement, all of the strands on the left edge of the rectangle must travel under the hole while all of the strands on the top edge of the rectangle must travel over the hole. The strands must continue to follow this pattern when translated on the cylinder and later the three-dimensional representation of the Klein bottle. The consistency of this hole placement is crucial in the formation of our Klein links; changing this placement would result in different crossings and therefore a different link.

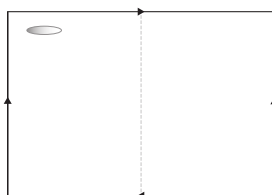


Figure 6: Hole placement on the rectangular representation of the Klein bottle.

Figure 7 demonstrates the construction of a $(3, 2)$ -Klein link from a $(3, 2)$ -torus knot; Figure 8 does a similar demonstration for the $(4, 3)$ -Klein link.

One must realize that this specific construction of the Klein bottle and our careful placement of the hole are crucial in guaranteeing that our Klein links are well defined. It can be verified that alternate hole placements create different crossings and produce different links. We have not yet explored alternative constructions of the Klein bottle or other hole placements; and we do recognize that possible alternative constructions and various hole

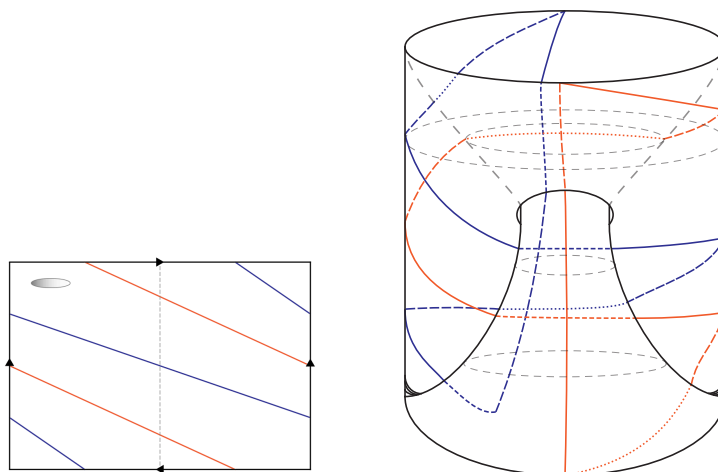


Figure 7: The creation of the $(3, 2)$ -Klein link.

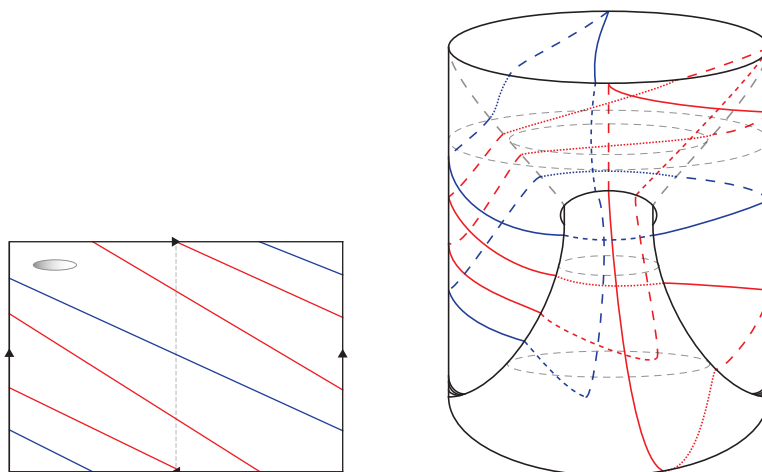


Figure 8: The creation of the $(4, 3)$ -Klein link.

placements need to be explored further. We believe it may be useful to look at the Klein bottle in its higher dimensional form, instead of the rectangular form that we use. Our Klein links are, in fact, links on a “punctured” Klein bottle. Again, the location of the “puncture” in relation to the link strands will be an important investigation. However, it should also be pointed out that this particular construction of the Klein bottle and our current hole placement prove to be particularly valuable in how these links lend themselves to braid representations.

In the next section, we introduce braid theory as another method of representing links in the plane. In Section 3, we demonstrate that Klein links have a predictable braid representation and, in Section 4, use this to show the equivalence of certain Klein links and that some Klein links have simpler braid forms. Finally, in Section 5, we summarize our results

and suggest further avenues of Klein link research.

2 Introduction to Braids

A braid is intuitively described as a collection of interwoven strings. We give here only a brief introduction to braids; a more thorough development of braids can be found in [1]. Given a braid, we obtain a link by closing the braid as shown in Figure 9. The **closed braid** is formed by attaching the bottom strands to the corresponding top strands. Conversely, according to a classical result called Alexander's Theorem, every link can be represented as a braid [1]. The significance of this correspondence is that questions of knot theory can be related to braids.

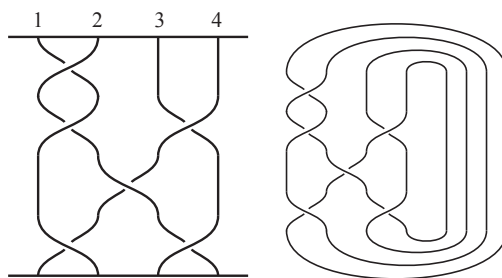


Figure 9: A mathematical braid and its closed braid representation.

Suppose we have a braid with n strings, numbered $1, 2, \dots, n$ from left to right. Then we can describe the braid algebraically using a finite set of generators. A **braid generator** visually corresponds to a crossing in a braid. In general, σ_i denotes the i th string of a braid crossing over the $(i+1)$ st string and σ_i^{-1} represents the i th string crossing under the $(i+1)$ st string, as shown in Figure 10.

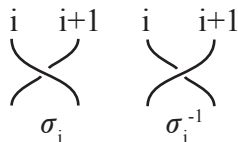


Figure 10: Braid generators.

The composition of braid generators forms a **braid word**, which serves as a name for the particular braid. The braid word, when read from left to right, represents the corresponding crossings in a braid progressing from top to bottom. For example, the braid in Figure 9 has braid word $\sigma_1^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_1\sigma_3^{-1}$.

More importantly, braid generators generate a group of n -string braids under the composition operation with identity 1 [1, 4]. Including the standard group axioms, there are three types of geometric manipulations that we can apply to alter the appearance of a braid

without changing the braid type, depicted in Figure 11. Each of these moves naturally correspond to algebraic manipulations of the braid word:

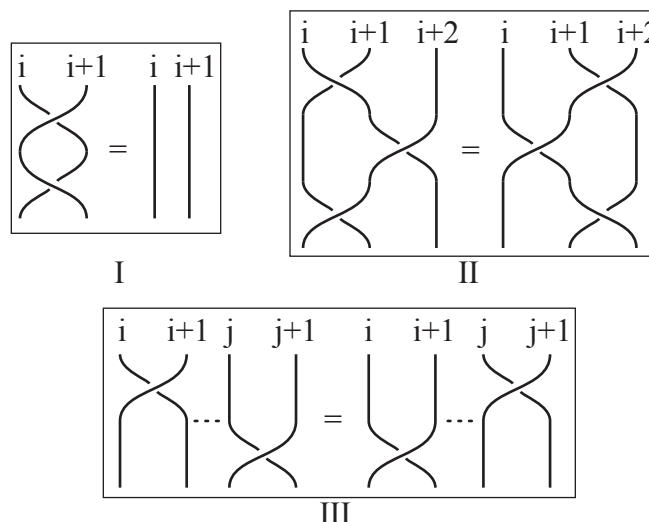


Figure 11: Three geometric braid operations that preserve braid type.

Move 1 For all i , $\sigma_i \sigma_i^{-1} = 1 = \sigma_i^{-1} \sigma_i$.

Move 2 For all $i < n$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Move 3 When $|i - j| > 1$, $\sigma_i \sigma_j = \sigma_j \sigma_i$.

Each of these braid moves preserve the braid type as well as the link type of the closed braid [1]. However, there are two additional operations which preserve only the link type of the closed braid and are depicted in Figure 12:

Conjugation For any braid on n strings with braid word B , $B = \sigma_i B \sigma_i^{-1} = \sigma_i^{-1} B \sigma_i$ for $1 \leq i \leq n$.

Stabilization For an n -string braid word B , $B = B \sigma_n = B \sigma_n^{-1}$.

Note that we have used equality to relate the braid words before and after each operation. For the three braid moves, this is technically a braid equivalence statement, but it is a statement regarding closed braid equivalence for the final two operations. Given that we are more interested in the closed braid form, we will use equality to relate braid words according to closed braid equivalence. The significance of these five operations is summarized in a classical result known as Markov’s Theorem: any two braids with equivalent closed braid representations are related by a finite sequence of braid moves, conjugations, and stabilizations [1].

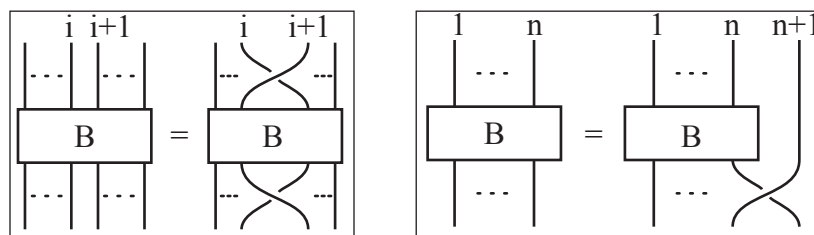


Figure 12: Conjugation (left) and stabilization (right).

The final definition we require for braids, although it truly extends to all links, is the notion of a mirror image. Given a braid word B , the mirror image of B replaces every generator with its inverse. In classical knot theory, this can create an important distinction between links: some links are equivalent to their mirror image while others are not [1].

3 Representing Klein Links as Braids

A commonly-stated connection between braids and torus knots is that the (m, n) -torus knot has a predictable braid word, $T(m, n) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^m$, which is depicted in Figure 13 [1]. We are able to establish this result relatively simply: identify the vertical edges of the rectangular representation of a torus knot to form a cylinder with strands on it. The remaining edges of the diagram are now represented as circles and, pinching them into lines, form the bars of a braid. Removing the remainder of the cylinder, the strands laying on the cylinder become strings of a braid and we obtain a braid representation of a torus knot.

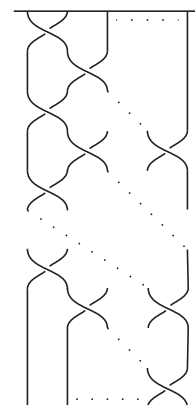


Figure 13: The general braid representation for $T(m, n)$.

Analogously, we can use the rectangular diagram for a Klein bottle to develop the braid word which corresponds to the (m, n) -Klein link. First, we consider the rectangular diagram that corresponds to the (m, n) -torus knot, which consists of m identification points on the

vertical edges of the rectangle and n on the horizontal edges. By identifying the vertical edges, we have recreated the braid word representation for the (m, n) -torus knot.

Changing the orientation of the bottom horizontal edge to create the diagram for a (m, n) -Klein link, we must twist the bottom bar of the braid 180° . In order to stay consistent with Klein link formation, the bar is twisted to the left (clockwise as we look down the cylinder). This composes the braid word for the (m, n) -torus knot with the half-twist,

$$\prod_{i=1}^{n-1} (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_i^{-1}).$$

(Note that this is the mirror image of the twist, Δ_n , given in [4].)

This composition is depicted in Figure 14. Thus a braid word for the (m, n) -Klein link is

$$K(m, n) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^m (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_1^{-1}) (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_3^{-1} \sigma_2^{-1}) \cdots \sigma_{n-1}^{-1}.$$

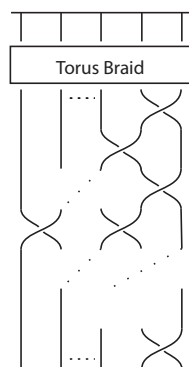


Figure 14: Twist used to present $K(m, n)$.

This formation of a Klein link allows us to see its relation to what are called twisted torus knots. Introduced in [2], the twisted torus knot has additional full twists of a subset of strings at the bottom of the n -string torus knot braid. The Klein link as a braid is a related case where at the bottom of the torus knot braid all of the strings are twisted 180° (one-half of a full twist).

4 Results

Having introduced Klein links and their braid word representation, we will demonstrate several patterns that arise due to the Klein link construction. First, we will show an immediate relationship between certain torus knots and Klein links.

Theorem 4.1. *A $(m, 2)$ -Klein link is equivalent to a $(m - 1, 2)$ -torus knot.*

Proof. By definition, $K(m, 2) = (\sigma_1)^m (\sigma_1^{-1}) = \sigma_1^{m-1} = T(m - 1, 2)$. □

To cleanly establish deeper results involving Klein links, we require additional notation.

Definition 4.1. For $1 \leq l \leq k$, let $\Gamma_{l,k} = \sigma_l \sigma_{l+1} \sigma_{l+2} \cdots \sigma_k$.

As a consequence, it naturally follows from group properties that $\Gamma_{l,k}^{-1} = \sigma_k^{-1} \sigma_{k-1}^{-1} \cdots \sigma_l^{-1}$. Using this notation, the general form of the braid word for the (m, n) -Klein link becomes

$$K(m, n) = \Gamma_{1,n-1}^m \Gamma_{1,n-1}^{-1} \Gamma_{2,n-1}^{-1} \cdots \Gamma_{n-1,n-1}^{-1}.$$

Another consequence of this notation makes use of the fact that braids have a group structure.

Lemma 4.2. For $1 \leq k \leq n$, $\Gamma_{1,n} \Gamma_{k,n}^{-1} = \Gamma_{1,k-1}$.

Proof. By definition and repeated application of braid move 1,

$$\Gamma_{1,n} \Gamma_{k,n}^{-1} = (\sigma_1 \sigma_2 \cdots \sigma_n) (\sigma_n^{-1} \sigma_{n-1}^{-1} \cdots \sigma_k^{-1}) = \sigma_1 \sigma_2 \cdots \sigma_{k-1} = \Gamma_{1,k-1}.$$

□

In the case $k = 1$, the previous Lemma would produce the empty braid word. With this property, we can demonstrate a recursive relation for the $(1, n)$ -Klein links and establish the equivalence of certain Klein links. We begin by showing that the $(1, n)$ -Klein link is related to the $(0, n - 1)$ -Klein link.

Theorem 4.3. If $n > 1$, then the $(1, n)$ -Klein link is the disjoint union of the unknot and the $(0, n - 1)$ -Klein link.

Proof. As shown in Figure 15, we can apply Lemma 4.2 to obtain

$$K(1, n) = \Gamma_{1,n-1} \Gamma_{1,n-1}^{-1} \Gamma_{2,n-1}^{-1} \cdots \Gamma_{n-1,n-1}^{-1} = \Gamma_{2,n-1}^{-1} \cdots \Gamma_{n-1,n-1}^{-1}.$$

When the braid is closed, the first string forms the unknot and so it can be removed without affecting the remaining braid. Relabeling the generators, we obtain

$$\Gamma_{1,n-2}^{-1} \cdots \Gamma_{n-2,n-2}^{-1} = K(0, n - 1).$$

Therefore, the $(1, n)$ -Klein link is the disjoint union of the unknot and the $(0, n - 1)$ -Klein link. □

Our next major result simplifies $K(n, n)$ significantly by repeatedly applying the following lemma.

Lemma 4.4. Let B be the braid word for an n -string braid which consists of braid generators using strings numbered $k + 1$ or higher, where k is fixed with $2 \leq k \leq n - 1$. Then

$$\Gamma_{1,k-1} B = B \Gamma_{1,k-1}.$$

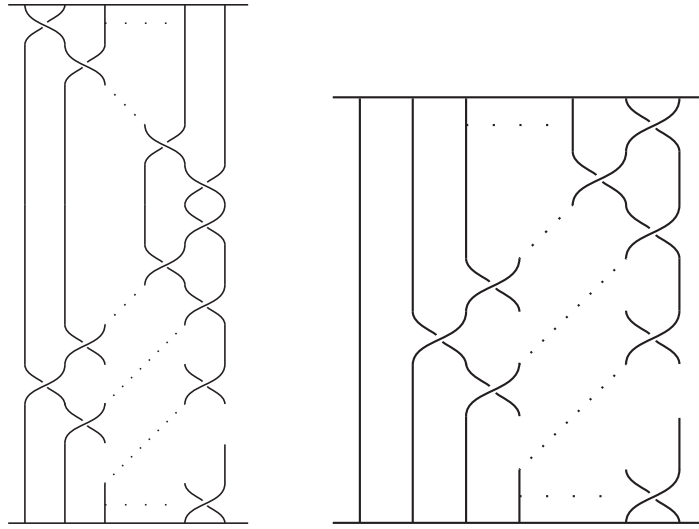


Figure 15: $K(1, n)$ represented in general braid form (left) and as a simplified braid (right).

Proof. Since every braid generator in B uses strings numbered $k + 1$ or greater and $|(k + 1) - i| > 1$ for $i = 1, 2, \dots, k - 1$, we may interchange σ_i with any braid generator in B by braid move 2. Then, in the first step, we can move σ_{k-1} past B to obtain

$$\Gamma_{1,k-1}B = \Gamma_{1,k-2}\sigma_{k-1}B = \Gamma_{1,k-2}B\sigma_{k-1}.$$

Similarly, in the j th step for $j \geq 2$, we can move another generator from $\Gamma_{1,k-j}$ past B to obtain

$$\begin{aligned} \Gamma_{1,k-j}B\Gamma_{k-j+1,k-1} &= \Gamma_{1,k-j-1}\sigma_{k-j}B\Gamma_{k-j+1,k-1} = \Gamma_{1,k-j-1}B\sigma_{k-j}\Gamma_{k-j+1,k-1} \\ &= \Gamma_{1,k-j-1}B\Gamma_{k-j,k-1}. \end{aligned}$$

After the $(k - 1)$ st iteration, we have $\Gamma_{1,k-1}B = B\Gamma_{1,k-1}$. □

While Lemma 4.4 is a specific case of the commutativity of certain composed braid words, generalizing braid move 3, its particular statement is useful in our next theorem.

Theorem 4.5. *For all $n \geq 0$,*

$$K(n, n) = \Gamma_{1,n-1}\Gamma_{1,n-2}\cdots\Gamma_{1,1} = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})(\sigma_1\sigma_2\cdots\sigma_{n-3})\cdots(\sigma_1).$$

Proof. Let us consider the (n, n) -Klein link. By Lemma 4.2,

$$K(n, n) = \Gamma_{1,n-1}^n\Gamma_{1,n-1}^{-1}\Gamma_{2,n-1}^{-1}\cdots\Gamma_{n-1,n-1}^{-1} = \Gamma_{1,n-1}^{n-1}\Gamma_{2,n-1}^{-1}\cdots\Gamma_{n-1,n-1}^{-1}.$$

Applying Lemma 4.2 again, $\Gamma_{1,n-1}\Gamma_{2,n-1}^{-1} = \Gamma_{1,1}$. Take $B_2 = \Gamma_{3,n-1}^{-1}\cdots\Gamma_{n-1,n-1}^{-1}$. Then by Lemma 4.4, $\Gamma_{1,1}B_2 = B_2\Gamma_{1,1}$ and so

$$K(n, n) = \Gamma_{1,n-1}^{n-2}B_2\Gamma_{1,1}.$$

In general, set $B_k = \Gamma_{k+1,n-1}^{-1} \cdots \Gamma_{n-1,n-1}^{-1}$. Then by Lemmas 4.2 and 4.4,

$$\Gamma_{1,n-1} \Gamma_{k,n-1}^{-1} B_k = \Gamma_{1,k-1} B_k = B_k \Gamma_{1,k-1}.$$

Hence in the k th step, the (n, n) -Klein link has a braid representation of the form

$$\Gamma_{1,n-1}^{n-k} B_k \Gamma_{1,k-1} \Gamma_{1,k-2} \cdots \Gamma_{1,1}.$$

So in the $(n - 1)$ st step, $B_{n-1} = 1$ and so $K(n, n) = \Gamma_{1,n-1} \Gamma_{1,n-2} \cdots \Gamma_{1,1}$ as desired. \square

Our final two results make use of Theorem 4.5 to demonstrate the equivalence, or near equivalence, of certain Klein links. We go on to relate the $(0, n)$ - and the (n, n) -Klein links.

Theorem 4.6. *The $(0, n)$ -Klein link and the (n, n) -Klein link are mirror images of the same link.*

Proof. Applying Theorem 4.5, $K(n, n) = \Gamma_{1,n-1} \Gamma_{1,n-2} \cdots \Gamma_{1,1}$. If we consider the current projection of the braid and one obtained by looking at the braid from the back, we are looking at the same braid with relatable generators (see Figure 16).

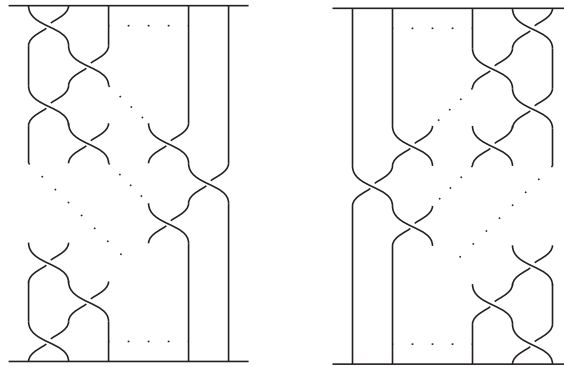


Figure 16: Two braid representations of $K(n, n)$.

When the braid is closed, this alternate projection does not change the resulting link. The braid word associated with this alternate projection is

$$K(n, n) = (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1) (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_2) (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_3) \cdots \sigma_{n-1}.$$

Now consider the mirror image of this braid word:

$$(\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_1^{-1}) (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_2^{-1}) (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_3^{-1}) \cdots \sigma_{n-1}^{-1} = K(0, n).$$

Therefore, the $(0, n)$ -Klein link and the (n, n) -Klein link are mirror images of the same link. \square

Finally, we conclude by showing that the (n, n) - and $(n, n - 1)$ -Klein links are equivalent.

Theorem 4.7. *The (n, n) -Klein link and the $(n, n - 1)$ -Klein link are equivalent links.*

Proof. Successively applying Theorem 4.5 and conjugation, a braid representation of the (n, n) -Klein link is

$$\begin{aligned} K(n, n) &= \Gamma_{1,n-1}\Gamma_{1,n-2}\cdots\Gamma_{1,1} = \Gamma_{1,n-2}^{-1}(\Gamma_{1,n-1}\Gamma_{1,n-2}\cdots\Gamma_{1,1})\Gamma_{1,n-2} \\ &= (\Gamma_{1,n-2}^{-1}\Gamma_{1,n-1})(\Gamma_{1,n-2}\cdots\Gamma_{1,1})\Gamma_{1,n-2}. \end{aligned}$$

Now we apply Lemma 4.2 and stabilization to the resulting braid word to obtain

$$\begin{aligned} K(n, n) &= (\Gamma_{1,n-2}^{-1}\Gamma_{1,n-1})(\Gamma_{1,n-2}\cdots\Gamma_{1,1})\Gamma_{1,n-2} = \sigma_{n-1}(\Gamma_{1,n-2}\cdots\Gamma_{1,1})\Gamma_{1,n-2} \\ &= (\Gamma_{1,n-2}\cdots\Gamma_{1,1})\Gamma_{1,n-2}. \end{aligned}$$

Finally, we use conjugation again to obtain $K(n, n) = \Gamma_{1,n-2}(\Gamma_{1,n-2}\cdots\Gamma_{1,1})$.

For the $(n, n - 1)$ -Klein link, we use group properties and Theorem 4.5 to get

$$\begin{aligned} K(n, n - 1) &= \Gamma_{1,n-2}^n\Gamma_{1,n-1}^{-1}\Gamma_{2,n-2}^{-1}\cdots\Gamma_{n-1,n-1}^{-1} \\ &= \Gamma_{1,n-2}(\Gamma_{1,n-2}^{n-1}\Gamma_{1,n-1}^{-1}\Gamma_{2,n-2}^{-1}\cdots\Gamma_{n-1,n-1}^{-1}) \\ &= \Gamma_{1,n-2}K(n - 1, n - 1) \\ &= \Gamma_{1,n-2}(\Gamma_{1,n-2}\Gamma_{1,n-3}\cdots\Gamma_{1,1}). \end{aligned}$$

Thus $K(n, n) = K(n, n - 1)$. □

5 Conclusions

Our recent results have established a framework for viewing Klein links that uses the plane instead of 3-dimensional space. We have shown that, similar to (m, n) -torus knots, (m, n) -Klein links have a predictable braid word and we have used this to show the equivalency of a small class of Klein links. Based on established similarities to torus knots, we expect that other link invariants should be formulaic for Klein links as well. This includes crossing number, linking number and braid index.

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