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Cold positions of the restricted Wythoff's game

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Abstract. Wythoff's game is a kind of 2-pile Nim game, which admits taking the same number of stones from both piles. It differs only a little from the 2-pile Nim game, but their winning strategies are quite different from each other. Amazingly the winning strategy of Wythoff's game is directly related to a real number, specifically the golden ratio. In this paper we add two restrictions to this game, and investigate the winning strategy of the revised game.

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1 Introduction

Nim games are mathematical two player games in which two players take at least one stone alternately. The most simple Nim game is the 1-pile Nim game, in which two players take stones alternately from a pile. The next simple example of Nim games is the 2-pile Nim game, in which two players can take stones from one of two piles but can not take stones from both piles each round. For a natural number $n \geqq 3$, the n-pile Nim game can be defined similarly. Namely, it is a Nim game in which two players can take stones only from one of n piles each round.

It is said that the origin of Nim games is in ancient China, and there are various kinds of Nim games now (cf. [2], [3]). Among them Wythoff's game [1] which is a kind of the 2-pile Nim game that is especially interesting; its winning strategy is known very well and has a beautiful structure (cf. [1]). In Wythoff's game, two players can take stones not only from one of two piles, but also from both piles, provided that the numbers of stones taken from each pile must be equal. When one of two piles has no stones in it, Wythoff's game is equivalent to the 1-pile Nim game above. The aim of this paper is to investigate the winning strategy of a certain 2-pile Nim game which is similar to Wythoff's game.

Before we discuss about winning strategies of Nim games, we have to define the victory or defeat. A Nim game is called *of regular type* when a player who takes the last stone wins. Otherwise it is called *of misère type*. In this paper we investigate winning strategies of Nim games which are of regular type. From now on we denote $(m_1, m_2, \cdots, m_n)$ for n piles of stones which consist of $m_1, m_2, \cdots, m_n$ stones and call it *a position*. Since we do not consider the order of n piles, we identify $(m_1, m_2, \cdots, m_n)$ $(m_1 \geqq m_2 \geqq \cdots \geqq m_n)$ with all positions which are realized from it by permutations. For simplicity we define two terms below.

Definition 1.1. When a player can definitely win by taking stones suitably, we call such a state *a hot position*. Otherwise we call such a state *a cold position*.

In short, the winning strategy is nothing but the method of confronting the opponent with cold positions. Therefore our aim is equivalent to determining cold positions completely.

We can find cold positions for Wythoff's game easily as below. First, $(0, 0)$ is a cold position by the rule of this game. Therefore we see that $(n, 0), (0, n), (n, n)$ are hot positions for every $n \in \mathbb{N}$. Thus $(1, 2), (2, 1)$ must be cold positions and so $(1+n, 2), (1, 2+n), (1+n, 2+n), (2+n, 1), (2, 1+n), (2+n, 1+n)$ are hot positions for every $n \in \mathbb{N}$. Therefore $(3, 5), (5, 3)$ must be cold positions. In a similar way, we see that $(4, 7), (7, 4), (6, 10), (10, 6), (8, 13), (13, 8)$, and so on, are cold positions. In Figure 1 we represent a part of these cold positions by dots.

In 1907, Wythoff [1] determined all cold positions of this game completely in his paper. Amazingly, we see that the winning strategy of it is directly related to the golden ratio as below.

[1]The name of this game comes from a Dutch mathematician Willem Abraham Wythoff. Wythoff's game was played in ancient China, and it was called jiǎn shizǐ.

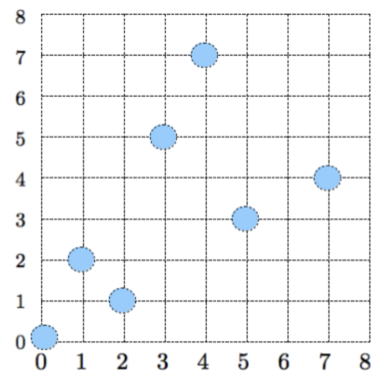

Figure 1: cold positions of Wythoff's game

Theorem 1.1 (cold positions of Wythoff's game)**.** *The set of cold positions of Wythoff's game $CP_{Wythoff}$ is*

$$CP_{Wythoff} = \Big\{(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \mid i = 0, 1, 2, \cdots \Big\}$$

where $\phi = (1+\sqrt{5})/2 = 1.61803398875\cdots$.

For $n \in \mathbb{N}$, we add the following restrictions to Wythoff's game, and investigate how cold positions appear.

(1) The number of stones one can take from a pile is fewer than n.

(2) The numbers of stones one can take from each pile are fewer than n.

We call Wythoff's game with these restrictions *n_Wythoff's game*. The following is our main result.

Theorem 1.2 (cold positions of n_Wythoff's game)**.** *The set of cold positions of n_Wythoff's game $CP_{n\_Wythoff}$ is*

$$CP_{n\_Wythoff} = \Big\{a_1(n+1, 0) + a_2(0, n+1) + a_3(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \\ \mid a_1, a_2 \in \mathbb{Z}_{\geqq 0}, a_3 \in \{0, 1\}, i = 1, 2, \cdots, m\Big\}$$

where $m = max\Big\{i \in \mathbb{Z}_{\geqq 0} \mid \lfloor \phi i \rfloor + i < n+1\Big\}$.

Here is an outline of the contents of the individual sections. In Section 2 we introduce previously-known results with proofs and provide the necessary background. In Section 3 we prove our main result. In Section 4 we state the difference between $CP_{Wythoff}$ and $CP_{n\_Wythoff}$, and conclude our paper.

2 Background

In this section we introduce some winning strategies for various kinds of Nim games. And then we introduce the winning strategy of Wythoff's game.

In general a Nim game has two patterns, one of which admits taking an arbitrary number of stones and the other does not. We first treat the former case, and then the latter.

The 1-pile Nim game is a game which the first player definitely wins, so it is a trivial game.

Lemma 2.1 (the cold position of the 1-pile Nim game)**.** (0) *is the only cold position of the 1-pile Nim game.*

Proof. Needless to say, (0) is a cold position. For any $n \in \mathbb{N}$, the first player can take n stones, so (n) is a hot position. □

The 2-pile Nim game is also a trivial game.

Lemma 2.2 (cold positions of the 2-pile Nim game)**.** *The set of cold positions of the* 2*-pile Nim game is*

$$\left\{(m, m) \in \mathbb{Z}^2_{\geqq 0} \mid m = 0, 1, 2, \cdots\right\}.$$

Proof. By the rule of the 2-pile Nim game, we see that $(0, 0)$ is a cold position easily. For $m \in \mathbb{N}$, the first player cannot help making (m, m) into (m, n) or (n, m), where n is a natural number such that $n < m$. The second player can make (m, n), (n, m) into (n, n). By repeating this process, the second player can definitely win. Thus we see that (m, m) is a cold position. For (m, n) such that $m \neq n$, we may assume that $m > n$ since we do not consider the order of two piles. In this case, the first player can make (m, n) into a cold position (n, n), therefore we see that (m, n) is a hot position. Thus the claim is proved. □

Compared to the 1-pile and the 2-pile Nim games, the 3-pile Nim game is rather complicated. Before we state the winning strategy of it, we look at two examples.

Example 2.1. $(3, 2, 1)$ is a cold position. In fact possible positions which one can realize from $(3, 2, 1)$ are $(2, 2, 1), (3, 2, 0), (0, 2, 1), (3, 1, 1), (3, 0, 1), (1, 2, 1)$. Among them, one can make $(2, 2, 1), (3, 2, 0)$ into $(2, 2, 0)$, and this is a cold position as we have seen in Theorem 2.2. Similarly, one can make $(0, 2, 1), (3, 1, 1)$ into a cold position $(0, 1, 1)$, and make $(3, 0, 1), (1, 2, 1)$ into a cold position $(1, 0, 1)$. Thus all of the positions $(2, 2, 1), (3, 2, 0), (0, 2, 1), (3, 1, 1), (3, 0, 1), (1, 2, 1)$ are hot positions, and we see $(3, 2, 1)$ is a cold position.

Example 2.2. $(3, 3, 1)$ is a hot position. In fact, one can make $(3, 3, 1)$ into $(3, 2, 1)$, and this is a cold position as we have seen in the above.

In order to state the winning strategy of the 3-pile Nim game, it is convenient to use the binary digital sum $\oplus$. Here the binary digital sum $\oplus$ is a operation for natural numbers which make $a_1, a_2, \cdots \in \mathbb{N}$ into binary numbers, sum each place of them modulo 2, and finally make it into a decimal number. For example $5 = 101_{(2)}, 7 = 111_{(2)}$, and by summing each place of them modulo 2 we obtain $010_{(2)}$. $010_{(2)}$ equals to 2 in the decimal system and hence $5 \oplus 7 = 2$. Let us look at one more example. Since $4 = 100_{(2)}, 6 = 110_{(2)}, 9 = 1001_{(2)}$, by summing each place of them modulo 2 we obtain $1011_{(2)}$. Thus $4 \oplus 6 \oplus 9 = 11$.

By using the binary digital sum $\oplus$, we can state the winning strategy of the 3-pile Nim game as below.

Theorem 2.1 (cold positions of the 3-pile Nim game)**.** *The set of cold positions of the* 3*-pile Nim game is*

$$\left\{(l, m, n) \in \mathbb{Z}^3_{\geqq 0} \mid l \oplus m \oplus n = 0\right\}.$$

We will give a proof of this theorem in a more general situation later. Here we confirm Example 2.1 and 2.2 by using this theorem.

Example 2.3. In the binary system $3, 2$ and 1 are represented as

$$3 = 11_{(2)}, 2 = 10_{(2)}, 1 = 01_{(2)}.$$

Therefore,

$$3 \oplus 2 \oplus 1 = 00_{(2)} = 0.$$

Thus we see that $(3, 2, 1)$ is certainly a cold position.

On the other hand,

$$3 \oplus 3 \oplus 1 = 01_{(2)} = 1 \neq 0,$$

thus we see that $(3, 3, 1)$ is a hot position.

Generalizing the number of piles, we state the winning strategy of the n-pile Nim game next.

Theorem 2.2 (cold positions of the n-pile Nim game)**.** *The set of cold positions of the n-pile Nim game is*

$$\left\{(m_1, m_2, \cdots, m_n) \in \mathbb{Z}^n_{\geqq 0} \mid m_1 \oplus m_2 \oplus \cdots \oplus m_n = 0\right\}.$$

Proof. Needless to say, $(0, 0, \cdots, 0)$ is a cold position and $0 \oplus 0 \oplus \cdots \oplus 0 = 0$. Assuming that $m_1 \oplus m_2 \oplus \cdots \oplus m_n = 0$, we have

$$m_1 \oplus \cdots \oplus m_{i-1} \oplus m_{i+1} \oplus \cdots \oplus m_n = m_i.$$

Thus if $m_i \neq 0$ then for any $m_i' < m_i$

$$m_1 \oplus \cdots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \cdots \oplus m_n = m_i \oplus m_i' \neq 0.$$

On the other hand, if $m_1 \oplus m_2 \oplus \cdots \oplus m_n = s \neq 0$ then there exists i such that $s \oplus m_i = m_i^{'} < m_i$. For this $m_i^{'}$ we have

$$\begin{aligned} m_1 \oplus & \cdots \oplus m_{i-1} \oplus m_i^{'} \oplus m_{i+1} \oplus \cdots \oplus m_n \\ &= m_i^{'} \oplus (m_1 \oplus \cdots \oplus m_{i-1} \oplus m_{i+1} \oplus \cdots \oplus m_n) \\ &= m_i^{'} \oplus (s \oplus m_i) = m_i^{'} \oplus m_i^{'} = 0. \end{aligned}$$

Therefore, if $(m_1, m_2, \cdots, m_n)$ such that $m_1 \oplus m_2 \oplus \cdots \oplus m_n = 0$ is the first position, the second player can definitely win by repeating the above process, and thus we see that such a position is a cold position. On the other hand, if $(m_1, m_2, \cdots, m_i, \cdots, m_n)$ such that $m_1 \oplus m_2 \oplus \cdots \oplus m_i \oplus \cdots \oplus m_n \neq 0$ $(m_i \neq 0)$ is the first position, the first player can make it into $(m_1, m_2, \cdots, m_i^{'}, \cdots, m_n)$ such that $m_1 \oplus m_2 \oplus \cdots \oplus m_i^{'} \oplus \cdots \oplus m_n = 0$ $(m_i^{'} < m_i)$ as above. Thus by repeating the above process, the first player can definitely win, and we see that such a position is a hot position. Thus the claim is proved. □

So far we have stated winning strategies for Nim games which admit taking an arbitrary number of stones. The following is the winning strategy for a Nim game which restricts the number of stones one can take each round. From now on we call n-pile Nim game in which one can take fewer than q stones *q_n-pile Nim game*. For q_n-pile Nim game, we obtain the following result by using Theorem 2.2.

Theorem 2.3 (cold positions of the q_n-pile Nim game)**.** *The set of cold positions of the q_n-pile Nim game is*

$$\left\{(m_1, m_2, \cdots, m_n) \in \mathbb{Z}^n_{\geqq 0} \mid \overline{m}_1 \oplus \overline{m}_2 \oplus \cdots \oplus \overline{m}_n = 0\right\}$$

where $\overline{m}_i$ means the remainder which we obtain by dividing m_i by $q+1$.

Proof. When the first player takes $i (1 \leqq i \leqq q)$ stones from a pile, the next player can take $q+1-i (1 \leqq q+1-i \leqq q)$ stones from the same pile. Thus integral multiples of $q+1$ do not concern the winning strategy at all. Therefore, in the q_n-pile Nim game, $(m_1, m_2, \cdots, m_n)$ is a cold position if and only if $(\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n)$ is a cold position. Furthermore, since $\overline{m}_i \leqq q$ for any i, $(\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_n)$ is a cold position in the q_n-pile Nim game if and only if it is a cold position in the n-pile Nim game. Thus the claim follows from Theorem 2.2. □

Winning strategies of various 1-pile Nim games are well known. Here we introduce some of them. Let $p_1, p_2, \cdots$ denote numbers of stones one can take at a time in a 1-pile Nim game. The next theorem is fundamental.

Theorem 2.4. *Assume that there exists $q \in \mathbb{N}$ such that $p_1 = 1, p_2 = 2, \cdots, p_q = q$ and all of $p_{q+1}, p_{q+2}, \cdots$ are not divisible by $q+1$. Then the set of cold positions of this 1-pile Nim game equals to that of the q_1-pile Nim game.*

Proof. Since one can take $1, 2, \cdots, q$ stones but can not take integral multiples of $q+1$ stones, this game is substantially equivalent to the q_1-pile Nim game. Thus cold positions of them naturally coincide. □

Example 2.4. Suppose that $p_i = n^{i-1}(i = 1, 2, \cdots)$. When $n = 2$, $p_1 = 1, p_2 = 2$ and all of $p_3, p_4, \cdots$ are not divisible by 3, so the set of cold positions of this 1-pile Nim game equals to that of the 2_1-pile Nim game. When n is a odd number greater than 3, $p_1 = 1$ and all of $p_2, p_3 \cdots$ are not divisible by 2, so the set of cold positions of this 1-pile Nim game equals to that of the 1_1-pile Nim game.

Example 2.5. When p_i are 1 and prime numbers, namely $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5, \cdots$, the set of cold positions of this 1-pile Nim game equals to that of the 3_1-pile Nim game.

At last we introduce the winning strategy of Wythoff's game. As we stated in a previous section, it is a kind of 2-pile Nim game which admits taking the same number of stones from both piles. See [1] for a detailed proof.

Theorem 2.5 (cold positions of Wythoff's game)**.** *The set of cold positions of Wythoff's game $CP_{Wythoff}$ is*

$$CP_{Wythoff} = \Big\{(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \mid i = 0, 1, 2, \cdots \Big\}$$

where $\phi = (1 + \sqrt{5})/2 = 1.61803398875 \cdots$.

3 The restricted Wythoff's game

In this section we state our main result. For $n \in \mathbb{N}$, we add the following restrictions to Wythoff's game, and investigate its cold positions.

(1) The number of stones one can take from a pile is fewer than n.

(2) The numbers of stones one can take from each pile are fewer than n.

We call Wythoff's game with these restrictions n_Wythoff's game. Although we only changed the number of stones which we can take each round, we shall see that the set of its cold positions is quite different from that of Wythoff's game. Let $CP_{n\_Wythoff}$ denotes the set of cold positions of n_Wythoff's game.

For the simplicity of notations we define three subsets of $\mathbb{Z}^2_{\geqq 0}$ below:

$$A^{(n)} = \Big\{a_1(n+1, 0) + a_2(0, n+1) + a_3(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \\ \mid a_1, a_2 \in \mathbb{Z}_{\geqq 0}, a_3 \in \{0, 1\}, i = 1, 2, \cdots, m\Big\}$$

where $m = max\Big\{i \in \mathbb{Z}_{\geqq 0} \mid \lfloor \phi i \rfloor + i < n + 1\Big\}$, and for $k, l \in \mathbb{N}$

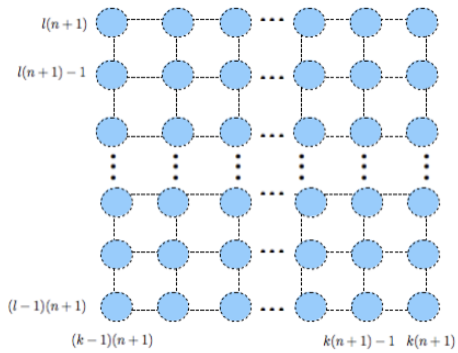

Figure 2: $F^{(n)}_{k,l}$

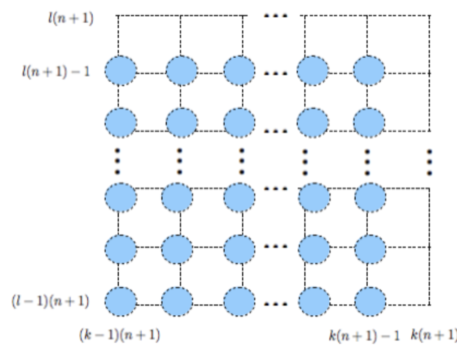

Figure 3: $I^{(n)}_{k,l}$

$$F^{(n)}_{k,l} = \Big\{(a,b) \in \mathbb{Z}^2_{\geqq 0} \mid (k-1)(n+1) \leqq a \leqq k(n+1), (l-1)(n+1) \leqq b \leqq l(n+1)\Big\},$$
$$I^{(n)}_{k,l} = \Big\{(a,b) \in \mathbb{Z}^2_{\geqq 0} \mid (k-1)(n+1) \leqq a < k(n+1), (l-1)(n+1) \leqq b < l(n+1)\Big\}.$$

$F^{(n)}_{k,l}$ and $I^{(n)}_{k,l}$ represent sets of lattice points as above. $I^{(n)}_{k,l}$ is obtained by removing points in the upper end and the right end of $F^{(n)}_{k,l}$. Using $F^{(n)}_{k,l}$, we divide $\mathbb{Z}^2_{\geqq 0}$ as below, and we shall prove our main result by induction. $I^{(n)}_{k,l}$ will simplify our discussions.

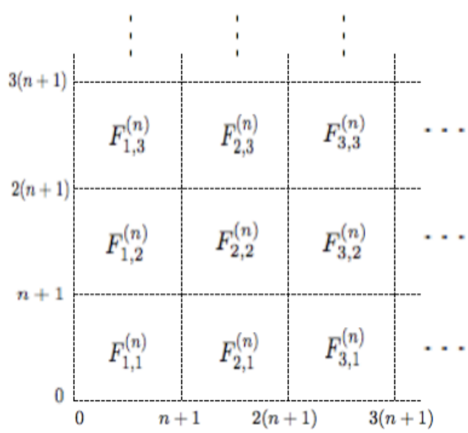

Figure 4: $\mathbb{Z}^2_{\geqq 0} = \bigcup_{k,l=1}^{\infty} F^{(n)}_{k,l}$

First, we investigate cold positions of n_Wythoff's game in $F^{(n)}_{1,1}$.

Proposition 3.1. $CP_{n\_Wythoff} \cap F^{(n)}_{1,1} = A^{(n)} \cap F^{(n)}_{1,1}$.

Proof. Since one can take fewer than n stones from a pile $(0,0),(n+1,0),(0,n+1) \in CP_{n\_Wythoff} \cap F_{1,1}^{(n)}$. And for $i \in \mathbb{N}$ such that $1 \leqq i \leqq n$, one can make $(n+1-i,n+1-i),(n+1,n+1-i),(n+1-i,n+1)$ into $(0,0),(n+1,0),(0,n+1)$ respectively, so $(n+1,n+1) \in CP_{n\_Wythoff} \cap F_{1,1}^{(n)}$. The rest of the cold positions can exist in $CP_{n\_Wythoff} \cap I_{1,1}^{(n)}$, but by the definition of $I_{1,1}^{(n)}$, $CP_{n\_Wythoff} \cap I_{1,1}^{(n)} = CP_{Wythoff} \cap I_{1,1}^{(n)}$. Namely,

$$CP_{n\_Wythoff} \cap I_{1,1}^{(n)} = \Big\{(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \mid i = 0,1,2,\cdots,m\Big\}.$$

Therefore,

$$\begin{aligned}
CP_{n_Wythoff} \cap F_{1,1}^{(n)} &= \Big\{(0,0),(n+1,0),(0,n+1),(n+1,n+1)\Big\} \\
&\qquad \cup \Big\{(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \mid i = 0,1,2,\cdots,m\Big\} \\
&= \Big\{a_1(n+1,0) + a_2(0,n+1) + a_3(\lfloor \phi i \rfloor + i, \lfloor \phi i \rfloor) \in \mathbb{Z}^2_{\geqq 0} \\
&\qquad \mid a_1,a_2,a_3 \in \{0,1\}, i = 1,2,\cdots,m\Big\} \\
&= A^{(n)} \cap F_{1,1}^{(n)}.
\end{aligned}$$

□

As an example we draw a figure of $CP_{5\_Wythoff} \cap F_{1,1}^{(5)}$ below. In this figure, dots represent elements of $CP_{5\_Wythoff} \cap F_{1,1}^{(5)}$.

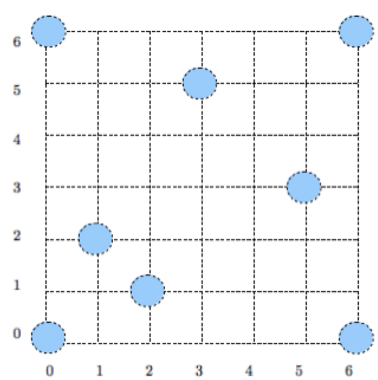

Figure 5: $CP_{5\_Wythoff} \cap F_{1,1}^{(5)}$

Before we state the theorem, we present a lemma as a preparation.

Lemma 3.1. *For distinct* $(a, b), (a^{'}, b^{'}) \in A^{(n)} \cap I_{k,l}^{(n)}$*, we have*

$$b^{'} - b \not\equiv a^{'} - a \pmod{n+1}.$$

Proof. By the definition of $A^{(n)}$ we may assume that $k, l = 1$. Since $(\lfloor\phi \cdot 0\rfloor + 0, \lfloor\phi \cdot 0\rfloor) = (0, 0)$,

$$A^{(n)} \cap I_{1,1}^{(n)} = \Big\{(\lfloor\phi i\rfloor + i, \lfloor\phi i\rfloor) \in \mathbb{Z}_{\geqq 0}^2 \mid i = 0, 1, \cdots, m\Big\}.$$

For distinct $(\lfloor\phi i\rfloor + i, \lfloor\phi i\rfloor), (\lfloor\phi j\rfloor + j, \lfloor\phi j\rfloor) \in A^{(n)} \cap I_{1,1}^{(n)}$, we assume that

$$\lfloor\phi j\rfloor - \lfloor\phi i\rfloor \equiv (\lfloor\phi j\rfloor + j) - (\lfloor\phi i\rfloor + i) \pmod{n+1}.$$

Then we have

$$i - j \equiv 0 \pmod{n+1}.$$

Since $0 \leqq i, j \leqq m < n + 1$ we must have $i = j$, and this is a contradiction. □

The following is our main result.

Theorem 3.1 (cold positions of n_Wythoff' s game)**.** *The set of cold positions of* n*_Wythoff's game* $CP_{n\_Wythoff}$ *is*

$$CP_{n\_Wythoff} = \Big\{a_1(n+1, 0) + a_2(0, n+1) + a_3(\lfloor\phi i\rfloor + i, \lfloor\phi i\rfloor) \in \mathbb{Z}_{\geqq 0}^2 \mid a_1, a_2 \in \mathbb{Z}_{\geqq 0}, a_3 \in \{0, 1\}, i = 1, 2, \cdots, m\Big\}$$

where $m = max\Big\{i \in \mathbb{Z}_{\geqq 0} \mid \lfloor\phi i\rfloor + i < n + 1\Big\}$.

Proof. Our goal is to prove that $CP_{n\_Wythoff} = A^{(n)}$. First, we have

$$CP_{n\_Wythoff} \cap F_{1,1}^{(n)} = A^{(n)} \cap F_{1,1}^{(n)}$$

by Proposition 3.1. We assume that

$$CP_{n\_Wythoff} \cap F_{k,1}^{(n)} = A^{(n)} \cap F_{k,1}^{(n)}$$

for $k \in \mathbb{N}$ and show

$$CP_{n\_Wythoff} \cap F_{k+1,1}^{(n)} = A^{(n)} \cap F_{k+1,1}^{(n)}.$$

For any $(a, b) \in A^{(n)} \cap F_{k+1,1}^{(n)}$, $(a - i, b)$ is a hot position for any $i = 1, 2, \cdots, n$ by the above assumption. $(a, b - i)$ is also a hot postion for any $i = 1, 2, \cdots, b$ since if $(a, b - i)$ is a cold position then $(a - (n + 1), b - i)$ must be a cold position but this contradicts the above assumption. We see that $(a - i, b - i)$ is also a hot postion for any $i = 1, 2, \cdots, b$ by Lemma 3.1 as below. Assume that $(a - i, b - i)$ is a cold position. If $(a - i, b - i) \in F_{k,1}^{(n)}$ then by the above assumption $(a - i, b - i) \in A^{(n)} \cap F_{k,1}^{(n)}$. For $(a - (n + 1), b) \in A^{(n)} \cap F_{k,1}^{(n)}$, we have

$$\{b-(b-i)\}-\{(a-(n+1))-(a-i)\}=n+1,$$

and this contradicts Lemma 3.1. If $(a-i,b-i)\in F^{(n)}_{k+1,1}$ then by the above assumption $(a-i-(n+1),b-i)\in A^{(n)}\cap F^{(n)}_{k,1}$. For $(a-(n+1),b)\in A^{(n)}\cap F^{(n)}_{k,1}$, we have

$$\{b-(b-i)\}-\{(a-(n+1))-(a-i-(n+1))\}=0,$$

and this also contradicts Lemma 3.1. Thus $(a-i,b-i)$ is also a hot position for any $i=1,2,\cdots,b$. Therefore $(a,b)\in A^{(n)}\cap F^{(n)}_{k+1,1}$ is a cold position, and we have

$$CP_{n\_Wythoff}\cap F^{(n)}_{k+1,1}\supset A^{(n)}\cap F^{(n)}_{k+1,1}.$$

If $(a,b)\notin A^{(n)}\cap F^{(n)}_{k+1,1}$ is a cold position then $(a-(n+1),b)\notin A^{(n)}\cap F^{(n)}_{k,1}$ must be a cold position but this contradicts the above assumption. Therefore

$$CP_{n\_Wythoff}\cap F^{(n)}_{k+1,1}=A^{(n)}\cap F^{(n)}_{k+1,1}.$$

Thus by k-induction, for any $k\in\mathbb{N}$ we have

$$CP_{n\_Wythoff}\cap F^{(n)}_{k,1}=A^{(n)}\cap F^{(n)}_{k,1}.$$

Next, we assume that

$$CP_{n\_Wythoff}\cap F^{(n)}_{k,l}=A^{(n)}\cap F^{(n)}_{k,l}$$

for $k,l\in\mathbb{N}$ and show

$$CP_{n\_Wythoff}\cap F^{(n)}_{k,l+1}=A^{(n)}\cap F^{(n)}_{k,l+1}.$$

When $k=1$, we can show this equality in a similar way to the above discussion. Thus we may assume $k\geqq 2$. For any $(a,b)\in A^{(n)}\cap F^{(n)}_{k,l+1}$, $(a,b-i)$ is a hot position for any $i=1,2,\cdots,n$ by the above assumption. $(a-i,b)$ is also a hot position for any $i=1,2,\cdots,n$ since if it is a cold position then $(a-i,b-(n+1))$ must be a cold position but this contradicts the above assumption. Next we show that $(a-i,b-i)$ is also a hot position for any $i=1,2,\cdots,n$. Assume that $(a-i,b-i)$ is a cold position. If $(a-i,b-i)\in F^{(n)}_{k,l+1}$ then $(a-i,b-i-(n+1))\in F^{(n)}_{k,l}$ must be a cold position. By the above assumption $(a-i,b-i-(n+1))\in A^{(n)}\cap F^{(n)}_{k,l}$. For $(a,b-(n+1))\in A^{(n)}\cap F^{(n)}_{k,l}$ we have

$$\{(b-(n+1))-(b-i-(n+1))\}-\{a-(a-i)\}=0$$

and this contradicts Lemma 3.1. If $(a-i,b-i)\in F^{(n)}_{k-1,l+1}$ then $(a-i+(n+1),b-i-(n+1))\in A^{(n)}\cap F^{(n)}_{k,l}$ by the above assumption. For $(a,b-(n+1))\in A^{(n)}\cap F^{(n)}_{k,l}$ we have

$$\{(b-(n+1))-(b-i-(n+1))\}-\{a-(a-i+(n+1))\}=2(n+1)$$

and this also contradicts Lemma 3.1. If $(a-i,b-i)\in F^{(n)}_{k-1,l}$ then $(a-i+(n+1),b-i)\in A^{(n)}\cap F^{(n)}_{k,l}$ by the above assumption. For $(a,b-(n+1))\in A^{(n)}\cap F^{(n)}_{k,l}$ we have

$$\{(b-(n+1))-(b-i)\}-\{a-(a-i+(n+1))\}=0$$

and this also contradicts Lemma 3.1. If $(a-i,b-i)\in F_{k,l}^{(n)}$ then $(a-i,b-i)\in A^{(n)}\cap F_{k,l}^{(n)}$ by the above assumption. For $(a,b-(n+1))\in A^{(n)}\cap F_{k,l}^{(n)}$ we have

$$\{(b-(n+1))-(b-i)\}-\{a-(a-i)\}=-(n+1)$$

and this also contradicts Lemma 3.1. Thus we see that $(a-i,b-i)$ is also a hot position for any $i=1,2,\cdots,n$. Thus we have

$$CP_{n\_Wythoff}\cap F_{k,l+1}^{(n)}\supset A^{(n)}\cap F_{k,l+1}^{(n)}.$$

If $(a,b)\notin A^{(n)}\cap F_{k,l+1}^{(n)}$ is a cold position then $(a,b-(n+1))\notin A^{(n)}\cap F_{k,l}^{(n)}$ must be a cold position but this contradicts the above assumption. Hence we have

$$CP_{n\_Wythoff}\cap F_{k,l+l}^{(n)}=A^{(n)}\cap F_{k,l+1}^{(n)}.$$

Thus by l-induction, for any $k,l\in\mathbb{N}$ we have

$$CP_{n\_Wythoff}\cap F_{k,l}^{(n)}=A^{(n)}\cap F_{k,l}^{(n)}.$$

Therefore,

$$\begin{aligned}CP_{n\_Wythoff}&=CP_{n\_Wythoff}\cap\Big(\bigcup_{k,l=1}^{\infty}F_{k,l}^{(n)}\Big)\\&=\bigcup_{k,l=1}^{\infty}CP_{n\_Wythoff}\cap F_{k,l}^{(n)}\\&=\bigcup_{k,l=1}^{\infty}A^{(n)}\cap F_{k,l}^{(n)}\\&=A^{(n)}\cap\Big(\bigcup_{k,l=1}^{\infty}F_{k,l}^{(n)}\Big)\\&=A^{(n)}.\end{aligned}$$

□

4 Conclusions

As we have seen in the previous section, $CP_{n\_Wythoff}$ has a doubly periodic structure. Namely, $CP_{n\_Wythoff}\cap F_{k,l}^{(n)}$ is just a copy of $CP_{n\_Wythoff}\cap F_{1,1}^{(n)}$ for any $k,l\in\mathbb{N}$. On this point $CP_{n\_Wythoff}$ differs greatly from $CP_{Wythoff}$. But as n increases this defference gets smaller and smaller. We describe the figure of $CP_{n\_Wythoff}\cap F_{1,1}^{(n)}$ for $n=5,7$ below.

In this paper we treated Nim games which are of regular type. We would like to investigate winning strategies of Nim games which are of misère type next.

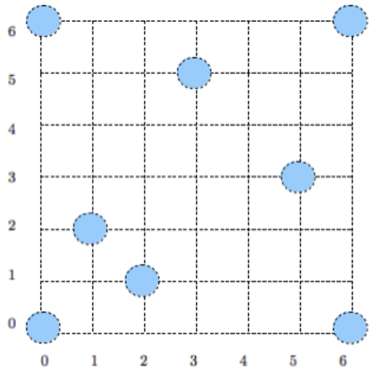

Figure 6: $CP_{5\_Wythoff} \cap F_{1,1}^{(5)}$

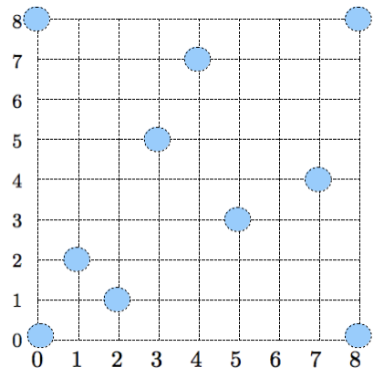

Figure 7: $CP_{7\_Wythoff} \cap F_{1,1}^{(7)}$

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