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The Edge Slide Graph of the 3-cube

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THE EDGE SLIDE GRAPH OF THE
3-CUBE

Lyndal Henden^a

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THE EDGE SLIDE GRAPH OF THE 3-CUBE

Lyndal Henden

Abstract. The goal of this paper is to study the spanning trees of the 3-cube by understanding their *edge slide graph*. A spanning tree of a graph G is a minimal set of edges that connects all vertices. An edge slide occurs in a spanning tree of the 3-cube when a single edge can be slid across a 2-dimensional face to form another spanning tree. The edge slide graph is the graph whose vertices are the spanning trees, with an edge between two vertices if the spanning trees are related by a single edge slide.

This report completely determines the edge slide graph of the 3-cube. The edge slide graph of the 3-cube has twelve components isomorphic to the 4-cube, and three other components, mutually isomorphic, with 64 vertices each. The main result is to determine the structure of the three components that each have 64 vertices and we also describe their symmetries. Some partial results on the 4-cube are also provided.

Acknowledgements: I would like to thank Dr. Christopher Tuffley for his help with this paper.

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1 Introduction

The goal of this paper is to study the spanning trees of the 3-cube by understanding their *edge slide graph*. The 3-cube is the graph on the left in Figure 1, whose vertices and edges are the vertices and edges of an ordinary cube or die, and a spanning tree of a graph is a minimal set of edges that connects all of the vertices. In the case of the cube we may use the structure of the cube to define an *edge slide* operation transforming one tree into another: two trees are related by an edge slide if it is possible to transform one tree into the other by “sliding” an edge across a two-dimensional face. We may then look at the graph whose vertices are the spanning trees, with an edge between two trees if they are related by a single edge slide.

The 3-cube is the third member of an infinite family of graphs $\{Q_n\}$, where the n -cube Q_n consists of the edges and vertices of an n -dimensional cube. The 3-cube Q_3 and the 4-cube Q_4 can be seen in Figure 1. The number of spanning trees of Q_n may be found using Kirchhoff’s Matrix Tree Theorem, and is known to be

$$|\text{Tree}(Q_n)| = 2^{2^n - n - 1} \prod_{k=1}^n k^{\binom{n}{k}} \quad (1)$$

(see for example Stanley [3]). However, it is an open problem to find a bijective proof of the formula for the number of spanning trees of Q_n . Tuffley [4] has recently found a combinatorial proof for counting spanning trees in the case where $n = 3$ using edge slides.

There are difficulties when we try to generalise Tuffley’s results for Q_3 to higher dimensions. Edge slides can be defined for higher dimensional cubes, however, some of the properties used to count the spanning trees of Q_3 break down, and the argument cannot

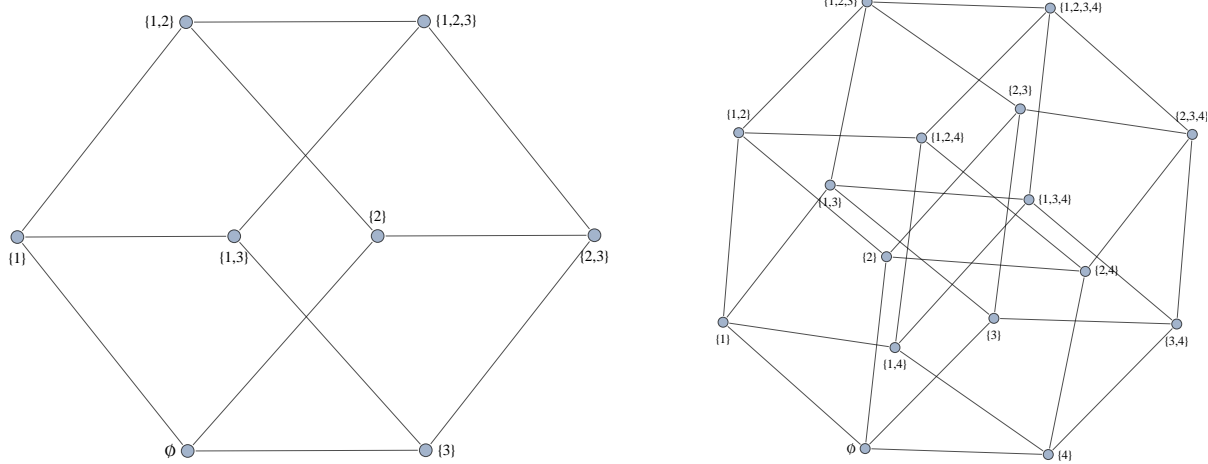


Figure 1: The 3-cube Q_3 on the left, and the 4-cube Q_4 on the right.

readily be extended to higher dimensions. Our hope is that an understanding of the edge slide graph will give some insight into counting the spanning trees of Q_n , for $n \geq 4$.

In this paper we completely determine the structure of the edge slide graph of Q_3 . We find that this has twelve components isomorphic to Q_4 , and three mutually isomorphic components that each have 64 vertices. In Section 2 we define terms and determine the structure of the edge slide graph of Q_2 , while in Section 3 we determine the structure of the edge slide graph of Q_3 . Finally, in Section 4 we present partial results on Q_4 .

2 Preliminaries

2.1 Edge slides and the edge slide graph

Definition 1. The n -cube is the graph Q_n that has 2^n vertices, the subsets of $\{1, 2, \dots, n\}$. There is an edge between two vertices s and t if they differ by adding or removing one element. If s and t differ by adding or removing i , we will say that the edge $\{s, t\}$ is in *direction* i . The 3-cube appears in Figure 1.

Definition 2. A *spanning tree* of a graph G is a minimal set of edges that connects all vertices. See Figure 2 for an example. If G has v vertices then a spanning tree will have $v - 1$ edges (see for example Agnarsson and Greenlaw [1, p.98]).

Definition 3. An *edge slide* occurs in a spanning tree T_1 of Q_n when a single edge can be slid across a two dimensional face to get a second spanning tree T_2 in Q_n . More precisely, an edge slide is the operation of switching between two edges of the form $(s, s \cup \{i\})$ and $(s \cup \{j\}, s \cup \{i, j\})$, where $s \subseteq \{1, 2, \dots, n\}$, $i, j \in \{1, 2, \dots, n\}$, and $i, j \notin s$. An example is shown in Figure 3.

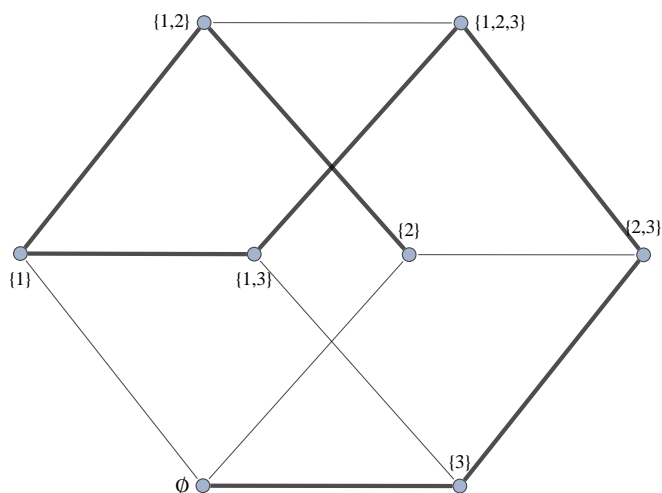


Figure 2: The bold edges form a spanning tree of the 3-cube.

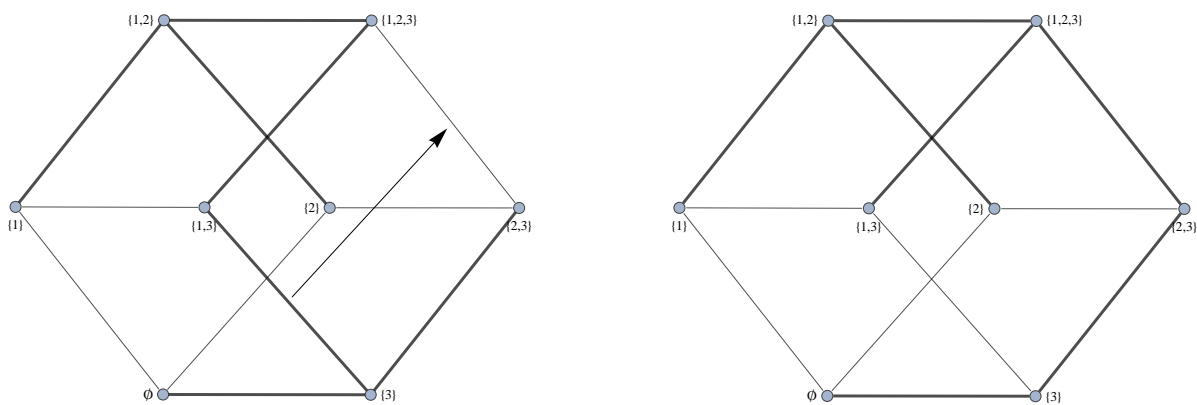


Figure 3: An edge slide from T_1 to T_2 . This slide is an upward slide, because the edge moves from vertices $\{1,3\}$ and $\{3\}$ to vertices $\{1,2,3\}$ and $\{2,3\}$, and the number of elements in each vertex has increased.

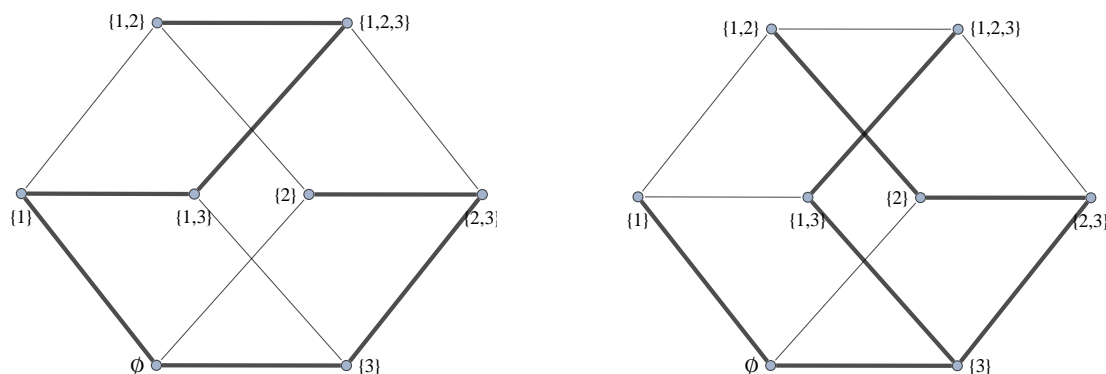


Figure 4: Spanning trees with different signatures. The tree on the left has one edge in direction 1, two edges in direction 2, and four edges in direction 3; its signature is $(1,2,4)$. The tree on the right has three edges in direction 1 and two edges in each of directions 2 and 3; the signature is $(3,2,2)$.

A *downward edge slide* occurs when the number of elements in both vertices joined by the edge decreases when the edge is slid. In contrast, an *upward edge slide* occurs when the number of elements in each vertex increases when the edge is slid, as seen in Figure 3. We define the *upper* and *lower* faces of Q_n with respect to direction i as the subgraphs induced by the vertices that respectively do and do not contain i . Then a downward edge slide in direction i moves an edge from the upper to the lower face with respect to direction i , while an upward edge slide moves an edge from the lower to the upper face.

Definition 4. We define the *edge slide graph* of Q_n , $\mathcal{E}(Q_n)$, to be the graph whose vertices are the spanning trees of Q_n , with an edge between T_1 and T_2 if they are related by a single edge slide.

Thus, for example the edge slide graph of Q_3 will have an edge between the vertices corresponding to the trees in Figure 3.

2.2 Signatures

Definition 5. If a spanning tree T of Q_n has k_i edges in direction i then we define the *signature* of T to be (k_1, k_2, \dots, k_n) . Figure 4 shows two spanning trees with different signatures.

Lemma 1 (Tuffley [4]). *A spanning tree with k_i edges in direction i will have at least $k_i - 1$ edges that can be slid in direction i , or exactly $k_i - 1$ if $n = 2$ or 3 .*

Lemma 2. *Trees with different signatures belong to different components.*

For example, a spanning tree with the signature $(1,3,3)$ will not be in the same component as a spanning tree with the signature $(3,2,2)$.

Proof. An edge in direction j that can be slid in direction i will only ever slide to another edge in direction j that is not already in the spanning tree. Therefore the signature will not change, hence spanning trees with different signatures will not be connected and will belong to different components of $\mathcal{E}(Q_n)$. \square

Definition 6. Given a signature (k_1, k_2, \dots, k_n) , we define $\mathcal{E}(k_1, k_2, \dots, k_n)$ to be the subgraph of $\mathcal{E}(Q_n)$ induced by the trees with signature (k_1, k_2, \dots, k_n) . By Lemma 2, the graph $\mathcal{E}(k_1, k_2, \dots, k_n)$ consists of one or more connected components of $\mathcal{E}(Q_n)$.

Lemma 3. *If two signatures (k_1, k_2, \dots, k_n) and $(\ell_1, \ell_2, \dots, \ell_n)$ are related by a permutation, then $\mathcal{E}(k_1, k_2, \dots, k_n)$ and $\mathcal{E}(\ell_1, \ell_2, \dots, \ell_n)$ are isomorphic.*

Proof. Each permutation σ of $\{1, \dots, n\}$ induces an automorphism of Q_n , and this in turn induces an automorphism of $\mathcal{E}(Q_n)$. This automorphism carries trees with signature (k_1, k_2, \dots, k_n) to trees with the signature $(k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(n)})$. \square

In view of Lemma 3 it makes sense to consider signatures up to permutation.

2.3 Orientations and upright trees

It will be useful to orient the edges of a spanning tree of Q_n . For this purpose we will consider all spanning trees to be rooted at $\emptyset \in V(Q_n)$. Edges can then be classified as *upwards* or *downwards* as follows.

Definition 7. Let T be a spanning tree of Q_n and let $\{s, t\}$ be an edge of T . Suppose that t is on the path from s to the root (the empty set). Then $\{s, t\}$ is an *upward edge* if s is a subset of t . If t is a subset of s then $\{s, t\}$ is a *downward edge*.

Definition 8. A spanning tree with all downward edges is called an *upright tree*. An example appears in Figure 5.

Given an upright spanning tree T , the first edge on the path from a non-root vertex s to the root must be in a direction belonging to s . This choice of direction out of s gives us a function $f_T : V(Q_n) \setminus \{\emptyset\} \rightarrow \{1, 2, \dots, n\}$ such that $f(s) \in s$ for all s . It can be shown that this function determines T , and moreover that any such function corresponds to an upright tree (see Lemma 4.2 of Tuffley [4] for the case $n = 3$). It follows that the number of upright trees is

$$\prod_{k=1}^n k^{\binom{n}{k}} \quad (2)$$

which we see appears in the formula (1) for counting trees of Q_n in Section 1.

We will use the functions defined above to label trees in Section 3.2. We note that it is only necessary to specify the function values of vertices of cardinality at least two. It can be shown that every spanning tree is connected to at least one upright tree either directly or by a series of downward edge slides (Tuffley [4, Corollary 5.3]), and we will use this to show that $\mathcal{E}(3, 2, 2)$ and $\mathcal{E}(2, 2, 4, 7)$ are connected in Sections 3.2 and 4.

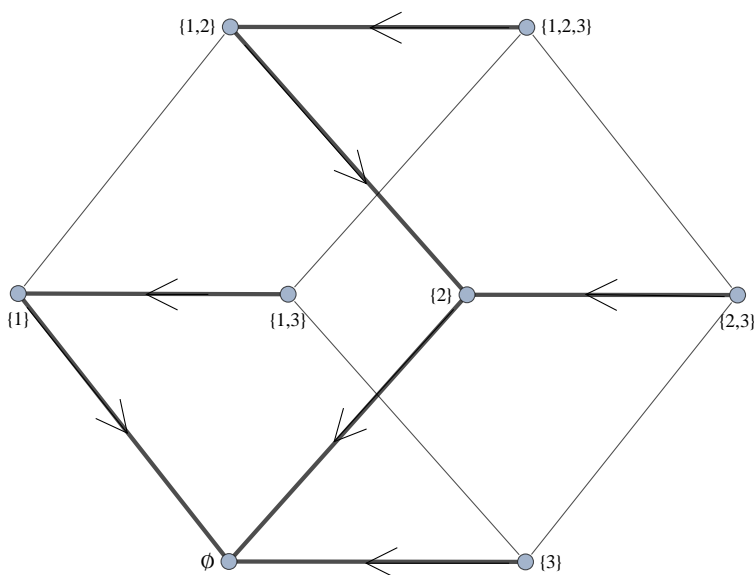


Figure 5: An upright tree for Q_3 . This corresponds to choosing direction 3 from $\{1, 2, 3\}$, direction 1 from $\{1, 2\}$, direction 3 from $\{1, 3\}$, direction 3 from $\{2, 3\}$, direction 1 from $\{1\}$, direction 2 from $\{2\}$, and direction 3 from $\{3\}$.

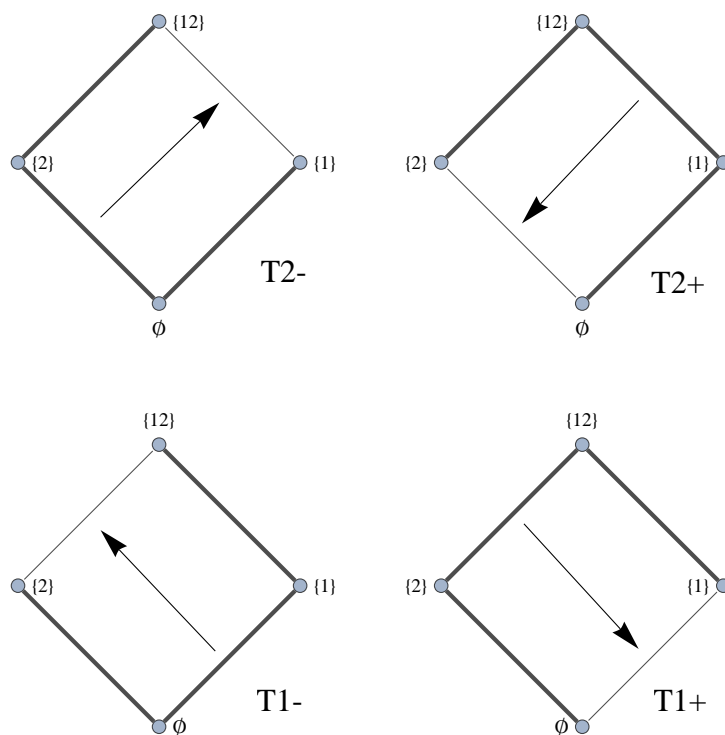
2.4 The edge slide graph of the 2-cube, Q_2

Spanning trees of the 2-cube, Q_2 , have possible signatures of $(1, 2)$ or $(2, 1)$, where there is one edge in direction 1 and two edges in direction 2, or two edges in direction 1 and one edge in direction 2. The 2-cube has four spanning trees, and each spanning tree has only one possible edge slide. This can be seen in Figure 6.

There are two upright trees for Q_2 which are T_{1-} and T_{2-} , as in Figure 6. The spanning tree T_{1+} is related to the upright tree T_{1-} by a single edge slide, and the spanning tree T_{2+} is related to the upright tree T_{2-} by a single edge slide; therefore $\mathcal{E}(Q_2)$ contains two components. Each component has two vertices and one edge joining them. Figure 7 shows the two components of $\mathcal{E}(Q_2)$.

3 The edge slide graph of the 3-cube

The signature (k_1, k_2, k_3) of a spanning tree of Q_3 satisfies $k_1 + k_2 + k_3 = 7$ and $1 \leq k_i \leq 4$, so there are only three different signatures of Q_3 up to permutation, namely $(1, 2, 4)$, $(1, 3, 3)$ and $(3, 2, 2)$. By Lemma 3 we may therefore understand $\mathcal{E}(Q_3)$ by understanding each of the subgraphs $\mathcal{E}(1, 2, 4)$, $\mathcal{E}(1, 3, 3)$, and $\mathcal{E}(3, 2, 2)$.

Figure 6: Spanning trees with edge slides for Q_2 .Figure 7: Edge slide graph for Q_2 .

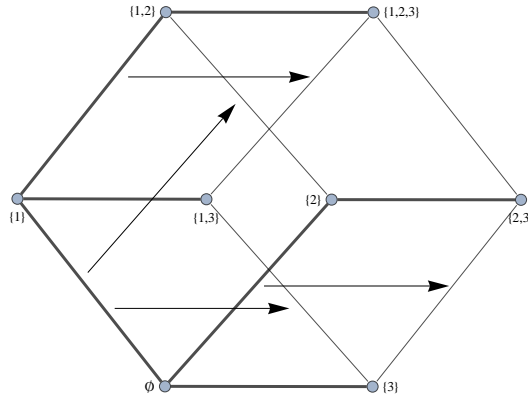


Figure 8: Edge slides for an upright tree with signature (1,2,4).

3.1 Signatures (1, 2, 4) and (1, 3, 3)

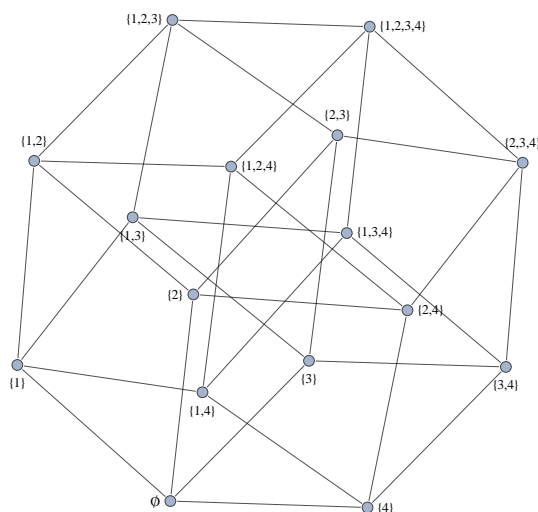
Let T be a spanning tree of Q_3 of signature (1,2,4). Then T consists of two spanning trees of Q_2 joined by an edge in direction 1. These two spanning trees of Q_2 lie in the upper and lower faces of Q_3 with respect to direction 1. All four edges in direction 3 belong to T and hence none of these edges is able to be slid. This means that the trees in the upper and lower faces both have the signature $(k_2, k_3) = (1,2)$. The edges in direction 2 may be slid in direction 3 only, and the single edge in direction 1 can always be slid in both directions 2 and 3. This means there are a total of four possible single edge slides, and they can all be made independently of each other as seen in Figure 8. This gives us

$$2^2 \cdot 4$$

spanning trees. Therefore, $\mathcal{E}(1, 2, 4)$ has 16 vertices and each vertex has four edges attached. The graph $\mathcal{E}(1, 2, 4)$ is isomorphic to the 4-cube, Q_4 , which is shown in Figure 9. In view of Lemma 3, the edge slide graph $\mathcal{E}(1, 2, 4)$ up to permutation will be isomorphic to the 4-cube.

A spanning tree T with the signature (1,3,3) is also made up of two spanning trees belonging to Q_2 with a single edge in direction 1 joining them. However, the trees in the upper and lower faces now have different signatures. One has the signature $(k_2, k_3) = (1,2)$ and the other has the signature $(k_2, k_3) = (2,1)$. We are unable to slide edges of T in direction 1, just as in the case of signature (1,2,4), and as a consequence these upper and lower signatures cannot be changed by an edge slide. This breaks $\mathcal{E}(1, 3, 3)$ into two disjoint subgraphs, one containing the spanning trees with upper signature $(k_2, k_3) = (1,2)$ and lower signature $(k_2, k_3) = (1,2)$, and the other containing the trees with upper signature $(k_2, k_3) = (2,1)$ and lower signature $(k_2, k_3) = (1,2)$.

For a tree of upper signature (1,2) we can choose to slide the direction-2 edge in the upper face or not, choose to slide the direction-3 edge in the lower face or not, and choose to slide the single edge in direction 1 in either of directions 2 or 3. These four slides may

Figure 9: The 4-cube, Q_4 .

all be made independently. This gives us 16 spanning trees and the edge slide graph for the signature $(1,3,3)$ for trees with upper signature $(1,2)$ is isomorphic to Q_4 .

Similarly, a tree of upper signature $(2,1)$ can have a direction-3 edge slid in the tree in the upper face, a direction-2 edge slid in the tree in the lower face, and the single edge in direction 1 can be slid in either of directions 2 or 3. These four slides may again all be made independently, so the edge slide graph for the signature $(1,3,3)$, when the tree in the upper face has the signature $(2,1)$, is isomorphic to the 4-cube.

3.2 Signature $(3,2,2)$

Spanning trees of Q_3 with the signature $(3,2,2)$ have four possible edge slides: two in direction 1, and one in each of directions 2 and 3. Figure 10 shows this.

Unlike signatures $(1,2,4)$ and $(1,3,3)$, there are multiple edges in each direction so we find that there are four upright trees for the signature $(3,2,2)$ as seen in Figure 11. As discussed in Section 2.3, each upright tree may be labelled by listing which direction is chosen from the subset of vertices $\{1, 2, 3\}$, $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. We will list the choices in this order, and represent the label as a four digit number. For example, if direction 3 was chosen from $\{1, 2, 3\}$, direction 2 from $\{1, 2\}$, direction 1 from $\{1, 3\}$ and direction 3 from $\{2, 3\}$ then the label would be 3213. The labels for the upright trees with signatures $(3,2,2)$ are 1132, 3112, 1213, and 2113, as indicated in Figure 11.

Each tree can be associated with a canonically chosen upright tree, in such a way that sixteen trees are associated with each upright tree. The sixteen trees may be obtained from the upright tree as follows. First we decide whether or not to carry out each of the direction-1 edge slides; these commute so we get a total of four spanning trees. From here we decide whether or not to carry out the direction-2 edge slides in each of the four spanning trees,

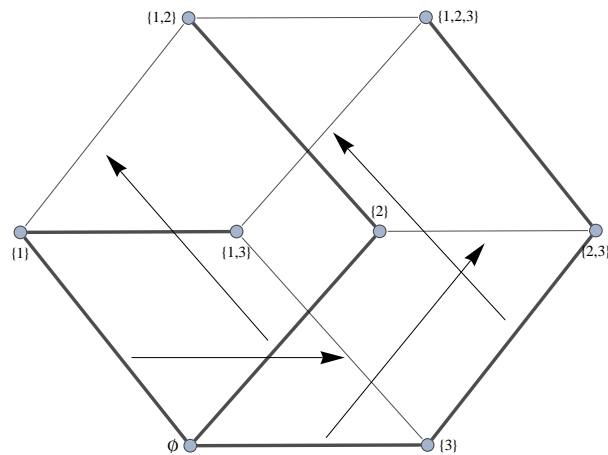


Figure 10: Edge slides for an upright tree with signature (3,2,2).

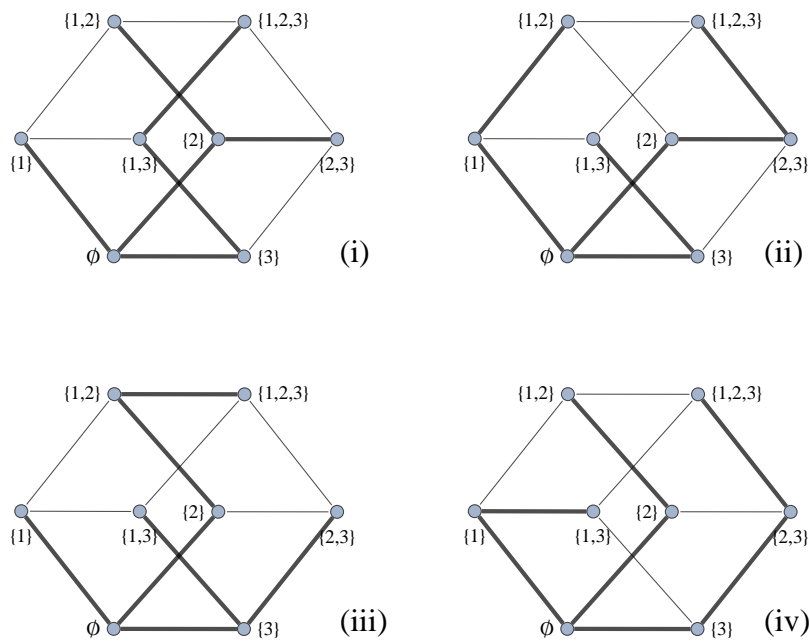


Figure 11: The four upright trees with signatures (3,2,2). The labels are (i) 2113, (ii) 1213, (iii) 3112, and (iv) 1132.

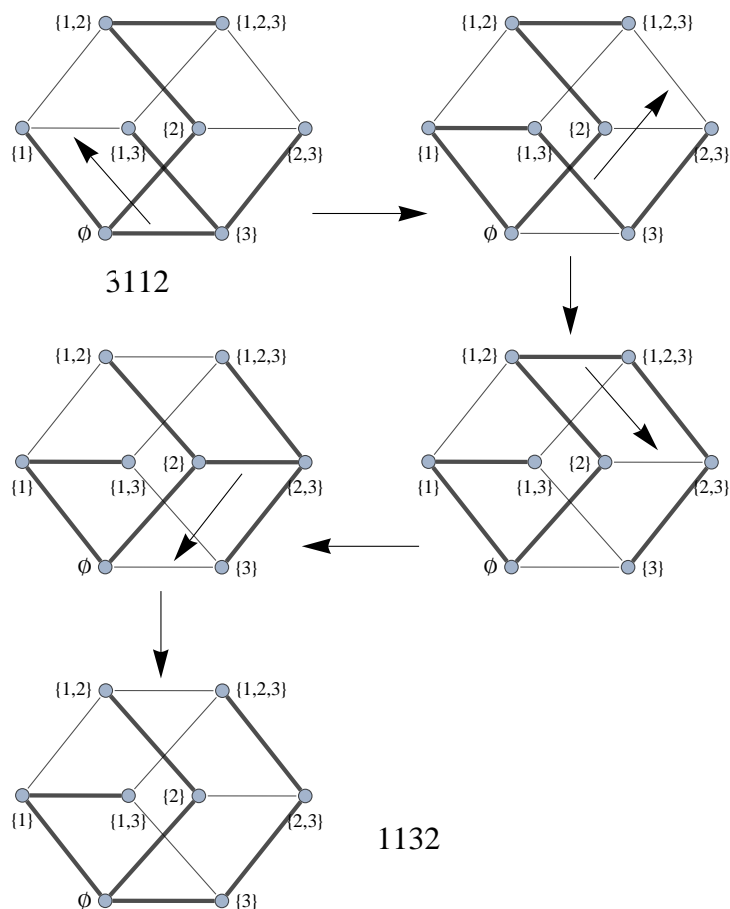


Figure 12: A series of edge slides from upright tree 3112 to upright tree 1132.

giving eight spanning trees. Next, we decide whether or not to carry out the direction-3 edge slide in each spanning tree, giving a total of sixteen spanning trees. We will always get a total of 16 distinct spanning trees for each upright tree belonging to Q_3 ; refer to Tuffley [4, Section 4.2] for details.

Starting with an upright tree with the signature $(3,2,2)$ we show that it is possible to get to any other upright tree with the same signature. Figures 12, 13 and 14 show this through a series of edge slides. Because of this, the graph $\mathcal{E}(3,2,2)$ contains all four upright trees and their associated spanning trees so will have a total of 64 vertices.

The graph $\mathcal{E}(3,2,2)$ was determined by recording which trees were connected by edge slides and then it was drawn using computer software. The results are shown in Figures 18, 19 and 20 on pages 86, 87 and 88 respectively. The program `neato` from the Graphviz package [2] was used to draw Figure 18, which allowed for coloured edges that represented the different edge slides performed and helped identify the structure of the graph. Black edges represent edge slides in direction 1, blue edges represent edge slides in direction 2, and red

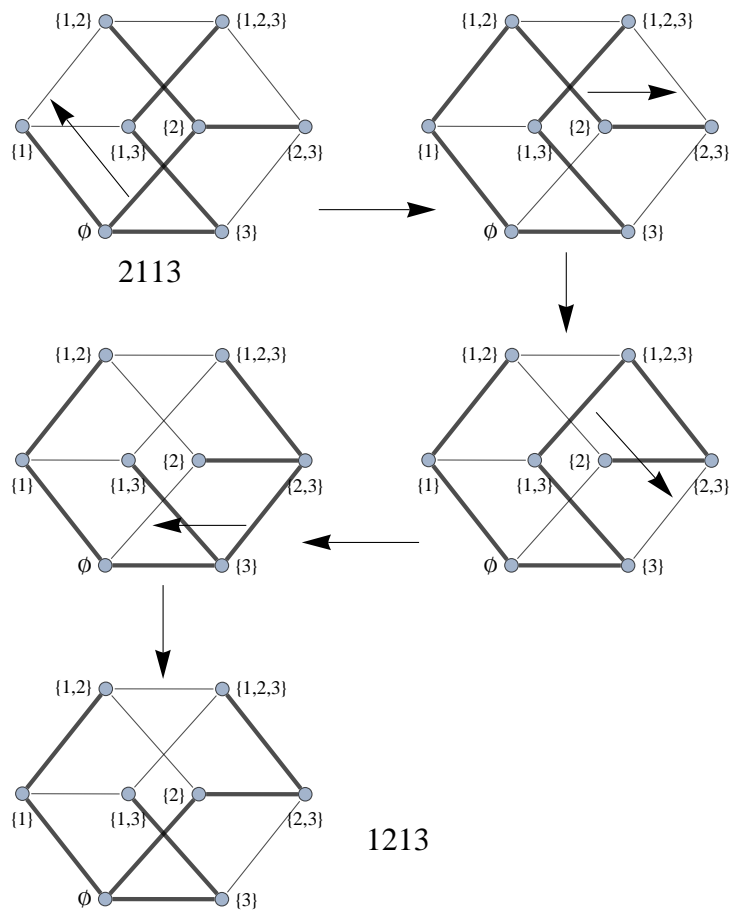


Figure 13: A series of edge slides from upright tree 2113 to upright tree 1213.

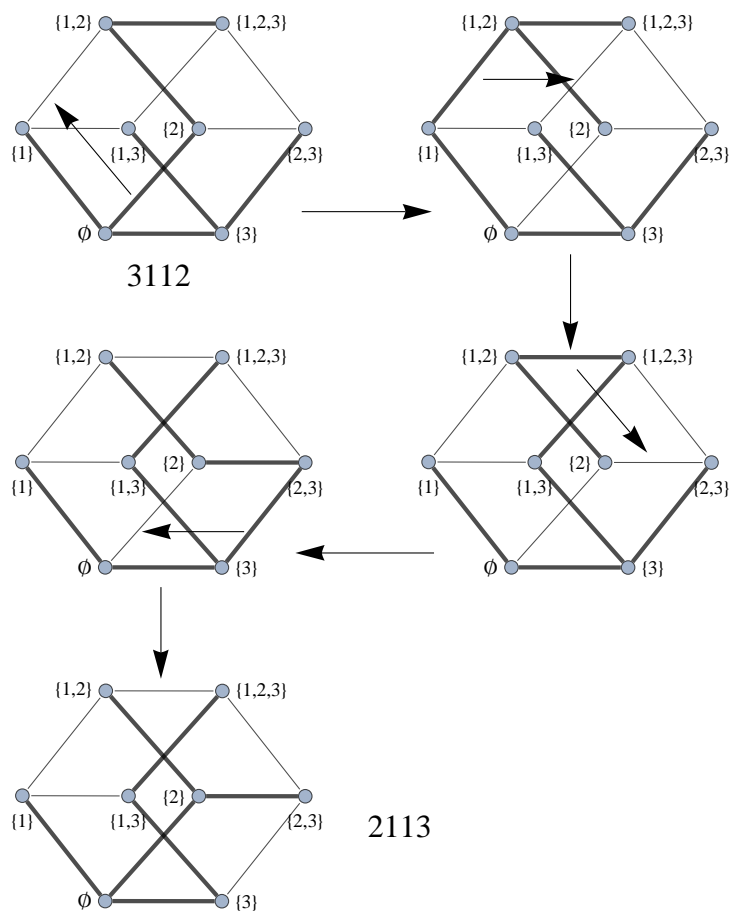


Figure 14: A series of edge slides from upright tree 3112 to upright tree 2113.

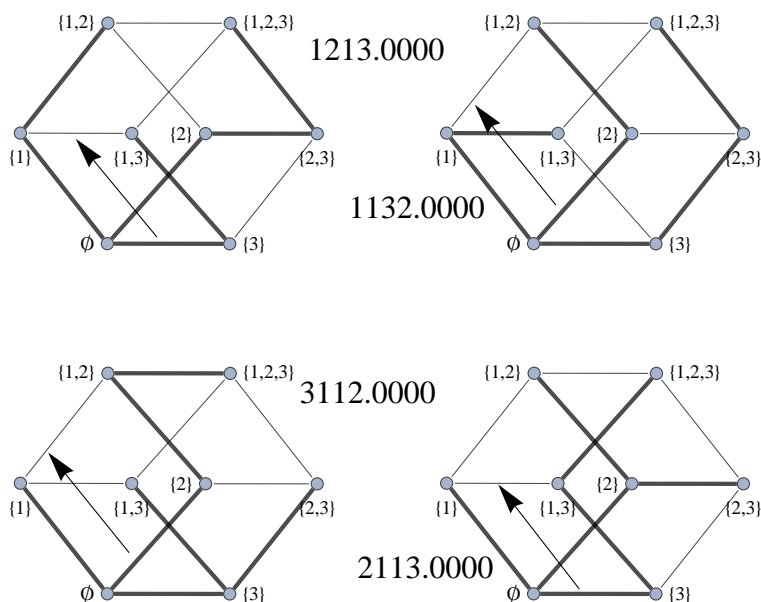


Figure 15: Edge slides in direction 1 that are considered to be the first digit after the decimal point in the label. When this edge slide is performed the labels for the spanning trees will be 1213.1000, 1132.1000, 3112.1000 and 2113.1000.

edges represent edge slides in direction 3. Wolfram Mathematica 8.0 [5] was the computer software used to produce Figures 19 and 20, which allowed for a 3-dimensional view of the graph.

Each vertex has a label with the first four digits representing the upright tree it came from and the second set of four digits representing which edge slides have been made. If an edge has been slid then the number 1 indicates this, otherwise 0 indicates no movement. The first two digits of this number represent whether or not the edge slides in direction 1 have been made. Figure 15 shows which edge slide in direction 1 represents the first digit in this number. The third digit represents whether or not the edge slide in direction 2 has been made, and the fourth digit represents whether or not the edge slide in direction 3 has been made. For example, 1132.0011 stands for the 1132 upright tree with the edge slide in direction 2 having been made, followed by the edge slide in direction 3 in the resulting tree.

3.2.1 The Structure of $\mathcal{E}(3, 2, 2)$

In order to understand the structure of $\mathcal{E}(3, 2, 2)$, it is helpful to break the graph down into smaller sections.

Although it is hard to tell from Figure 18, the graph $\mathcal{E}(3, 2, 2)$ can be broken down into three sections. We can see that there are two meshes, one on top of the other, in the middle of the graph. Each mesh makes a pattern of thirteen squares and can be seen in Figure 21.

Around the outside of the meshes there are two cycles made up of twelve vertices, with red and blue edges joining them only. These cycles alternate between the two meshes. These cycles pass through four cubes that both meshes are also joined to. It is hard to recognise the cubes as they have been twisted because of the cycles alternating between the meshes. The spanning trees in each of the two cycles are

$$\begin{aligned} &2113.1010 \text{ --- } 2113.1011 \text{ --- } 1213.0101 \text{ --- } 1213.0100 \text{ --- } 1213.0110 \text{ --- } 1213.0111 \text{ ---} \\ &1132.1001 \text{ --- } 1132.1000 \text{ --- } 1132.1010 \text{ --- } 1132.1011 \text{ --- } 2113.1001 \text{ --- } 2113.1000 \end{aligned}$$

and

$$\begin{aligned} &3112.1010 \text{ --- } 3112.1011 \text{ --- } 1213.1001 \text{ --- } 1213.1000 \text{ --- } 1213.1010 \text{ --- } 1213.1011 \text{ ---} \\ &1132.0101 \text{ --- } 1132.0100 \text{ --- } 1132.0110 \text{ --- } 1132.0111 \text{ --- } 3112.1001 \text{ --- } 3112.1000. \end{aligned}$$

One of the cubes, C_1 , belonging to $\mathcal{E}(3, 2, 2)$ consists of spanning trees generated from the 1132 upright tree only and is the very bottom twisted cube in Figure 18. Another cube, C_2 , has spanning trees generated from the 1213 upright tree only and can be seen on the right-hand side of this figure. The third cube, C_3 , is made up of spanning trees from the 3112 and 2113 upright trees and is the top twisted cube in Figure 18. Lastly, cube four, C_4 , has spanning trees from the 3112, 2113, and 1132 upright trees and is on the left-hand side of Figure 18. We note that the map associating a tree to an upright tree (i.e. to the label) breaks the symmetry between directions 2 and 3, and so there is no reason to expect symmetry between labels 2 and 3 in the graph.

The structure of $\mathcal{E}(3, 2, 2)$ is not so obvious, however the symmetry group of $\mathcal{E}(3, 2, 2)$ contains a subgroup G isomorphic to $\mathbb{Z}_2 \times D_4$, where D_4 denotes the dihedral group of order 8. This subgroup consists of the symmetries of $\mathcal{E}(3, 2, 2)$ induced by symmetries of Q_3 that preserve the signature $(3, 2, 2)$. These are generated by the reflections σ_i , which swap the “upper” and “lower” faces with respect to direction i , and $\tau_{2,3}$, which exchanges directions 2 and 3. Note that $i = 1, 2$ or 3 .

These generators satisfy the relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \\ \tau_{2,3} \sigma_1 &= \sigma_1 \tau_{2,3}, \\ \sigma_2 \tau_{2,3} &= \tau_{2,3} \sigma_3, \\ \sigma_i^2 &= \tau_{2,3}^2 = 1. \end{aligned}$$

Since σ_1 commutes with the other three generators, we can split the group as

$$\begin{aligned} G &= \langle \sigma_1 \rangle \times \langle \sigma_2, \sigma_3, \tau_{2,3} \rangle \\ &\cong \mathbb{Z}_2 \times \langle \sigma_2, \sigma_3, \tau_{2,3} \rangle. \end{aligned}$$

Furthermore $\langle \sigma_2, \sigma_3, \tau_{2,3} \rangle$ is isomorphic to D_4 , with the isomorphism given by Figure 16. The

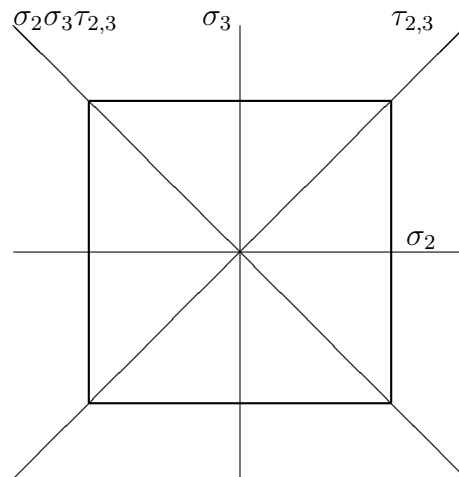


Figure 16: The isomorphism between $\langle \sigma_2, \sigma_3, \tau_{2,3} \rangle$ and D_4 . Each group element may be mapped to the reflection in the corresponding line.

stabilisers of C_1, C_2, C_3 and C_4 are found to be

$$\begin{aligned} \text{stab}(C_1) &= \{1, \sigma_1, \sigma_3, \sigma_1\sigma_3\} \\ \text{stab}(C_2) &= \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\} \\ \text{stab}(C_3) &= \{1, \sigma_1, \sigma_3, \sigma_1\sigma_3\} \\ \text{stab}(C_4) &= \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\}. \end{aligned}$$

In fact we believe that the group $\mathbb{Z}_2 \times D_4$ is the entire automorphism group, based on looking at smaller parts of Figure 18. Each of the two meshes is made up of thirteen squares and each mesh has a centre square made up of four trees. The trees in the centre square of one of the meshes are 2113.0100 — 2113.0101 — 2113.0111 — 2113.0110, and the trees in the other centre square are 3112.0100 — 3112.0101 — 3112.0111 — 3112.0110. Any symmetry of the graph must map each centre square either to itself or to the other. It seems likely that any automorphism is determined by its action on these two squares, in which case the automorphism group will be the product of the symmetry group of a square by an automorphism exchanging the two squares. This would give $\mathbb{Z}_2 \times D_4$ as the entire automorphism group.

4 Some partial results on the 4-cube.

In this section we outline some partial results for the 4-cube, Q_4 , which is shown in Figure 9.

The number of edges in a spanning tree belonging to Q_4 is given by

$$k_1 + k_2 + k_3 + k_4 = 15,$$

where k_i is the number of edges in direction i . Note that $1 \leq k_i \leq 8$. We find that there are twenty different signatures for Q_4 up to permutation. If a 1 occurs in the signature, then no edge slides in the corresponding direction can be made and the edge slide graph for that signature may be understood in terms of “upper” and “lower” signatures from Q_3 , in a similar manner to the graph $\mathcal{E}(1, 3, 3)$. Putting such trees aside we consider two signatures in which every entry is at least 2. Our results are to determine the number of upright trees for each of the signatures $(2,2,4,7)$ and $(2,3,3,7)$, and to determine that $\mathcal{E}(2, 2, 4, 7)$ is connected.

To find all the upright trees for the signature $(2,2,4,7)$ we systematically consider all cases for the direction of the edge selected from the vertex $\{1, 2, 3, 4\}$. We outline the four cases below, and in addition illustrate one (direction 3) in Figure 17.

1. If we choose an edge in direction 1 from the vertex $\{1, 2, 3, 4\}$ we are forced to choose the only other edge in direction 1 from the vertex $\{1\}$. As a result we are left with only one choice for where the seven edges in direction 4 can go. There is also only one choice for where the edges in directions 2 and 3 can go, so there is only one upright tree possible when the edges in direction 1 selected are at these vertices.
2. The same argument applies when an edge in direction 2 is selected from the vertex $\{1, 2, 3, 4\}$, so there is only one upright tree possible when direction 2 is selected from this vertex.
3. When an edge in direction 3 is chosen from the vertex $\{1, 2, 3, 4\}$, there is still only one choice of where the edges in direction 4 can go. However, there are multiple choices for where the remaining edges in directions 1, 2 and 3 can go. As a result, we find that there are four upright trees when an edge in direction 3 is selected from the vertex $\{1, 2, 3, 4\}$. These four upright trees are shown in Figure 17 where the red edges represent the multiple choices of where the remaining edges in directions 1, 2 and 3 can go.
4. When an edge in direction 4 is chosen from the vertex $\{1, 2, 3, 4\}$, another must be chosen from vertex $\{4\}$, leaving five edges in direction 4 that can rotate about the six vertices $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 4\}$, $\{2, 4\}$ and $\{3, 4\}$. In each case this leaves multiple choices of where the edges in directions 1, 2 and 3 can go, and we find that there are a total of eighteen upright trees when direction 4 is selected from the vertex $\{1, 2, 3, 4\}$.

Thus, altogether there are a total of $1 + 1 + 4 + 18 = 24$ upright trees with the signature $(2,2,4,7)$. We may use a similar approach to find the upright trees with the signature $(2,3,3,7)$, and find that there are a total of 38 upright trees with this signature.

The signature $(2,2,4,7)$ has fewer upright trees than the signature $(2,3,3,7)$ so we chose to study this in more detail. We used a similar approach to that used for signature $(3,2,2)$ to show that it is possible to go from an upright tree with the signature $(2,2,4,7)$ to any other upright tree with this signature. Since every spanning tree can be connected to an upright tree by a series of downward slides and all 24 upright trees belong in the same component $\mathcal{E}(Q_4)$, then $\mathcal{E}(2, 2, 4, 7)$ is connected.

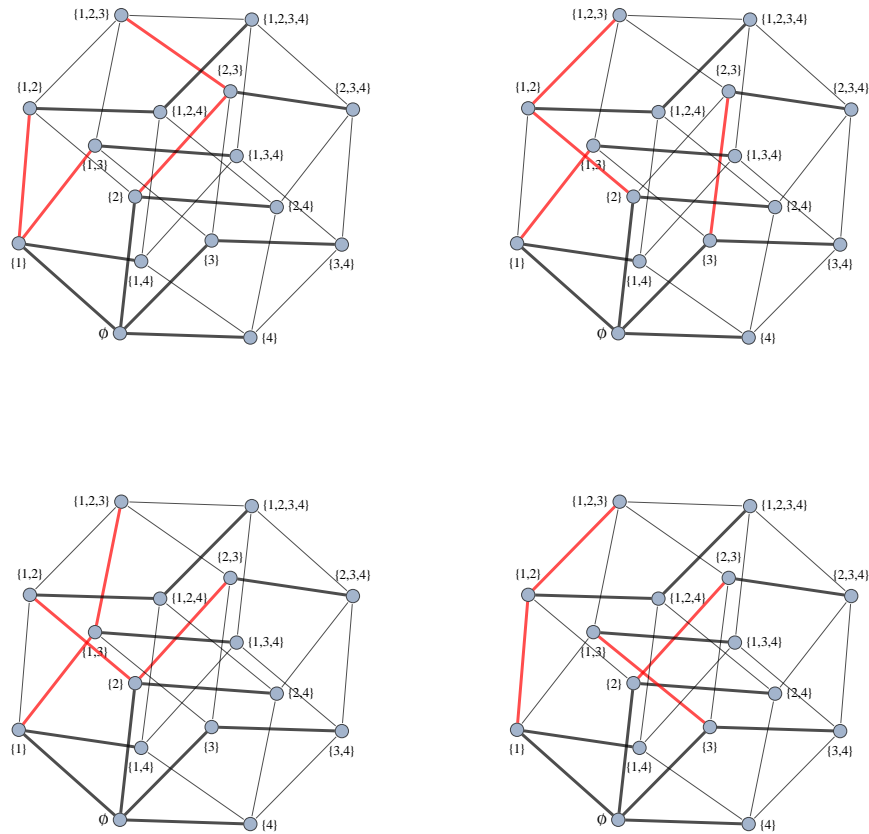


Figure 17: The four possible upright trees with signature $(2,2,4,7)$ when an edge in direction 3 is chosen from the vertex $\{1,2,3,4\}$. The thick black edges represent edges that are fixed and the red edges represent the multiple choices of where the remaining edges in directions 1, 2 and 3 can go.

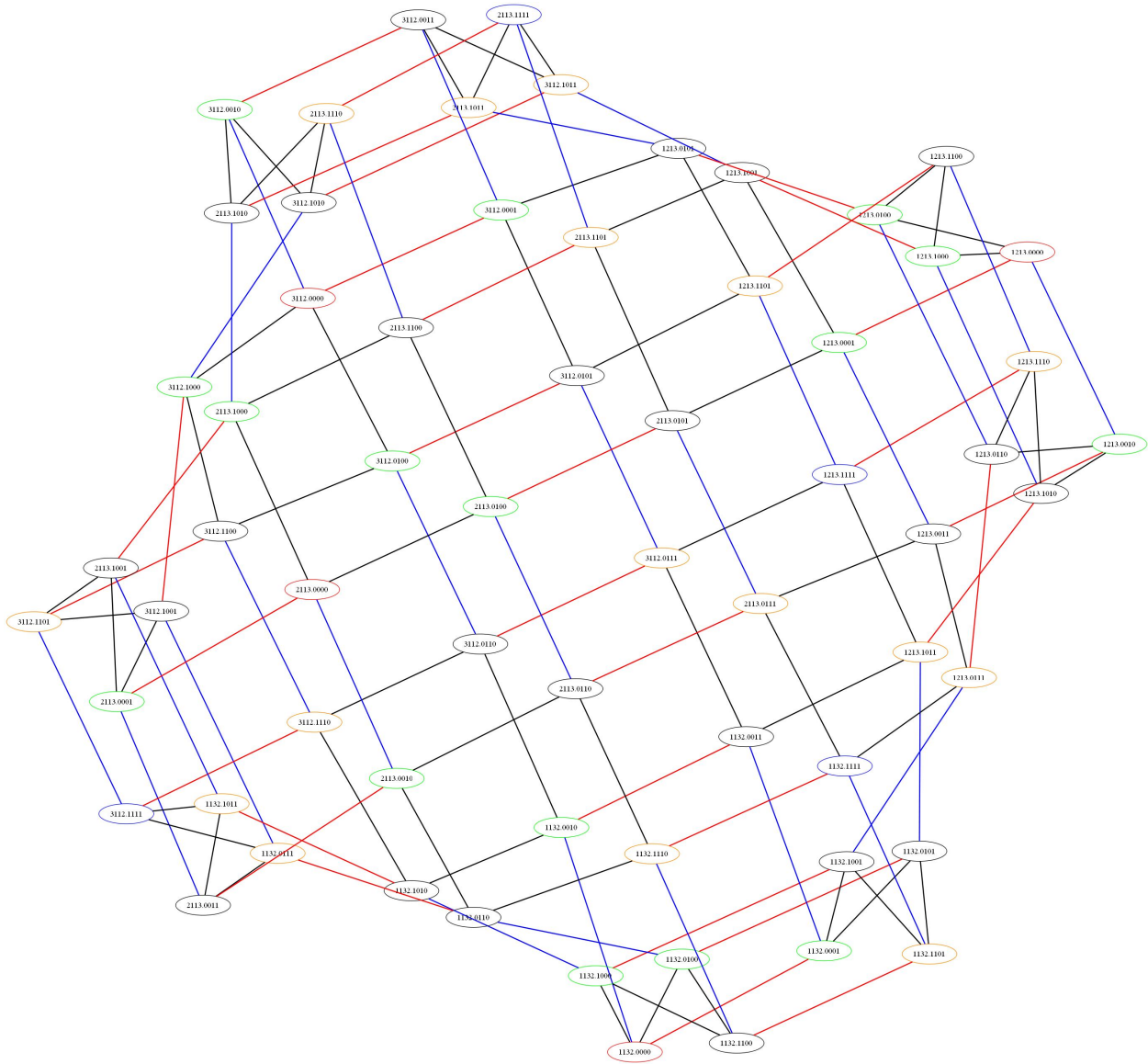


Figure 18: The graph $\mathcal{E}(3, 2, 2)$, drawn using the layout program *neato* from the *Graphviz* package [2].

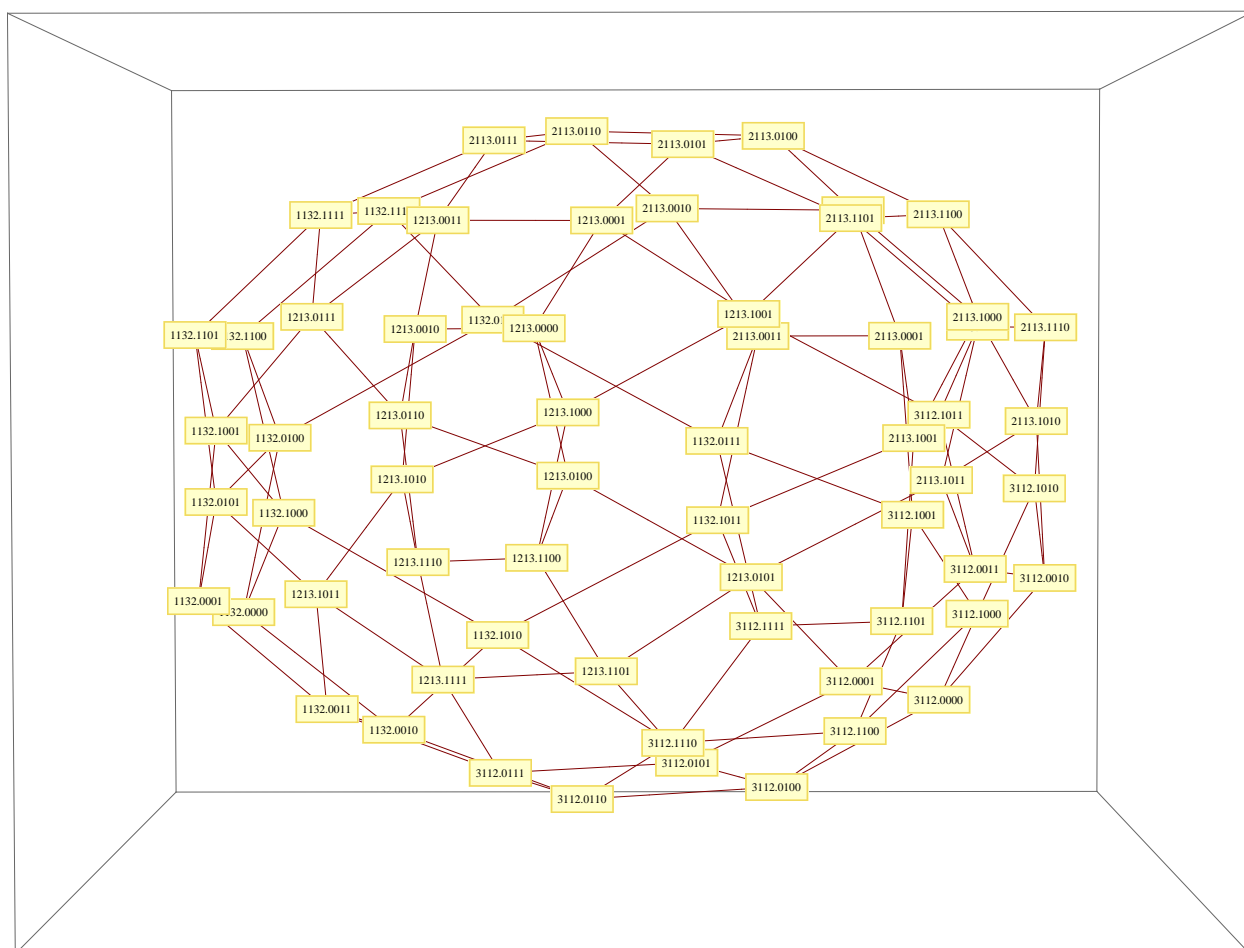


Figure 19: A 3D version of $\mathcal{E}(3, 2, 2)$, drawn by Mathematica 8.0 [5].

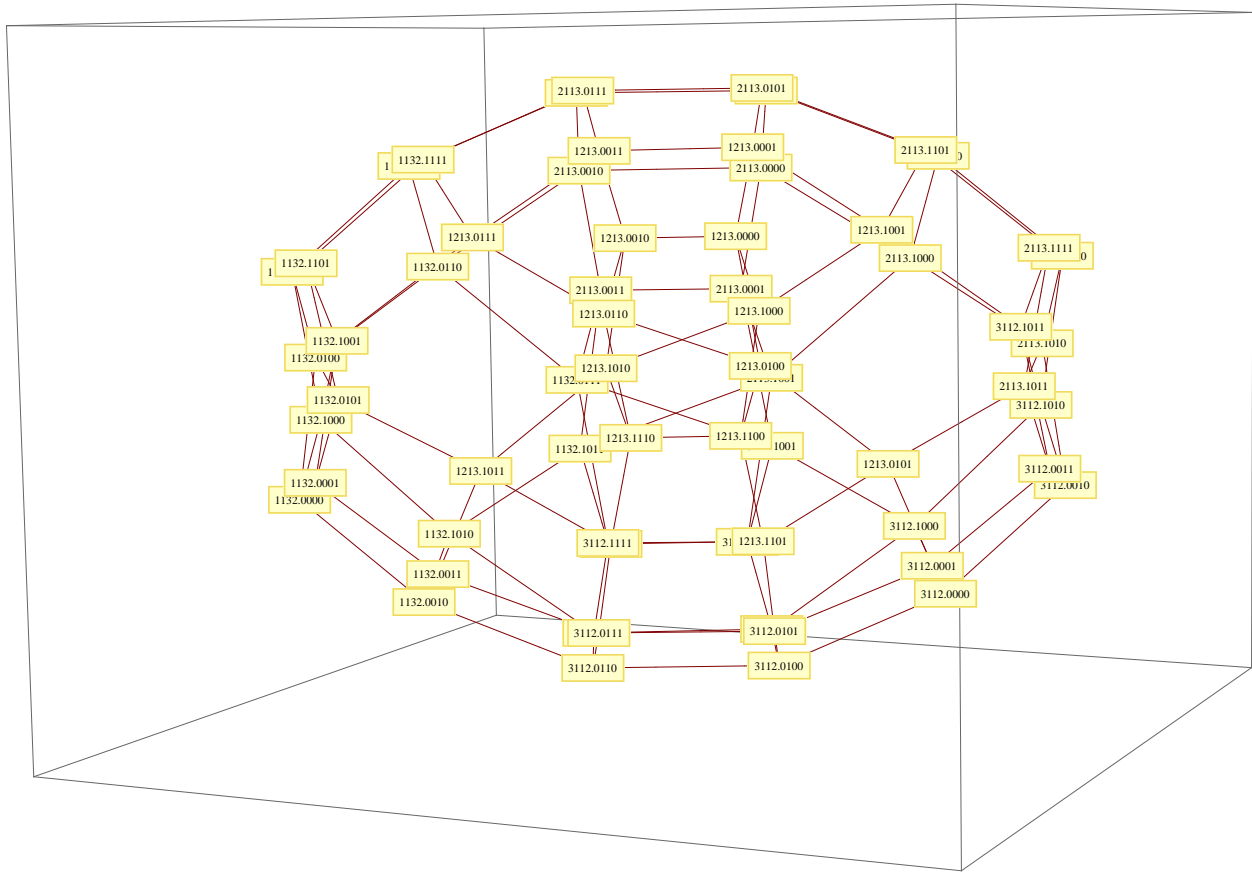


Figure 20: A second 3D version of $\mathcal{E}(3, 2, 2)$.

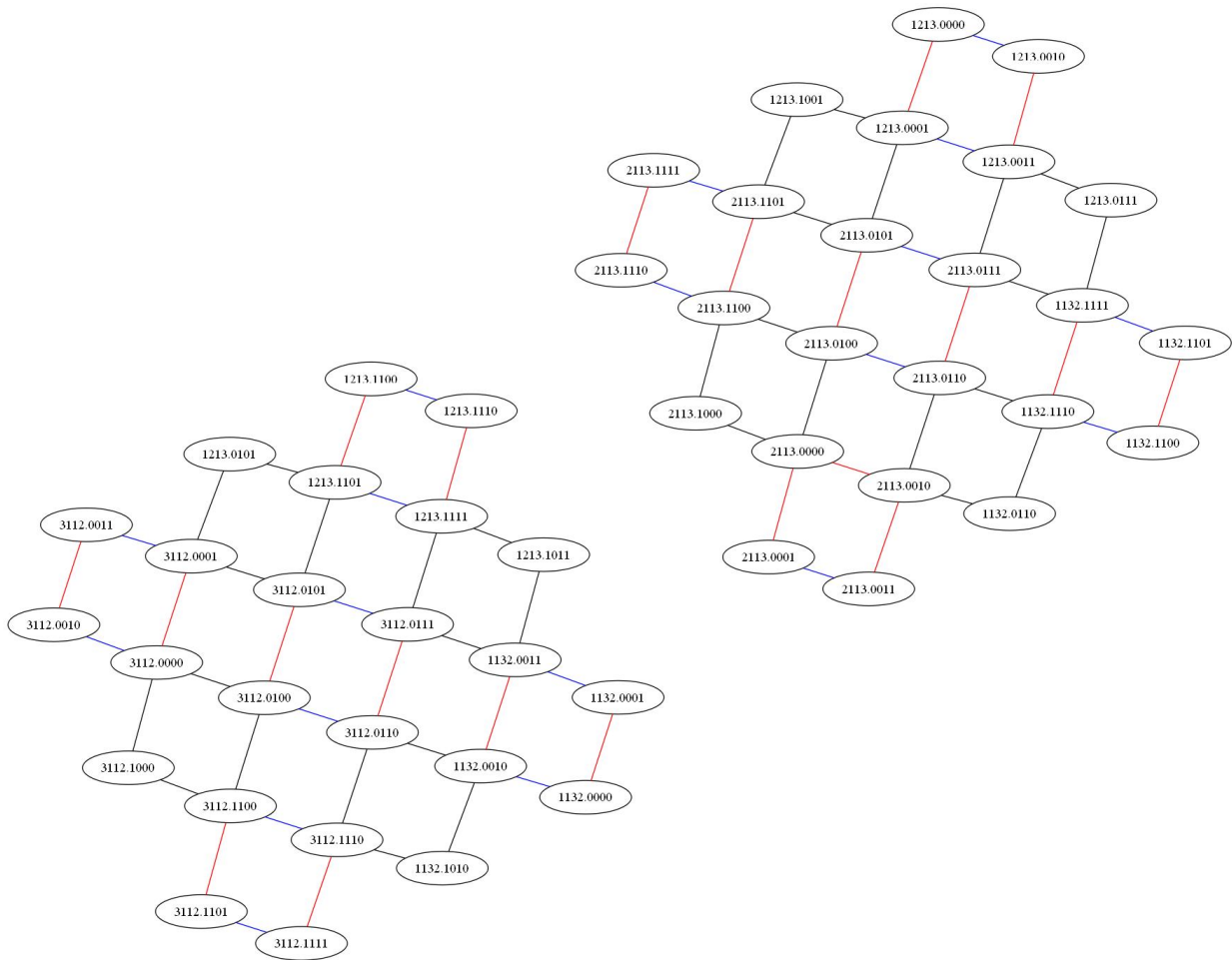


Figure 21: The two meshes from $\mathcal{E}(3, 2, 2)$.

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