Rose-Hulman Undergraduate Mathematics Journal

Volume 12
Issue 2
Article 5

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Rose-Hulman Undergraduate Mathematics Journal

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Volume 12, No. 2, Fall 2011

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Abstract. The degree chromatic polynomial $P_m(G, k)$ of a graph G counts the number of k-colorings in which no vertex has m adjacent vertices of its same color. We prove Humpert and Martin's conjecture on the leading terms of the degree chromatic polynomial of a tree.

Acknowledgements: I would like to thank Federico Ardila for bringing this problem to my attention, and for helping me improve the presentation of this note. I would also like to acknowledge the support of the SFSU-Colombia Combinatorics Initiative.

1 Introduction

George David Birkhoff defined the chromatic polynomial of a graph to attack the renowned four color problem. The chromatic polynomial $P(G, k)$ counts the k-colorings of a graph G in which no two adjacent vertices have the same color [3].

Given a graph G, Humpert and Martin defined its m-chromatic polynomial $P_m(G, k)$ to be the number of k-colorings of G such that no vertex has m adjacent vertices of its same color. They proved this is indeed a polynomial. When $m = 1$, we recover the usual chromatic polynomial of the graph $P(G, k)$.

The chromatic polynomial is of the form

$$P(G, k) = k^n - ek^{n-1} + o(k^{n-1})$$

where n is the number of vertices and e the number of edges of G. For $m > 1$ the formula is no longer true, but Humpert and Martin conjectured the following formula when the graph is a tree T:

$$P_m(T, k) = k^n - \sum_{v \in V(T)} \binom{d(v)}{m} k^{n-m} + o(k^{n-m}) \tag{1}$$

where $d(v)$ is the degree of v. Note that (1) is not true for $m = 1$ —we will see why in the course of proving Theorem 1.

The goal of this paper is to prove this conjecture in Theorem 1. In section 2 we discuss the basic concepts required to understand the theorem, while in section 3 we provide the proof.

2 Background

A *finite graph* G is an ordered pair (V, E), where V is a finite set of *vertices* and E is a set of *edges*, which are 2-element subsets of V.

Figure 1 shows the graphic representation of graph.

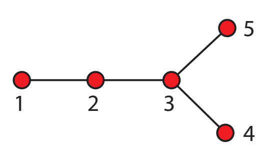

Figure 1: Graphic representation of a graph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$.

We now present some basic definitions of graph theory.

Definition 1. The *degree* of a vertex v is the number of edges which contain v, and is denoted as $d(v)$. Two vertices $p, q \in V$ are said to be *adjacent* if the pair $\{p, q\} \in E$. A *path* is a sequence of vertices $v_0, v_1, \ldots, v_k$ where v_i is adjacent to v_{i+1} for $0 \leq i \leq k-1$. A *cycle* is a path $v_0, \ldots, v_k$ with $v_0 = v_k$. A graph is *connected* if for any pair of vertices there exists a path containing both of them. A *tree* is a connected graph with no cycles.

It is easy to see that the graph in Figure 1 is actually a tree.

A *coloring* of a graph is an assignment of colors to each of its vertices. If σ is a coloring, we denote by $\sigma(v)$ the color assigned to the vertex v. A k-coloring is one in which $\sigma(v) \in \{1, 2, \ldots, k\}$ for all v, i.e. we may use at most k different colors. A graph with n vertices clearly has k^n different k-colorings, as each of its n vertices has k possible choices for its color.

A coloring is called *proper* if there is no edge connecting any two identically colored vertices. Figure 2 shows all of these colorings with $k = 3$ for a 3-vertex tree.

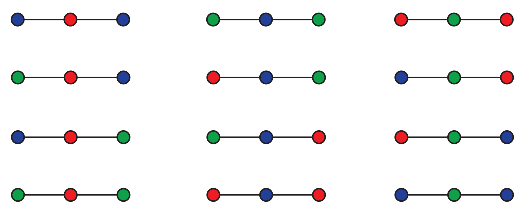

Figure 2: Proper 3-colorings of a tree with 3 vertices.

The *chromatic polynomial* of a graph $P(G, k)$ counts the proper k-colorings of G. It is well-known to be a monic polynomial in k of degree n, the number of vertices.

Example 1. The chromatic polynomial of a tree T with n vertices is $P(T, k) = k(k-1)^{n-1}$. To prove this, fix an initial vertex v_0. There are k possible choices for its color $\sigma(v_0)$. Then, consider a vertex v_1 adjacent to v_0. There are $k-1$ ways to choose $\sigma(v_1)$, as it has to be different from $\sigma(v_0)$. Now, consider a vertex v_2 adjacent to v_0 or to v_1. Notice it cannot be adjacent to both of them, or there would be cycle. Thus, there are also $k-1$ possible choices for $\sigma(v_2)$. If we repeat this algorithm, we will always have a vertex adjacent to exactly one of the previously colored vertices, so it can be colored in $k-1$ ways. The result follows after repeating this procedure $n-1$ times.

3 Results

Now, we prove the conjecture stated by Humpert and Martin.

Theorem 1 ([1, 2], Conjecture)**.** *Let T be a tree with n vertices and let m be an integer with $1 < m < n$. Then the equation (1) holds, where $P_m(G, k)$ counts the number of k-colorings of T in which no vertex has m adjacent vertices of its same color.*

Proof. For a given coloring of T, say vertices v_1 and v_2 are "friends" if they are adjacent and have the same color. For each v, let A_v be the set of colorings such that v has at least m friends. We want to find the number of colorings which are not in any A_v, and we will use the inclusion-exclusion principle. As the total number of k-colorings is k^n, we have

$$P_m(T,k) = k^n - \sum_{v \in V} |A_v| + \sum_{v_1, v_2 \in V} |A_{v_1} \cap A_{v_2}| - \dots$$

We first show that $|A_v| = \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$. Let $A_v^{(l)}$ be the set of k-colorings such that v has exactly l friends. In order to obtain a coloring in $A_v^{(l)}$, we may choose the l friends in $\binom{d(v)}{l}$ ways, the color of v and its friends in k ways, the color of the remaining adjacent vertices to v in $(k-1)^{d(v)-l}$ ways, and the color of the rest of the vertices in $k^{n-1-d(v)}$ ways. Then

$$\begin{aligned} |A_v| = \sum_{l=m}^{n-1} |A_v^{(l)}| &= \sum_{l=m}^{n-1} \binom{d(v)}{l} k^{n-d(v)} (k-1)^{d(v)-l} \\ &= \binom{d(v)}{m} k^{n-m} + o(k^{n-m}). \end{aligned}$$

To complete the proof, it is sufficient to see that for any set S of at least 2 vertices $|\bigcap_{v \in S} A_v| = o(k^{n-m})$; clearly we may assume $S = \{v_1, v_2\}$. Consider the following cases:
Case 1 (v_1 and v_2 are not adjacent). Split A_{v_1} into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \neq v_2.$$

Note that each equivalence class C consists of k colorings, which only differ in the color of v_2. In addition, for each C at most $\frac{d(v_2)}{m}$ of its colorings are in A_{v_2}, as if $\sigma \in A_{v_2}$ there must be m vertices adjacent to v_2 with the color $\sigma(v_2)$. Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| \leq \sum_C \frac{d(v_2)}{m} = \frac{|A_{v_1}|}{k} \cdot \frac{d(v_2)}{m}.$$

It follows that $\frac{|A_{v_1} \cap A_{v_2}|}{|A_{v_1}|}$ goes to 0 as k goes to infinity, so $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$.
Case 2 (v_1 and v_2 are adjacent). Let W be the set of adjacent vertices to v_2 other than v_1. They are not adjacent to v_1 as T has no cycles. Split A_{v_1} into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \notin W.$$

Each equivalence class C consists of $k^{|W|}$ colorings, which may only differ in the colors of the vertices in W. If v_1 and v_2 are friends in the colorings of C, then a coloring in $|C \cap A_{v_2}|$ must contain at least $m-1$ vertices in W of the same color as v_2. Therefore

$$|C \cap A_{v_2}| = \sum_{l=m-1}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Notice that here we are using $m \geq 2$ so that $l \geq 1$. Otherwise, if v_1 and v_2 are not friends in the colorings of C, then

$$|C \cap A_{v_2}| = \sum_{l=m}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Therefore

$$\begin{aligned} |A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| &< \sum_C 2^{|W|} k^{|W|-1} \\ &= \frac{|A_{v_1}|}{k^{|W|}} \cdot 2^{|W|} k^{|W|-1} = \frac{|A_{v_1}| \cdot 2^{|W|}}{k} \end{aligned}$$

and $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$ follows as in the first case.

This completes the proof of the theorem. □

4 Conclusions

In conclusion, the degree-chromatic polynomial is a natural generalization of the usual chromatic polynomial, and it has a very particular structure when the graph is a tree. The leading terms of the chromatic polynomial are determined by the number of edges. Likewise, when $m \geq 2$, the leading coefficients of the degree chromatic polynomial $P_m(G)$ can be described easily in terms of G, but now they depend on the degree of the vertices of G.

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