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Laplacians of Covering Complexes

Richard Gustavson

Abstract. The Laplace operator on a simplicial complex encodes information about the adjacencies between simplices. A relationship between simplicial complexes does not always translate to a relationship between their Laplacians. In this paper we look at the case of covering complexes. A covering of a simplicial complex is built from many copies of simplices of the original complex, maintaining the adjacency relationships between simplices. We show that for dimension at least one, the Laplacian spectrum of a simplicial complex is contained inside the Laplacian spectrum of any of its covering complexes.

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1 Introduction

The combinatorial Laplacian of a simplicial complex has been extensively studied both in geometry and combinatorics. Combinatorial Laplacians were originally studied on graphs, beginning with Kirchhoff and his study of electrical networks in the mid-1800s. Simplicial complexes can be viewed as generalizations of graphs, and the graph Laplacian was likewise generalized to the combinatorial Laplacian of simplicial complexes, which is studied here. The study of the Laplacian of simplicial complexes is relatively recent, beginning in the mid-1970s [1]. See [5] for a more detailed history of the Laplacian.

The Laplacian has been shown to have many interesting properties. Most intriguing is the fact that some types of simplicial complexes, namely chessboard complexes [6], matching complexes [2], matroid complexes [8], and shifted complexes [5], all have integer Laplacian spectra. Matroid and shifted complexes also satisfy a recursion formula for calculating the Laplacian spectrum in terms of certain subcomplexes [4].

The motivation for this paper comes from the search for relationships between the Laplacians of different simplicial complexes. It also comes out of the study of covering complexes, the simplicial analogue of topological covering spaces, and their properties. Covering complexes have many applications outside of combinatorial theory; for example, the theory of covering complexes can be used to show the famous result that subgroups of free groups are free. See [12] for more information about covering complexes.

The combinatorial Laplace operator is not a topological invariant; thus even simplicial maps that preserve the underlying topological structure of a simplicial complex might change the Laplacian. Like topological spaces, complexes can have coverings. Our goal in this paper is to determine the relationship between the Laplacian of a simplicial complex and the Laplacians of its coverings. Our main theorem is the following relationship between the spectra of the two Laplacians.

Theorem. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\Delta}_d$ and Δ_d be the d^{th} Laplacian operators of $\widetilde{K}$ and K, respectively. Then for all $d \geq 1$,* $\mathrm{Spec}(\Delta_d) \subseteq \mathrm{Spec}(\widetilde{\Delta}_d)$.

Our goal in this paper is to prove this theorem. In Section 2, we give the definition of a simplicial complex and some terms associated with it. We introduce the boundary operator and its adjoint and provide a formulaic construction of the adjoint. In Section 3, we use the boundary operator to define the Homology groups $H_d(K)$ and the Laplace operator $\Delta_d(K)$. We then prove the main result of Combinatorial Hodge Theory, which says $H_d(K) \cong \ker(\Delta_d(K))$.

We introduce the notion of a covering complex in Section 4. After proving some simple relationships between a complex and its coverings, we prove our main theorem, that for dimension at least one, the Laplacian spectrum of a covering complex contains the Laplacian spectrum of the original complex.

2 Abstract Simplicial Complexes

This section is devoted to definitions and basic facts about simplicial complexes. The definitions here will be used throughout the paper. See [9,11] for more information about abstract simplicial complexes.

Definition. *An* ***abstract simplicial complex*** *K is a collection of finite sets that is closed under set inclusion, i.e. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.*

We will usually drop the word "abstract," and occasionally the word "simplicial," and just use the term "simplicial complex" or "complex." In addition, in this paper we will only deal with finite abstract simplicial complexes, i.e. only the case where $|K| < \infty$. A set $\sigma \in K$ is called a *simplex* of K. The *dimension* of a simplex σ is one less than the number of elements of σ. The *dimension* of K is the largest dimension of all of the simplices in K, or is infinite if there is no largest simplex. Since in this paper we will only discuss finite complexes, all complexes in this paper will have finite dimension. We call σ a *d-simplex* if it has dimension d.

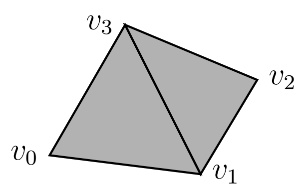
The *p-skeleton* of K, written $K^{(p)}$, is the set of all simplices of K of dimension less than or equal to p. The non-empty elements of the set $K^{(0)}$ are called the *vertices* of K. Occasionally we will refer to the 1-simplices as *edges*. According to the definition of a simplicial complex, the empty set $\emptyset$ be in K for all K, since $\emptyset \subseteq \sigma$ for all $\sigma \in K$ by basic set theory. We say that $\emptyset$ has dimension -1.

A simplicial complex K is *connected* if, for every pair of vertices $u, v \in K^{(0)}$, there is a sequence of vertices $\{v_1, \ldots, v_n\}$ in K such that $v_1 = u$, $v_n = v$, and $\{v_i, v_{i+1}\}$ is an edge in K for all $i = 1, \ldots, n-1$.

Definition. *Let K and L be two abstract simplicial complexes. A map $f : K^{(0)} \to L^{(0)}$ is called a* ***simplicial map*** *if whenever $\{v_0, \ldots, v_d\}$ is a simplex in K, then $\{f(v_0), \ldots, f(v_d)\}$ is a simplex in L.*

While a simplicial map f maps the vertices of K to the vertices of L, we will often speak of f as mapping K to L and write $f : K \to L$; thus if $\sigma \in K$ is a simplex, we will write $f(\sigma)$. Notice that if σ is a d-simplex in K, then $f(\sigma)$ need not be a d-simplex in L, as $f(\sigma)$ might in fact be of a lower dimension.

Example 2.1. Let K be the following simplicial complex:



K is a connected 2-dimensional complex. We list all of its simplices so the reader has a better understanding for the definition of a simplicial complex.

-1-simplices	0-simplices	1-simplices	2-simplices
$\emptyset$	$\{v_0\}$	$\{v_0, v_1\}$	$\{v_0, v_1, v_3\}$
	$\{v_1\}$	$\{v_0, v_3\}$	$\{v_1, v_2, v_3\}$
	$\{v_2\}$	$\{v_1, v_2\}$	
	$\{v_3\}$	$\{v_1, v_3\}$	
		$\{v_2, v_3\}$	

We see that K has a single -1-simplex (the empty set), four 0-simplices (the four vertices), five 1-simplices (the edges), and two 2-simplices. The way we listed the elements of the 1-simplices and 2-simplices was completely arbitrary; for example, we could have written $\{v_1, v_0\}$ instead of $\{v_0, v_1\}$. We make this distinction clear in the following discussion.

Given a d-simplex σ, there are $(d+1)!$ ways of ordering (i.e., listing) the $d+1$ vertices composing σ. We want a way to distinguish between possible orderings. Recall that a permutation is a bijection from a set to itself. A permutation of a finite set is called even if it consists of an even number of transpositions, i.e. interchanges of pairs of elements (see [3] for more information on permutations).

We define an equivalence relation on the set of orderings of σ as follows: we say that two orderings are equivalent if there is an even permutation sending one to the other. It is easy to check (see [11]) that this is an equivalence relation, and for $d > 0$, there are exactly two equivalence classes for each simplex σ. We call each of these equivalence classes an *orientation* of σ, and a simplex with an orientation is called an *oriented simplex*. An *oriented simplicial complex* K is one for which we have chosen an orientation for each of its simplices.

Given an oriented simplicial complex K, let $C_d(K)$ be the set of all formal $\mathbb{R}$-linear combinations of oriented d-simplices of K. The set $C_d(K)$ is then a vector space over $\mathbb{R}$ with the oriented d-simplices as a basis. Each element of $C_d(K)$ is called a *d-chain* (see [11] for a more formal construction of the d-chains). We will write C_d instead of $C_d(K)$ when the simplicial complex K is clear. In particular, note that $C_{-1}(K) = \mathbb{R}$ for all K, as $\emptyset$ is the only (-1)-simplex of K for all K.

Example 2.2. Let K be the two-dimensional simplicial complex from Example 2.1. We can orient the 1-simplices as they are written in the table above, giving oriented 1-simplices $[v_0, v_1]$, $[v_0, v_3]$, $[v_1, v_2]$, $[v_1, v_3]$, and $[v_2, v_3]$. Then

$$C_1(K) = a_1[v_0, v_1] + a_2[v_0, v_3] + a_3[v_1, v_2] + a_4[v_1, v_3] + a_5[v_2, v_3]$$

where the $a_i \in \mathbb{R}$. We can also orient the 2-simplices as in the table above, giving $[v_0, v_1, v_3]$, $[v_1, v_2, v_3]$, so we get

$$C_2(K) = b_1[v_0, v_1, v_3] + b_2[v_1, v_2, v_3]$$

with the $b_i \in \mathbb{R}$. We will look at this complex in future examples, and in all future examples we will use the orientations of the 1-simplices and 2-simplices given above. However, we could have given another orientation for both the 1-simplices and the 2-simplices, giving, for example, $[v_1, v_0]$, $[v_3, v_0]$, $[v_2, v_1]$, $[v_3, v_1]$, and $[v_3, v_2]$ as the oriented 1-simplices and $[v_0, v_1, v_3]$ and $[v_1, v_3, v_2]$ as the oriented 2-simplices. Notice, for example, that by the equivalence relation given above, the orderings $[v_0, v_1, v_3]$ and $[v_1, v_3, v_0]$ are equivalent.

Since the oriented d-simplices form a basis for C_d, we can define linear functions on C_d by defining how they act on the oriented d-simplices. We now define perhaps the most important linear function on C_d, the boundary operator.

Definition. *The **boundary operator** $\partial_d : C_d(K) \to C_{d-1}(K)$ is the linear function defined for each oriented d-simplex $\sigma = [v_0, \ldots, v_d]$ by*

$$\partial_d(\sigma) = \partial_d[v_0, \ldots, v_d] = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \widehat{v}_i, \ldots, v_d]$$

where $[v_0, \ldots, \widehat{v}_i, \ldots, v_d]$ is the subset of $[v_0, \ldots, v_d]$ obtained by removing the vertex v_i.

Example 2.3. We give some examples of calculating the boundary operator ∂_d. Let K be the simplicial complex defined in Example 2.1, with the orientation given in Example 2.2. Then

$$\begin{aligned} \partial_2[v_0, v_1, v_3] &= [v_1, v_3] - [v_0, v_3] + [v_0, v_1] \\ \partial_1[v_0, v_1] &= [v_1] - [v_0] \\ \partial_0[v_0] &= \emptyset. \end{aligned}$$

Notice that if $\sigma = [v] \in C_0(K)$, then $\partial_0(\sigma) = \emptyset \in C_{-1}(K)$, and since there are no simplices of dimension -2, $\partial_{-1}(\emptyset) = 0$ always. If $f : K \to L$ is a simplicial map, we define a homomorphism $f_\# : C_d(K) \to C_d(L)$ by defining it on basis elements (i.e. oriented simplices) as follows:

$$\begin{aligned} f_\#(\sigma) &= f_\#([v_0, \ldots, v_d]) \\ &= \begin{cases} [f(v_0), \ldots, f(v_d)], & \text{if } f(v_0), \ldots, f(v_d) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

We call the family $\{f_\#\}$ the *chain map induced by the simplicial map f.*

Technically speaking, each $f_\#$ acts only on one d-chain C_d. When we want to specify which dimension we are working with, we shall write f_d instead of $f_\#$. Chain maps have the special property that they commute with the boundary operator.

Lemma 2.4. *The homomorphism $f_\#$ commutes with the boundary operator ∂, that is,*

$$f_{d-1} \circ \partial_d = \partial_d \circ f_d.$$

Proof. Since both $f_\#$ and ∂ are linear, we need only show that the equation holds for basis elements. A simple computation gives

$$\begin{aligned} f_{d-1}(\partial_d[v_0, \ldots, v_d]) &= f_{d-1}\left(\sum_{i=0}^{d}(-1)^i[v_0, \ldots, \widehat{v}_i, \ldots, v_d]\right) \\ &= \sum_{i=0}^{d}(-1)^i f_{d-1}[v_0, \ldots, \widehat{v}_i, \ldots, v_d] \\ &= \partial_d(f_d[v_0, \ldots, v_d]). \end{aligned}$$

□

Since we are assuming that $|K| < \infty$, $C_d(K)$ is a finite dimensional vector space for all d, so we can define an inner product $\langle\,\rangle_d$ on $C_d(K)$ as follows: Let $\sigma_1, \ldots, \sigma_n$ be the oriented d-simplices of simplicial complex K, and let $a, b \in C_d$ be arbitrary elements of C_d, which we can write as:

$$a = \sum_{i=1}^{n} a_i \sigma_i \qquad\qquad b = \sum_{i=1}^{n} b_i \sigma_i$$

where the $a_i, b_i \in \mathbb{R}$ (this is possible since the σ_i form a basis for C_d). Then the inner product of a and b is given by

$$\langle a, b\rangle_d = \sum_{i=1}^{n} a_i b_i.$$

It is easy to check that this definition satisfies the properties of an inner product.

Example 2.5. Let K be the simplicial complex defined in Example 2.1. Let $a, b \in C_1$ be defined by $a = 2[v_0, v_1] - 3[v_1, v_2] + 4[v_2, v_3]$, $b = -[v_0, v_3] + 5[v_1, v_2] + [v_2, v_3]$. Then

$$\langle a, b\rangle_1 = 2 \cdot 0 + 0 \cdot (-1) + (-3) \cdot 5 + 4 \cdot 1 = -11.$$

Since each boundary operator $\partial_d : C_d \to C_{d-1}$ is a linear map, we can associate to it its adjoint operator $\partial_d^* : C_{d-1} \to C_d$ as the unique linear operator that satisfies

$$\langle \partial_d(a), b\rangle_{d-1} = \langle a, \partial_d^*(b)\rangle_d$$

where $\langle\,\rangle_{d-1}$ and $\langle\,\rangle_d$ are the inner products on C_{d-1} and C_d, respectively, and $a \in C_d, b \in C_{d-1}$. Since ∂_d and ∂_d^* are both linear, they both have associated matrices, which we call $\mathcal{B}_d$ and $\mathcal{B}_d^T$, respectively (where here, $\mathcal{B}_d^T$ is the transpose of $\mathcal{B}_d$, as ∂_d^* is the adjoint of ∂_d).

We now give a way of calculating ∂_d^*. Let $S_d(K)$ be the set of all oriented d-simplices of the simplicial complex K (i.e. the set of basis elements of $C_d(K)$), and let $\tau \in S_{d-1}(K)$. Then define the two sets

$$S_d^+(K,\tau) = \{\sigma \in S_d(K) | \text{the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } +1\}$$
$$S_d^-(K,\tau) = \{\sigma \in S_d(K) | \text{the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } -1\}.$$

Notice that S_d^+ and S_d^- are only defined for $d \geq 0$, and since $S_{-1}(K) = \{\emptyset\}$ with $\partial_0(\sigma) = \emptyset$ for all $\sigma \in S_0(K)$, when $d = 0$ we have $S_0^+(K,\emptyset) = S_0(K)$ and $S_0^-(K,\emptyset) = \emptyset$. We now give an explicit formula for calculating ∂_d^*, which we will use later on in proving Theorem 4.4.

Theorem 2.6. *Let ∂_d^* be the adjoint of the boundary operator ∂_d, and let $\tau \in S_{d-1}(K)$. Then*

$$\partial_d^*(\tau) = \sum_{\sigma' \in S_d^+(K,\tau)} \sigma' \; - \sum_{\sigma'' \in S_d^-(K,\tau)} \sigma'' \; .$$

Proof. Let $f : C_{d-1} \to C_d$ be defined on basis elements by

$$f(\tau) = \sum_{\sigma' \in S_d^+(K,\tau)} \sigma' \; - \sum_{\sigma'' \in S_d^-(K,\tau)} \sigma'' \; .$$

We show that $\langle \partial_d(a), b \rangle_{d-1} = \langle a, f(b) \rangle_d$, for $a \in C_d, b \in C_{d-1}$, for then the function f will satisfy the requirements for the adjoint operator, and since the adjoint is unique, we will have $f = \partial_d^*$. First observe that f is linear, so (since ∂_d is also linear) we only need to show $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$ for σ, τ basis elements, i.e. $\sigma \in S_d(K), \tau \in S_{d-1}(K)$.

Look at the term $\langle \partial_d(\sigma), \tau \rangle_{d-1}$. As τ is a single simplex, $\langle \partial_d(\sigma), \tau \rangle_{d-1} \neq 0$ if and only if τ is in the sum $\partial_d(\sigma)$, that is, if and only if $\tau \subseteq \sigma$. Since the coefficient of every term of ∂_d is ± 1, we see that

$$\langle \partial_d(\sigma), \tau \rangle_{d-1} = \begin{cases} 1, & \tau \subseteq \sigma \text{ and the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } +1 \\ -1, & \tau \subseteq \sigma \text{ and the coefficient of } \tau \text{ in } \partial_d(\sigma) \text{ is } -1 \\ 0, & \tau \nsubseteq \sigma \end{cases}$$

Now look at the term $\langle \sigma, f(\tau) \rangle_d$. Since σ is a single simplex, $\langle \sigma, f(\tau) \rangle_d \neq 0$ if and only if σ is in the sum $f(\tau)$, that is, if and only if $\sigma \supseteq \tau$. Since the coefficient of every term of $f(\tau)$ is ± 1, we see that

$$\langle \sigma, f(\tau) \rangle_d = \begin{cases} 1, & \sigma \supseteq \tau \text{ and the coefficient of } \sigma \text{ in } f(\tau) \text{ is } +1 \\ -1, & \sigma \supseteq \tau \text{ and the coefficient of } \sigma \text{ in } f(\tau) \text{ is } -1 \\ 0, & \sigma \nsupseteq \tau \end{cases}$$

By the definition of f, however, we have that the coefficient of τ in $\partial_d(\sigma)$ is $+1$ if and only if the coefficient of σ in $f(\tau)$ is $+1$, and the coefficient of τ in $\partial_d(\sigma)$ is -1 if and only if the coefficient of σ in $f(\tau)$ is -1. Thus we have that $\langle \partial_d(\sigma), \tau \rangle_{d-1} = \langle \sigma, f(\tau) \rangle_d$, so by definition of the adjoint, $f = \partial_d^*$. $\square$

Example 2.7. Let K be the simplicial complex from Example 2.1. Then

$$\begin{aligned}
\partial_0^*(\emptyset) &= [v_0] + [v_1] + [v_2] + [v_3] \\
\partial_1^*[v_1] &= [v_0, v_1] - [v_1, v_2] - [v_1, v_3] \\
\partial_2^*[v_1, v_3] &= [v_0, v_1, v_3] - [v_1, v_2, v_3].
\end{aligned}$$

3 Homology Groups and the Laplacian

With all of this information at hand, we can define the Homology groups of a simplicial complex. First we need a lemma:

Lemma 3.1. *If K is a simplicial complex, the composition $\partial_{d-1} \circ \partial_d = 0$.*

Proof. A simple computation on basis elements gives

$$\begin{aligned}
\partial_{d-1}(\partial_d(\sigma)) &= \partial_{d-1}\left(\sum_{i=0}^{d}(-1)^i[v_0, \ldots, \widehat{v_i}, \ldots, v_d]\right) \\
&= \sum_{j<i}(-1)^i(-1)^j[v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_d] \\
&\qquad\qquad + \sum_{j>i}(-1)^{j-1}(-1)^i[v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_d] \\
&= \sum_{j<i}(-1)^i(-1)^j[v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_d] \\
&\qquad\qquad + \sum_{i>j}(-1)^{i-1}(-1)^j[v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_d] \\
&= 0.
\end{aligned}$$

□

As a result, we see that $\operatorname{im}(\partial_{d+1}) \subseteq \ker(\partial_d)$. Thus if we think of $\ker(\partial_d)$ and $\operatorname{im}(\partial_{d+1})$ as groups (they are both abelian groups, since they are vector spaces), we can define the d^{th} *homology group* $H_d(K)$ as the quotient group $H_d(K) = \ker(\partial_d)/\operatorname{im}(\partial_{d+1})$.

The homology groups of a complex are a topological invariant, that is, if K and K' are homeomorphic as topological spaces, then $H_d(K) = H_d(K')$; see [11] for a proof. We will see how the homology groups are related the the Laplace operator shortly.

We now define the combinatorial Laplace operator and the Laplacian spectrum for a simplicial complex.

Definition. *Let K be a finite oriented complex. The* ***d^{th} combinatorial Laplacian*** *is the linear operator $\Delta_d : C_d(K) \to C_d(K)$ given by*

$$\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d.$$

The d^{th} *Laplacian matrix* of K, denoted $\mathcal{L}_d$, with respect to the standard bases for C_d and C_{d-1}, is the matrix representation of Δ_d, given by

$$\mathcal{L}_d = \mathcal{B}_{d+1}\mathcal{B}_{d+1}^T + \mathcal{B}_d^T\mathcal{B}_d.$$

Note that the combinatorial Laplacian is actually a set of operators, one for each dimension in the complex. Since the product of a matrix and its transpose is symmetric, both $\mathcal{B}_d^T\mathcal{B}_d$ and $\mathcal{B}_{d+1}\mathcal{B}_{d+1}^T$ are symmetric, and thus so is $\mathcal{L}_d$. As a result, $\mathcal{L}_d$ is real diagonalizable, so the Laplacian Δ_d has a complete set of real eigenvalues. The d^{th} *Laplacian spectrum* of a finite oriented simplicial complex K, denoted $\mathrm{Spec}(\Delta_d(K))$, is the multiset of eigenvalues of the Laplacian $\Delta_d(K)$.

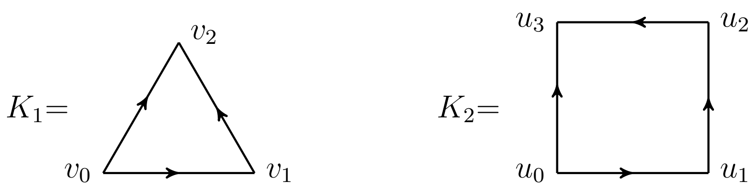
The Laplacian acts on an oriented simplicial complex. However, simplicial complexes are not naturally oriented. Notice that when we constructed the boundary operator, and thus the Laplacian, we gave the simplicial complex an arbitrary orientation. This might lead one to believe that the same simplicial complex could produce different Laplacian spectra for different orientations of its simplices. However, this is not the case, as is shown in the following theorem. See [7] for the proof.

Theorem 3.2. *Let K be a finite simplicial complex. Then Spec$(\Delta_d(K))$ is independent of the choice of orientation of the d-simplices of K.*

As a result, we can speak of the Laplacian spectrum of a simplicial complex without regard to its orientation.

Every simplicial complex can be embedded in $\mathbb{R}^n$ for some n, and thus can be considered a topological space (see [11] for the proof). It is possible for two different simplicial complexes to embed in $\mathbb{R}^n$ as the same topological space; any cycle, for example, is homeomorphic to a circle. A natural question to ask, then, is whether the Laplacian is a topological invariant; that is, whether different simplicial complexes that are homeomorphic as topological spaces have the same Laplacian. The answer is no, the Laplacian is not a topological invariant. We show this with an example.

Example 3.3. Let K_1 and K_2 be the following one-dimensional oriented simplicial complexes:



Notice that both K_1 and K_2 are graphs, and the edges of the graph are exactly the 1-simplices of the complexes. Clearly K_1 and K_2 are topologically equivalent; they are both cycles, and thus both homeomorphic to the circle. However, it is easily seen that

$$\mathcal{L}_1(K_1) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \qquad \mathcal{L}_1(K_2) = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}.$$

Thus the Laplacian operators on K_1 and K_2 are not the same. Even the spectra of the two Laplacians are not the same, as we have $\mathrm{Spec}(\Delta_1(K_1)) = \{0, 3, 3\}$ and $\mathrm{Spec}(\Delta_1(K_2)) = \{0, 2, 2, 4\}$.

It should be noted, however, that the kernel of the Laplacian is a topological invariant, which we now show. The following argument is inspired by [13].

Lemma 3.4. *The kernel of the Laplacian can be characterized by*

$$\ker(\Delta_d) = \{a \in C_d \mid \partial_d(a) = \partial^*_{d+1}(a) = 0\}.$$

Proof. First assume that $\partial_d(a) = \partial^*_{d+1}(a) = 0$. Then by definition

$$\Delta_d(a) = \partial_{d+1}(\partial^*_{d+1}(a)) + \partial^*_d(\partial_d(a)) = \partial_{d+1}(0) + \partial^*_d(0) = 0,$$

so $a \in \ker(\Delta_d)$. Now suppose $a \in \ker(\Delta_d)$. Then since $\partial_{d+1}(\partial^*_{d+1}(a)) + \partial^*_d(\partial_d(a)) = 0$, we have

$$\begin{aligned} 0 &= \langle \partial_{d+1}(\partial^*_{d+1}(a)) + \partial^*_d(\partial_d(a)), a\rangle \\ &= \langle \partial_{d+1}(\partial^*_{d+1}(a)), a\rangle + \langle \partial^*_d(\partial_d(a)), a\rangle \text{ , since the inner product is bilinear} \\ &= \langle \partial^*_{d+1}(a), \partial^*_{d+1}(a)\rangle + \langle \partial_d(a), \partial_d(a)\rangle \text{ , since } \partial_d, \partial^*_d \text{ are adjoint operators.} \end{aligned}$$

Since $\langle b, b\rangle > 0$ for all $b \neq 0$, this means that $\partial_d(a) = \partial^*_{d+1}(a) = 0$, completing the proof. □

Recall that if V is a vector space with inner product $\langle\,\rangle$ and U is a subspace of V, then the subspace $U^\perp = \{v \in V \mid \langle v, u\rangle = 0 \text{ for all } u \in U\}$ is called the *orthogonal complement* of U. We can always decompose V as the direct sum $V = U \oplus U^\perp$. Since $\ker(\Delta_d) \subseteq C_d(K)$, it too has an orthogonal complement $(\ker(\Delta_d))^\perp$ such that $C_d(K) = \ker(\Delta_d) \oplus (\ker(\Delta_d))^\perp$.

Lemma 3.5. *The orthogonal complement of* $\ker(\Delta_d) \subseteq C_d(K)$ *is*

$$(\ker(\Delta_d))^\perp = \mathrm{im}(\Delta_d).$$

Proof. First we show $\mathrm{im}(\Delta_d)) \subseteq (\ker(\Delta_d))^\perp$. To do this, we must show that if $a \in \mathrm{im}(\Delta_d)$, then $\langle a, b\rangle = 0$ for all $b \in \ker(\Delta_d)$. Since $a \in \mathrm{im}(\Delta_d)$, there is a $c \in C_d$ such that $\Delta_d(c) = a$. Thus for all $b \in \ker(\Delta_d)$ we have

$$\begin{aligned} \langle a, b\rangle = \langle \Delta_d(c), b\rangle &= \langle c, \Delta_d(b)\rangle \text{ , since } \Delta_d \text{ is symmetric} \\ &= \langle c, 0\rangle \text{ , since } b \in \ker(\Delta_d) \\ &= 0. \end{aligned}$$

Thus $\mathrm{im}(\Delta_d) \subseteq (\ker(\Delta_d))^\perp$. Now we show the opposite inclusion. Since Δ_d is a symmetric linear map on C_d, by the Spectral Theorem (see [10], Theorem 15.7.1) there is a complete set of orthonormal eigenvectors of Δ_d, i.e. there exist $v_1, \ldots, v_n \in C_d$ such that $\langle v_i, v_j \rangle = \delta_{ij}$ and $\Delta_d(v_i) = \lambda_i v_i$ for some eigenvalue $\lambda_i \in \mathbb{R}$. Without loss of generality we can assume $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_i \neq 0$ for $i = k+1, \ldots, n$, i.e. $v_1, \ldots, v_k$ form a basis for $\ker(\Delta_d)$ and $v_{k+1}, \ldots, v_n$ form a basis for $(\ker(\Delta_d))^\perp$.

For any $a \in C_d$, we can write $a = \sum \alpha_i v_i$ with $\alpha_i \in \mathbb{R}$. If $a \in (\ker(\Delta_d))^\perp$, then $\langle a, b \rangle = 0$ for all $b \in \ker(\Delta_d)$. In particular, $\langle a, v_i \rangle = 0$ for all $i = 1, \ldots, k$. But $\langle a, v_i \rangle = \alpha_i$, so this means $\alpha_i = 0$ for all $i = 1, \ldots, k$, so we can write

$$a = \sum_{i=k+1}^{n} \alpha_i v_i.$$

We claim that $a \in \mathrm{im}(\Delta_d)$. Define $c \in C_d$ by

$$c = \sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} v_i.$$

The vector c is well-defined, since $\lambda_i \neq 0$ for all $i = k+1, \ldots, n$. Then

$$\begin{aligned} \Delta_d(c) &= \Delta_d \left(\sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} v_i \right) = \sum_{i=k+1}^{n} \Delta_d \left(\frac{\alpha_i}{\lambda_i} v_i \right) \\ &= \sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} \Delta_d(v_i) = \sum_{i=k+1}^{n} \frac{\alpha_i}{\lambda_i} \lambda_i v_i \\ &= \sum_{i=k+1}^{n} \alpha_i v_i = a. \end{aligned}$$

Thus $(\ker(\Delta_d))^\perp \subseteq \mathrm{im}(\Delta_d)$, so they are equal. □

As a result, we can decompose $C_d(K)$ as the direct sum $C_d(K) = \ker(\Delta_d) \oplus \mathrm{im}(\Delta_d)$. In fact, we can go further, as is seen in the following lemma:

Lemma 3.6. *The space of d-chains $C_d(K)$ can be decomposed as*

$$C_d(K) = \ker(\Delta_d) \oplus \mathrm{im}(\partial_{d+1}) \oplus \mathrm{im}(\partial_d^*).$$

Proof. We have shown that we can decompose C_d as $C_d = \ker(\Delta_d) \oplus \mathrm{im}(\Delta_d)$. Thus it suffices to show that $\mathrm{im}(\Delta_d) = \mathrm{im}(\partial_{d+1}) \oplus \mathrm{im}(\partial_d^*)$. First we show that $\mathrm{im}(\partial_{d+1})$ and $\mathrm{im}(\partial_d^*)$ are orthogonal, so that $\mathrm{im}(\partial_{d+1}) \oplus \mathrm{im}(\partial_d^*)$ is well defined; i.e. we must show that if $a \in \mathrm{im}(\partial_{d+1})$ and $b \in \mathrm{im}(\partial_d^*)$, then $\langle a, b \rangle = 0$. Since $a \in \mathrm{im}(\partial_{d+1})$, there is an $a' \in C_{d+1}$ such that

$\partial_{d+1}(a') = a$. Similarly, since $b \in \text{im}(\partial_d^*)$, there is a $b' \in C_{d-1}$ such that $\partial_d^*(b') = b$. We then have

$$\begin{aligned}\langle a, b\rangle &= \langle \partial_{d+1}(a'), \partial_d^*(b')\rangle \\ &= \langle \partial_d(\partial_{d+1}(a')), b'\rangle \text{ , since } \partial_d, \partial_d^* \text{ are adjoint operators} \\ &= \langle 0, b'\rangle \text{ , by Lemma 3.1} \\ &= 0.\end{aligned}$$

Thus the direct sum $\text{im}(\partial_{d+1}) \oplus \text{im}(\partial_d^*)$ is well-defined. Now we show $\text{im}(\Delta_d) \subseteq \text{im}(\partial_{d+1}) \oplus \text{im}(\partial_d^*)$. Let $a \in \text{im}(\Delta_d)$, so there is a $b \in C_d$ such that $\Delta_d(b) = a$. By definition $\Delta_d(b) = \partial_{d+1}(\partial_{d+1}^*(b)) + \partial_d^*(\partial_d(b))$. Setting $\alpha = \partial_{d+1}^*(b)$ and $\beta = \partial_d(b)$, this becomes $a = \Delta_d(b) = \partial_{d+1}(\alpha) + \partial_d^*(\beta)$, so $a = u+v$ for $u \in \text{im}(\partial_{d+1})$ and $v \in \text{im}(\partial_d^*)$, so $\text{im}(\Delta_d) \subseteq \text{im}(\partial_{d+1}) \oplus \text{im}(\partial_d^*)$.

Now we show that $\ker(\Delta_d)$ is orthogonal to both $\text{im}(\partial_{d+1})$ and $\text{im}(\partial_d^*)$, i.e. if $v \in \ker(\Delta_d)$, $a \in \text{im}(\partial_{d+1})$, and $b \in \text{im}(\partial_d^*)$, then $\langle v, a\rangle = \langle v, b\rangle = 0$. Since $a \in \text{im}(\partial_{d+1})$, there is an $a' \in C_{d+1}$ such that $\partial_{d+1}(a') = a$, and since $b \in \text{im}(\partial_d^*)$, there is a $b' \in C_{d-1}$ such that $\partial_d^*(b') = b$. Thus

$$\begin{aligned}\langle v, a\rangle &= \langle v, \partial_{d+1}(a')\rangle \\ &= \langle \partial_{d+1}^*(v), a'\rangle \text{ , since } \partial_{d+1}, \partial_{d+1}^* \text{ are adjoint operators} \\ &= \langle 0, a'\rangle \text{ , by Lemma 3.4} \\ &= 0 \\ \langle v, b\rangle &= \langle v, \partial_d^*(b')\rangle \\ &= \langle \partial_d(v), b'\rangle \text{ , since } \partial_d, \partial_d^* \text{ are adjoint operators} \\ &= \langle 0, b'\rangle \text{ , by Lemma 3.4} \\ &= 0\end{aligned}$$

As a result, $\text{im}(\partial_{d+1})$ and $\text{im}(\partial_d^*)$ are both contained in the orthogonal complement of $\ker(\Delta_d)$, and thus so is their direct sum, i.e. $\text{im}(\partial_{d+1}) \oplus \text{im}(\partial_d^*) \subseteq (\ker(\Delta_d))^\perp = \text{im}(\Delta_d)$, the last equality by Lemma 3.5. Thus we have $\text{im}(\Delta_d) = \text{im}(\partial_{d+1}) \oplus \text{im}(\partial_d^*)$, so $C_d = \ker(\Delta_d) \oplus \text{im}(\Delta_d) = \ker(\Delta_d) \oplus \text{im}(\partial_{d+1}) \oplus \text{im}(\partial_d^*)$. $\square$

Lemma 3.7. *The kernel of the map ∂_d can be decomposed as*

$$\ker(\partial_d) = \ker(\Delta_d) \oplus \text{im}(\partial_{d+1}).$$

Proof. First observe that both $\ker(\Delta_d)$ and $\text{im}(\partial_{d+1})$ are contained in $\ker(\partial_d)$, the former by Lemma 3.4 and the latter by Lemma 3.1, so $\ker(\Delta_d) \oplus \text{im}(\partial_{d+1}) \subseteq \ker(\partial_d)$ (note that this is a valid direct sum by the previous lemma).

We now must show that $\ker(\partial_d) \subseteq \ker(\Delta_d) \oplus \text{im}(\partial_{d+1})$. We do this by showing that $\ker(\partial_d)$ is orthogonal to $\text{im}(\partial_d^*)$. Let $v \in \ker(\partial_d)$ and $u \in \text{im}(\partial_d^*)$, so $\partial_d(v) = 0$ and there is a $w \in C_{d-1}$ with $\partial_d^*(w) = u$. Then

$$\langle v, u\rangle = \langle v, \partial_d^*(w)\rangle = \langle \partial_d(v), w\rangle = \langle 0, w\rangle = 0.$$

Thus $\ker(\partial_d) \subseteq (\text{im}(\partial_d^*))^\perp = \ker(\Delta_d) \oplus \text{im}(\partial_{d+1})$, completing the proof. $\square$

We can now prove the Combinatorial Hodge Theory. With all that we have done to this point, the proof is now trivial.

Theorem 3.8. *If K is a simplicial complex, then* $\ker(\Delta_d(K)) \cong H_d(K)$.

Proof. By definition $H_d(K) = \ker(\partial_d)/\operatorname{im}(\partial_{d+1})$. Thus by Lemma 3.7, we have

$$H_d(K) = \ker(\partial_d)/\operatorname{im}(\partial_{d+1}) = (\ker(\Delta_d) \oplus \operatorname{im}(\partial_{d+1}))/\operatorname{im}(\partial_{d+1}) \cong \ker(\Delta_d).$$

$\square$

Since the homology group $H_d(K)$ is a topological invariant, by Theorem 3.8 this means that $\ker(\Delta_d(K))$ is a topological invariant as well. In light of Example 3.3, $\ker(\Delta_d(K))$ is most likely the only topological invariant of the Laplacian.

4 Covering Complexes

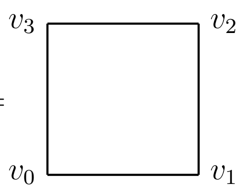
We can think of simplicial complexes as topological spaces. As we have just shown, however, the Laplacian is not a topological invariant. Thus, while two complexes might be topologically homeomorphic, they could have very different Laplacian spectra. Our goal in this section is to show that if two simplicial complexes are related by a covering map, then their Laplacian spectra are also related. We begin with the definition of a covering complex. This definition comes from [12], and is similar to the definition of a topological covering space.

Definition. *Let K be a simplicial complex. A pair $(\widetilde{K}, p)$ is a **covering complex** of K if:*

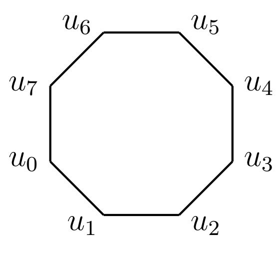
1. *$\widetilde{K}$ is a connected simplicial complex.*
2. *$p : \widetilde{K} \to K$ is a simplicial map.*
3. *For every simplex $\sigma \in K$, $p^{-1}(\sigma)$ is a union of pairwise disjoint simplices, $p^{-1}(\sigma) = \bigcup \widetilde{\sigma}_i$, with $p|_{\widetilde{\sigma}_i} : \widetilde{\sigma}_i \to \sigma$ a bijection for each i.*

Example 4.1. Let K and $\widetilde{K}$ be the following simplicial complexes:

$K=$



$\widetilde{K}=$



We see that $\widetilde{K}$ is connected. Define the map $p : \widetilde{K}^{(0)} \to K^{(0)}$ by

$$p(u_i) = \begin{cases} v_i & 0 \leq i < 4 \\ v_{i-4} & 4 \leq i \leq 7 \end{cases}$$

One can easily check that p is a simplicial map and that condition (3) above is satisfied, so that $(\widetilde{K}, p)$ is a covering complex of K.

Covering complexes are the simplicial complex equivalent of the covering spaces of a topological space. The reason we require $\widetilde{K}$ to be connected is to exclude the trivial case where $\widetilde{K}$ is the disjoint union of some number of copies of K.

Since a covering is a simplicial map, there is a chain map associated to it. Let $(\widetilde{K}, p)$ be a covering of an oriented complex K. Define the *chain covering map* $p_\# : C_d(\widetilde{K}) \to C_d(K)$ to be the chain map induced by the covering map p. Notice that by definition of p, if $\sigma = \{v_0, \ldots, v_d\} \in S_d(\widetilde{K})$, then $p(\sigma) = \{p(v_0), \ldots, p(v_d)\} \in S_d(K)$ (i.e. the $p(v_i)$ are distinct), so we can define $p_\#$ on basis elements by

$$p_\#(\sigma) = p_\#([v_0, \ldots, v_d]) = [p(v_0), \ldots, p(v_d)].$$

Again, if we want to specify which dimension the chain covering acts on, we will write p_d instead of $p_\#$. By Lemma 2.4, we see that $p_\#$ commutes with the boundary operator ∂. Normally, a chain map will not commute with the adjoint boundary operator. We now show, however, that for the chain covering they do commute. We will use the following lemma to show that the Laplacian Δ_d commutes with the chain covering $p_\#$ when $d \geq 1$.

Lemma 4.2. *If $d \geq 1$, the adjoint boundary operator ∂^* commutes with the chain covering $p_\#$, that is,*

$$p_d \circ \partial_d^* = \partial_d^* \circ p_{d-1}.$$

Proof. Since both ∂^* and $p_\#$ are linear, we only need to look at one basis element $\tau \in S_{d-1}(\widetilde{K})$, that is we must show

$$p_d \circ \partial_d^*(\tau) = \partial_d^* \circ p_{d-1}(\tau).$$

Using the formula for ∂^* from Theorem 2.6, we see that

$$\begin{aligned} p_d \circ \partial_d^*(\tau) &= p_d \left(\sum_{\sigma' \in S_d^+(\widetilde{K}, \tau)} \sigma' - \sum_{\sigma'' \in S_d^-(\widetilde{K}, \tau)} \sigma'' \right) \\ &= \sum_{\sigma' \in S_d^+(\widetilde{K}, \tau)} p_d(\sigma') - \sum_{\sigma'' \in S_d^-(\widetilde{K}, \tau)} p_d(\sigma''), \end{aligned}$$

the last step because $p_\#$ is linear. In addition, we see that

$$\partial_d^* \circ p_{d-1}(\tau) = \sum_{\eta' \in S_d^+(K, p_{d-1}(\tau))} \eta' - \sum_{\eta'' \in S_d^-(K, p_{d-1}(\tau))} \eta''.$$

First we show that for every $\sigma \in S_d(\widetilde{K})$ such that $\sigma \supseteq \tau$, there is exactly one $\eta \in S_d(K)$ such that $\eta \supseteq p(\tau)$ and $p(\sigma) = \eta$, and conversely (i.e. for every $\eta \in S_d(K)$ with $\eta \supseteq p(\tau)$ there is exactly one $\sigma \in S_d(\widetilde{K})$ with $\sigma \supseteq \tau$ and $p(\sigma) = \eta$).

Pick a $\sigma \in S_d(\widetilde{K})$ with $\sigma \supseteq \tau$. Since $p|_\sigma$ is a bijection, $p(\sigma)$ is unique and is in $S_d(K)$. But $\tau \subseteq \sigma$, so $p(\tau) \subseteq p(\sigma)$, so $\eta = p(\sigma)$ is the unique d-simplex satisfying the requirements.

Now pick an $\eta \in S_d(K)$ with $\eta \supseteq p(\tau)$. Look at $p^{-1}(\eta) = \bigcup \sigma_i$ with $\sigma_i \cap \sigma_j = \emptyset$ if $i \neq j$ and $p|_{\sigma_i}$ a bijection. Since $p(\tau) \subseteq \eta$, $p^{-1}(p(\tau)) \subseteq p^{-1}(\eta)$. But $\tau \subseteq p^{-1}(p(\tau))$, so $\tau \subseteq p^{-1}(\eta)$. Since τ is a simplex and $\dim(\tau) \geq 0$ (as $\tau \in S_{d-1}(\widetilde{K})$ and $d \geq 1$), τ is connected, so it lies in exactly one of the σ_i in the inverse image of η. Call this unique simplex $\sigma \in S_d(\widetilde{K})$. Then $p(\sigma) = \eta$ and $\tau \subseteq \sigma$, with σ clearly unique by construction.

Now observe that since $p_\#$ simply assigns an orientation to each simplex in addition to performing the action of p, the above statement also holds for $p_\#$, i.e. if $\tau \in C_{d-1}(\widetilde{K})$ a basis element, then for every $\sigma \in C_d(\widetilde{K})$ a basis element such that $\sigma \supseteq \tau$, there is exactly one $\eta \in C_d(K)$ a basis element such that $\eta \supseteq p_{d-1}(\tau)$ and $p_d(\sigma) = \eta$, and for every $\eta \in C_d(K)$ a basis element with $\eta \supseteq p_{d-1}(\tau)$ there is exactly one $\sigma \in C_d(\widetilde{K})$ a basis element with $\sigma \supseteq \tau$ and $p_d(\sigma) = \eta$. Thus we see that for every term in $p_d \circ \partial_d^*(\tau)$, there is exactly one term in $\partial_d^* \circ p_{d-1}(\tau)$, and vice-versa; we now show that these terms are equal.

First, pick a $\sigma \in S_d^+(\widetilde{K}, \tau)$. We know that $p(\sigma) = \eta$ for some $\eta \in C_d(K)$. There are two cases, corresponding to $p_\#$ either preserving the orientation of σ or reversing it:

1. $p_d(\sigma) = \eta$
2. $p_d(\sigma) = -\eta$

First we show (1). By definition, $\sigma \in S_d^+(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^+(K, p_{d-1}(\tau))$. Now η is a basis element of $C_d(K)$, so $p_d(\sigma)$ is also. Thus the coefficient of the basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is $+1$, and the coefficient of basis element $p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is $+1$, proving case (1).

Now we prove the case for (2). By definition, $\sigma \in S_d^+(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $-\eta \in S_d^+(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^-(K, p_{d-1}(\tau))$ (this last equivalence is because switching the orientation switches the sign). Now η is a basis element of $C_d(K)$, so $-p_d(\sigma)$ is a basis element of $C_d(K)$. Thus the coefficient of basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is -1, and the coefficient of non-basis element $p_d(\sigma)$ in $p_d \circ \partial_d^*(\tau)$ is $+1$. But we want everything in terms of basis elements, so the coefficient of basis element $-p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is -1, proving case (2).

Now pick a $\sigma \in S_d^-(\widetilde{K}, \tau)$. We know that $p(\sigma) = \eta$ for some $\eta \in C_d(K)$. There are two cases, corresponding to $p_\#$ either preserving the orientation of σ or reversing it:

1. $p_d(\sigma) = \eta$
2. $p_d(\sigma) = -\eta$

First we show (1). By definition, $\sigma \in S_d^-(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^-(K, p_{d-1}(\tau))$. Now η is a basis element of $C_d(K)$, so $p_d(\sigma)$ is also. Thus the coefficient of the basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is -1, and the coefficient of basis element $p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is -1, proving case (1).

Now we prove the case for (2). By definition, $\sigma \in S_d^-(\widetilde{K}, \tau)$ if and only if $p_d(\sigma) \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $-\eta \in S_d^-(K, p_{d-1}(\tau))$, which is true if and only if $\eta \in S_d^+(K, p_{d-1}(\tau))$ (this last equivalence is because switching the orientation switches the sign). Now η is a basis element of $C_d(K)$, so $-p_d(\sigma)$ is a basis element of $C_d(K)$. Thus the coefficient of basis element η in $\partial_d^* \circ p_{d-1}(\tau)$ is $+1$, and the coefficient of non-basis element $p_d(\sigma)$ in $p_d \circ \partial_d^*(\tau)$ is -1. But we want everything in terms of basis elements, so the coefficient of basis element $-p_d(\sigma) = \eta$ in $p_d \circ \partial_d^*(\tau)$ is $+1$, proving case (2).

Thus we have that $p_d \circ \partial_d^*(\tau) = \partial_d^* \circ p_{d-1}(\tau)$, completing the proof. □

With Lemmas 2.4 and 4.2, we can now prove that the Laplacian and the chain covering commute.

Theorem 4.3. *If $d \geq 1$, the Laplacian Δ_d commutes with the chain covering $p_\#$, that is,*

$$\Delta_d \circ p_d = p_d \circ \Delta_d.$$

Proof. By definition, we have that $\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d$. By Lemma 2.4, we know that $\partial_d \circ p_d = p_{d-1} \circ \partial_d$, and by Lemma 4.2, since $d \geq 1$ we know that $\partial_d^* \circ p_{d-1} = p_d \circ \partial_d^*$. Thus we have that

$$\begin{aligned}
(\partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d) \circ p_d &= (\partial_{d+1} \circ \partial_{d+1}^*) \circ p_d + (\partial_d^* \circ \partial_d) \circ p_d \\
&= \partial_{d+1} \circ (\partial_{d+1}^* \circ p_d) + \partial_d^* \circ (\partial_d \circ p_d) \\
&= \partial_{d+1} \circ (p_{d+1} \circ \partial_{d+1}^*) + \partial_d^* \circ (p_{d-1} \circ \partial_d) \\
&= (\partial_{d+1} \circ p_{d+1}) \circ \partial_{d+1}^* + (\partial_d^* \circ p_{d-1}) \circ \partial_d \\
&= (p_d \circ \partial_{d+1}) \circ \partial_{d+1}^* + (p_d \circ \partial_d^*) \circ \partial_d \\
&= p_d \circ (\partial_{d+1} \circ \partial_{d+1}^*) + p_d \circ (\partial_d^* \circ \partial_d) \\
&= p_d \circ (\partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d).
\end{aligned}$$

Thus $\Delta_d \circ p_d = p_d \circ \Delta_d$, completing the proof. □

We can now prove that the spectrum of a covering complex contains the spectrum of the original complex when the dimension is at least one. Recall that the d^{th} *Laplacian spectrum* of a simplicial complex K, denoted $\mathrm{Spec}(\Delta_d(K))$, is the multiset of eigenvalues of the Laplacian $\Delta_d(K)$.

Theorem 4.4. *Let $(\widetilde{K}, p)$ be a covering complex of simplicial complex K, and let $\widetilde{\Delta}_d$ and Δ_d be the Laplacian operators of $\widetilde{K}$ and K, respectively. Then for all $d \geq 1$, Spec$(\Delta_d) \subseteq$ Spec$(\widetilde{\Delta}_d)$.*

Proof. By definition the map $p_\#$ is surjective. Let $\ker(p_\#)$ be the kernel of $p_\#$. Then $\widetilde{\Delta}_d$ carries $\ker(p_\#)$ to itself, for if $\sigma \in \ker(p_\#)$, then $p_\#(\sigma) = 0$, which implies $\Delta_d(p_\#(\sigma)) = 0$, which by Theorem 4.3 (since $d \geq 1$) implies that $p_\#(\widetilde{\Delta}_d(\sigma)) = 0$, which implies that $\widetilde{\Delta}_d(\sigma) \in \ker(p_\#)$.

Choose a basis $v_1, \dots, v_k$ for $\ker(p_\#)$, and choose $u_1, \dots, u_j \in C_d(\widetilde{K})$ so that $v_1, \dots, v_k$, $u_1, \dots, u_j$ is a basis for $C_d(\widetilde{K})$. Then $p_\#(u_1), \dots, p_\#(u_j)$ is a basis for $C_d(K)$. (To see why, suppose not; then we can write $\sum \alpha_i p_\#(u_i) = 0$ with the α_i not all 0. But then since $p_\#$ is linear, that means $p_\#(\sum \alpha_i u_i) = 0$, which means $\sum \alpha_i u_i \in \ker(p_\#)$, a contradiction, since we chose the u_i so that this would not be true.) Let M be the matrix for Δ_d with respect to the basis $p_\#(u_1), \dots, p_\#(u_j)$, and let N be the matrix for $\widetilde{\Delta}_d$ restricted to $\ker(p_\#)$, with respect to $v_1, \dots, v_k$. Then (by [10] Theorem 14.3.7) the matrix for $\widetilde{\Delta}_d$ with respect to the basis $v_1, \dots, v_k, u_1, \dots, u_j$ is in the block form

$$\Gamma = \begin{bmatrix} N & * \\ 0 & M \end{bmatrix}.$$

We then see that

$$\det(\Gamma - \lambda I) = \det \begin{bmatrix} N - \lambda I & * \\ 0 & M - \lambda I \end{bmatrix} = \det(N - \lambda I)\det(M - \lambda I).$$

As a result, the characteristic polynomial of M divides the characteristic polynomial of Γ, so we have $\operatorname{Spec}(\Delta_d) \subseteq \operatorname{Spec}(\widetilde{\Delta}_d)$. □

Example 4.5. Let K and $\widetilde{K}$ be the two complexes from Example 4.1. We want to verify Theorem 4.4 with K and $\widetilde{K}$. For dimension greater than one the result is trivial, as neither K nor $\widetilde{K}$ have any simplices of dimension greater than one, so their Laplacian spectra will both be empty. Thus the only case we have to consider is dimension 1. One can easily compute that

$$\operatorname{Spec}(\Delta_1) = \{0, 2, 2, 4\}$$
$$\operatorname{Spec}(\widetilde{\Delta}_1) = \{0, 2 - \sqrt{2}, 2 - \sqrt{2}, 2, 2, 2 + \sqrt{2}, 2 + \sqrt{2}, 4\}$$

We see that, as predicted, $\operatorname{Spec}(\Delta_1) \subseteq \operatorname{Spec}(\widetilde{\Delta}_1)$. Notice that if we compute Δ_0 and $\widetilde{\Delta}_0$, we will se that $\operatorname{Spec}(\Delta_0)$ contains two 4's, while $\operatorname{Spec}(\widetilde{\Delta}_0)$ contains only one 4, so Theorem 4.4 does not necessarily hold for dimension 0.

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