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Counting Modular Tableaux

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Abstract. In this paper we provide a bijection between all modular tableaux of size kn and all partitions of n labeled with k colors. This bijection consists of a new function proven in this paper composed with mappings given by Garrett and Killpatrick in [3] and Stanton and White in [4]. We also demonstrate the novel construction and proof of a mapping essentially equivalent to Stanton and White's, but more useful for the purposes of the bijection mentioned above. By using the generating function for the number of k-colored partitions of n in conjunction with our bijection, we can count the number of modular tableaux of size kn.

1 Introduction

We solve the problem of counting the number of modular tableaux by establishing a one-to-one correspondence between modular tableaux of size kn and k-colored partitions of n. This correspondence, actually a mapping composed of several bijections, establishes the equality of the size of its domain and codomain. Using generating functions to count k-colored partitions then allows us to count modular tableaux.

In [3], Garrett and Killpatrick prove a bijection Φ between modular tableaux of size kn and k-rim hook tableau shapes of size kn. Stanton and White's paper [4] establishes the equality of the number of k-rim hook tableaux and the number of k-tuples of standard tableaux of total size kn using the mapping Π. We complete the composition by giving a bijection α between k-tuples of standard tableau shapes of total size kn and k-colored partitions of n.

The only problem with the composition of these mappings is that the shapes constituting both the domain and the codomain of Stanton and White's bijection are tableaux, while the other two bijections use shapes without content in the relevant sets. Though we show that this problem can easily be rectified as it stands, we also provide a bijection Π' that functions similarly to Stanton and White's Π (though Π' is defined much differently) and maps between tableau shapes, instead of tableaux with content. This bijection composes nicely with the first (Φ) and last (α) bijections of the composition. Figure 1 shows the domain and codomain of each bijection with an example object from each set, followed by a description of each set. When the three bijections are composed, they provide a one-to-one correspondence between modular tableaux of size kn and k-colored partitions of n.

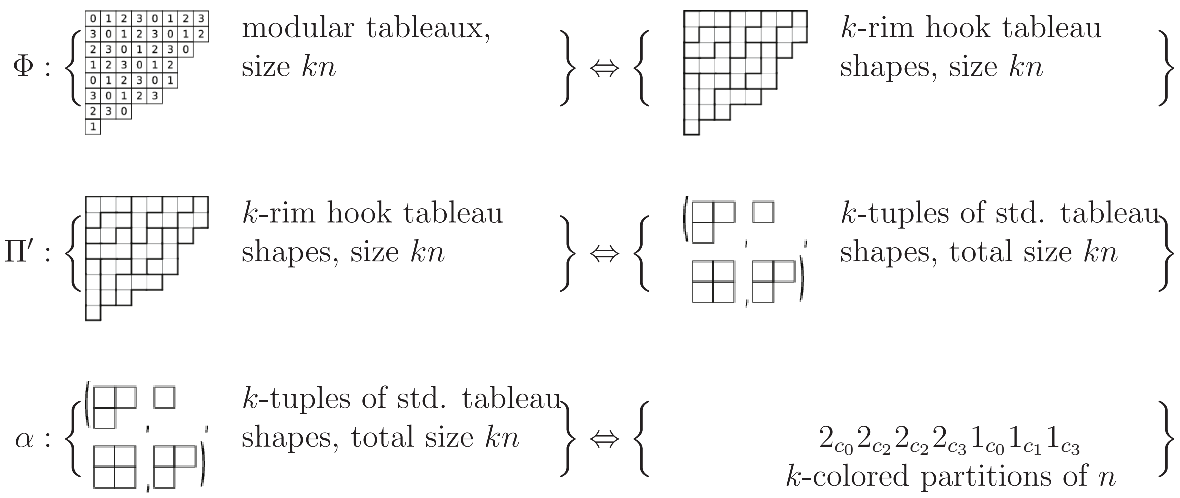

Figure 1: An illustration of the final composition, $\Phi \circ \Pi' \circ \alpha$, presented in this paper.

Sections 2 and 3 give a brief introduction to partition and tableau theory, along with a theorem relating modular tableaux to their conjugates (Theorem 3.2). An explanation of Stanton and White's mapping follows in Section 4, which provides the context for Section 5's related mapping. This related mapping is our first main result (Theorem 5.11). With our newly-defined function Π' in hand, we go on to show our second result, the final bijection in the composition (Theorem 6.1).

2 Partitions and Tableaux

Let n be a positive integer. A partition λ of n is a non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots \lambda_k)$ that sum to n. A Ferrers shape graphically represents this partition using a collection of cells constructed such that row i is λ_i cells long. We write the shape of a partition λ as $1^{i_1}2^{i_2}\cdots m^{i_m}$, where $m = \lambda_1$ and there are i_j j's in the partition λ. When its cells are each contain some symbol, called their content and usually an integer, the Ferrers shape is called a tableau. We call a tableau with integer content that increases west to east and north to south a standard tableau. In a tableau, if there are i_j cells with content j, then the tableau has content $1^{i_1}2^{i_2}\cdots m^{i_m}$. Figure 2 illustrates a partition, its Ferrers shape, and a standard tableau of the same shape. For more comprehensive introductions to partition and tableau theory see [1] and [2], respectively.

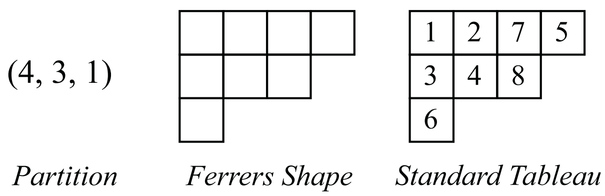

Figure 2: A partition $\lambda = (4, 3, 1)$, its Ferrers shape, and a standard tableau of shape λ.

We now proceed to define several additional terms useful for the purposes of this paper. Figure 3 gives an illustration of the following two definitions: shell and diagonal. In Figure 3, the diagonals of the Ferrers shape have been labeled with their integer values. The label of the dashed line cutting through any cell is the diagonal of that cell.

Definition 2.1. We refer to each cell in a Ferrers shape F with an ordered pair (c, r) representing the cell's column and row respectively, which both equal zero in the northwesternmost cell. A cell (c, r) in F has **shell** $sh_F((c, r)) = min(c, r)$.

Definition 2.2. The **diagonal** of a cell c in a shape F, denoted $\text{diag}_F(c)$, is an integer value describing the location of cell c within the shape. The central diagonal (northwest-to-southeast running), on which the northwesternmost cell of the shape lies, is numbered 0. Adjacent diagonals to the right are incremented by 1 and diagonals below are decremented by 1.

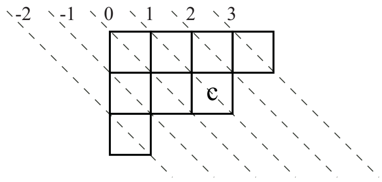

Figure 3: An example showing the shell $(\min(2, 1) = 1)$ and diagonal (1) of cell c.

Definition 2.3. A Ferrers shape S' is **conjugate** to a Ferrers shape S when S' is obtained by reflecting S over the 0 diagonal. A tableau T' is conjugate to a tableau T if the shape of T' is conjugate to the shape of T and the content of cell (j, i) of T' is equal to the content of cell (i, j) of T for all i and j.

2.1 Hooks and Orientation

Definition 2.4. In a shape, two cells are contiguous if they share an edge. Within a Ferrers shape or a tableau, a **hook** is a set of contiguous cells with no two cells on the same diagonal. If the hook is in a tableau, its cells must have identical content. Figure 4 gives an example of a typical hook within a Ferrers shape of the partition $\lambda = (5, 5, 4, 4, 2)$.

The head cell of a hook is the northeasternmost cell in the hook, and the tail cell of a hook is the southwesternmost cell in the hook. The diagonal of the head cell of a hook is the hook's diagonal. A hook has an illegal head if its head's row is not 0 and the cell directly north of its head is not part of the shape. Similarly, a hook has an illegal tail if its tail's column is not 0 and the cell directly west of its tail is not part of the shape. A collection of hooks is a Ferrers shape λ if and only if no hook in λ has an illegal tail or an illegal head. Figure 5 gives examples of legal and illegal heads and tails: by the definitions above the cells

	*	*	*	
*	*			

Figure 4: A Ferrers shape of $\lambda = (5, 5, 4, 4, 2)$, containing a hook with content *.

in Figure 5 (i) with content * are a hook with a legal head and a legal tail. In Figure 5 (ii) the cells with content * are a hook with illegal head and an illegal tail. In the latter case, the head of the hook (in position $(4, 3)$) is illegal because it is not in row 0, and the cell directly north of its head (in position $(4, 2)$) is not in the collection of cells.

i)

a	a	a	a	c
a	b	b	c	c
b	b	c	c	
b	*	*	*	
*	*			

ii)

a	a	a	a	c
a	b	b	c	c
b	b	c	c	
b	*	*	*	*
	*			

Figure 5: Examples of legal and illegal heads and tails.

In some shape λ, two hooks are overlapping if they share at least one diagonal. Two hooks are head-to-tail if (1) they are not overlapping, and (2) the head cell of one hook and the tail cell of the other hook are contiguous. Separated hooks are neither head-to-tail nor overlapping.

Definition 2.5. The **outside border** of λ is all of the cells not in λ but below and to the right of the cells in λ, as well as all of the cells not in λ but in row 0 or column 0. In Figure 6, the outside border of the tableau consists of the cells outlined in black, including the cells from the top row and left column that do not belong to λ. Also, any set of contiguous cells in the outside border of a shape constitutes a hook. This is called a rim hook outside λ, as discussed below.

A hook τ with cells only lying on the outside border of some shape λ, where τ and λ taken together form another shape λ', is called a rim hook outside λ. For example, in Figure 5 (i), the hook with content * is a rim hook outside of the rest of the tableau. Repeatedly adjoining rim hooks with strictly increasing content to an empty (size 0) shape λ results in a rim hook tableau.

Definition 2.6. If the rim hook tableau consists entirely of rim hooks of size k, it is a $\mathbf{k-}$**rim hook tableau**. Notice that, because of their construction method, k-rim hook tableaux will have content weakly increasing to the south and east. Figure 7 gives an example where $k = 5$. Note that the tableau can be constructed by adding the 4 5-rim hooks to an empty shape one at a time, in alphabetical order of content.

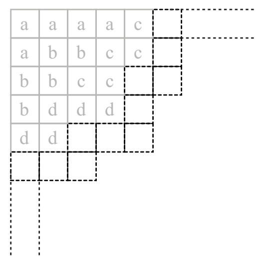

Figure 6: An example showing the outside border of a tableau of shape λ.

Definition 2.7. The **orientation** $i \geq 0$ of a k-rim hook is the number of the diagonal mod k on which the head cell of the k-rim hook lies. A k-rim hook tableau is said to be oriented if all of its hooks have the same orientation, i.

Take the 5-rim hook with content b in Figure 7 as an example. Its head cell is at position $(2, 1)$ with diagonal 1; thus the 5-rim hook has orientation 1, because $1 \mod 5 = 1$.

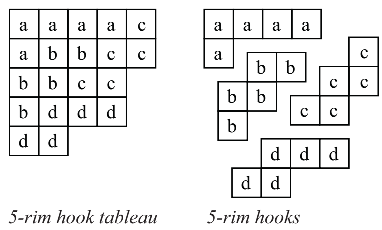

Figure 7: A 5-rim hook tableau and its constituent 5-rim hooks.

3 Modular Tableaux

In this section we discuss modular tableaux (also called balanced tableaux) of size kn, which, through a result proven by Garrett and Killpatrick [3], are in one-to-one correspondence with the distinct shapes of k-rim hook tableaux of size kn.

Definition 3.1. A (size kn) **modular tableau** for k, a, and b, where $(0 \leq a, b \leq n)$, is a tableau with the following properties.

1. Any given cell $c_{i,j} = (i, j)$ has content $ai + bj \mod k$.

2. The tableau contains n 0's, n 1's, …, and n $k - 1$'s.

For the purposes of this paper, assume that $k = a + b$ (Figure 8). Note that this condition ensures that all cells in a given diagonal will have identical content. This restriction is in place because the relevant part of our final bijection restricts modular tableaux with this criterion. See the concluding paragraph for discussion of the $k \neq a+b$ case. A theorem regarding the conjugates of modular tableaux follows from this definition (Theorem 3.2).

0	1	2	3	0	1	2	3
3	0	1	2	3	0	1	2
2	3	0	1	2	3	0	
1	2	3	0	1	2		
0	1	2	3	0	1		
3	0	1	2	3			
2	3	0					
1							

Figure 8: A modular tableau of size 44 with parameters $a = 1, b = 3, k = 4$, and $n = 11$.

Given a modular tableau λ and its conjugate tableau λ', we define a new tableau $\widetilde{\lambda'}$ by modifying the content of the tableau λ'. To obtain $\widetilde{\lambda'}$, we replace the content $m = ai + bj \mod k$ of a given cell (j, i) in λ' with the content $m' = aj + bi \mod k$ in $\widetilde{\lambda'}$. Given these tableaux (λ, λ', and $\widetilde{\lambda'}$) we offer the following theorem:

Theorem 3.2. If a tableau λ is modular for k, a, and b with $k = a + b$, then the following two conditions hold:

1. Its conjugate tableau λ' is modular for k, b, and a.
2. The tableau $\widetilde{\lambda'}$ is modular for k, a, and b.

Proof. Let λ be a modular tableau for k, a, and b with $k = a+b$. Then the content m of cell (i, j) in λ (and accordingly, cell (j, i) in the conjugate tableau λ') is given by $ai+bj \mod k$. Hence, λ' is, by definition, modular for k, b, and a: the same parameters as for λ, with a and b switched.

Our second claim is less trivially proven but more useful than the first. Because a given cell (j, i) in $\widetilde{\lambda'}$ has content $m' = aj + bi \mod k$, the first condition of $\widetilde{\lambda'}$ being modular for k, a, and b is met. It only remains to show that $\widetilde{\lambda'}$ contains n 0's, n 1's, ..., and n $k-1$'s. To this end, note that

$$m + m' = (ai + bj) + (aj + bi) = i(a + b) + j(a + b) = (i + j)k \equiv 0 \mod k.$$

Therefore m and m' are additive inverses, and since additive inverses are unique in $\mathbb{Z}_k$, $\widetilde{\lambda'}$ has the same number of cells as λ for each content $0, 1, \dots, k-1$. By definition, $\widetilde{\lambda'}$ is modular for k, a, and b.

$\square$

In practice, to obtain the tableau $\widetilde{\lambda'}$ one simply exchanges the content of each cell in λ' with its additive inverse. Figure 9 illustrates the progression from λ to λ' to $\widetilde{\lambda'}$. By Theorem 3.2, λ' is modular for k, b, and a, and $\widetilde{\lambda'}$ is modular for k, a, and b.

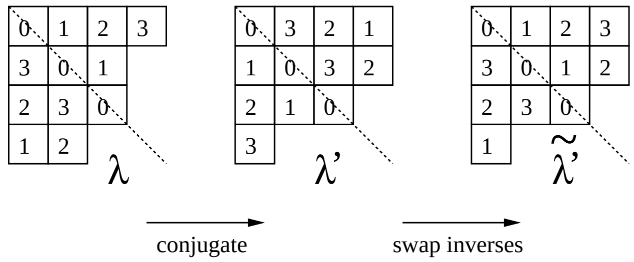

Figure 9: The tableau λ is modular for $k = 4, a = 1, b = 3$. Here, we show λ, λ', and $\widetilde{\lambda'}$.

This theorem, particularly the second part, is useful in counting the number of modular tableaux of size kn with $k = a + b$. In effect, it tells us that given a modular tableau λ, there exists a tableau that (1) has a shape conjugate to λ and (2) is also modular for k, a, and b (Figure 10). Note that the shapes on the right in Figure 10 are conjugate to those on the left.

We move ahead now in our consideration of modular tableaux to a main result of Garrett and Killpatrick in [3]. Garrett and Killpatrick present a bijection (which we will label) Φ between all modular tableaux of size kn for some k, a and b, where $k = a + b$, and the distinct shapes of all k-rim hook tableaux of size kn. Given a modular tableau λ, $\Phi(\lambda)$ is the k-rim hook tableau shape λ, which can be filled with n hooks of size k. Note that the power of the bijection here lies not with its actual function (which is quite trivial) but with the fact that all modular tableau shapes are also k-rim hook tableau shapes. The inverse mapping, of course, also preserves shape. Thus, any modular tableau of size kn has the shape of a k-rim hook tableau with n hooks. We underscore the fact that the bijection Φ establishes the equality of the number of distinct shapes of k-rim hook tableaux of size kn and the number of modular tableaux of size kn, in anticipation of composing this bijection with two others to prove our main result. The next section begins to deal with the first of these other mappings.

4 Stanton and White: k-tuples of standard tableaux

In [4], Stanton and White present the mapping Π, a bijection between k-rim hook tableaux and k-tuples of standard tableaux with distinct entries (Corollary 23). Here, we present an explanation of the mapping. Refer to [4] for a proof and a more extensive discussion.

In our case, the two sets above (k-rim hook tableaux and k-tuples of standard tableaux) have all content and k-rim hook structure removed. The bijection Π can still be applied to these two sets, given (1) an algorithm to fill an empty shape of a k-rim hook tableau with

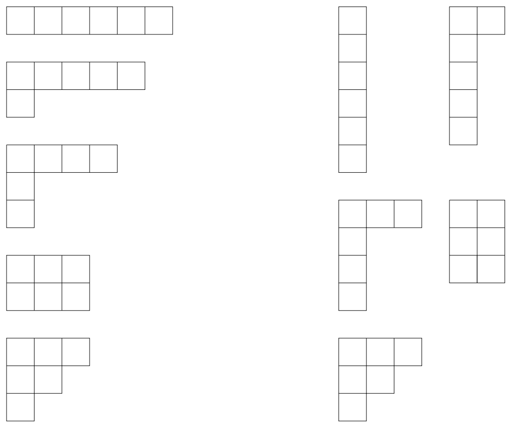

Figure 10: All modular tableau shapes for $k = 3, a = 1, b = 2$.

k-rim hooks and content and (2) an algorithm to fill a k-tuple of standard tableaux with distinct entries. Garrett and Killpatrick present a solution for the former problem in [3]. The latter can be accomplished simply by filling each standard tableau in turn with consecutive positive integers, moving across rows and then down columns.

Example 4.1. We will use tableau T (Figure 11) as a running example throughout this section.

T:

1	1	1	5	5	6	6	10
1	2	2	5	6	6	10	10
2	2	3	5	8	9	10	
3	3	3	8	8	9		
4	7	7	8	9	9		
4	7	11	11	11			
4	7	11					
4							

Figure 11: The 4-rim hook example tableau T.

4.1 Manipulating k-Rim Hook Tableaux

The mapping Π is actually a composition of two main functions, E and Γ. We begin by describing the E function through a discussion of its three constituents, Ch, SE and Era.

The function Ch_c acts on a k-rim hook tableau P with content $1^k 2^k \cdots m^k$, changing P into a new k-rim hook tableau Q. Specifically, Ch_c adjusts the content of the cells of the k-rim hook with content c, assigning c to a value between c and $c+1$. We note that tableaux traditionally require integer contents; however for ease of description, we let the symbols in the tableau be non-integer rationals as well. Therefore we have $c \in \mathbb{Q}$.

Let Q be a k-rim hook tableau with content $1^k 2^k \cdots (i-1)^k c^k (i+1)^k \cdots m^k$, where c holds a value between i and $i+1$, and m is the greatest content in Q. As usual, all cells in a given k-rim hook hold the same value, the content of each k-rim hook is distinct, and the content of the cells in the tableau weakly increases to the south and east in the tableau. Note that the tableau Q contains m k-rim hooks.

The function SE_c maps Q to Q', another k-rim hook tableau of the same shape but with content $1^k 2^k \cdots (i-1)^k (i+1)^k \cdots m^k c^k$ (where $c > m$) as described below. In each iteration of the process, the function adds 1 to the value of c, changing the content of the tableau. But the content cannot be changed without also adjusting other content, as the tableau must have weakly decreasing content. This step of adjusting other content, split into three cases, is performed first. For each case, refer to Figure 12 for examples. In Figure 12, the original content of the tableau is $c^4 2^4 3^4 \cdots 11^4$. Several consecutive steps of the process are shown, each of which is an example of one of the three cases enumerated below.

1. Adjust appropriate content of the tableau.
    1. If (the k-rim hooks with content) c and $i+1$ overlap, SE_c swaps the contents c and $i+1$ along the diagonals of Q' common to c and $i+1$.
    2. If c and $i+1$ are separated, SE_c leaves the tableau unchanged.
    3. Otherwise, c and $i+1$ are head to tail. In this case, SE_c swaps all contents c and $i+1$.
2. Now SE_c adds 1 to the value of c, adjusting the content of the k-rim hook with content c, and thus creating a different k-rim hook tableau.

This two-step process repeats until the content c is the largest in the tableau.

Note that SE_c essentially migrates the k-rim hook with content c from its original position in Q to a position adjacent to the outside border of Q'. Note also that SE_c preserves orientations: that is, if a k-rim hook r has orientation i in Q, r will also have orientation i in Q'.

Next, we define one more mapping: $Era_c(P)$. Let P be any k-rim hook tableau. The mapping $Era_c(P)$ erases the k-rim hook with content c in P by removing all of its cells from P. Thus $Era_c(P)$ will be a valid k-rim hook tableau if c is the largest content in P: in this case, the k-rim hook r with content c is a k-rim hook outside the shape $P - r$ (P, without any of the cells in r).

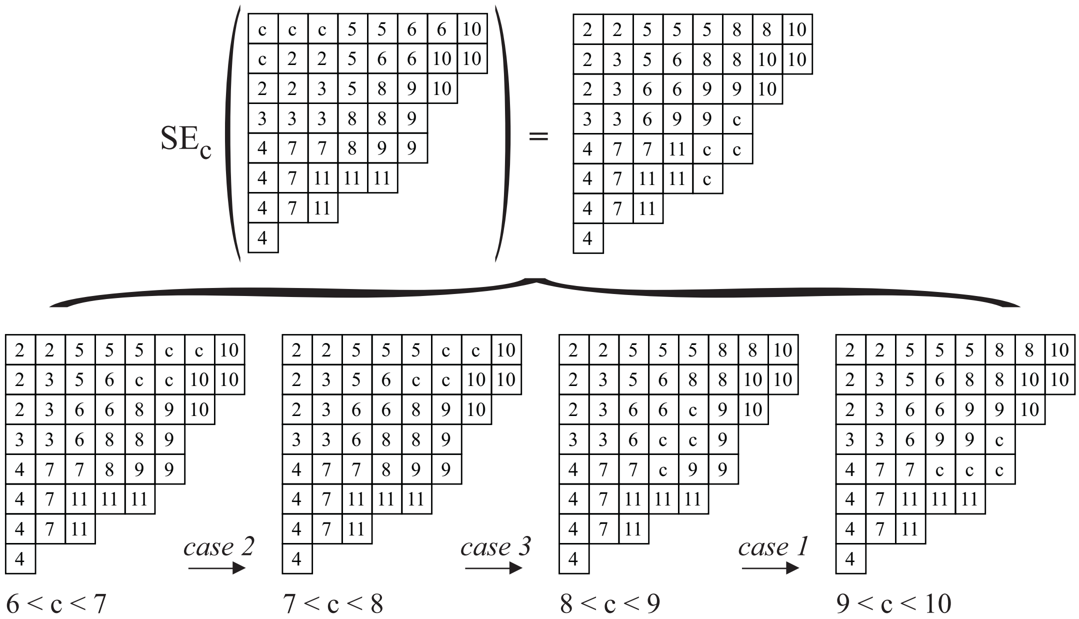

Figure 12: An example of the function $SE_c(T)$.

4.2 Evacuating k-Rim Hook Tableaux

Let P be any k-rim hook tableau. Combining SE_c, Ch_c, and Era_c, for some content c in P, the mapping $E_c(P)$ effectively removes (or "evacuates") the k-rim hook with content c in P and repositions any other affected k-rim hooks so that $E_c(P)$ yields a k-rim hook tableau, but with k fewer cells. We define $E_c(P)$ as follows:

$$E_c(P) = Era_c \circ SE_c \circ Ch_c(P).$$

The mapping $E_c(P)$ first changes the content of the cells in the k-rim hook with original content c, then migrates this changed k-rim hook using SE_c. Finally, as c is then the greatest content of P, its k-rim hook can be erased, creating a new k-rim hook tableau P' (Figure 13).

2	2	5	5	5	8	8	10
2	3	5	6	8	8	10	10
2	3	6	6	9	9	10	
3	3	6	9	9			
4	7	7	11				
4	7	11	11				
4	7	11					
4							

Figure 13: Example 4.1 continued: $E_c(T)$, $c = 1$.

We know that $E_c(P)$ preserves the orientation of each of the k-rim hooks in P. Suppose that the k-rim hooks in P that have orientation i $(0 \leq i \leq k-1)$ have content $c_1, c_2, \ldots, c_l$ $(1 \leq c_1 < c_2 < \cdots < c_l \leq m)$. Then the k-rim hook tableau given by

$$E_{(i)}(P) = E_{c_1} \circ E_{c_2} \circ \cdots \circ E_{c_l}(P)$$

has no k-rim hooks of orientation i. Stanton and White prove that the E_{c_j}'s can be composed in any order without changing the resulting tableau (Stanton and White, Lemma 10 and Lemma 12). Regardless of the order of composition of the $E_{(j)}$'s in the final evacuation algorithm,

$$E_{\widetilde{(i)}}(P) = E_{(0)} \circ E_{(1)} \circ \cdots \circ E_{(i-1)} \circ E_{(i+1)} \circ \cdots \circ E_{(k-1)}(P),$$

the result will be the same. The mapping $E_{\widetilde{(i)}}(P)$ evacuates all k-rim hooks without orientation i, producing an i-oriented k-rim hook tableau (Figure 14). With the mapping $E_{\widetilde{(i)}}$ in hand, we note a crucial result from Stanton and White (Lemma 15): two k-rim hook tableaux P and Q have the same shape if and only if $E_{\widetilde{(i)}}(P)$ and $E_{\widetilde{(i)}}(Q)$ have the same shape for all i.

1	1	1	6	6	6	6
1	8	8	8			
7	8					
7						
7						
7						

Figure 14: Example 4.1 continued: $E_{\widetilde{(i)}}(T)$ with $i = 2$.

4.3 Completing the Bijection Π

Let P be an i-oriented k-rim hook tableau. Then the mapping $\Gamma(P)$ produces a standard tableau: for every k-rim hook r with content c in P, if $\mathrm{diag}_P(r) = ak+i$ (with $i \geq 0$) for some integer a, then $\mathrm{diag}_{\Gamma(P)}(r) = a$ (Figure 15). If P (for example) has content $1^k 2^0 3^k 4^k 5^0 6^0 7^0 8^k$, then $\Gamma(P)$ has content $1^1 2^0 3^1 4^1 5^0 6^0 7^0 8^1$. Moreover, it is not hard to prove that two i-oriented k-rim hook tableaux P and Q have the same shape if and only if $\Gamma(P)$ and $\Gamma(Q)$ have the same shape (Stanton and White, Lemma 16). With this result, we can combine Γ and $E_{\widetilde{(i)}}$ to arrive at the desired bijection Π.

1	6
7	8

Figure 15: Example 4.1 continued: the composition $\Gamma \circ E_{\widetilde{(i)}}(T)$ for $i = 2$.

The mapping $\Pi(P)$ transforms any k-rim hook tableau P of size kn into a distinct k-tuple of standard tableaux of total size n (Figure 16). Let $0 \leq i \leq k-1$. Then the mapping $\Pi_{(i)}(P) = \Gamma \circ E_{\widetilde{(i)}}(P)$ yields the standard tableau λ^i from the k-tuple:

$$\Pi(P) = (\lambda^0, \lambda^1, \ldots, \lambda^{k-1}) \text{ where } \lambda^i = \Pi_{(i)}(P) = \Gamma \circ E_{\widetilde{(i)}}(P).$$

Let R be the set of all distinct shapes of k-rim hook tableaux of size kn, and let S be the set of all the distinct k-tuples of shapes of standard tableaux that have sizes totalling n. By previous results, two k-rim hook tableaux P and Q have the same shape if and only if $\Pi_{(i)}(P)$ and $\Pi_{(i)}(Q)$ have the same shape for all i. Therefore, the mapping Π is one-to-one with R as its domain and S as its codomain. Stanton and White show that Π is a bijection from k-rim hook tableaux to k-tuples of standard tableaux with distinct content (Corollary 23). Thus Π is onto. These two results imply that Π is also a bijection between R and S. Using these two sets is important for the eventual composition of Π with Garrett and Killpatrick's Φ, above, and our α, below. Note that, though we have just proven the equality of the sizes of the sets R and S, Π does not biject between them directly. To do so (as we will need to do for Theorem 6.1), it is necessary to use either mappings to add and remove content (specified at the beginning of this section) from the domain and range of Π) or to provide a new bijection entirely without content. We pursue the latter approach below.

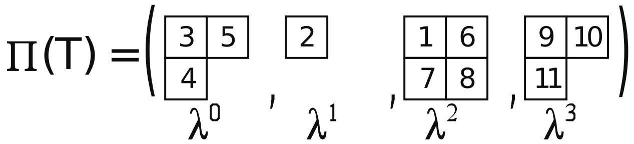

Figure 16: Example 4.1 continued: $\Pi(T) = (\lambda^0, \lambda^1, \lambda^2, \lambda^3)$.

5 Mapping using k-Rim Hook Shapes: Π'

The function $\Pi = \Gamma \circ E$, explained above, maps each k-rim hook tableau of size kn to a k-tuple of standard tableaux of total size n. As mentioned above, we will introduce the effectively similar but much differently defined Π' in this section to facilitate function composition with the mappings Φ (Section 3) and α (Section 6). Additionally, Π' proves the equality of the sizes of the domain and codomain of Stanton and White's Π in a new manner. The function Π' maps from size kn k-rim hook shapes, instead of k-rim hook tableaux, to k-tuples of Ferrers shapes (essentially 1-rim hook tableaux) with total size kn. We will define Π' using the Γ function (Section 4.3) composed with a new function E'. Thus $\Pi' = \Gamma \circ E'$.

5.1 Evacuating k-Rim Hooks

Though the function E' in Π' takes the place of E in Π, E' is defined much differently than its analogue. In the description of E' below, we define and compose the functions NW and Era, and when showing E'^{-1}, the functions NW and Com are required. We begin with a construction of E'.

Let P be any k-rim hook shape (Figure 17). Then the mapping $Era_{\widetilde{(i)}}(P)$ erases all of the cells of all k-rim hooks of orientation $\neq i$ in P (Figure 18). The mapping NW moves all (remaining) cells diagonally northwest as far as possible (Figure 19). Let i be an orientation such that $0 \leq i < k-1$. Using these two mappings, we can define the mapping $E'_{\widetilde{(i)}}(P)$, which removes all cells belonging to k-rim hooks of orientation $\neq i$ in P, and then creates a k-rim hook shape of orientation i with the remaining cells, as follows:

$$E'_{\widetilde{(i)}}(P) = NW \circ Era_{\widetilde{(i)}}(P).$$

By evacuating from a k-rim hook shape all of the k-rim hooks that have orientation not equal to some integer i, the function $E'(P)$ maps k-rim hook shapes of size kn to k-tuples of i-oriented k-rim hook shapes of total size kn. Thus we define $E'(P)$ by

$$E'(P) = (E'_{\widetilde{(0)}}(P), E'_{\widetilde{(1)}}(P), \ldots, E'_{\widetilde{(k-1)}}(P)).$$

Let c be any given cell in $Era_{\widetilde{(i)}}(P)$, and let r be the k-rim hook that contains c. Let the nonnegative integer m be the number of cells s in $Era_{\widetilde{(i)}}(P)$ where $sh_P(s) < sh_P(c)$ and $\text{diag}_P(s) = \text{diag}_P(c)$. Then $sh_{E'_{\widetilde{(i)}}(P)}(c) = m$. For example, within the 4-rim hook r in Figure 18, the cell with content c satisfies $diag_P(c) = 1$. To determine $sh_{E'_{\widetilde{(2)}}(P)}(c)$, we find the number m of cells directly northwest of c. Here, s is the only such cell, so $m = 1$.

Now all of the remaining k-rim hooks have the same orientation. Given any integer n, the number of cells in $Era_{\widetilde{(i)}}(P)$ with diagonal $j + n$ is equal for each $0 \leq j < k$. In other words, the number m will be equal for each c in k-rim hook r. Thus, $sh_{E'_{\widetilde{(i)}}(P)}(c) = m$ for *all* cells c in r (Figures 18 and 19).

Example 5.1. Shown below, T is a 4-rim hook shape with 44 cells, an element of the domain of Π'. Note that the shape T is the same as the shape of the tableau in Example 4.1 above. We use T as a running example throughout this section.

T:

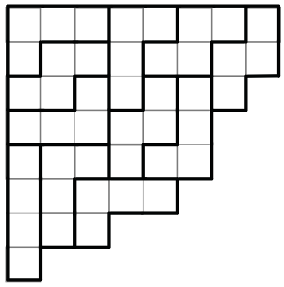

Figure 17: An example 4-rim hook shape T, used throughout this section.

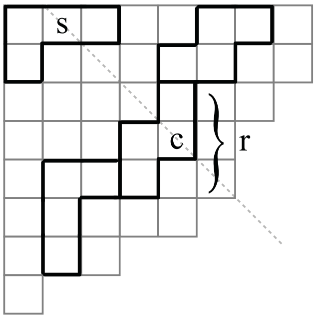

Figure 18: Example 5.1 continued: the collection of cells $Era_{(\tilde{2})}(T)$.

Let P be any k-rim hook tableau. For any diagonal d in P, let the sets S_{d+1} and S_d consist of all the cells in P with diagonals $d+1$ and d, respectively. Let n_d be the number of cells in P with diagonal d that are head cells of some k-rim hook (with diagonal d) in P. Let the set S'_d consist of all the cells in S_d that are not head cells of some k-rim hook (with diagonal d) in P. So $n_d = |S_d| - |S'_d|$.

Lemma 5.2. Let $A = \{0, 1\}$. Let $\delta_+ = |S_{d+1}| - |S_d|$ and $\delta_- = |S_{d-1}| - |S_d|$. Then there are n_{d+k} and n_{d-k} k-rim hooks of diagonal $d+k$ and $d-k$, respectively, where:

1. $n_{d+k} = n_d + \delta_+$.
2. $n_{d-k} = n_d + \delta_-$.
3. $\delta_+ \in -A$ if $d \geq 0$; $\delta_+ \in A$ if $d < 0$.
4. $\delta_- \in -A$ if $d - k + 1 < 0$; $\delta_- \in A$ if $d - k + 1 \geq 0$.

Proof. Because P is a shape, $d \geq 0$ implies $\delta_+ = 0$ or -1 and $d < 0$ implies $\delta_+ = 0$ or 1 (3). By substitution in the definitions above, $|S_{d+1}| - |S'_d| = n_d + \delta_+$. If any cell in S_{d+1} is not a tail cell of some k-rim hook (which would have diagonal $d+k$), then its k-rim hook must contain a cell in S'_d as well. Moreover, no cell in S'_d is a head cell (by assumption), so each must be in the same k-rim hook as some cell in S_{d+1}. In other words, the non-tail cells in S_{d+1} are all paired with a distinct cell in S'_d. Because the sets differ in size by $n_d + \delta_+$, there must be exactly $n_d + \delta_+$ cells in S_{d+1} that are tail cells of k-rim hooks (with diagonal $d+k$).

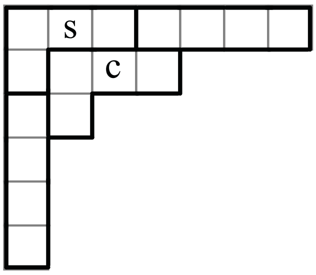

Figure 19: Example 5.1 continued: $NW \circ Era_{(\tilde{2})}(T) = E'_{(\tilde{2})}(T)$.

This conclusion implies (1). Parts (2) and (4) follow from similar arguments concerning the tails of the k-rim hooks, or alternatively from the application of the proof above to the conjugate of P. For a concrete application of this lemma, see Examples 5.3 and 5.4 and the accompanying Figures 20 and 21.

□

Example 5.3. The collection of cells below (Figure 20) contains k-rim hooks of orientation 1 from some shape with $k = 4$. All 4-rim hooks are shown that have diagonal 5 or -3. We use Lemma 5.2 to find how many 4-rim hooks of diagonal 9, 1, and -7 must belong in the original shape. We know $n_5 = 2$ and $n_{-3} = 1$.

Let $d = 5$. By Lemma 5.2 (3) and (4), we have $\delta_+ \in -A$ and $\delta_- \in A$. By (1), $n_9 = n_5 + \delta_+ = 1$ or 2. By (2), $n_1 = n_5 + \delta_- = 2$ or 3.

Let $d = -3$. Using Lemma 5.2 again, we get $\delta_+ \in A$ and $\delta_- \in -A$. Then $n_1 = 1$ or 2 and $n_{-7} = 0$ or 1. By the conclusions from both values of d, we know that n_1 must be 2.

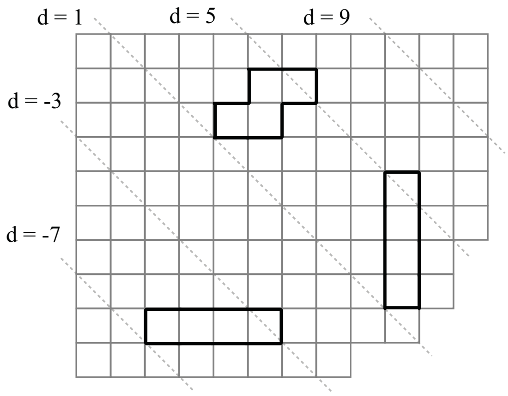

Figure 20: Applying Lemma 5.2, we find that $n_9 = 1$ or 2, $n_1 = 2$, and $n_{-7} = 0$ or 1.

Example 5.4. In Figure 21 below, $k = 4$. Let $d = 5$. Then $n_d = n_5 = 2$. To show an example of Lemma 5.2 (1), we must find $n_{5+k} = n_9$. From the proof of 5.2, we know that there must be exactly $n_d + \delta_+ = n_{d+k}$ cells in S_{d+1} that are tail cells of k-rim hooks with diagonal $d+k$. Therefore, exactly n_9 of the cells with content c_1 through c_6 (in S_5) will be tail cells of the n_9 4-rim hooks with diagonal 9. But c_7, c_9, c_{10}, and c_{12} cannot be head cells, because (as mentioned above) all the 4-rim hooks with diagonal 5 are shown. If none of these four cells are head cells, then the 4-rim hooks of which they are a part must also each contain exactly one of c_1 through c_6. Therefore $n_9 \leq 2 = 6 - 4$. But if any cell c_1 through c_6 is not a tail cell, its 4-rim hook will also contain a cell in diagonal 5. Therefore, $n_9 = 2$. Lemma 5.2 (1) finds this same result using δ_+, which, in this case, is $6 - 6 = 0 \in -A$, so $n_9 = n_5 + 0 = 2$.

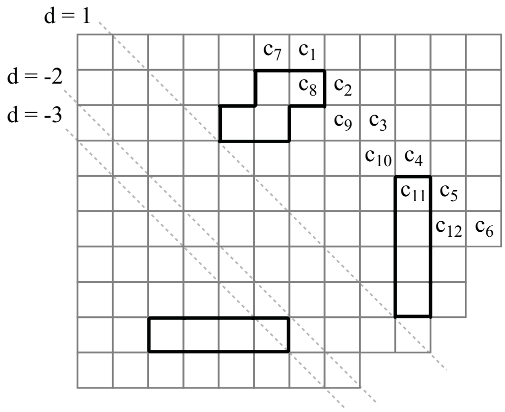

Figure 21: Example of Lemma 5.2 (1) when $d = 5$.

Corollary 5.5. If $d \geq 0$ and $\delta_+ = -1$, then $n_d \geq 1$. If $d < 0$ and $\delta_+ = 1$, then $n_{d+1} \geq 1$.

Proof. The proof of Lemma 5.2 states that when $d \geq 0$ there must be $n_d + \delta_+$ cells in S_{d+1} that are tail cells of k-rim hooks (with diagonal $d+k$). Of course, there cannot be -1 such cells, as there would be if $\delta_+ = -1$ and $n_d = 0$. Therefore, $n_d \geq 1$. Likewise, if $\delta_+ = 1$ and $d < 0$, $n_{d+1} \geq 1$ because $n_d \geq 0$. Note that a similar corollary can be drawn for δ_-. $\square$

Next consider each nonempty set Sh_m of the remaining (i-oriented) k-rim hooks that have shell $m \geq 0$ in $E'_{\widetilde{(i)}}(P)$. Let $|Sh_m| = q$ and $0 \leq j < q$. Order the k-rim hooks in Sh_m by their diagonal such that for all k-rim hooks $r_{m_j} \in Sh_m$ $(0 < j < q)$, $\text{diag}_{E'_{\widetilde{(i)}}(P)}(r_{m_j}) > \text{diag}_{E'_{\widetilde{(i)}}(P)}(r_{m_{j-1}})$. No two k-rim hooks in Sh_m have the same diagonal, as they are all in the same shell and therefore cannot overlap. Then the ordered set $\{\text{diag}_{E'_{\widetilde{(i)}}(P)}(r_{m_j})\} = k\{c_j\} + i$ for some integers $c_j \in \mathbb{Z}$.

Suppose, toward contradiction, that $c_0 > 0$ (i.e. no k-rim hook in Sh_m contains a cell with diagonal 0). By Lemma 5.2 (2) and (4), there is another k-rim hook in Sh_m that has diagonal $k(c_0 - 1) + i$ and is head to tail with Sh_{m_j}. Thus c_0 is not the smallest value in Sh_m, a contradiction. So $c_0 \leq 0$.

Likewise, assume $c_{q-1} < 0$. Then by Lemma 5.2 (1) and (3), $Sh_{m_{q-1}}$ is head-to-tail with another k-rim hook in Sh_m with diagonal $k(c_0 + 1) + i$, and c_{q-1} is not the largest value in Sh_m. Therefore, $c_{q-1} \geq 0$, the set $\{c_j\}$ consists entirely of consecutive integers, and $0 \in \{c_j\}$.

Thus in $E'_{\widetilde{(i)}}(P)$, each Sh_m is a collection of cells made up entirely of contiguous k-rim hooks, making it an i-oriented k-rim hook shape of size kq.

Lemma 5.6. For any $m \geq 0$, when $|Sh_m| > 0$ and $|Sh_{m+1}| > 0$, $Sh_m \cup Sh_{m+1}$ is an i-oriented k-rim hook shape.

Proof. Suppose $max(\{\text{diag}(Sh_m)\}) < max(\{\text{diag}(Sh_{m+1})\})$. Then the rim hook with the greatest diagonal, $r_{max} \in Sh_{m+1}$, would not satisfy $sh_{E'_{\widetilde{(i)}}(P)}(r_{max}) = m + 1$ because the diagonal of the head cell of r_{max} does not include a cell in shell m, by hypothesis. Thus we have $sh_{E'_{\widetilde{(i)}}(P)}(r_{max}) \leq m$, a contradiction.

Suppose $max(\{\text{diag}(Sh_m)\}) = max(\{\text{diag}(Sh_{m+1})\}) = d$. Then there are $n_d \geq 2$ k-rim hooks of diagonal d in P. By Lemma 5.2, there must be at least $n_d - 1$ k-rim hooks of diagonal $d + k$ in P. But then Sh_m contains a k-rim hook with diagonal $d + k$, and $max(\{\text{diag}(Sh_m)\}) > d$, a contradiction.

Therefore, $max(\{\text{diag}(Sh_m)\}) > max(\{\text{diag}(Sh_{m+1})\})$. Moreover, $|Sh_m| > 1$ because $|Sh_{m+1}| > 0$ and because each shell must contain a k-rim hook with a cell on the zero diagonal. So $max(\{\text{diag}(Sh_m)\}) > 0$, and there is a cell in shell m directly north of the head cell of the k-rim hook with the greatest diagonal in shell $m + 1$. Thus there are no illegal heads in $Sh_m \cup Sh_{m+1}$.

Similar arguments applying to the inequality relating $min(\{\text{diag}(Sh_m)\})$ and $min(\{\text{diag}(Sh_{m+1})\})$ prove that there are no illegal tails in $Sh_m \cup Sh_{m+1}$ either. Therefore, $Sh_m \cup Sh_{m+1}$ is a shape, and because both Sh_m and Sh_{m+1} consist entirely of i-oriented k-rim hooks, $Sh_m \cup Sh_{m+1}$ is an i-oriented k-rim hook tableau. □

Lemma 5.7. For any orientation i and any k-rim hook shape P, $E'_{\widetilde{(i)}}(P)$ is an i-oriented k-rim hook shape.

Proof. Consider each nonempty set Sh_m of the (i-oriented) k-rim hooks that have shell $m \geq 0$ in $E'_{\widetilde{(i)}}(P)$. Let the set containing k-rim hooks in the greatest shell be denoted Sh_t. We know that Sh_0 is an i-oriented k-rim hook shape. By Lemma 5.6, $Sh_0 \cup Sh_1$ and $Sh_1 \cup Sh_2$ are also i-oriented k-rim hook shapes. Thus $Sh_0 \cup Sh_1 \cup Sh_2$ forms an i-oriented k-rim hook shape. Likewise, adding the cells of Sh_j to the i-oriented k-rim hook shape $Sh_0 \cup Sh_1 \cup \cdots \cup Sh_{j-1}$ creates a new i-oriented k-rim hook shape. When $j = t$, we see that $Sh_0 \cup Sh_1 \cup \cdots \cup Sh_t$ is an i-oriented k-rim hook shape as well. □

Lemma 5.7 shows that $E'_{\widetilde{(i)}}(P)$ is an i-oriented k-rim hook shape (see Figure 19 for an example). Also by Lemma 5.7, $E'(P)$ results in a k-tuple of i-oriented k-rim hook shapes of the same total size as P. Figure 22 gives an example of $E'(P)$, mapping the 4-rim hook shape T (from Example 5.1) to a 4-tuple of i-oriented 4-rim hook shapes, for $0 \leq i < 4$.

It is obvious but important to note that the set of all diagonals of the k-rim hooks in P is identical to the set of all diagonals of the k-rim hooks in $E'(P)$ (in other words, the k-rim hooks do not change diagonal as they are moved around by E').

5.2 The Inverse Evacuation Algorithm, E'^{-1}

Let $Q = (P_0, P_1, \ldots, P_{k-1})$ be a k-tuple of i-oriented k-rim hook shapes P_i for all $0 \leq i < k$ (Figure 22). The inverse evacuation algorithm, E'^{-1}, will rebuild the 4-rim hook shape T (Figure 17) from these i-oriented k-rim hook shapes (Figure 22). The mapping $Com(Q)$ composes these tableaux into one collection of cells as follows: start with a collection of cells of size 0. Then, for consecutive values of i, starting with $i = 0$, place P_i such that (1) its northwesternmost cell lies on the 0-diagonal and (2) P_i occupies the lowest unused shell or shells (Figure 23).

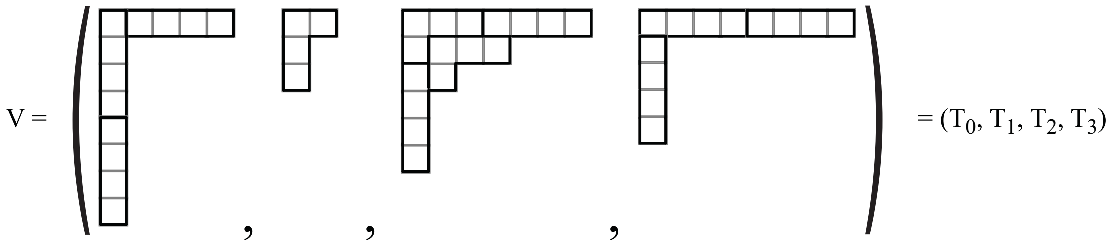

Figure 22: Example 5.1 continued: $E'(T) = V = (T_0, T_1, T_2, T_3)$.

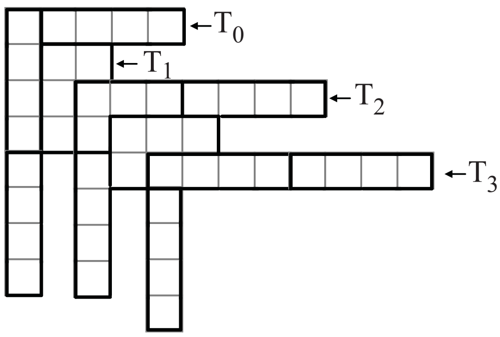

Figure 23: Example 5.1 continued: $Com(V)$, with V as in Figure 22.

We then define the mapping $E'^{-1} = NW \circ Com$, using the function NW. In Figure 24, composing the mapping NW (see Section 5.1) with $Com(V)$ (Figure 23) results in $E'^{-1}(V) = NW \circ Com(V)$, which has the same shape as T from Figure 17.

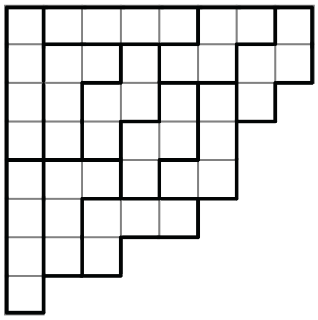

Figure 24: Example 5.1 continued: $E'^{-1}(V) = NW \circ Com(V)$.

Lemma 5.8. For any $Q = (P_0, P_1, \ldots, P_{k-1})$ as defined above, $E'^{-1}(Q)$ is a k-rim hook shape of the same size as the total size of the P_i.

Proof. We know that the 0-oriented k-rim hook shape (which occupies shells $0, 1, \ldots$ in Com) is a shape, and therefore has no illegal heads or tails. Now assume that, for orientation $0 < i < k$, $E'^{-1}((P_0, P_1, \ldots, P_{i-1})) = T$ is a shape. Then consider the collection of cells $E'^{-1}((P_0, P_1, \ldots, P_i)) = T'$, formed by shoving all cells from the i-oriented k-rim hook shape P_i northwest along their diagonals (the mapping NW accomplishes this). Consider the cells in P_i by shell, starting with P_i's lowest shell, denoted s (Example 5.9 provides an instance of the following discussion).

There is a cell on the 0-diagonal of P_i by Lemma 5.6. This cell (s, s) will not move under NW because Com assures that the cell $(s-1, s-1)$ is part of T. Consider the cells in the same shell, (c, s), for column numbers $s < c \leq m$, where m is the easternmost column in P_i. To preserve the k-rim hooks, each cell (c, s) can be shoved at most one more unit northwest than its neighbor cell $(c-1, s)$. Assume (without loss of generality) that $(c-1, s)$ cannot be shoved northwest at all; it is shoved 0 units. If (c, s) could be shoved more than one unit northwest, then the cell $(c-2, s-1)$ would be an illegal head. Since the cells in shells $< s$ constituted the shape T, this would be a contradiction. So, as the preceding proof applies to all consecutive shells of P_i, all of the k-rim hooks with cells on diagonals ≥ 0 will be intact in T', and (obviously) their diagonals and orientations will not change (Figure 25, (1)).

Finally, consider the northeasternmost cell p in P_i, with position (m, s) in $Com(Q)$.

In T', $p = (m-n, s-n)$ for some integer n. Suppose that, in T', p is an illegal head. Necessarily then, $(m-n, s-n-1) \notin T$. Note that, by definition of NW, the cell $(m-n-1, s-n-1) \in T$, and also $(m-n-1, s-n-2) \in T$ because T is a shape by assumption. Let d be the diagonal of p ($d = (m-n) - (s-n)$). We know $d > 0$ because $m - n > s - n$. Let the sets S_{d+1} and S_d consist of all the cells in T with diagonal $d+1$ and d respectively, as in Corollary 5.5 above. Then we have $|S_{d+1}| - |S_d| = -1 = \delta_+$, because the cells $(m-n-1, s-n-2), (m-n-1, s-n-1) \in T$, and T is a shape.

By Corollary 5.5, n_d, the number of k-rim hooks of diagonal d in T, satisfies $n_d \geq 1$.

Moreover, $d \mod k = i$, because p is the head cell of an i-oriented k-rim hook in P_i, so there are n_d k-rim hooks of orientation i in T. But because the shell s of P_i is the lowest shell of P_i and no oriented k-rim hook shape in Q besides P_i contains any k-rim hooks of orientation i, this is a contradiction.

Therefore, (m, s) is not an illegal head in T'. Because the cells in P_i remain contiguous, the only possible illegal head in T' is (m, s). So T' has no illegal heads (Figure 25 (2)). A similar argument (or the same argument, considering the conjugates of T and P_i) shows that T' will have no illegal tails, that all k-rim hooks with cells on diagonals less than 0 will be intact in T', and (obviously) that their diagonals and orientations will not change in the shell s. Therefore, T' is a shape.

By induction, the function $E'^{-1}(Q)$ results in a k-rim hook shape. Since $E'^{-1}(Q)$ first composes all of the cells from all P_i into one collection of cells, $E'^{-1}(Q)$ must have the same size as the total size of all P_i. □

Example 5.9. Below (Figure 25) is an example of the impossibility of NW creating illegal heads or illegal tails. The cell c_1, under NW, cannot be shoved more than one unit diagonally beyond the position of its neighboring cell c_0. Were it was shoved more than one unit beyond c_0, leaving c_0 as an illegal head, the cells c_3, c_4, and c_5 would not exist, and c_2 would be an illegal head in the original shape. The algebra in the discussion above can be applied to the case displayed here, where the shell and column of the cell in question (c_1) are $s = 2$ and $c = 4$, respectively.

Also note that if r_Q is a k-rim hook in P_i for some i, then a k-rim hook $r_{E'^{-1}(Q)}$ consists of exactly the same cells as r_Q. Moreover, $\text{diag}_Q(r_Q) = \text{diag}_{E'^{-1}(Q)}(r_{E'^{-1}(Q)})$.

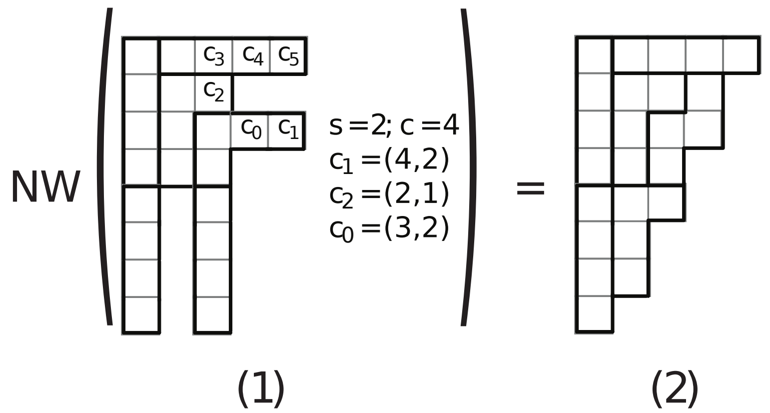

Figure 25: An illustration that NW results in no illegal heads or tails.

Lemma 5.10. The mapping E' is a bijection between the set of all k-rim hook shapes of size kn and the set of all k-tuples of shapes of i-oriented k-rim hook tableaux (one for each $0 \leq i < k$) with total size kn.

Proof. Since the diagonals of each k-rim hook in Q completely determine the P_i for all i, and since these diagonals do not change either in E' or in E'^{-1}, Stanton and White's Lemma 15 implies that $E' \circ E'^{-1} = \epsilon = E'^{-1} \circ E'$. As shown above, both E' and its inverse are well-defined functions between the two finite sets (Lemmas 5.7 and 5.8). Therefore, E' is a bijection. □

Note that the domain of E' is all k-rim hook *shapes*, not k-rim hook tableaux. This fact is one of the primary characteristics that differentiate Π' from Π.

5.3 Completing the Bijection Π'

Having proven that E' is a bijection, we are ready to compose E' with Γ to complete our proof of the mapping Π'. Define Γ as in Section 4.3. Note that, though the running example in Section 4 was a tableau, the content of the tableau (which sets it apart from a mere shape) is not necessary for the functioning of Γ. So here we consider as Γ's domain all shapes of i-oriented k-rim hook tableaux of size kn, and for its codomain the shapes of standard tableaux of size n. As before, Γ is a bijection between these two sets.

The function Π' itself is defined as before and has the same properties as before:

$$\Pi'(P) = (\lambda^0, \lambda^1, \ldots, \lambda^{k-1}) \text{ where } \lambda^i = \Pi'_{(i)}(P) = \Gamma \circ E'_{\widetilde{(i)}}(P).$$

Theorem 5.11. The mapping Π', as defined above, is a bijection between all modular tableau shapes of size kn and all k-tuples of shapes of standard tableaux with total size n.

Proof. Both Γ and $E'_{\widetilde{(i)}}$ are bijections, as shown above, so their composition is a bijection as well. Note that the function Π found by Stanton and White and described above (Section 4) is proven with Π': by adding appropriate content to the k-rim hook shapes in the domain of Π' and preserving the content of each cell throughout the bijection, Π' can be used to demonstrate Π. □

6 Counting Modular Tableaux with Colored Partitions

In order to reach our final result, we define the bijection α between the set of k-tuples of λ^is (standard tableau shapes) and the set of k-colored partitions of n. Note that, in a k-colored partition of n, multiple rows of the same length hold a combination, not a permutation of colors. Thus, swapping rows of the same length but of different colors in a k-colored partition of n does not result in a different k-colored partition of n.

The bijection α maps any k-tuple of standard tableau shapes to a distinct colored partition of n by coloring each row of the standard tableau shape λ^i with color i and then creating a partition of n composed of all the row lengths and their colors from the k-tuple. For example, if λ^0 has a Ferrers shape of $2^1 3^1$ then the standard tableau shape λ^0 contributes

$3_{c_0}\ 2_{c_0}$ (where c_i is a distinct color for each i) to the partition. Given this definition of α, we can also define its inverse. The function α^{-1} maps any k-colored partition of n to a distinct k-tuple of standard tableaux by separating the parts of the partition based on their color and creating a distinct standard tableau shape out of each monochromatic group of row lengths. Thus α^{-1} results in a k-tuple of standard tableau shapes of total size n. Because α and α^{-1} are well-defined, α is a bijection. Below, See Figure 26, below, for an example. The partition on the right is colored with colors c_i for $0 \leq i < 4$.

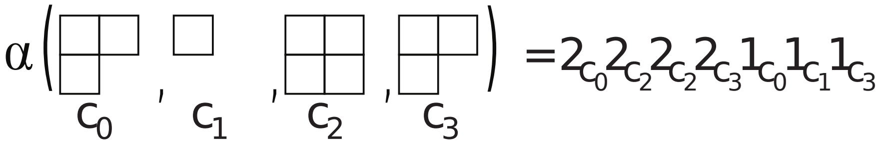

Figure 26: An illustration of $\alpha \circ \Pi(T)$ with $\Pi(T)$ given as in Figure 16.

With α in hand, we are now ready to compose the bijections Φ, Π', and α to attain our final result, which allows us to count modular tableaux by counting partitions:

Theorem 6.1. The number of modular tableaux of size kn is equal to the number of k-colored partitions of n for all $k, n \in \mathbb{N}$.

Proof. Because Φ, Π', and α are bijections, their composition is also a bijection. Therefore the two sets (the domain of Φ and the codomain of α) have equal size. $\square$

Generating functions can be used to count the number of k-colored partitions of n. See [1] for a thorough introduction to generating functions; here, we give a cursory explanation. The number of k-colored partitions of n is

$$\left\langle q^n \;\middle|\; \prod_{m=1}^{\infty} \left(\frac{1}{1-q^m}\right)^k \right\rangle.$$

That is, the coefficient of the q^n term in the expansion of

$$\prod_{m=1}^{\infty} \left(\frac{1}{1-q^m}\right)^k$$

is the number of k-colored partitions of n. For example, let $k = n = 3$. Then the number of 3-colored partitions of 3 is 22, the coefficient of the q^3 term in the equation

$$\prod_{m=1}^{\infty} \left(\frac{1}{1-q^m}\right)^3 = 1 + 3x + 9x^2 + 22x^3 + 51x^4 + O(x^5).$$

Having established the equality of the number of k-colored partitions of n and the number of modular tableaux of size kn (for $a + b = k$) in Theorem 6.1, we can use the generating function above to enumerate the latter set as well.

This result also prompts and renews interest in related questions. Here we offer three subjects for further exploration. The first deals with tableaux that are modular when we do not consider a collection of cells, known as a *core*, which remains after all possible k-rim hooks have been removed from the tableau. For tableaux containing cores, we propose that a result similar to Theorem 6.1 can be proven. A second open question arises from patterns we have noted in the number of modular tableaux with other combinations of a and b, where $a + b \neq k$. Some behave similarly to the $a + b = k$ case, while others exhibit recurring sequences that are unexplained. Explaining these sequences with a combinatorial mapping should be possible; an attempt could follow the general structure of our results above. Finally, though the composition given in Theorem 6.1 suffices to show the desired correspondence, it seems plausible that a simpler combinatorial mapping exists. Research could continue in this area as well.

References

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