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CUT-SETS AND CUT-VERTICES IN THE ZERO-DIVISOR GRAPH OF $\prod_{i=1}^{m} \mathbb{Z}_{n_i}$

B. COTÉ, C. EWING, M. HUHN, C. M. PLAUT, D. WEBER

Abstract. We examine minimal sets of vertices which, when removed from a zero-divisor graph, separate the graph into disconnected subgraphs. We classify these sets for all direct products of $\Gamma\left(\prod_{i=1}^{m} \mathbb{Z}_{n_i}\right)$.

1. Introduction and Definitions

All rings in the paper are commutative with unity. An element $a \in R$ is a *zero-divisor* if there exists a nonzero $r \in R$ such that $ar = 0$; we denote the set of all zero-divisors in R as $Z(R)$. For a graph G, define $V(G)$ as the set of vertices in G, and $E(G)$ as the set of edges in G. We define a *path* between two elements $a_1, a_m \in V(G)$ to be an ordered sequence of distinct vertices $\{a_1, a_2, \ldots, a_n\}$ of G such that there is an edge incident to a_{i-1} and a_i, denoted a_{i-1} — a_i for each i. For $x, y \in V(G)$, the number of edges crossed to get from x to y in a path is called the *length* of the path; the length of the shortest path between x and y, if it exists, is called the *distance* between x and y and is denoted $d(x, y)$. If such a path does not exist then $d(x, y) = \infty$. The *diameter* of a graph is $diam(G) = max\{d(x, y) \mid, y \in V(G)\}$. A graph is *connected* if a path exists between any two distinct vertices.

A *zero-divisor graph*, denoted $\Gamma(R)$, is a graph whose vertices are all the nonzero zero-divisors of R. Two vertices a and b are connected by an edge in $\Gamma(R)$ if and only if $ab = 0$. In R, we define the annihilator of a, $ann(a)$, by $ann(a) = \{b \in R \mid ba = 0\}$, so that the neighbors of a in $\Gamma(R)$ are the nonzero elements of $ann(a)$. A vertex a is looped if and only if $a^2 = 0$. By [1], we know that $\Gamma(R)$ is always connected and $diam(\Gamma(R)) \leq 3$ for any ring R.

Definition 1.1. A vertex, a, in a connected graph G is a *cut-vertex* if G can be expressed as a union of two subgraphs X and Y such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a\}$, $X \backslash \{a\} \neq \emptyset$, and $Y \backslash \{a\} \neq \emptyset$.

Definition 1.2. A set $A \subseteq Z(R)^*$, where $Z(R)^* = Z(R) \backslash \{0\}$, is said to be a *cut-set* if there exist $c, d \in Z(R)^* \backslash A$ where $c \neq d$ such that every path in $\Gamma(R)$ from c to d involves at least one element of A, and no proper subset of A satisfies the same condition.

Another way to define a cut-set is as a set of vertices $\{a_1, a_2, a_3, \ldots\}$ in a connected graph G where G can be expressed as a union of two subgraphs X and Y

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such that $E(X) \neq \emptyset$, $E(Y) \neq \emptyset$, $E(X) \cup E(Y) = E(G)$, $V(X) \cup V(Y) = V(G)$, $V(X) \cap V(Y) = \{a_1, a_2, a_3, \ldots\}$, $X \backslash \{a_1, a_2, a_3, \ldots\} \neq \emptyset$, $Y \backslash \{a_1, a_2, a_3, \ldots\} \neq \emptyset$, and no proper subset of $\{a_1, a_2, a_3, \ldots\}$ also acts as a cut-set for any choice of X and Y. A cut-vertex can be thought of as a cut-set with only one element. For a cut-set A in $\Gamma(R)$, a vertex $a \notin A$ is said to be *isolated*, or an *isolated point*, if $ann(a) \backslash \{0\} \subseteq A$.

Example 1.3. Consider $\Gamma(\mathbb{Z}_{12})$ shown in Figure 1. In this graph, 6 is a cut-vertex isolating 2 and 10. In addition, $\{4, 8\}$ is a cut-set isolating 3 and 9.

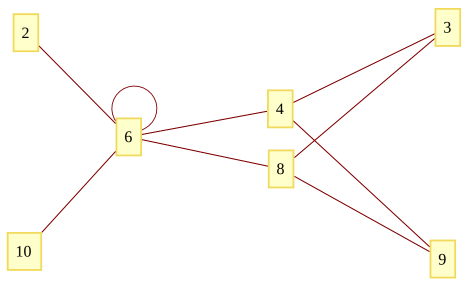

FIGURE 1. $\Gamma(\mathbb{Z}_{12})$ generated using [6].

Example 1.4. Consider $\Gamma(\mathbb{Z}_{30})$ shown in Figure 2. In this graph, 15 is a cut-vertex isolating 2 among other vertices. Observe that the set $\{6, 12, 18, 24\}$ is a cut-set isolating 5 and 25. In addition, $\{10, 20\}$ is a cut-set isolating 3, 9, 21 and 27.

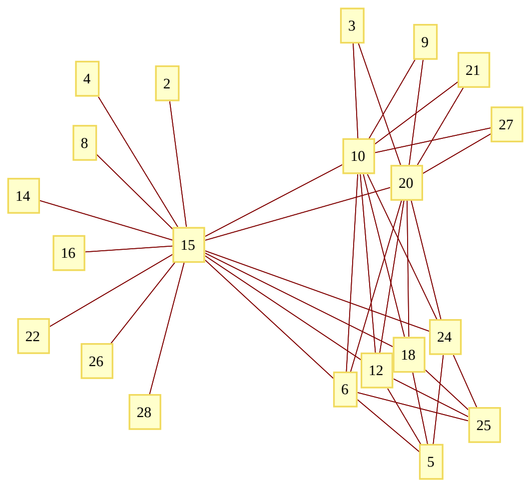

FIGURE 2. $\Gamma(\mathbb{Z}_{30})$ generated using [6].

The study of cut-vertices in a zero-divisor graph began in [3], where it was proven that if a vertex, a, is a cut-vertex of $\Gamma(R)$ for any commutative ring, R, then $\{0, a\}$

forms an ideal in R. We will generalize this notion and expand on many of the results from [3].

This paper will classify cut-vertices and cut-sets of zero-divisor graphs of finite commutative rings of the form $\Pi(\mathbb{Z}_{n_i})$. In section 2 we classify $\Gamma(\mathbb{Z}_n)$, and apply our findings to cut-vertices of $\Gamma(\Pi(\mathbb{Z}_{n_i}))$. Section 3 classifies cut-sets of $\Gamma(\Pi(\mathbb{Z}_{n_i}))$ by examining cut-sets of $\Gamma(\mathbb{Z}_n)$.

2. Cut-Vertices in $\Gamma\left(\prod_{i=1}^{m} \mathbb{Z}_{n_i}\right)$

This section begins with an examination of cut-vertices in the ring $\mathbb{Z}_n$ in preparation for generalizing to direct products.

Recall that for a commutative ring R, if $a, b \in R^*$, where R^* is $R\backslash\{0\}$, such that $ab = 0$, then $a(-b) = 0$. Also, in $\mathbb{Z}_n$, let $p \in \mathbb{Z}$ be a prime that divides n. Then $ann(p) \subseteq ann(ap)$ for any $a \in \mathbb{Z}$.

Theorem 2.1. *An element a is a cut-vertex of $\Gamma(\mathbb{Z}_n)$ if and only if $2a = n$ with $n \geq 6$.*

Proof. ($\Rightarrow$) Observe that $\Gamma(\mathbb{Z}_n)$ has no cut-vertex for $n < 6$ [5]. So let $n \geq 6$, and assume that a is a cut vertex of $\Gamma(\mathbb{Z}_n)$. Then $\Gamma(\mathbb{Z}_n)$ is split into two subgraphs X and Y, which are distinct except for their common vertex a. Let $V(X) = \{a, x_1, x_2, \ldots, x_m\}$ and $V(Y) = \{a, y_1, y_2, \ldots, y_l\}$. Since a is a cut vertex, there exists some $x_i \in V(X)$ and some $y_j \in V(Y)$ such that $x_i — a — y_j$. Since $x_i a = 0$, $x_i(-a) = 0$. Similiarly, $y_j a = 0$ and $y_j(-a) = 0$, so there is a path $x_i — a — y_j$. Since a is a cut-vertex, $a = -a$ or $2a = 0$ in $\mathbb{Z}_n$. Thus, $2a = n$.

($\Leftarrow$) It suffices to show $ann(2) = \{0, a\}$ in $\mathbb{Z}_{2a}$ with $a \geq 3$. Assume $2m = 0$. This implies $m = 0$ or $m = a$. Since $a \neq 2$, 2 is a vertex isolated by the cut-vertex a. $\square$

Theorem 2.2. *Let $\mathbb{Z}_n \times \mathbb{Z}_m \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $(a, 0)$ is a cut-vertex of $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ if and only if $2a = n$.*

Proof. ($\Rightarrow$) Assume $(a, 0)$ is a cut-vertex of $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$ separating subgraphs X and Y. There exists some $(x_{i_1}, x_{i_2}) \in V(X)$ and some $(y_{j_1}, y_{j_2}) \in V(Y)$ such that $(x_{i_1}, x_{i_2}) — (a, 0) — (y_{j_1}, y_{j_2})$. But then, $(x_{i_1}, x_{i_2}) — (-a, 0) — (y_{j_1}, y_{j_2})$. Since $(a, 0)$ is a cut-vertex, $(a, 0) = (-a, 0)$, which means $a = -a$. This implies $2a = 0$, in $\mathbb{Z}_n$, so $2a = n$.

($\Leftarrow$) Assume $2a = n$. In the case that $n = 2$, there is a cut-vertex at $(1, 0)$ since it is the only element adjacent to $(0, 1)$ and it is also adjacent to $(0, 2)$ since $\mathbb{Z}_n \times \mathbb{Z}_m \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For the last case assume that $n > 2$ and consider $ann((2, 1))$. Clearly, $ann((2, 1)) = \{(0, 0), (a, 0)\}$. Therefore, $(a, 0)$ isolates $(2, 1)$ and is a cut-vertex by definition. $\square$

Theorem 2.3. *Consider $R = \prod_{i=1}^{m} \mathbb{Z}_{n_i}$ for $m \geq 3$. Then $(0, 0, \ldots, a_i, \ldots, 0)$ is a cut-vertex of $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_i} \times \cdots \times \mathbb{Z}_{n_m})$ if and only if $2a_i = n_i$.*

Proof. ($\Rightarrow$) Assume $(0, 0, \ldots, a_i, \ldots, 0)$ is a cut-vertex of $\Gamma(R)$. Then $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_i} \times \cdots \times \mathbb{Z}_{n_m})$ is split into two subgraphs X and Y, which are distinct except for their common vertex $(0, 0, \ldots, a_i, \ldots, 0)$. There exists some

$(x_1,\ x_2,\dots,x_i,\dots,x_m) \in V(X)$ and $(y_1,y_2,\dots,y_i,\dots,y_m) \in V(Y)$ such that $(x_1,x_2,\dots,x_i,\dots,x_m)$ — $(0,0,\dots,a_i,\dots,0)$ — $(y_1,y_2,\dots,y_i,\dots,y_m)$. Hence the following path must also exist: $(x_1,x_2,\dots,x_i,\dots,x_m)$ — $(0,0,\dots,-a_i,\dots,0)$ — $(y_1,y_2,\dots,y_i,\dots,y_m)$. Since $(0,0,\dots,a_i,\dots,0)$ is a cut-vertex, $(0,0,\dots,a_i,\dots,0) = (0,0,\dots,-a_i,\dots,0)$, which implies $a_i = -a_i$, or $2a_i = 0$ in $\mathbb{Z}_{n_i}$. Therefore $2a_i = n_i$.

($\Leftarrow$) Without loss of generality, let $n_1 = 2a_1$. It suffices to show that $ann((2,1,\dots,1)) = \{(0,\dots,0),(a_1,0,\dots,0)\}$. Assume $(2,1,\dots,1)(b,0,\dots,0) = (0,\dots,0)$. This implies $b = 0$ or $b = a_1$. □

The next theorem shows that for $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is any finite commutative ring, all cut-vertices are of the form $(0,0,\dots,0,a_i,0,\dots,0)$

Theorem 2.4. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. If a is a cut-vertex of $\Gamma(R)$ then there exists some i, $1 \le i \le n$, such that $a = (0,\dots,0,a_i,0,\dots,0)$.*

Proof. Let a be a cut-vertex of R with $a = (a_1,a_2,\dots,a_n)$. Assume $a_i, a_j \neq 0$ with $i \neq j$. Since a is a cut-vertex, there exists α, $\beta \in Z(R)$ such that the only path between them is α — a — β. Consider the ring element $b = (0,\dots,0,a_i,0,\dots,0)$. Then α — b — β, a contradiction. □

The next corollary follows from Theorems 2.3 and 2.4.

Corollary 2.5. *Let $R = \prod_{i=1}^{m} \mathbb{Z}_{n_i}$ for $m \ge 3$. Then $a = (a_1,a_2,\dots,a_m)$ is a cut-vertex if and only if $a = (0,0,\dots,0,a_i,0,\dots,0)$ where $2a_i = n_i$, for some i, $1 \le i \le m$.*

Proof. Let $R = \prod_{i=1}^{m} \mathbb{Z}_{n_i}$ for $m \ge 3$.

($\Rightarrow$) Let $a = (a_1,a_2,\dots,a_m)$ be a cut-vertex. Then by Theorem 2.4, $a = (0,\dots,0,a_i,0,\dots,0)$ for some $1 \le i \le m$. Since $a = (0,\dots,0,a_i,0,\dots,0)$ is a cut-vertex, then by Theorem 2.3, $2a_i = n_i$.

($\Leftarrow$) Let $a = (0,0,\dots,0,a_i,0,\dots,0)$ where $2a_i = n_i$. Then by Theorem 2.3 a is a cut-vertex. □

3. Cut-Sets in $\Gamma\left(\prod_{i=1}^{m} \mathbb{Z}_{n_i}\right)$

In this section we generalize the idea of a cut-vertex to that of a cut-set. Many results on cut-vertices generalize to cut-sets, and we may consider all theorems in the previous section as corollaries to the following theorems on cut-sets.

Note that when $n = p$, p a prime, the ring $\mathbb{Z}_n$ is a field so $\Gamma(\mathbb{Z}_n)$ is empty. When $n = 2p$, $p > 2$, $\Gamma(\mathbb{Z}_n)$ is a star-graph, where the only cut-set is $A = ann(2)\backslash\{0\} = \{p\}$. For example, Figure 3 shows $\Gamma(\mathbb{Z}_{14})$. Notice that $ann(p)\backslash\{0\} = V(\Gamma(\mathbb{Z}_n))\backslash\{p\}$. When $n = p^2$, $\Gamma(\mathbb{Z}_n)$ is a complete graph; whence there are no cut-sets.

Theorem 3.1. *Let $n \in \mathbb{Z}^+$ such that $n \neq p, 2p, p^2$ for any prime p. A set A is a cut-set of $\Gamma(\mathbb{Z}_n)$ if and only if $A = ann(p)\backslash\{0\}$ for some prime p which divides n.*

Proof. ($\Leftarrow$) Let $A = ann(p)\backslash\{0\}$ for some prime $p \in \mathbb{Z}$ that divides n. Observe that $p \notin ann(p)$ since $n \neq p^2$. Then p is only connected to A in $\Gamma(\mathbb{Z}_n)$, so when A is removed, p is isolated.

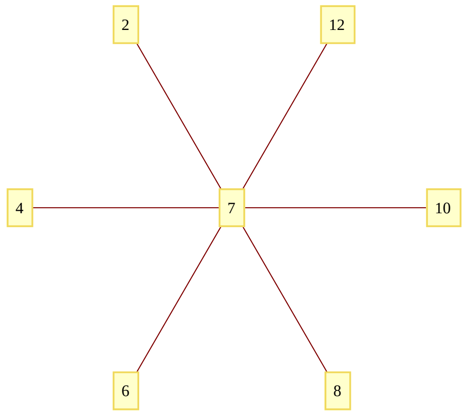

FIGURE 3. $\Gamma(\mathbb{Z}_{14})$ generated using [6].

Notice that $n-p \neq p$, and $n-p$ is connected to all elements in A, but $(n-p)p \neq 0$ since $n \neq p^2$. This implies that A splits $\Gamma(\mathbb{Z}_n)$ into two subgraphs.

Suppose some subset B of A splits the graph similarly, and let $a \in A \backslash B$. Then $p — a — (n-p)$, a contradiction.

($\Rightarrow$) Assume A is a cut-set of $\Gamma(\mathbb{Z}_n)$ separating subgraphs X and Y. Take any $x \in X \backslash A$ and $y \in Y \backslash A$ where $x — a_i$ and $y — a_j$, with $a_i, a_j \in A$. Rewrite x and y as $x = rp_x$ and $y = qp_y$ where p_x and p_y are primes dividing n.

First assume that $p_x \neq p_y$, where p_x does not divide y and p_y does not divide x, and take any nonzero element of $ann(p_x) = \{k(n/p_x) \mid k \in \mathbb{Z}_{p_x}\}$, say, $c(n/p_x)$. This is a multiple of p_y, so we have that $ann(p_y) \subseteq ann(c(n/p_x))$ and that $x \in ann(c(n/p_x))$, since x is a multiple of p_x. Thus for any nonzero $\beta \in ann(p_y)$ we have $y — \beta — c(n/p_x) — x$. Since A is a cut-set separating X and Y, $\beta \in A$ or $c(n/p_x) \in A$. If $\beta \in A$ this implies that $ann(p_y) \backslash \{0\} \subseteq A$, and if $c(n/p_x) \in A$ this implies that $ann(p_x) \backslash \{0\} \subseteq A$. Since p_x does not divide y and p_y does not divide x, x is not connected to any element of $ann(p_y) \backslash \{0\}$ and y is not connected to any element of $ann(p_x) \backslash \{0\}$. If $ann(p_y) \backslash \{0\} \subseteq A$, then since x is connected to an element of A and not to any element of $ann(p_y) \backslash \{0\}$, there is at least one additional element in A. Similarly, if $ann(p_x) \backslash \{0\} \subseteq A$, then since y is connected to an element of A and not to any element of $ann(p_x) \backslash \{0\}$, there is at least one additional element in A. Since $ann(p_x) \backslash \{0\}$ and $ann(p_y) \backslash \{0\}$ are cut-sets, in either case A would contain a cut-set as a proper subset, a contradiction.

We may therefore assume that $p_x = p_y$. Then since $ann(p_x) \subseteq ann(x)$ and $ann(p_x) \subseteq ann(y)$, for any nonzero $\alpha \in ann(p_x)$ we have $x — \alpha — y$, which implies that $ann(p_x) \backslash \{0\} \subseteq A$. Since $ann(p_x) \backslash \{0\} \subseteq A$ and $ann(p_x) \backslash \{0\}$ is a cut-set by the first direction of this proof, then $ann(p_x) \backslash \{0\} = A$ by definition of a cut-set. $\square$

Lemma 3.2. *In $\mathbb{Z}_n$, let $p \in \mathbb{Z}$ be a prime that divides n. Then $|ann(p)| = p$.*

Proof. In the ring $\mathbb{Z}_n$, $ann(p) = \{a(n/p) \mid a \in \mathbb{Z}_p\}$ since $p(n/p) = 0$, $(p+1)(n/p) = n/p$, and so on. There are p distinct elements in this set. $\square$

Because every cut-set of $\Gamma(\mathbb{Z}_n)$ is an annihilator of some prime that divides n, by Lemma 3.2, the size of any cut-set in $\Gamma(\mathbb{Z}_n)$ is known. The following lemmas and corollaries will be useful in the proofs of some of the next theorems in this section.

Lemma 3.3. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. If $a \in R$ with $a = (a_1, \ldots, a_i, \ldots, a_n)$ and $a' = (a_1, \ldots, 0, \ldots, a_n)$ then $ann(a) \subseteq ann(a')$.*

Proof. Let $b \in ann(a)$. Then $b_l a_l = 0$ for all $1 \leq l \leq n$, and $b_i \cdot 0 = 0$. Thus $b \in ann(a)$. $\square$

Corollary 3.4. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. Let A be a cut-set of R. If $a \in A$ with $a = (a_1, \ldots, a_i, \ldots, a_n)$ and $a' = (a_1, \ldots, 0, \ldots, a_n)$, then $a' \in A$.*

Proof. Assume $a' \notin A$. Observe that for $b \in Z(R)^*$, if $b - a$, then $b - a'$ by Lemma 3.3. Thus $A \backslash \{a\}$ is (or contains) a cut-set - a contradiction. $\square$

Lemma 3.5. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. If $a \in R$ with $a = (a_1, \ldots, a_i, \ldots, a_n)$ and $a' = (a_1, \ldots, 1, \ldots, a_n)$ then $ann(a') \subseteq ann(a)$.*

Proof. Let $b \in ann(a')$. Then $b_l a_l = 0$ for all $1 \leq l \leq n$ and in particular $b_i = 0$. This implies $b_i a_i = 0$, and thus $b \in ann(a)$. $\square$

Theorem 3.6. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ with $n \geq 2$. If A is a cut-set of $\Gamma(R)$ then there exists some i, $1 \leq i \leq n$ such that $a = (0, \ldots, 0, a_i, 0, \ldots, 0)$ for every $a \in A$.*

Proof. Let A be a cut-set of R which splits $\Gamma(R)$ into X and Y. Without loss of generality, assume there exists some $b = (b_1, b_2, \ldots, b_n) \in A$ with $b_1 \neq 0$, $c = (c_1, c_2, \ldots, c_n) \in A$ with $c_2 \neq 0$.

Consider the set of all elements in $Z(R)$ with a 0-entry in the 1^{st} position; let this set be denoted by $\{0_1\}$. Let 1_1 denote the element $(1, 0, \ldots, 0)$ and similarly denote 1_2, and so on. Notice $ann(1_1) = \{0_1\}$. Denote by $(1_1)^*$ all elements with 0 everywhere except the 1^{st} position; i.e., $(1_1)^*$ is the ideal generated by 1_1, omitting 0. If $1_1 \in A$ then $(1_1)^* \in A$ since any element which annhilates 1_1 also annhilates any element in $(1_1)^*$.

Consider the element $\overline{1_1} = (0, 1, \ldots, 1)$. Notice $ann(\overline{1_1}) = (1_1)^* \cup \{0\}$. Therefore, $(1_1)^*$ forms a cut-set by isolating $\overline{1_1}$. Notice $(1_1)^* \subsetneq A$ since $c \in A$, a contradiction. Therefore, $1_1 \notin A$ and we can similarly show $1_i \notin A$ for every $1 \leq i \leq n$. Without loss of generality, 1_1,$1_2 \in X \backslash A$ since $1_1 - 1_2$. Similarly every $1_i \in X \backslash A$ for $1 \leq i \leq n$. This implies that if $y \in Y \backslash A$ then $y_i \neq 0$ for any $1 \leq i \leq n$, since otherwise it would be connected to an element in $X \backslash A$, namely 1_i.

Consider $\alpha = (b_1, 0, \ldots, 0)$ and $\beta = (0, c_2, 0, \ldots, 0)$. Notice $\alpha, \beta \in A$ by Corollary 3.4 since $b, c \in A$. This implies b_1 and c_2 are not units, since otherwise we reach the contradiction shown in the previous paragraph.

Since $\alpha \in A$, there exists $y = (y_1, y_2, \ldots, y_n) \in Y \backslash A$ such that $y - \alpha$. Clearly $ann(y) \backslash \{0\} \subseteq Y$. Consider the element $y' = (y'_1, y'_2, \ldots, y'_n)$ where $y'_2 = 1$ and $y'_i = y_i$ for all $i \neq 2$. By Lemma 3.5, $ann(y') \backslash \{0\} \subseteq ann(y) \backslash \{0\}$. Also, $ann(y') \backslash \{0\} \subseteq A$ since any element which annihilates y' must have a zero in the second position. Therefore, $ann(y') \backslash \{0\} \subsetneq A$ since $\beta \notin ann(y') \backslash \{0\}$ but $\beta \in A$. Thus, $ann(y') \backslash \{0\}$ forms a cut-set, a contradiction of the minimality of A. Therefore, all elements in A must be of the form $(0, \ldots, 0, a_i, 0, \ldots, 0)$. $\square$

Theorem 3.7. *A set A of $V\left(\Gamma\left(\prod_{i=1}^{m} \mathbb{Z}_{n_i}\right)\right)$ with $m \geq 2$ is a cut-set if and only if $A = \{(0, 0, \ldots, a_{i_1}, \ldots, 0), (0, 0, \ldots, a_{i_2}, \ldots, 0), \ldots, (0, 0, \ldots, a_{i_k}, \ldots, 0)\}$ where $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} = ann(p)\backslash\{0\}$ for some prime $p \in \mathbb{Z}$ such that $p|n_i$ in $\mathbb{Z}_{n_i}$.*

Proof. ($\Leftarrow$) First note that the case where $n_i = p$ is included in this theorem because the nonzero annihilators of p are exactly the nonzero elements of $\mathbb{Z}_{n_i}$. Without loss of generality, let p be a prime such that p divides n_1 and let $ann(p) = \{0, a_{1_1}, a_{1_2}, \ldots, a_{1_k}\}$ in $\mathbb{Z}_{n_1}$. Because $ann((p, 1, \ldots, 1)) = \{(0, \ldots, 0), (a_{1_1}, 0, \ldots, 0), (a_{1_2}, 0, \ldots, 0), \ldots, (a_{1_k}, 0, \ldots, 0)\}$, when $A = \{(a_{1_1}, 0, \ldots, 0), (a_{1_2}, 0, \ldots, 0), \ldots, (a_{1_k}, 0, \ldots, 0)\}$ is removed from the graph, $(p, 1, \ldots, 1)$ is isolated. Further, $Z\left(\prod_{i=1}^{m} \mathbb{Z}_{n_i}\right)^* \neq A \cup \{(p, 1, \ldots, 1)\}$ since $(0, 1, 0, \ldots, 0)$ is outside of A but distinct from $(p, 1, \ldots, 1)$. Because $ann(p) = \{0, a_{1_1}, a_{1_2}, \ldots, a_{1_k}\}$ in $\mathbb{Z}_{n_1}$, we see that for any $1 \leq l, q \leq k$, $ann(a_{1_l}) = ann(a_{1_q})$ in $\mathbb{Z}_{n_1}$, implying that no proper subset of A will act as a cut-set.

($\Rightarrow$) Let A be a cut-set of $\Gamma(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m})$ and let X and Y be two subgraphs created when A is removed. Take any $(x_1, x_2, \ldots, x_m) \in X\backslash A$ and $(y_1, y_2, \ldots, y_m) \in Y\backslash A$ such that $(x_1, x_2, \ldots, x_m) — (a_1, a_2, \ldots, a_m)$ and $(y_1, y_2, \ldots, y_m) — (b_1, b_2, \ldots, b_m)$ for some $(a_1, a_2, \ldots, a_m), (b_1, b_2, \ldots, b_m) \in A$. Because any $(u_1, u_2, \ldots, u_m)$ where each u_i is a unit contains only the zero element in its annihilator, we know that at least one of the x_i must be a zero-divisor of the corresponding $\mathbb{Z}_{n_i}$, and similarly for the y_i. Since we also know that all elements of A are zero in every component position but, say, the ith position by Theorem 3.6, the ith component position of both $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ must contain a zero-divisor. We may therefore assume without loss of generality that both x_1 and y_1 are zero-divisors of $\mathbb{Z}_{n_1}$. Since this is the case we can rewrite $x_1 = rp_{x_1}$ and $y_1 = qp_{y_1}$ for $r, q \in \mathbb{Z}_{n_1}$ and primes $p_{x_1}, p_{y_1} \in \mathbb{Z}$ dividing n_1.

First assume that $p_{x_1} \neq p_{y_1}$, where p_{x_1} does not divide y_1 and p_{y_1} does not divide x_1. Then by Theorem 3.1 we know that for any nonzero $\beta \in ann(p_{y_1})$ and any nonzero $c(n/p_{x_1})$, $y_1 — \beta — c(n/p_{x_1}) — x_1$ in $\Gamma(\mathbb{Z}_{n_1})$. Therefore we have that $(y_1, y_2, \ldots, y_m) — (\beta, 0, \ldots, 0) — (c(n/p_{x_1}), 0, \ldots, 0) — (x_1, x_2, \ldots, x_m)$. This implies that $(\beta, 0, \ldots, 0) \in A$, which would imply inclusion of all such $(\beta, 0, \ldots, 0)$ in A, or $(c(n/p_{x_1}), 0, \ldots, 0) \in A$, which would imply a similar inclusion. Since p_{x_1} does not divide y_1 and p_{y_1} does not divide x_1, $(x_1, x_2, \ldots, x_m)$ is not connected to elements of the form $(\beta, 0, \ldots, 0)$ and $(y_1, y_2, \ldots, y_m)$ is not connected to elements of the form $(c(n/p_{x_1}), 0, \ldots, 0)$. However, since we know that both $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ are connected to elements in A and that the set of all nonzero $(\beta, 0, \ldots, 0)$ and the set of all nonzero $(c(n/p_{x_1}), 0, \ldots, 0)$ are cut-sets by the first direction, we see that A contains a proper subset that is a cut-set, a contradiction.

We may therefore assume that $p_{x_1} = p_{y_1}$. Then since $ann(p_{x_1}) \subseteq ann(x_1)$ and $ann(p_{x_1}) \subseteq ann(y_1)$, then for any nonzero $\alpha \in ann(p_{x_1})$, $(x_1, x_2, \ldots, x_m) — (\alpha, 0, \ldots, 0) — (y_1, y_2, \ldots, y_m)$. We then have that $(\alpha, 0, \ldots, 0) \subseteq ann((x_1, x_2, \ldots, x_m))$ and $(\alpha, 0, \ldots, 0) \subseteq ann((y_1, y_2, \ldots, y_m))$, meaning that $(\alpha, 0, \ldots, 0) \subseteq A$, but since the set of all nonzero $(\alpha, 0, \ldots, 0)$ where $\alpha \in ann(p_{x_1})$ is a cut-set by the first direction, then $A = \{(\alpha, 0, \ldots, 0) \mid \alpha \neq 0,\ \alpha \in ann(p_{x_1})\}$. $\square$

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References

[1] Anderson, David F. and Livingston, Philip S., The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434-447.
[2] Axtell, M., Baeth, N., Stickles, J., Cut Vertices and Zero-Divisor Graphs, submitted.
[3] Axtell, M., Stickles, J., and Trampbachls, W., Zero-divisor ideals and realizable zero-divisor graphs, *Involve* **2(1)** (2009), 17-27.
[4] Dummit, David S., and Foote, Richard M., *Abstract Algebra, 3rd Ed.* (2004), John Wiley and Sons, Inc.
[5] Redmond, S., On Zero-divisor graphs of small finite commutative rings, *Discrete Math.* **307(9-10)** (2007), 1155-1166.
[6] MATHEMATICA, Wolfram Software.