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Abstract. The article introduces Ahlfors' generalization of Schwarz' lemma. With this powerful geometric tool of complex functions in one variable, we are able to prove some theorems concerning the size of images under holomorphic mappings, including celebrated Picard's theorems. The article concludes with a brief insight into the theory of Kobayashi hyperbolic complex manifolds. Although this survey paper does not contain any new results, it may be useful for the beginner in complex analysis to better understands the concept of hyperbolicity in complex geometry.

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1 Introduction

Since 1881, when famous French mathematician **Henry Poincaré** (1854–1912) connected hyperbolic geometry on a disc with complex analysis, it had been known that the Poincaré metric (1) has constant Gaussian curvature -1 (Definition 2). This is also why the metric is traditionally considered hyperbolic. Surprisingly, it was not until 1938 that a Finnish mathematician **Lars V. Ahlfors** (1907–1996), one of the first two Fields medalists, realized that the Schwarz-Pick lemma (Theorem 6) was a consequence of the negative curvature of the Poincaré metric. His result, published in the paper *An extension of Schwarz's lemma* [Ahl38], is known as Ahlfors' lemma (Theorem 8) or the Schwarz-Pick-Ahlfors lemma in full. According to Ahlfors himself, he published the result because the lemma provides a relatively simple proof of Bloch's theorem (Theorem 3) with a very good estimation of the constant B from Bloch's theorem. He also said that his lemma "is an almost trivial fact that anybody who sees the need could prove it at once" [Kra04, p. 70]. In the last section of this paper we will see the influence of Ahlfors' lemma in the theory of hyperbolic complex manifolds.

It seems rather interesting that although Ahlfors' lemma is comparatively old and straightforward, it is rarely presented in (mostly undergraduate) textbooks on complex analysis of one variable, Ahlfors' classic itself [Ahl79] being no exception. On the other hand, it is included in a book by Narasimhan [NN01], who proves and uses it further to prove two theorems of Picard (Theorems 4 and 5). That said, Narasimhan's book does not contain a proof of Bloch's theorem, one of Ahlfors' own applications, and only briefly addresses the importance of completeness of metrics. For instance, the latter property is crucial for the proof of the Normality theorem (Theorem 13) and consequently for the proof of the Big Picard theorem.

There are two reasons for writing this article. The first reason is to provide a proof of the original version of Ahlfors' lemma and then use it to prove Bloch's and Picard's theorems. Here, the word "original" means that we proved Ahlfors' lemma for a supporting pseudometric (Definition 3). This is presented in the way that is useful for the second reason, which is an introduction to the theory of hyperbolic complex manifolds. Here, we introduce the Kobayashi pseudodistance and related hyperbolic manifolds. We mentioned some basic generalizations of Picard's theorems to complex manifolds via hyperbolicity. The article can thus serve as a motivation for that interesting and still developing area of complex geometry.

The article is self-contained, very little of elementary complex analysis is assumed. Section 2 provides a background material where we state classical theorems concerning this paper, namely Schwarz's lemma, Bloch's theorem and Picard's theorems. Section 3 starts with the definition of the Poincaré metric on a disc, we calculate the corresponding distance and state the Schwarz-Pick lemma. In Section 4 we prove some properties of inner distances, the most important of which is the Hopf-Rinow theorem (Theorem 7). Here we introduce the notion of b-completeness of a domain (Definition 1), which could be understood as the boundary of a domain being "infinitely far away" from every inner point. In Section 5 we prove Ahlfors' lemma and Section 6 gives some applications of it: a proof of Bloch's theorem, two theorems of Landau (Theorems 9 and 10), Schottky's theorem (Theorem 11) and Pi-

card's theorems. Section 7 concludes the article with some properties of hyperbolic complex manifolds, especially those connected with Picard's theorems.

2 Preliminaries

This preliminary section is a review of some fundamental definitions and theorems that are used in this paper. Also some historical comments are added.

Let $\mathbb{C}$ be the complex plane, $\mathbb{D}_r := \{z \in \mathbb{C}\colon |z| < r\}$ an open disc with radius $r > 0$ and $\mathbb{D}_r^* := \mathbb{D}_r \setminus \{0\}$. Also $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{D}^* := \mathbb{D}_1^*$. A domain $\Omega \subseteq \mathbb{C}$ is, by definition, an open and connected set. The family of holomorphic functions on the domain Ω is denoted by $\mathcal{O}(\Omega)$, and the family of holomorphic mappings from a domain Ω_1 to domain Ω_2 by $\mathcal{O}(\Omega_1, \Omega_2)$. Biholomorphic map is, by definition, a holomorphic map with holomorphic inverse.

Theorem 1 (Schwarz's lemma)**.** *Assume $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $f(0) = 0$.*

(a) *Then $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$ and $|f'(0)| \leq 1$.*

(b) *If $|f'(0)| = 1$, or if $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}^*$, then there exists $a \in \partial\mathbb{D}$ such that $f(z) = az$.*

In the years 1869–1870, the German mathematician **Hermann A. Schwarz** (1843–1921) was trying to offer an ultimate proof of the celebrated Riemann mapping theorem. In his thesis (1851), Riemann proposed a theorem which stated that every simply connected domain $\Omega \subset \mathbb{C}$ is biholomorphic to $\mathbb{D}$. Résumé of the second chapter of Schwarz's lecture entitled "Zur Theorie der Abbildung" [Sch90, pp. 109-111] is: *Let $f\colon \mathbb{D} \to f(\mathbb{D}) \subset \mathbb{C}$ be a biholomorphic mapping with $f(0) = 0$. Assume that ρ_1 (resp. ρ_2) is the minimum (resp. the maximum) distance from 0 to the boundary of $f(\mathbb{D})$. Then $\rho_1|z| \leq |f(z)| \leq \rho_2|z|$ and $\rho_1 \leq |f'(0)| \leq \rho_2$.* Schwarz derived the first inequality from examining real part of $\log(f(z)/z)$ and the second inequality from the Cauchy integral representation theorem for $f'(0)$. It is clear that part (a) of Theorem 1 follows from this result. The present form, proof and name of the lemma were written in 1912 by a Greek-German mathematician **Constantin Carathéodory** (1873–1950). He popularized the lemma through various problems of conformal mappings. For proof of the lemma, Carathéodory used the maximum principle on an auxiliary function $f(z)/z$. Since $f(0) = 0$, this function is holomorphic on $\mathbb{D}$.

Schwarz's lemma could be very easily reformulated for discs of arbitrary radii. Supposing that $f \in \mathcal{O}(\mathbb{D}_{r_1}, \mathbb{D}_{r_2})$, where $r_1, r_2 > 0$ and $f(0) = 0$. Then the mapping $F(z) := r_2^{-1} f(r_1 z)$ meets the conditions of the lemma, so we get $|F(z)| \leq |z|$ for $z \in \mathbb{D}$. Then $|f(z)| \leq (r_2/r_1)|z|$ for $z \in \mathbb{D}_{r_1}$. Let f be the entire function, i.e. holomorphic on $\mathbb{C}$ such that $f(\mathbb{C}) \subset \mathbb{D}_{r_2}$ for a fixed $r_2 > 0$. The radius r_1 can be arbitrary large, so we get $f(z) \equiv 0$. This is the content of the following well-known theorem.

Theorem 2 (Liouville)**.** *Every bounded entire function is constant.*

The connection between Schwarz's lemma and Liouville's theorem is a wonderful and simple example of *Bloch's principle: nihil est in infinito quod non prius fuerit in finito*, which can be translated as *there is nothing in the infinite which was not first in the finite*. In consequence, for a global result like Liouville's, there must be a more powerful local result, such as Schwarz's. A French mathematician **André Bloch** (1893–1948) published his principle in the paper in 1926. Let $\mathbb{D}_r(z_0) := \{z \in \mathbb{C} \colon |z - z_0| < r\}$ be an open disc with radius $r > 0$ and centre $z_0 \in \mathbb{C}$. We denote by $\mathcal{A}(\Omega)$ the set of all continuous functions on $\overline{\Omega}$ which are holomorphic on Ω. In 1924, Bloch proved

Theorem 3. *There is a universal constant $B > 0$ with the property that for every value of $0 < R < B$, every function $f \in \mathcal{A}(\mathbb{D})$ with $|f'(0)| = 1$ maps a domain $\Omega \subset \mathbb{D}$ biholomorphically onto $\mathbb{D}_R(z_0)$ for some $z_0 \in f(\mathbb{D})$.*

We have named the discs from the theorem simple ("schlicht") discs. Bloch's theorem is interesting because it guarantees the existence of simple discs with a fixed radius in the image of "quite a large family" of holomorphic functions on a disc. In accordance with his principle, Bloch derived the following celebrated global result from his "local" theorem.

Theorem 4 (The Little Picard Theorem)**.** *Any entire function whose range omits at least two distinct values is a constant.*

The above theorem is a remarkable generalization of Liouville's theorem. It is simple to find entire functions whose range is the entire $\mathbb{C}$; nonconstant polynomials, for instance. The exponential function is an example of an entire function whose range omits only one value, namely zero. But there does not exist a nonconstant entire function whose range omits 0 and 1. The latter statement is actually equivalent to the Little Picard theorem since $(b - a)z + a$ is a biholomorphic mapping between $\mathbb{C} \setminus \{0, 1\}$ and $\mathbb{C} \setminus \{a, b\}$, where $a \neq b$. Theorem 4 was proved in 1879 by **Charles É. Picard** (1856–1941), by using arguments based on the modular function. A modular function $\lambda(z)$ is a covering map from the upper halfplane $\mathbb{H} := \{z \in \mathbb{C} \colon \Im(z) > 0\}$ onto $\mathbb{C} \setminus \{0, 1\}$. The function $g(z) := (\mathrm{i} - z)(\mathrm{i} + z)^{-1}$ maps $\mathbb{H}$ biholomorphically onto $\mathbb{D}$. If f is an entire function whose range omits 0 and 1, then f can be lifted by λ to $\widetilde{f} \in \mathcal{O}(\mathbb{C}, \mathbb{H})$, i.e. $\lambda \circ \widetilde{f} = f$. Since $g \circ \widetilde{f}$ is constant according to Liouville's theorem, $\widetilde{f}$ is constant and therefore f is also constant. The nontrivial and technically challenging part of the proof is the construction of such λ. One can find a construction, where Theorem 4 is proved in that way [Ahl79, §7.3.4]. This is why mathematicians searched for "elementary" proofs that avoid modular function.

The name of Theorem 4 suggests that a similar theorem named after Picard exists.

Theorem 5 (The Big Picard Theorem)**.** *In the neighborhood of an isolated essential singularity a holomorphic function takes every value in $\mathbb{C}$ infinitely often with no more than one exception.*

Similarly to the relation between Liouville's theorem and the Little Picard theorem, there is a weaker and more accessible theorem in the case of the Big Picard theorem. We know that a holomorphic function on $\Omega \setminus \{a\}$ has in a one and only one type of isolated singularities:

removable singularity, pole and essential singularity. In the latter case, the limit $\lim_{z\to a}|f(z)|$ does not exist and this happens if and only if the image of the neighborhood of the point a is dense in $\mathbb{C}$. This proposition is also known as the Sohocki-Casorati-Weierstrass theorem [BG91, Proposition 2.4.4]. Theorem 5 can be reformulated as a *meromorphic extension: if a holomorphic function in the neighborhood of an isolated essential singularity omits two distinct values, then the singularity is removable or it is a pole.* In this case, a function becomes meromorphic.

3 The Poincaré metric on a disc

In this section the Poincaré metric on a disc is introduced and the corresponding distance is calculated in order to apply the Schwarz-Pick lemma.

Introduce $\mathbb{R}^+ := \{x \in \mathbb{R}\colon x > 0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. The Poincaré metric on $\mathbb{D}_r$ is

$$d\rho_r^2 := \frac{4r^2|\mathrm{d}z|^2}{(r^2-|z|^2)^2}. \tag{1}$$

This is a form of a Hermitian pseudometric, which is on domain $\Omega \subseteq \mathbb{C}$ defined by

$$ds_\Omega^2 := 2\lambda(z)|\mathrm{d}z|^2 \tag{2}$$

where $\lambda(z) \in \mathcal{C}^2(\Omega, \mathbb{R}_0^+)$ is twice real-differentiable function with $\lambda(z) = \overline{\lambda(z)}$ and $Z(\lambda) := \{z \in \Omega\colon \lambda(z) = 0\}$ is a discrete set. If $Z(\lambda) = \varnothing$, then ds_Ω^2 is said to be a Hermitian metric. We can observe that (1) is really a Hermitian metric. For the sake of simplicity, let us say $d\rho^2 := d\rho_1^2$.

Let Ω_1 and Ω_2 be two domains on $\mathbb{C}$ and $f \in \mathcal{O}(\Omega_1, \Omega_2)$. The pullback of arbitrary pseudometric $ds_{\Omega_2}^2$ is defined by $f^*(ds_{\Omega_2}^2) := 2\lambda(f(z))|f'(z)|^2|\mathrm{d}z|^2$, which is a pseudometric on Ω_1. Poincaré noticed that for the Möbius transformation

$$\varphi_a(z) := \frac{z-a}{1-\bar{a}z},$$

where $a \in \mathbb{D}$, there is $\varphi_a \in \mathrm{Aut}(\mathbb{D})$ and $\varphi_a^*(d\rho^2) = d\rho^2$. By $\mathrm{Aut}(\mathbb{D})$ a family of holomorphic automorphisms of a disc is denoted. Therefore Möbius transformations are isometries for the Poincaré metric. In 1884 Poincaré proved that

$$\mathrm{Aut}(\mathbb{D}) = \{e^{\mathrm{i}a}\varphi_b(z)\colon a \in \mathbb{R}, b \in \mathbb{D}\}.$$

This follows from Schwarz's lemma [BG91, Examples 2.3.12].

A pseudodistance can always be assigned to a Hermitian pseudometric. The process is described in what follows. A pseudodistance differs from the distance in metric spaces only in that the distance between two different points might be zero.

Let $\Omega \subseteq \mathbb{C}$ be an arbitrary domain and $x, y \in \Omega$ arbitrary points. The mapping $\gamma\colon [0,1] \to \Omega$ is called $\mathcal{C}^n$-path from x to y for $n \geq 0$ if $\gamma(t)$ is n-times differentiable mapping and

$\gamma(0) = x, \gamma(1) = y$. In the case $n = 0$ we speak about $\mathcal{C}$-paths. The concatenation of $\mathcal{C}^n$-paths γ_1 from x to y and γ_2 from y to z is $\mathcal{C}$-path

$$(\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(2t), & t \in [0, 1/2] \\ \gamma_2(2t-1), & t \in [1/2, 1] \end{cases}$$

from x to z. Piecewise $\mathcal{C}^n$-path γ from x to y is $\gamma := \gamma_1 * \cdots * \gamma_k$ where $\gamma_1, \ldots, \gamma_k$ are $\mathcal{C}^n$-paths and $\gamma(0) = x, \gamma(1) = y$.

Assume that domain Ω is equipped with a Hermitian pseudometric ds^2_Ω. Let $\gamma\colon [0,1] \to \Omega$ be a piecewise $\mathcal{C}^1$-path from x to y. The length of γ is defined by

$$L_{ds^2_\Omega}(\gamma) := \int_0^1 \sqrt{2(\lambda \circ \gamma)}|\dot{\gamma}|\mathrm{d}t.$$

The pseudodistance between the points is defined by $d_\Omega(x,y) := \inf L_{ds^2_\Omega}(\gamma)$, where the infimum goes through all piecewise $\mathcal{C}^1$-paths γ from x to y.

Let domains $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be equipped with pseudometrics $ds^2_{\Omega_1} = 2\lambda_1(z)|\mathrm{d}z|^2$ and $ds^2_{\Omega_2} = 2\lambda_2(z)|\mathrm{d}z|^2$. Let there be points $x, y \in \Omega_1$, $f \in \mathcal{O}(\Omega_1, \Omega_2)$ and a piecewise $\mathcal{C}^1$-path $\gamma\colon [0,1] \to \Omega_1$ from x to y. Assume that $f^*(ds^2_{\Omega_2}) \leq ds^2_{\Omega_1}$. Then $f(\gamma)$ is a piecewise $\mathcal{C}^1$-path from $f(x)$ to $f(y)$. We have

$$d_{\Omega_2}(f(x), f(y)) \leq \int_0^1 \sqrt{2(\lambda_2 \circ f \circ \gamma)}|f'(\gamma)||\dot{\gamma}|\mathrm{d}t \leq \int_0^1 \sqrt{2(\lambda_1 \circ \gamma)}|\dot{\gamma}|\mathrm{d}t. \tag{3}$$

Because this is valid for every such path, it follows

$$d_{\Omega_2}(f(x), f(y)) \leq d_{\Omega_1}(x, y). \tag{4}$$

If $f \in \mathcal{O}(\Omega_1, \Omega_2)$ is a biholomorphic mapping and f is an isometry for pseudometrics i.e. $f^*(ds^2_{\Omega_2}) = ds^2_{\Omega_1}$, then we can, with similar inequality as (3), but on inverse mapping f^{-1}, obtain $d_{\Omega_2}(f(x), f(y)) = d_{\Omega_1}(x, y)$. In this case f is also an isometry for the induced pseudodistances.

To the Poincaré metric on a disc we can explicitly write down the distance function between arbitrary points $p, q \in \mathbb{D}$. We denote it with $\rho(p,q)$ and call it the Poincaré distance. The proposition below will show that it is expressible with an area hyperbolic tangent

$$\operatorname{artanh}(x) := \frac{1}{2}\log\frac{1+x}{1-x},\ x \in (-1,1).$$

This function is increasing with zero at $x = 0$. It is also $\lim_{x\to -1}\operatorname{artanh}(x) = -\infty$ and $\lim_{x\to 1}\operatorname{artanh}(x) = \infty$.

Proposition 1. *For arbitrary points $p, q \in \mathbb{D}$, the Poincaré distance is*

$$\rho(p,q) = \log\frac{|1-\bar{p}q| + |p-q|}{|1-\bar{p}q| - |p-q|} = \log\frac{1+|\varphi_p(q)|}{1-|\varphi_p(q)|} = 2\operatorname{artanh}|\varphi_p(q)|. \tag{5}$$

Proof. The second and third equalities are clear from the definitions, so the first equality remains to be proved.

Bearing in mind that rotations and Möbius transformations are isometries for the Poincaré metric, it is sufficient to show that

$$\rho(0,a) = \log\frac{1+a}{1-a} \tag{6}$$

for every $a \in [0,1)$. Because

$$\rho\left(0, \left|\frac{p-q}{1-\bar{p}q}\right|\right) = \rho\left(\varphi_{-p}(0), \varphi_{-p}\left(\frac{p-q}{1-\bar{p}q}\right)\right) = \rho(p,q),$$

equation (5) follows from (6). Let $\gamma(t) := x(t) + \mathrm{i}y(t)$ be a piecewise $\mathcal{C}^1$-path from 0 to a and $\bar{\gamma}(t) := at$. Then

$$\begin{aligned} L_{d\rho^2}(\gamma) = \int_0^1 \frac{2\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}}{1-x^2(t)-y^2(t)}\mathrm{d}t &\geq \int_0^1 \frac{2\dot{x}(t)\mathrm{d}t}{1-x^2(t)} \\ &= \log\frac{1+x(t)}{1-x(t)}\Big|_{t=0}^{t=1} = \log\frac{1+a}{1-a} = L_{d\rho^2}(\bar{\gamma}). \end{aligned}$$

The inequality above becomes equality if and only if $y \equiv 0$. If $L_{d\rho^2}(\gamma) = \rho(0,a)$, then $L_{d\rho^2}(\bar{\gamma}) = \rho(0,a)$, which is equivalent to (6). ■

The above proof makes it evident that the shortest path in the Poincaré metric from 0 to $a \in [0,1)$ is a chord between those points. We call the shortest path in arbitrary metric a geodesic. Using a proper rotation and the Möbius transformation, we map this chord into the unique geodesic between arbitrary points on a disc. On Figure 1a we observe some geodesics through 0 and on Figure 1b an action of the Möbius transformation on previous geodesics. For a general domain and metric on it, the geodesic does not always exist; think about a nonconvex domain, equipped with the Euclidean metric. If it exists, it may not be the only one. Some of these possible domains and metrics are discussed in [KL07].

In 1916, **Georg A. Pick** (1859–1942) connected Schwarz's lemma and the Poincaré metric in the so-called Schwarz-Pick lemma. Observe that assumption about centrality condition $f(0) = 0$ is not necessary.

Theorem 6 (The Schwarz-Pick Lemma)**.** *Assume $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$.*

(a) *Then*

$$\rho(f(p),f(q)) \leq \rho(p,q) \tag{7}$$

for every $p,q \in \mathbb{D}$ and

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2} \tag{8}$$

for every $z \in \mathbb{D}$.

(b) *If $p \neq q$ exist such that the equality in* (7) *is valid or such z_0 exists that the equality in* (8) *is valid, then* $f \in \mathrm{Aut}\,(\mathbb{D})$.

A sketch of the proof for the Schwarz-Pick lemma may be as follows. Choose arbitrary $f \in \mathcal{O}\,(\mathbb{D}, \mathbb{D})$ and arbitrary points $p, q \in \mathbb{D}$. Define $F(z) := (\varphi_{f(p)} \circ f \circ \varphi_{-p})(z)$. Then $F \in \mathcal{O}\,(\mathbb{D}, \mathbb{D})$, $F(0) = 0$ and

$$F'(0) = \frac{f'(p)(1 - |p|^2)}{1 - |f(p)|^2}.$$

According to Schwarz's lemma we have $|\varphi_{f(p)}(f(z))| \leq |\varphi_p(z)|$. This is equivalent to (7). If $f(0) = 0$, then at (7) we get $\rho(0, f(z)) \leq \rho(0, z)$, which is equivalent to $|f(z)| \leq |z|$ and at (8) we get $|f'(0)| \leq 1$. Therefore a part (a) of Schwarz's lemma is equivalent to a part (a) of the Schwarz-Pick lemma. To prove that parts (b) of both lemmas are equivalent, note that if $f \in \mathrm{Aut}\,(\mathbb{D})$ and $f(0) = 0$, then f is rotation.

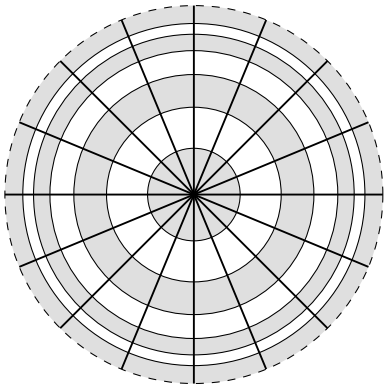

Figure 1a: Unit disc with geodesics through 0 and some balls with center 0 in the Poincaré metric.

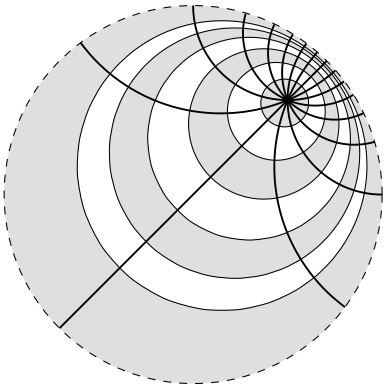

Figure 1b: Unit disc with geodesics through $(1 + \mathrm{i})/2$ and balls with center $(1 + \mathrm{i})/2$ in the Poincaré metric with the same radii as in Figure 1a.

4 Inner distances

What makes the Poincaré distance exceptional? We could, for example, introduce $\mu(p, q) := |\varphi_q(p)|$, which is a distance function on $\mathbb{D}$ with all the properties as ρ in the Schwarz-Pick lemma. This distance function is called the Möbius distance. But there is a crucial difference between ρ and μ: *in the Poincaré distance the boundary is infinitely far away from every point and the Möbius distance seemingly does not have that property.* On Figure 1a six concentric discs with a center 0 can be observed. These are balls in the Poincaré distance with radii $0.5, 1, 1.5, 2, 2.5, 3$. With the increasing of radii, circles are dense in the neighborhood of the boundary of a disc. This is even more evident if we choose the center of balls near the boundary, as in Figure 1b. It can be observed that closed balls in the Poincaré metric are compact. From the explicit expression for ρ we can prove that every Cauchy sequence

with respect to ρ is convergent in $\mathbb{D}$. We say that $(\mathbb{D}, \rho)$ is a complete metric space. Is there a connection among the infiniteness of a boundary, compactness of closed balls and completeness of a metric space? This question is dealt with in this section.

Let $\gamma\colon [0,1] \to \Omega$ be a piecewise $\mathcal{C}^n$-path from x to y and $\delta := \{0 = t_0 < t_1 < \cdots < t_k = 1\}$ partition of $[0,1]$ on k pieces. Length of γ in space (Ω, d_Ω) is defined by

$$L_{d_\Omega}(\gamma) := \sup_{\delta} \sum_{n=1}^{k} d_\Omega(\gamma(t_{n-1}), \gamma(t_n)).$$

We call $d^i_\Omega(x,y) := \inf L_{d_\Omega}(\gamma)$, where the infimum goes through all piecewise $\mathcal{C}^n$-paths γ from x to y, inner pseudodistance. It is not difficult to prove that this is indeed a pseudodistance. Because it is always $d_\Omega(x,y) \leq L_{d_\Omega}(\gamma)$, it follows $d_\Omega(x,y) \leq d^i_\Omega(x,y)$. If the opposite inequality is valid, then we call it d_Ω inner. In that case we have $d_\Omega = d^i_\Omega$.

Proposition 2. *Let $d_\Omega(p,q)$ be a pseudodistance, generated with (2). Then d_Ω is inner.*

Remark 1. The proof of this fact is very simple. Let $\gamma\colon [0,1] \to \Omega$ be a piecewise $\mathcal{C}^1$-path from p to q. Because we have

$$\sum_{n=1}^{k} d_\Omega(\gamma(t_{n-1}), \gamma(t_n)) \leq \sum_{n=1}^{k} \int_{t_{n-1}}^{t_n} \sqrt{2(\lambda \circ \gamma)}|\dot{\gamma}|\mathrm{d}t = L_{ds^2_\Omega}(\gamma)$$

for every partition δ, it follows $L_{d_\Omega}(\gamma) \leq L_{ds^2_\Omega}(\gamma)$. Therefore $d^i_\Omega(p,q) \leq d_\Omega(p,q)$.

We denote the ball with the center $x \in \Omega$ and its radius $r > 0$ with $B_{d_\Omega}(x,r)$. The closed ball will be $\overline{B}_{d_\Omega}(x,r)$.

Proposition 3. *Assume that d_Ω is a continuous inner distance. Then d_Ω is equivalent to the Euclidean topology on Ω.*

Proof. Choose an arbitrary $x \in \Omega$. Assume $d_\Omega\colon \{x\} \times \Omega \to [0,\infty)$ is a continuous function. The set $[0,r) \subset [0,\infty)$ is open. Because $B_{d_\Omega}(x,r) = \left(\mathrm{pr}_2 \circ d_\Omega^{-1}\right)([0,r))$ where pr_2 is a projection to the second component, every d_Ω-ball is open in the Euclidean topology.

Conversely, we will prove that every open set in Ω is open on d_Ω. Let $U \subset \Omega$ be an arbitrary neighborhood of a point $x \in \Omega$. We must show that $r > 0$ exists such that $B_{d_\Omega}(x,r) \subset U$. Choose a relatively compact neighborhood $U' \subset U$ of a point x. Define

$$r := d_\Omega(x, \partial U') = \inf_{y \in \partial U'} d_\Omega(x,y).$$

Because d_Ω is an inner distance, for every point $y \in B_{d_\Omega}(x,r)$ exists a piecewise $\mathcal{C}^n$-path γ from x to y such that $L_{d_\Omega}(\gamma) < r$. This means that $\gamma \subset B_{d_\Omega}(x,r)$. Hence $B_{d_\Omega}(x,r) \subset U'$, because contrary, for $y \in U \setminus \overline{U'}$ there will be $x' \in \partial U'$ such that $r + d_\Omega(x',y) \leq r$. As this is impossible, the proposition is thus proved. ∎

Remember that a complete metric space (X, d_Ω) means that every Cauchy sequence converges in d_Ω. If there is a continuous inner distance, then compactness of closed balls characterizes completeness of a metric space.

Theorem 7 (Hopf-Rinow)**.** *Assume that d_Ω is a continuous inner distance. Then (Ω, d_Ω) is a complete metric space if and only if every closed ball $\overline{B}_{d_\Omega}(x, r)$ is compact.*

Proof. The easy part of the proof is an implication from compactness of closed balls to completeness of a space and is valid without the assumption of innerness. Let every closed d_Ω-ball be compact. Because in a metric space every Cauchy sequence has one accumulation point at most and in a compact space every sequence has one accumulation point at least, it follows that (Ω, d_Ω) is complete.

Let (Ω, d_Ω) be a complete space. Fix $x_0 \in \Omega$. Then $r > 0$ exists such that $B_{d_\Omega}(x_0, r)$ is relatively compact. If we prove that this is true for all $r > 0$, our goal has been accomplished. Assuming the contrary, set

$$r_0 := \sup\left\{r \colon \overline{B}_{d_\Omega}(x_0, r) \text{ is compact}\right\}.$$

Then the set $\overline{B}_{d_\Omega}(x_0, r_0 - \varepsilon)$ is compact for all $\varepsilon > 0$. Therefore a sequence $\{y_i\}_{i=1}^n \subset \overline{B}_{d_\Omega}(x_0, r_0 - \varepsilon)$ exists such that

$$\overline{B}_{d_\Omega}(x_0, r_0 - \varepsilon) \subset \bigcup_{i=1}^{n} B_{d_\Omega}(y_i, \varepsilon).$$

We will demonstrate that $\{B_{d_\Omega}(y_i, 2\varepsilon)\}_{i=1}^n$ is an open cover of $B_{d_\Omega}(x_0, r_0)$. Let us take arbitrary

$$x \in B_{d_\Omega}(x_0, r_0) \setminus \overline{B}_{d_\Omega}(x_0, r_0 - \varepsilon).$$

By innerness a piecewise $\mathcal{C}^n$-path γ exists from x_0 to x such that $L_{d_\Omega}(\gamma) < r_0$. Then $t_0 \in (0, 1)$ and $y_j \in \{y_i\}_{i=1}^n$ exist such that $\gamma(t_0) \in \partial B_{d_\Omega}(x_0, r_0 - \varepsilon)$ and $\gamma(t_0) \in B_{d_\Omega}(y_j, \varepsilon)$. Then we have

$$\begin{aligned} L_{d_\Omega}\left(\gamma|_{[t_0,1]}\right) &= L_{d_\Omega}(\gamma) - L_{d_\Omega}\left(\gamma|_{[0,t_0]}\right) \\ &< r_0 - (r_0 - \varepsilon) = \varepsilon. \end{aligned}$$

This means that $d_\Omega(\gamma(t_0), x) < \varepsilon$ and $d_\Omega(x, y_j) < 2\varepsilon$. It follows

$$B_{d_\Omega}(x_0, r_0) \subset \bigcup_{i=1}^{n} B_{d_\Omega}(y_i, 2\varepsilon).$$

Then $n_1 \in \mathbb{N}$, $1 \le n_1 \le n$ exists such that $\overline{B}_{d_\Omega}(y_{n_1}, r_0/2)$ is not compact (we take $\varepsilon = r_0/2$). Set

$$r_1 := \sup\{r \colon \overline{B}_{d_\Omega}(y_{n_1}, r) \text{ is compact}\}.$$

We inductively continue this process as above. This is how a sequence of points $y_{n_k} \in B_{d_\Omega}(y_{n_{k-1}}, r_0 2^{1-k})$ is obtained, where $\overline{B}_{d_\Omega}(y_{n_k}, r_0 2^{-k})$ is not compact for every $k \in \mathbb{N}$. The nonconvergent sequence $\{y_{n_k}\}$ is Cauchy, which is in contradiction with the assumption of the completeness of domain Ω. The theorem is therefore proved. ∎

Let there be a domain $\Omega \subseteq \mathbb{C}$ and let us choose an arbitrary point $x \in \Omega$. The mapping $\gamma\colon [0,1) \to \Omega$ is a piecewise $\mathcal{C}^n$-path from x to $y \in \partial\Omega \cup \{\infty\}$ if for every $t_0 \in (0,1)$ mapping $\gamma|_{[0,t_0]}$ is a $\mathcal{C}^n$-path, $\gamma(0) = x$ and $\lim_{t\to 1} \gamma(t) = y$. In the introduction we announced the definition of b-completeness of a domain.

Definition 1. Domain Ω is b-complete with respect to the distance d_Ω if for arbitrary points $x \in \Omega$, $y \in \partial\Omega \cup \{\infty\}$ and for an arbitrary piecewise $\mathcal{C}^n$-path γ from x to y, it follows $\lim_{t\to 1} L_{d_\Omega}\left(\gamma|_{[0,t]}\right) = \infty$.

Letter "b" stands for "boundary". Intuitively speaking, (Ω, d_Ω) is b-complete if and only if the boundary is "infinitely far away" from every inner point.

Corollary 1. *Assume that d_Ω is a continuous inner distance. Then (Ω, d_Ω) is a complete metric space if and only if (Ω, d_Ω) is b-complete.*

Proof. Assume that Ω is not b-complete. Then there exists a piecewise $\mathcal{C}^n$-path $\gamma\colon [0,1) \to \Omega$ from $x \in \Omega$ to $y \in \partial\Omega$ such that $\lim_{t\to 1} L_{d_\Omega}(\gamma|_{[0,t]}) = r$ for some $r > 0$. For every sequence $\{y_n\} \subset \gamma([0,1))$, where $y_n \to y$, it follows that $d_\Omega(x, y_n) \leq r$ for every $n \in \mathbb{N}$. Because the closed ball $\overline{B}_{d_\Omega}(x, r)$ is not compact, Theorem 7 guarantees that (Ω, d_Ω) is not complete.

Assume that Ω is not a complete metric space. Then a Cauchy sequence $\{x_i\}_{i=1}^{\infty} \subset \Omega$ exists with the limit $x \in \partial\Omega$. Take arbitrary $\varepsilon \in (0,1)$. Since the sequence is Cauchy, then a subsequence $\{k_i\} \subset \mathbb{N}$ exists such that $d_\Omega(x_{k_i}, x_{k_{i+1}}) < \varepsilon^i$. Because d_Ω is an inner distance, there exist piecewise $\mathcal{C}^n$-paths γ_i with $\gamma_i(0) = x_{k_i}$ and $\gamma_i(1) = x_{k_{i+1}}$ such that

$$L_{d_\Omega}(\gamma_i) = d_\Omega(x_{k_i}, x_{k_{i+1}}) + \varepsilon^i < 2\varepsilon^i.$$

Define a piecewise $\mathcal{C}^n$-path $\gamma\colon [0,1) \to \Omega$ from x_{k_1} to x with $\gamma(t) := \gamma_i(2^i(t-1)+2)$ for $t \in [1-2^{1-i}, 1-2^{-i}]$. Take arbitrary $t_0 \in (0,1)$. Then $j \in \mathbb{N}$ exists such that $t_0 \in [1-2^{1-j}, 1-2^{-j}]$. Therefore

$$L_{d_\Omega}\left(\gamma|_{[0,t_0]}\right) < 2(\varepsilon + \varepsilon^2 + \cdots + \varepsilon^j) < \frac{2\varepsilon}{1-\varepsilon}$$

for $j > 1$. Since $\lim_{t\to 1} L_{d_\Omega}\left(\gamma|_{[0,t]}\right) < \infty$, the domain Ω is not b-complete. ∎

5 Ahlfors' generalization of the Schwarz-Pick lemma

As mentioned in the introduction, Ahlfors' generalization was based on curvature.

Definition 2. Gaussian curvature $K_{ds^2_\Omega}$ of a pseudometric (2) is defined by

$$K_{ds^2_\Omega}(z) := -\frac{1}{\lambda}\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda(z) \tag{9}$$

for $z \in \Omega \setminus Z(\lambda)$ and $-\infty$ for the rest of the points.

A simple calculation shows $K_{d\rho^2} \equiv -1$ for the Poincaré metric (1). It is worth mentioning that this curvature is indeed connected to Gaussian curvature of Riemannian metric on surfaces in real differential geometry. A Hermitian metric $2\lambda(z)|\mathrm{d}z|^2$ is a complex analogue of a Riemannian metric $E(x,y)\mathrm{d}x^2 + 2F(x,y)\mathrm{d}x\mathrm{d}y + G(x,y)\mathrm{d}y^2$ in real world. Since $z = x + \mathrm{i}y$, it is easy to accept that $F(x,y) = 0$ and $E(x,y) = G(x,y) = 2\lambda(x,y)$, so $ds^2 = 2\lambda(x,y)(\mathrm{d}x^2 + \mathrm{d}y^2)$. Let there be $u \in \mathcal{C}^2(\Omega, \mathbb{R}^+)$, where $\Omega \subset \mathbb{C}$. Then

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log u = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log u = \frac{u(u_{xx} + u_{yy}) - (u_x^2 + u_y^2)}{4u^2}. \tag{10}$$

If Gaussian curvature of ds^2 is calculated (see e.g. [Pre10, Corollary 10.2.3]), we get $K_{ds^2_\Omega} = (2\lambda)^{-2} K_{ds^2}$. The curvatures seem to be different, nevertheless, the sign does not change.

An important property of Gaussian curvature is invariance on the pullback, which explicitly means that for an arbitrary $f \in \mathcal{O}(\Omega_1, \Omega_2)$ there is

$$K_{ds^2_{\Omega_1}}(f(z)) = K_{ds^2_{\Omega_2}}(z)$$

where $ds^2_{\Omega_2}$ is an arbitrary Hermitian pseudometric on Ω_2 and $ds^2_{\Omega_1} := f^*\left(ds^2_{\Omega_2}\right)$. This can be easily seen from (9), using the chain rule and $f_{\bar{z}} \equiv 0$ since f is holomorphic.

We wish to have weaker assumptions for the function $\lambda(z)$. Assume that $\lambda(z)$ is only continuous function. Then $ds^2_\Omega = 2\lambda|\mathrm{d}z|^2$ is a continuous Hermitian metric.

Definition 3. A pseudometric $ds^2_{\mathrm{supp}} = 2\lambda_{\mathrm{supp}}(z)|\mathrm{d}z|^2$ is supporting pseudometric for ds^2 at $z_0 \in \Omega$ if there is a neighborhood $U \ni z_0$ in Ω such that $\lambda_{\mathrm{supp}} \in \mathcal{C}^2(U, \mathbb{R}^+_0)$ and $\lambda_{\mathrm{supp}}|_U \leq \lambda|_U$ with equality at z_0.

What seems particularly noteworthy is that we do not need a supporting pseudometric, defined on the whole domain Ω. When a supporting pseudometric exists for a continuous pseudometric, this is defined as local existence, which can change from point to point.

Theorem 8 (Ahlfors' lemma)**.** *Let Ω be a domain with a continuous Hermitian pseudometric ds^2_Ω, for which a supporting pseudometric ds^2_{supp} exists. Assume that $K_{ds^2_{\mathrm{supp}}}|_\Omega \leq L$ for some $L < 0$. Then for every $f \in \mathcal{O}(\mathbb{D}, \Omega)$ we have*

$$f^*(ds^2_\Omega) \leq |L|^{-1} d\rho^2 \tag{11}$$

where $d\rho^2$ is the Poincaré metric (1).

Proof. By assumptions there is a continuous Hermitian pseudometric $ds^2_\Omega = 2\lambda|\mathrm{d}z|^2$. Define

$$ds^2 := |L| f^*(ds^2_\Omega) = 2|L|\lambda(f) \cdot |f'|^2 \cdot |\mathrm{d}z|^2.$$

Then ds^2 is a continuous Hermitian pseudometric on $\mathbb{D}$. Define $\lambda_1 := |L|\lambda(f) \cdot |f'|^2$. The equation (11) is equivalent to $ds^2 \leq d\rho^2$.

For every $r \in \mathbb{R}^+$ define $\mu_r(z) := 2r^2(r^2 - |z|^2)^{-2}$ on $\mathbb{D}_r$. Hence $d\rho^2 = 2\mu_1(z)|\mathrm{d}z|^2$. Define the function $u_r(z) := \lambda_1(z)\mu_r^{-1}(z)$. Hence $ds^2 = u_r d\rho_r^2$. If we show that $u_1 \leq 1$ on $\mathbb{D}$, then $ds^2 \leq d\rho^2$.

Let us take arbitrary $r' \in (0,1)$. If we show that $u_{r'}(z) \leq 1$ for every $z \in \mathbb{D}_{r'}$, then $u_1 \leq 1$ on $\mathbb{D}$, because with $r' \to 1$ and fixed $z_0 \in \mathbb{D}_{r'}$ it follows $u_{r'}(z_0) \to u_1(z_0)$. Since λ_1 is bounded on $\mathbb{D}_{r'}$, from $|z| \to r'$ follows $u_{r'}(z) \to 0$. Function $u_{r'}$ is continuous, hence $z_0 \in \mathbb{D}_{r'}$ exists such that $\max u_{r'} = u_{r'}(z_0)$.

Let there be a supporting pseudometric ds^2_{supp} for ds^2_Ω at $f(z_0)$. Then $ds'^2_{\mathrm{supp}} := |L| f^*(ds^2_{\mathrm{supp}})$ is a supporting pseudometric for ds^2 at z_0, whose curvature is -1 at most. Then a neighborhood $U \ni z_0$ and $\lambda'_{\mathrm{supp}}(z) \in \mathcal{C}^2(U, \mathbb{R}_0^+)$ exist such that $\lambda'_{\mathrm{supp}}|_U \leq \lambda_1|_U$ with equality in z_0. Define function

$$v_r(z) := \frac{\lambda'_{\mathrm{supp}}(z)}{\mu_r^{-1}(z)} = \frac{\lambda'_{\mathrm{supp}}(z)}{\lambda_1(z)} u_r(z).$$

Hence $\max_{z \in U} v_{r'} = u_{r'}(z_0)$.

Although what follows is related to the theory of real functions, it is a crucial element of the proof. Let us have $u \in \mathcal{C}^2(\Omega, \mathbb{R}^+)$, where $\Omega \subset \mathbb{C}$ is a domain. Assume that a function u reaches its maximum at $(x_0, y_0) \in \Omega$. Because this point is singular, it follows $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$. But the point is maximum, so $u_{xx}(x_0, y_0) \leq 0$ and $u_{yy}(x_0, y_0) \leq 0$. By equation (10) we have

$$\left.\frac{\partial^2 \log u}{\partial z \partial \bar{z}}\right|_{z = x_0 + \mathrm{i}y_0} \leq 0.$$

Remember that the maximum of function $v_{r'}|_U$ is reached at point z_0. Hence

$$\begin{aligned} 0 \geq \left.\frac{\partial^2 \log v_{r'}|_U}{\partial z \partial \bar{z}}\right|_{z_0} &= \left.\frac{\partial^2 \log \lambda'_{\mathrm{supp}}}{\partial z \partial \bar{z}}\right|_{z_0} - \left.\frac{\partial^2 \log \mu_{r'}}{\partial z \partial \bar{z}}\right|_{z_0} \\ &= -\lambda'_{\mathrm{supp}}(z_0) K_{ds'^2_{\mathrm{supp}}}(z) - \mu_{r'}(z_0) \\ &= \mu_{r'}(z_0)\left(-v_{r'}(z_0) K_{ds'^2_{\mathrm{supp}}}(z) - 1\right) \\ &\geq \mu_{r'}(z_0)(v_{r'}(z_0) - 1). \end{aligned}$$

We get $v_{r'}(z_0) \leq 1$ and $u_{r'}(z_0) \leq 1$. Since z_0 is the maximum of $u_{r'}$, it follows $u_{r'}(z) \leq 1$ on $\mathbb{D}_{r'}$. ■

In the introduction we promised that Theorem 8 is original version of Ahlfors' lemma. However, Ahlfors proved his lemma for Riemann surfaces. These are one dimensional complex manifolds, so the proof is essentially the same as one above. Under originality we mean the concept of supporting pseudometric. Most authors prove Ahlfors' lemma without it, because for most applications twice-differentiable Hermitian pseudometrics would suffice.

Assume that $L = -1$ in Ahlfors' lemma. Then we have $f^*(ds^2_\Omega) \leq d\rho^2$. Therefore we can use inequality (4) and get

$$d_\Omega(f(p), f(q)) \leq \rho(p, q). \tag{12}$$

In the case of domain $(\mathbb{D}, d\rho^2)$, we get (7) of the Schwarz-Pick lemma. Inequality (8) is even more easily accessible; in (11) a proper metrics is put. It is a theorem from 1962 by M. Heins

that in the case of equality in (11) for one point only, it follows that there is equality on the whole domain Ω. This can be considered as the generalization of part (b) of the Schwarz-Pick lemma. Interested reader can find simplified proof due to D. Minda in the book [JP13, Proposition 1.2.1].

6 Applications

In this section we prove the theorems mentioned in the introduction. Firstly, we will prove Bloch's theorem and a familiar theorem due to Landau, which drops out the assumption about simple discs. Bloch's and Landau's theorems are also Ahlfors' examples of the applications of his lemma. Next, a complete Hermitian metric is constructed, i.e. an induced distance generates a complete metric, on domain $\mathbb{C} \setminus \{0, 1\}$, which satisfies the assumptions of Ahlfors' lemma. From that point, we are able to provide a proof of the Little Picard theorem. We use the nature of completeness of a space in studying the size of an image of a disc under a holomorphic mapping, which misses two distinct points. This result, named after Schottky is crucial for proving the Big Picard theorem.

6.1 The Bloch Theorem

Let there be $\mathscr{B} := \{f \in \mathcal{A}(\mathbb{D}) : |f'(0)| = 1\}$ where $\mathcal{A}(\mathbb{D})$ is the set of all continuous functions on $\overline{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$. Remember that Bloch's theorem (Theorem 3) guarantees the existence of simple discs with a fixed radius in the image $f(\mathbb{D})$, where $f \in \mathscr{B}$. Let $\mathrm{B}(f)$ be a supremum of all radii of simple discs in $f(\mathbb{D})$. We want to show that a constant $B > 0$ exists such that $\mathrm{B}(f) \geq B$ for every $f \in \mathscr{B}$.

Proof of Bloch's theorem. By $S := \{z \in \mathbb{D} \colon f'(z) = 0\}$ we denote a set of singular points. According to the open mapping theorem, $\Omega := f(\mathbb{D})$ is a domain and $f(\overline{\mathbb{D}}) \subseteq \overline{\Omega} \subset \mathbb{C}$. For every point $z \in \Omega$ there is a number $\rho(z)$ such that $\mathbb{D}_{\rho(z)}(z)$ is the largest simple disc. Therefore $\mathrm{B}(f) = \sup_{z\in\Omega} \rho(z)$ and $\mathrm{B}(f) < \infty$. On $\mathbb{D}$ we define a metric

$$\lambda(z) := \frac{A^2|f'(z)|^2}{2\rho(f(z))\left(A^2 - \rho(f(z))\right)^2}, \tag{13}$$

where A is a constant, which satisfies $A^2 > \mathrm{B}(f)$. Since ρ is a continuous function and $\rho(f(z)) = 0$ if and only if $z \in S$, then (13) is a continuous Hermitian metric at nonsingular points. We must care only at singular points. Take arbitrary $z_0 \in S$. We know that there is a neighborhood $U' \ni z_0$ on $\mathbb{D}$, $n \geq 1$ and biholomorphic function $\varphi(z)$ on U' such that $f(z) = f(z_0) + \varphi^n(z)$ on U' (see e.g. [BG91, Corollary 2.3.7]). Then there is a neighborhood $U \subset U'$ of point z_0 on $\mathbb{D}$ such that $\rho(f(z)) = |f(z) - f(z_0)|$ on U. Then the equation (13) can be rewritten as

$$\lambda(z) = \frac{A^2n^2|\varphi(z)|^{n-2}|\varphi'(z)|^2}{2(A^2 - |\varphi(z)|^n)^2}$$

for $z \in U$. Therefore (13) is a Hermitian pseudometric in the neighborhoods of singular points.

If we want to use Ahlfors' lemma, we need a supporting pseudometric for (13). Take an arbitrary nonsingular point $z_0 \in \mathbb{D} \setminus S$. Then $s_0 \in \overline{\mathbb{D}}$ exists such that the boundary of $\mathbb{D}_{\rho(f(z_0))}(f(z_0))$ contains a point $f(s_0)$. In the neighborhood U of a point z_0 it is $\rho(f(z)) \leq |f(z) - f(s_0)|$. On U define a Hermitian metric

$$\lambda_{\text{supp}}(z) := \frac{A^2|f'(z)|^2}{2|f(z) - f(s_0)|\left(A^2 - |f(z) - f(s_0)|\right)^2}. \tag{14}$$

The inequality $\lambda_{\text{supp}}(z) \leq \lambda(z)$ will be satisfied on U if $x(A^2 - x)^2$ is an increasing function on $[0, \mathrm{B}(f)]$. Since $\lambda_{\text{supp}}(z_0) = \lambda(z_0)$, metric (14), which has constant curvature -1, will be supporting for (13) at z_0. A quick calculation shows that the function is increasing on $[0, A^2/3]$. Therefore the metric is supporting if $A^2 > 3\mathrm{B}(f)$.

Let there be $f(0) = z_0$. By assumption $|f'(0)| = 1$, the upper bounds combined with Ahlfors' lemma give

$$3\mathrm{B}(f) < A^2 \leq 4\rho(z_0)(A^2 - \rho(z_0))^2 \leq 4\mathrm{B}(f)(A^2 - \mathrm{B}(f))^2.$$

Pushing A^2 toward $3\mathrm{B}(f)$, we get $\mathrm{B}(f) \geq \sqrt{3}/4$. Hence $B \geq \sqrt{3}/4$. ■

Edmund G. H. Landau (1877–1938) dropped the assumption about simple discs in Bloch's theorem.

Theorem 9. *Assume $f \in \mathcal{A}(\mathbb{D})$ and $|f'(0)| = 1$. Then a universal constant $L > 0$ exists such that in the image $f(\mathbb{D})$ a disc with the radius $R \geq L$ exists.*

Proof. Proving this theorem is very similar to proving Bloch's theorem. Let there be a real and positive function $\rho(z)$ such that $\mathbb{D}_{\rho(z)}(z)$ is the largest disc in $\Omega := f(\mathbb{D})$. Define $\mathrm{L}(f) := \sup_{z\in\Omega} \rho(z)$. Since we are not dealing with singular points, we take metrics

$$\lambda(z) := \frac{1}{2}\left(\rho(z)\log\frac{C}{\rho(z)}\right)^{-2} \quad \text{and} \quad \lambda_{\text{supp}}(z) := \frac{1}{2}\left(|z - s_0|\log\frac{C}{|z - s_0|}\right)^{-2}$$

on Ω. The metric $\lambda_{\text{supp}}(z)$ is defined on a neighborhood U of a point $z_0 \in \Omega$ and $s_0 \in \partial\mathbb{D}_{\rho(z_0)}(z_0) \cap \partial\Omega$ where it has constant curvature -1. Therefore λ_{supp} will be supporting for λ at z_0 if the inequality $\lambda_{\text{supp}}(z) \leq \lambda(z)$ is satisfied on U. This will be true if $x\log(Cx^{-1})$ is an increasing function on $[0, \mathrm{L}(f)]$. A function is increasing for $ex < C$, therefore the metric is supporting if $e\mathrm{L} < C$.

Assume $f(0) = z_0$. According to Ahlfors' lemma it follows

$$1 \leq \left(2\rho(z_0)\log\frac{C}{\rho(z_0)}\right)^2 \leq \left(2\mathrm{L}(f)\log\frac{C}{\mathrm{L}(f)}\right)^2.$$

Pushing C toward $e\mathrm{L}(f)$, we get $\mathrm{L}(f) \geq 1/2$ and hence $L \geq 1/2$. ■

The numbers B and L are called the Bloch and Landau constants. It follows from the definitions that $B \leq L$. Landau simplified Bloch's proof in 1926 and estimated $B \geq 1/16$. It is possible to prove Bloch's theorem with bound $B \geq \sqrt{3}/4$ without Ahlfors' lemma [Rem98, §10.1.4]. Standard proof of Landau's theorem with bound $L \geq 1/16$ could be found in Berenstein and Gay's textbook [BG91, Proposition 2.7.10]. The exact values of the Bloch and Landau constants are not known [Rem98, §10.1.5].

6.2 The Little Picard Theorem

The Little Picard theorem (Theorem 4) deals with domain $\mathbb{C} \setminus \{0, 1\}$. Therefore, the question whether a Hermitian metric with curvature, bounded with negative constant exists, is reasonable. If this is so, we can use Ahlfors' lemma.

Proof of the Little Picard theorem. Introduce $\mathbb{C}^{**} := \mathbb{C} \setminus \{0, 1\}$. Define

$$A_1(z) := \log \frac{C|z|^2}{1+|z|^2}, \quad A_2(z) := \log \frac{C|z-1|^2}{2(1+|z|^2)}, \quad A_3(z) := \log \frac{C}{1+|z|^2}$$

for a constant $C > 9$, which will be determined later. The expressions are well-defined on $\mathbb{C}^{**}$. We will prove that for

$$\lambda_{\mathbb{C}^{**}}(z) := \frac{4(1+|z|^2)}{|z|^2|z-1|^2 A_1^2(z) A_2^2(z) A_3^2(z)} \tag{15}$$

the metric $ds^2_{\mathbb{C}^{**}} := 2\lambda_{\mathbb{C}^{**}}(z)|dz|^2$ is a complete Hermitian metric on $\mathbb{C}^{**}$ with curvature $K(z) := K_{ds^2_{\mathbb{C}^{**}}}(z) < -1$.

From (15) we can see that in the neighborhood of a point $a \in \{0, 1\}$ it is

$$\lambda_{\mathbb{C}^{**}}(z) > \frac{A}{|z-a|^2 \log^2\left(\sqrt{C}|z-a|\right)}$$

for some constant $A > 0$ and

$$\lambda_{\mathbb{C}^{**}}(z) > \frac{B}{|z|^2 \log^2\left(\sqrt{2/C}|z|\right)} \tag{16}$$

for large $|z|$ and some constant $B > 0$. Let there be a polar presentation $\gamma(t) = r(t)\exp(\mathrm{i}\varphi(t)) + a$ of a piecewise $\mathcal{C}^1$-path $\gamma\colon [0, 1) \to \mathbb{C}^{**}$ from $\gamma(0)$ to $\lim_{t\to 1}\gamma(t) \in \{0, 1\}$. This means that for $t \to 1$ it follows $r(t) \to 0$. For the "point at infinity" we take the path $\gamma(t) = r(t)\exp(\mathrm{i}\varphi(t))$, where for $t \to 1$ it follows $r(t) \to \infty$. Since

$$\int_0^{t_0} \frac{2\sqrt{A}|\dot{r} + r\dot{\varphi}|\mathrm{d}t}{r|\log\sqrt{C}r|} \geq \int_0^{t_0} \frac{2\sqrt{A}\dot{r}\mathrm{d}t}{r\log\sqrt{C}r} = \sqrt{A}\log\frac{\log^2 Cr(t_0)}{\log^2 Cr(0)} \xrightarrow{t_0\to 1} \infty,$$

and the same result with inequality (16), $(\mathbb{C}^{**}, d_{\mathbb{C}^{**}})$ is b-complete. By Corollary 1 metric $ds^2_{\mathbb{C}^{**}}$ is a complete Hermitian metric.

The corresponding Gauss curvature is

$$\begin{aligned}K(z) = &- \frac{|z|^2|z-1|^2A_1^2(z)A_2^2(z)A_3^2(z)}{4(1+|z|^2)^3} - \frac{|z-1|^2A_2^2(z)A_3^2(z)(1+|z|^2A_1(z))}{2(1+|z|^2)^3} \\ &- \frac{|z|^2A_1^2(z)A_3^2(z)(|z+1|^2+|z-1|^2A_2(z))}{2(1+|z|^2)^3} \\ &- \frac{|z|^2|z-1|^2A_1^2(z)A_2^2(z)(|z|^2+A_3(z))}{2(1+|z|^2)^3}.\end{aligned}$$

It can be derived from the expression above that

$$\begin{aligned}&\lim_{z\to 0} K(z) = \lim_{z\to\infty} K(z) = -(1/2)\log^2(C/2)\log^2 C < -1, \\ &\lim_{z\to 1} K(z) = -(1/4)\log^4(C/2) < -1.\end{aligned}$$

For $r_1, r_2, r_3 > 0$ let us introduce a domain

$$X_{r_1,r_2,r_3} := \{z \in \mathbb{C}\colon 0 < |z| < r_1 \text{ or } 0 < |z-1| < r_2 \text{ or } r_3 < |z|\}.$$

Positive numbers r_1, r_2, r_3 exist such that $K(z) < -1$ for $z \in X_{r_1,r_2,r_3}$. Because $\mathbb{C}^{**} \setminus X_{r_1,r_2,r_3}$ is compact, there is a constant $C > 9$ such that $K(z) < -1$ for $z \in \mathbb{C}^{**} \setminus X_{r_1,r_2,r_3}$. Then $K(z) < -1$ on $\mathbb{C}^{**}$.

Let there be $r > 0$ and $f \in \mathcal{O}(\mathbb{D}_r, \mathbb{C}^{**})$. Then $f(rz) \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{**})$. According to Ahlfors' lemma it follows

$$r|f'(0)| \le \frac{2}{\lambda_{\mathbb{C}^{**}}(f(0))}. \tag{17}$$

We can now prove the Little Picard theorem. Assume that f is an entire function such that $f(\mathbb{C}) \subseteq \mathbb{C}^{**}$. Choose an arbitrary point $z_0 \in \mathbb{C}$ and introduce a function $g(z) := f(z+z_0)$. Let there be an increasing and unbounded sequence $\{r_n\}_{n=1}^{\infty}$ of positive real numbers and $g_n := g|_{\mathbb{D}_{r_n}}$. By equation (17) for every $n \in \mathbb{N}$ it follows

$$|f'(z_0)| = |g_n'(0)| \le \frac{2}{r_n\lambda_{\mathbb{C}^{**}}(g_n(0))} = \frac{2}{r_n\lambda_{\mathbb{C}^{**}}(f(z_0))} \xrightarrow{n\to\infty} 0,$$

since $g_n(0) = f(z_0)$ and $g_n'(0) = f'(z_0)$. Hence $f'(z_0) = 0$. Because z_0 was an arbitrary point, it follows $f' \equiv 0$ on $\mathbb{C}$. This means that f is a constant function. ■

By using inequality (17) we are able to provide a very easy proof of the following Landau theorem from 1904.

Theorem 10. *Assume that $f \in \mathcal{O}(\mathbb{D}_r, \mathbb{C}^{**})$ for some $r > 0$ and $f'(0) \neq 0$. Then there is a constant $C > 0$, depending only on $f(0)$ and $f'(0)$ such that $r \le C$.*

Proof. Inequality (17) suggests that a good choice for a constant is

$$C = \frac{2}{|f'(0)|\lambda_{\mathbb{C}^{**}}(f(0))},$$

which only depends on $f(0)$ and $f'(0)$. ■

Assume that $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is a power series expansion of f at 0. Then $f(0) = a_0$ and $f'(0) = a_1$. Theorem 10 has the following equivalent form: *if f omits 0 and 1 and $a_1 \neq 0$, then a constant $C(a_0, a_1) > 0$ exists such that the convergence radius of f is not greater than $C(a_0, a_1)$.* The story goes that Landau was reluctant to publish the above theorem because he found it too absurd [Rem98, §10.2.2].

6.3 The Schottky theorem

In 1904, the German mathematician **Friedrich H. Schottky** (1851–1935) studied the size of an image of a disc under a holomorphic mapping, which omits two distinct points on $\mathbb{C}$. In our proof completeness of a metric (15) is called for. The Hopf-Rinow characterization with closed balls can be used. The theorem is a bridge between the Little and Big Picard theorems.

Theorem 11. *Let R and C be positive real numbers. Assume that we have $f \in \mathcal{O}(\mathbb{D}_R, \mathbb{C}^{**})$ such that $|f(0)| < C$. Then for every $r \in (0, R)$ a constant M exists, depending only on R, r and C such that $|f(z)| \leq M$ for $|z| \leq r$.*

Proof. Take f and r from the theorem and set $a := f(0)$ and $g(z) := f(Rz)$. Then $g \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{**})$ and $g(0) = a$. Define $r' := \rho(0, r/R)$. Let $ds^2_{\mathbb{C}^{**}}$ be metric (15) and the $d_{\mathbb{C}^{**}}$ corresponding distance. Since $d_{\mathbb{C}^{**}}$ is complete, every closed ball $B := \overline{B}_{d_{\mathbb{C}^{**}}}(a, r')$ is compact. Hence, a constant M_1 exists, depending only on R, r and C such that $|z - a| < M_1$ for every $z \in B$. For every $f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{**})$ there is by (12)

$$d_{\mathbb{C}^{**}}(a, f(z)) \leq \rho(0, z) \leq r'$$

for $|z| \leq r$ and therefore $|f(z) - a| < M_1$. Since $|a| < C$, it follows $|f(z)| < C + M_1$ for $|z| \leq r$. The theorem follows after setting $M := M_1 + C$. ■

6.4 The Big Picard Theorem

While studying the properties of $\mathcal{O}(\Omega)$, we should introduce the concept of normal families: a family $\mathscr{F} \subset \mathcal{O}(\Omega)$ is normal if every sequence in $\mathscr{F}$ has a convergent subsequence in $\mathcal{O}(\Omega)$, where convergence is uniformly on compact sets. For $\mathscr{F}$ we say that it is bounded on Ω if for every compact set $K \subset \Omega$ a constant $C(K)$ exists such that

$$\sup_{f \in \mathscr{F}} \left(\sup_{z \in K} |f(z)| \right) \leq C(K). \tag{18}$$

A common sign for $\sup_{z \in K} |f(z)|$ is $\|f\|_K$. **Paul A. A. Montel** (1876–1975) proved the following theorem in 1907.

Theorem 12 (Montel)**.** *A family $\mathscr{F} \subset \mathcal{O}(\Omega)$ is bounded on Ω if and only if it is normal.*

The difficult part of proving Montel's theorem comes from the implication from a bound to normality, since the opposite direction is quite clear. Assume that a compact set $L \subset \Omega$ exists for which (18) is not true. Then a sequence $\{f_n\} \subset \mathscr{F}$ exists, such that $\|f_n\|_L \to \infty$ and does not have a convergent subsequence with the limit f, since $\|f_n\|_L - \|f - f_n\|_L \leq \|f\|_L$. The core of the problem is the celebrated Arzelá-Ascoli theorem [Ahl79, §5.5], which asserts that a family $\mathscr{F}$ of continuous functions on Ω is relatively compact if and only if the family is equicontinuous and $f(z_0)$ is relatively compact for every $z_0 \in \Omega$ and every $f \in \mathscr{F}$. The latter is satisfied since $\mathscr{F}$ is bounded and equicontinuity follows from the Cauchy inequality. By Weierstrass' theorem, which asserts that a family of holomorphic functions is closed in a family of continuous functions, we see that the Arzelá-Ascoli theorem implies Montel's theorem.

It is useful to expand the definition of normality in the direction that allows uniform convergence on compact sets to ∞. A closed family $\mathscr{F} \subset \mathcal{O}(\Omega_1, \Omega_2)$ is normal if every sequence in $\mathscr{F}$ has convergent subsequence or this sequence is compactly divergent. This means that for arbitrary compact sets $K \subset \Omega_1$ and $L \subset \Omega_2$ integer $N \in \mathbb{N}$ exists such that $f_n(K) \cap L = \varnothing$ for all $n > N$.

For the proof of the next theorem we need the classical result by A. Hurwitz [Ahl79, p. 178]: *Assume that $\{f_n\} \subset \mathcal{O}(\Omega)$ is a convergent sequence with the limit $f \in \mathcal{O}(\Omega)$. If $a \in \mathbb{C}$ exists such that $a \notin f_n(\Omega)$ for every $n \in \mathbb{N}$, then $a \notin f(\Omega)$ or $f \equiv a$.*

Theorem 13 (The Normality Theorem)**.** *Let there be $a, b \in \mathbb{C}$, $a \neq b$. Then the family $\mathscr{F} \subseteq \mathcal{O}(\Omega, \mathbb{C} \setminus \{a, b\})$ is normal for every domain $\Omega \subset \mathbb{C}$.*

Proof. Let there be $\mathscr{F} \subset \mathcal{O}(\Omega, \mathbb{C} \setminus \{0, 1\})$ and $\{f_n\}_{n=1}^{\infty} \subset \mathscr{F}$ is an arbitrary sequence. It is enough to show that for every point $x \in \Omega$ there is a neighborhood $U \subset \Omega$ such that the family $\{f_n|_U : n \in \mathbb{N}\}$ is normal.

Choose a fixed but arbitrary point $x \in \Omega$. An unbounded sequence $\{f_n(x)\}$ is compactly divergent. If the sequence $\{f_n(x)\}$ is bounded, then according to Schottky's theorem $C > 0$ exists such that $f_n(\mathbb{D}_r(x)) \subset \mathbb{D}_C$, where such $r > 0$ is chosen that $\mathbb{D}_r(x) \subset \Omega$. Since

$$\sup_{f \in \{f_n\}} \|f\|_{\overline{\mathbb{D}}_r(x)} \leq C,$$

according to Montel's theorem, a subsequence $\{f_{n1}\} \subset \{f_n\}$ exists such that f_{n1} uniformly converges to $f \in \mathcal{O}(\mathbb{D}_r(x))$ on compact sets. If $f(\mathbb{D}_r(x)) \subset \mathbb{C} \setminus \{0, 1\}$, the goal has been achieved. Therefore, let us have $z_0 \in \mathbb{D}_r(x)$ such that $f(z_0) \in \{0, 1\}$. Assume that $f \neq f(z_0)$. According to Hurwitz's theorem $N \in \mathbb{N}$ exists such that $f(z_0) \in f_{n1}(\mathbb{D}_r(x))$ for all $n > N$. Because this is not true, it follows that $f \equiv f(z_0)$. Therefore, the sequence is compactly divergent. ■

Montel proved the Normality theorem in 1912, which is why it is sometimes referred to as the Big Montel theorem.

The Normality theorem is used to prove the Big Picard theorem (Theorem 5). We are going to prove the sharper and not so widely known form of the theorem by **Gaston**

M. Julia (1893–1978) from 1924. For the formulation of the theorem the following "cone-shape" domain is needed

$$J(\zeta,\alpha) := \left\{t_1 e^{\mathrm{i}(\varphi_0+t_2\alpha)} \colon t_1 \in (0,1), t_2 \in (-1/2,1/2), e^{\mathrm{i}\varphi_0} = \zeta\right\} \subset \mathbb{D}^*.$$

The domain is a disc section with an angle α, which is symmetric on a chord with endpoints 0 and $\zeta \in \partial\mathbb{D}$.

Theorem 14. *Assume that $f \in \mathcal{O}(\mathbb{D}^*)$ with an essential singularity at 0. Then $\zeta \in \partial\mathbb{D}$ exists such that for all $\alpha > 0$ function f on $J(\zeta,\alpha)$ takes every value in $\mathbb{C}$ infinitely often with no more than one exception.*

Proof. Assume that for every $\zeta \in \partial\mathbb{D}$ there is $\alpha_\zeta > 0$ such that for some $a, b \in \mathbb{C}$, $a \neq b$ equations $f(z) = a$ and $f(z) = b$ have finite solutions on $J(\zeta,\alpha_\zeta)$. Since the boundary of a disc is a compact set, a sequence $\{\zeta_n\}_{n=1}^N$ exists such that $\{J(\zeta_n,\alpha_{\zeta_n})\}$ is a finite open cover of $\mathbb{D}^*$ with the previously mentioned property. Then there is $\varepsilon > 0$ such that $f(\mathbb{D}^*_\varepsilon) \subset \mathbb{C}\setminus\{a,b\}$.

Let $\mathbb{A}(r_1,r_2) := \{z \in \mathbb{C} \colon r_1 < |z| < r_2\}$ be an annulus. Define the family $\mathscr{F} := \{f_n(z)|_{\mathbb{A}(1/2,2)}\}$, where $f_n(z) := f(\varepsilon 2^{-n} z)$. Since

$$\mathscr{F} \subset \mathcal{O}(\mathbb{A}(1/2,2), \mathbb{C}\setminus\{a,b\}),$$

the Normality theorem guarantees that $\mathscr{F}$ is a normal family. Then a subsequence $\{f_{n1}\} \subset \{f_n\}$ exists such that $f_{n1} \to g \in \mathcal{O}(\mathbb{A}(1/2,2))$ or $\{f_n\}$ is compactly divergent. In the first case, the sequence $\{f_{n1}|_{\partial\mathbb{D}}\}$ is uniformly bounded. In the second case, the sequence $\left\{(f_n|_{\partial\mathbb{D}})^{-1}\right\}$ is uniformly bounded. Assume that we are dealing with the first case. Then there exists $M > 0$ such that $|f(z)| < M$ for every $|z| = \varepsilon 2^{-n1}$. According to the maximum principle $|f(z)| < M$ in the neighborhood of 0. This means that singularity is removable. This is in contradiction with the assumption of an essential singularity. In the second case, we get a removable singularity for $1/f$ in 0, which is also a contradiction. ■

7 A glimpse of hyperbolic complex manifolds

In this final section we briefly describe main properties of hyperbolic complex manifolds. We are especially interested on those properties which are in direct connection with Picard's theorems.

We begin with the notion of invariant pseudodistances. These are pseudodistances which can be constructed on the category of complex manifolds and they become isometries for biholomorphic mappings. In 1967, a Japanese mathematician **Shoshichi Kobayashi** (1932–2012) constructed one of those pseudodistances. For every $x, y \in M$ and every $f \in \mathcal{O}(M,N)$, where M and N are complex manifolds, the Kobayashi pseudodistance d_M^K has the following properties

$$d_N^K(f(x),f(y)) \leq d_M^K(x,y), \tag{19}$$

$$d_{\mathbb{D}}^K(x,y) = \rho(x,y). \tag{20}$$

Thus d_M^K is an invariant pseudodistance, which coincides with the Poincaré distance on a disc. Explicit construction is carried out by the so-called chain of holomorphic discs

$$\alpha : \begin{cases} p = p_0, p_1, \ldots, p_k = q \in M, \\ a_1, a_2, \ldots, a_k \in \mathbb{D}, \\ f_1, f_2, \ldots, f_k \in \mathcal{O}(\mathbb{D}, M) \end{cases}$$

between $p, q \in M$ where $f_n(0) = p_{n-1}$ and $f_n(a_n) = p_n$ for all $n \in \{1, \ldots, k\}$. Kobayashi pseudodistance is then defined as

$$d_M^K(p, q) := \inf_{\alpha} \left\{ \sum_{n=1}^{k} \rho(0, a_n) \right\}.$$

Thus the construction of d_M^K is in the spirit of Bloch's principle. Generally speaking, d_M^K is not a distance.

Example 1. We have $d_{\mathbb{C}}^K \equiv 0$. To see this, take a holomorphic mapping $f(z) := p + \varepsilon^{-1}(q - p)z$ from $\mathbb{D}$ into $\mathbb{C}$, where $p, q \in \mathbb{C}$ are arbitrary points and $\varepsilon > 0$ is an arbitrary small number. Then $f(0) = p$ and $f(\varepsilon) = q$. From (19) we get $d_{\mathbb{C}}^K(p, q) \leq 2\varepsilon$. Because mapping $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is surjective, it also follows $d_{\mathbb{C}^*}^K \equiv 0$.

A complex manifold is hyperbolic if the Kobayashi pseudodistance becomes a distance and complete hyperbolic if (M, d_M^K) is a complete metric space. Hyperbolic manifolds have several important properties, including the fact that the Kobayashi distance is inner, direct product of (complete) hyperbolic manifolds is (complete) hyperbolic and (complete) hyperbolicity is invariant with respect to unramified covering projections. The latter statement combined with the Poincaré-Koebe uniformization theorem asserts that Riemann surface is hyperbolic if and only if its universal cover is a disc. This is in agreement with traditional meaning of hyperbolic Riemann surfaces.

We call pseudodistances, which satisfy the properties (19) and (20), contractible pseudodistances. It can be shown that the Kobayashi pseudodistance is the largest among contractible pseudodistances. What is more, if a pseudodistance d_M satisfies $d_M(f(x), f(y)) \leq \rho(x, y)$ for all $x, y \in \mathbb{D}$ and all $f \in \mathcal{O}(\mathbb{D}, M)$, then $d_M \leq d_M^K$. Ahlfors' lemma implies that every planar domain (or more generally every Riemann surface), which carries a complete Hermitian metric of curvature not greater than -1, is complete hyperbolic. Therefore, the domain $\mathbb{C} \setminus \{0, 1\}$ is complete hyperbolic. From this it is easy to see that every domain $\Omega \subseteq \mathbb{C} \setminus \{a, b\}, a \neq b$, is complete hyperbolic; observe that every Cauchy sequence in Ω is also Cauchy in $\mathbb{C} \setminus \{a, b\}, a \neq b$. Of course in higher dimensions there exist hyperbolic domains which are not complete hyperbolic. Probably the simplest example is a punctured bidisc $\mathbb{D}_2 \setminus \{(0, 0)\}$.

Example 2. Denote punctured bidisc by X. Because X is bounded, it is hyperbolic. Define the following sequences $a_n := (0, \alpha_n)$, $b_n := (\alpha_n, 0)$ and $c_{n,m} := (\alpha_n, \alpha_m)$ where $\{\alpha_n\} \subset \mathbb{D}$

is a sequence with property $\rho(0,\alpha_n) = 2^{-n}$. Introducing domains $X_1 := \mathbb{D} \times \mathbb{D}^* \subset X$ and $X_2 := \mathbb{D}^* \times \mathbb{D} \subset X$ yields

$$d_X^K(a_n, b_n) \le d_{X_1}^K(a_n, c_{n,n}) + d_{X_2}^K(b_n, c_{n,n}),$$
$$d_X^K(b_n, a_{n+1}) \le d_{X_2}^K(b_n, c_{n,n+1}) + d_{X_1}^K(a_{n+1}, c_{n,n+1}).$$

Define $f_n \in \mathcal{O}(\mathbb{D}, X_1)$ with $f_n(z) := (z, \alpha_n)$ and $g_n \in \mathcal{O}(\mathbb{D}, X_2)$ with $g_n(z) := (\alpha_n, z)$. Then $d_{X_1}^K(a_n, c_{n,n}) \le \rho(0,\alpha_n)$ and $d_{X_1}^K(a_{n+1}, c_{n,n+1}) \le \rho(0,\alpha_n)$. Equivalently $d_{X_2}^K(b_n, c_{n,n}) \le \rho(0,\alpha_n)$ and $d_{X_2}^K(b_n, c_{n,n+1}) \le \rho(0,\alpha_n)$. Thus $d_X^K(a_n, b_n) \le 2^{1-n}$ and $d_X^K(b_n, a_{n+1}) \le 2^{1-n}$. Therefore $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence which converges to $(0,0) \notin X$.

Hyperbolicity is closely related to the Little Picard theorem. If we assume $f \in \mathcal{O}(\mathbb{C}, M)$, then we get $d_M^K(f(x), f(y)) = 0$ by (19). This shows that *every holomorphic map from* $\mathbb{C}$ *to a hyperbolic manifold is constant*. The converse of this statement is not true in general; however, it is true on compact complex manifolds in view of a fundamental theorem of Robert Brody from 1978. The image of holomorphic map from $\mathbb{C}$ is said to be entire curve. A complex manifold is said to be the Brody hyperbolic if all entire curves on it are constants. We prove that $\mathbb{C} \setminus \{a, b\}, a \ne b$ is also the Brody hyperbolic.

Our definition of normal families can be adapted to holomorphic mappings between complex manifolds. A complex manifold M is said to be taut if $\mathcal{O}(N, M)$ is a normal family for every complex manifold N. Taut manifolds are somewhere between complete hyperbolic and hyperbolic manifolds since it can be shown that completeness implies tautness and tautness implies hyperbolicity. The Hopf-Rinow theorem is crucial to prove this assertions. Therefore $\mathbb{C} \setminus \{a, b\}, a \ne b$ is taut domain which implies the Normality theorem.

Is there any generalization of the Big Picard theorem in the sense of hyperbolicity? The answer is yes and it goes through hyperbolic imbeddings. Let X be a relatively compact domain in complex manifold M. If for every $x, y \in \overline{X}$ there exist neighborhoods $U \ni x$ and $V \ni y$ in M such that $d_X^K(U \cap X, V \cap X) > 0$, then X is hyperbolically imbedded in M. Peter Kiernan coined this term in 1973 and proved that if X is hyperbolically imbedded domain in M, then every map $f \in \mathcal{O}(\mathbb{D}^*, X)$ has an extension to $\tilde{f} \in \mathcal{O}(\mathbb{D}, M)$. Since $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ is biholomorphic to $\mathbb{C} \setminus \{0, 1\}$ it follows from properties of the Hermitian metric, constructed in section 6.2, that $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ is hyperbolically imbedded in $\mathbb{CP}^1$.

We can consider a point in $\mathbb{CP}^1$ as a hyperplane. Mark L. Green proved: *complement of* $2n + 1$ *hyperplanes in general position in* $\mathbb{CP}^n$ *is complete hyperbolic and hyperbolically imbedded in* $\mathbb{CP}^n$. Main ideas in the proof of this theorem are:

(a) using Nevanlinna theory of meromorphic functions to study entire curves in complements of hyperplanes; here the most useful result is *Borel's lemma: if* $f_1 + \cdots + f_n \equiv 1$ *for entire functions* $f_1, \ldots, f_n$ *without zeros, then at least one function must be a constant*. Observe that this implies the Little Picard theorem.

(b) extending Brody's theorem to obtaining a criteria for hyperbolically imbedded complements of complex hypersurfaces in compact complex manifolds;

(c) connect hyperbolic imbeddings and complete hyperbolicity of such complements; it can be shown that *if a complement of a complex hypersurface is hyperbolically imbedded in a complex manifold, then this complement is complete hyperbolic.*

All results mentioned in this section can be found in Kobayashi's book [Kob10], still ultimate reference concerning hyperbolic complex spaces. The same author offers an excellent introduction to the subject [Kob05] while Krantz's book [Kra04] has similar approach to the subject as here. The historical aspect of invariant pseudodistances and hyperbolicity are described in Royden's paper [Roy88]. The greatness and beauty of invariant pseudodistances can be found in a comprehensive book [JP13].

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