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On Invariants for Spatial Graphs

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Abstract. We use combinatorial knot theory to construct invariants for spatial graphs. This is done by performing certain replacements at each vertex of a spatial graph diagram D, which results in a collection of knot and link diagrams in D. By applying known invariants for classical knots and links to the resulting collection, we obtain invariants for spatial graphs. We also show that for the case of undirected spatial graphs, the invariants we construct here satisfy a certain contraction-deletion recurrence relation.

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1 Introduction

The purpose of this paper is to create new invariants for spatial graphs using familiar concepts in knot theory. We also have a result that has a foundation in graph theory by implementing the contraction and deletion of a vertex. We create a multiset by performing certain replacements at each vertex of the spatial graph diagram. We then apply known invariants for classical knots to the multiset to obtain new invariants for spatial graphs. Most importantly, when we consider undirected spatial graph diagrams, the invariant we create satisfies a contraction-deletion recurrence relation. We will begin our discussion with some definitions.

Let Γ be a finite graph without vertices of valency (degree) zero or one. A *spatial embedding* of Γ is the image G of a topological embedding of Γ into 3-dimensional space $\mathbb{R}^3$ such that there is an orientation-preserving homeomorphism from $\mathbb{R}^3 \to \mathbb{R}^3$ that maps G into a polygonal graph in $\mathbb{R}^3$. A spatial embedding is also called a *spatial graph*. The embedding G is called an *r-component link* if the underlying graph Γ is homeomorphic to r disjoint circles, and is called a *knot* if Γ is homeomorphic to a circle. For our convenience, we consider a spatial graph G by ignoring the vertices of valency two.

Two spatial embeddings G and G' are called *ambient isotopic* (or *equivalent*) if there exists an orientation-preserving self-homeomorphism h on $\mathbb{R}^3$ such that $h(G) = G'$. A graph Γ is called *planar* if there exists an embedding of Γ into a plane. A spatial embedding G of a planar graph Γ is said to be *trivial* (or *unknotted*) if it is ambient isotopic to an embedding of Γ into an affine plane in $\mathbb{R}^3$.

We remark that since the 3-sphere S^3 is the one-point compactification of $\mathbb{R}^3$, formed by adjoining the point at infinity, we can also regard spatial graphs as graphs embedded in S^3. Spatial graphs are interesting objects and constitute one of the main research objects in knot theory. An excellent introduction to knot theory can be found, for example, in texts by Adams [1] and Kauffman [9]. A fundamental topological problem on spatial graphs is to determine whether two spatial embeddings of a graph Γ are equivalent or not. Similarly, a fundamental problem in knot theory is to find effective methods to differentiate knots or links.

A *diagram* of a spatial graph G is a regular projection of G onto a plane together with over/under information at each double point in the projection. Recall that a projection is called *regular* if its multiple points consists of finitely many transverse double points away from the vertices. The double points in a spatial graph diagram are called *crossings*.

It is well-known, see Kauffman's *Invariants on graphs in three-space* [7] for example, that two spatial graphs G and G' are ambient isotopic if and only if there is a finite sequence of planar isotopies and generalized Reidemeister moves I-V taking a diagram of G into a diagram of G'. The generalized Reidemeister moves are depicted in Figure 1. Note that the moves I-III are the *Reidemeister moves* which, together with planar isotopies, are sufficient to relate diagrams of equivalent knots or links.

Note that the generalized Reidemeister moves define an equivalence relation on the set of spatial graph diagrams; therefore, we can regard a spatial graph as the equivalence class

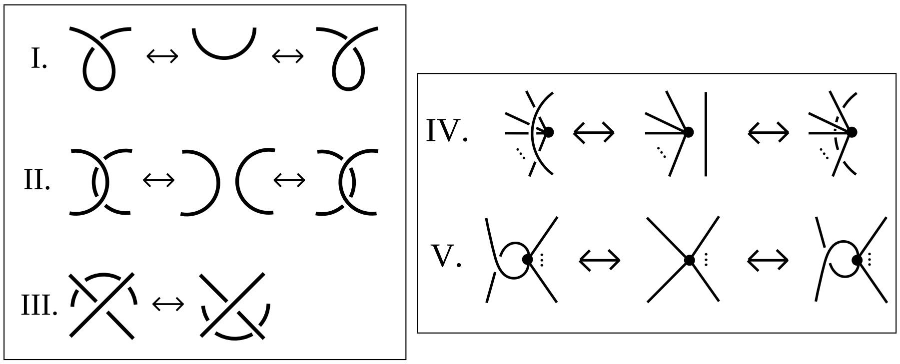


Figure 1: The generalized Reidemeister moves I-V

of spatial graph diagrams.

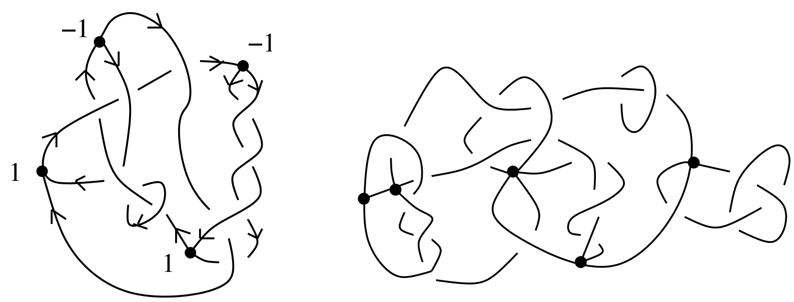
An *ambient isotopy invariant* for spatial graphs (or knots/links) is a quantity defined for each spatial graph (or knot/link) which is the same for ambient isotopic spatial graphs (or knots/links). To show that a quantity is indeed an invariant for spatial graphs (or knots/links) one needs to verify that it is not changed by the generalized Reidemeister moves I-V (or the Reidemeister moves I-III). If such a quantity upholds all of the Reidemeister moves except for the type I move, then the corresponding invariant is called a *regular isotopy invariant*.

Isotopy invariants are quite useful tools in distinguishing spatial graphs (or knots/links). When searching for new invariants for spatial graphs, the most challenging task is to obtain the invariance under the move of type V. In addition, most of the known isotopy invariants are not complete in the sense that two spatial graphs (knots/links) that are not equivalent may have the same invariant. However, it should be clear that if two spatial graphs (knots/links) have distinct invariants then they are not equivalent.

The purpose of this paper is to construct isotopy invariants for spatial graphs. To reach this goal, we model techniques from a construction by Kauffman in *Invariants on graphs in three-space* [7] in which certain replacements are performed at each vertex in a diagram D of a spatial graph G to obtain a knot or link diagram, which we call a *state* of D. In Section 2, by performing all possible replacements at the vertices, one at the time, we obtain a collection of all of the states corresponding to the original diagram D that is an ambient isotopy invariant for G. The replacements at the vertices are done in such a manner that the resulting states contain diagrams of constituent knots or links of the spatial graph G. By applying known knot or link polynomial invariants to all of the states corresponding to the diagram D, we construct various polynomial invariants for the spatial graph G. We can regard the resulting invariants as extensions to spatial graphs of known knot and link invariants.

Although we model our approach on Kauffman's construction, we will record more information about the constituent knots or links of a spatial graph for our invariant. Kauffman's invariant only lists one of each type of knot that occurs after all resolutions have been performed. In our method, we will keep track of how many times a knot appears by creating a multiset of knots and links for a given spatial graph.

We consider both undirected and directed spatial graphs. For undirected graphs we allow vertices of any valency, while for directed graphs we restrict to regular graphs (all vertices have the same valency). We should note that for directed graphs, valency is the in degree plus the out degree. For directed regular graphs with vertices of even valency, we impose that the edges are oriented such that the degree (in-degree minus out-degree) at each vertex is zero, and if vertices have odd valency, we require that the degree at a vertex is either 1 or -1 and that two adjacent vertices have opposite degrees. Recall that a graph will contain an even number of vertices of odd valency, and therefore, our directed regular graphs of odd valency will have the same number of vertices of degree 1 as of degree -1. Figure 2 illustrates (a) a directed trivalent spatial graph diagram, with the degrees of the vertices specified, and (b) an undirected spatial graph diagram with vertices of different valency.



(a) (b)

Figure 2: Examples of spatial graph diagrams: (a) directed; (b) undirected

For the case of undirected spatial graphs, we also show how to modify our collection of associated constituent knots and links of the original spatial graph in Section 3, so that we arrive at polynomial invariants for spatial graphs that satisfy a contraction-deletion recurrence relation. This allows us to compute the polynomial of a spatial graph in terms of polynomials of graphs with vertices of smaller valency.

2 First approach

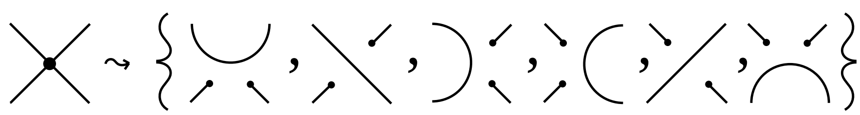
The method we use to construct our invariants is based on the work by Kauffman in *Invariants on graphs in three-space* [7]. Let G be a spatial graph with diagram D. We start at a vertex in D and consider every way to join two strands that meet at that vertex, leaving the other strands free. For undirected graphs, since any two strands can be joined, there are $\frac{n(n-1)}{2}$ different possible ways to resolve the n-valent vertex. Thus, we have created $\frac{n(n-1)}{2}$ diagrams just from resolving one vertex of valency n. We continue this process at each vertex of the graph diagram and complete the process by discarding all of the arcs that were created and keep the closed curves. In particular, if any diagram only contains arcs, the entire diagram is discarded. Therefore, our collection only contains knots and links after we

Figure 3: Unoriented and oriented 4-valent vertices

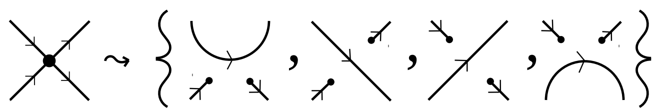
have resolved each vertex and discarded the arcs. We will denote this multiset by $\mathcal{C}(D)$, and refer to its elements as the *states* of the diagram D.

We perform a similar process for directed spatial graphs. However, this time we can only resolve each vertex in such a way as to maintain global compatibility of the orientations of the strands. This means that there will exist diagrams that contain contradictory orientations that must be discarded along with the diagrams that only contain arcs. Thus, after performing our replacements at each vertex in D and keeping those diagrams that result in globally compatible orientations throughout the diagram, we will obtain a collection of oriented knots and links associated to the original diagram D. We will also denote the resulting multiset by $\mathcal{C}(D)$.

To illustrate this procedure at a vertex, we consider unoriented and oriented 4-valent vertices as shown in Figure 3. We need to consider each way to join two strands that are incident with the vertex, and there are $\frac{4(4-1)}{2} = 6$ resolutions corresponding to an unoriented vertex of valency four, as shown below:

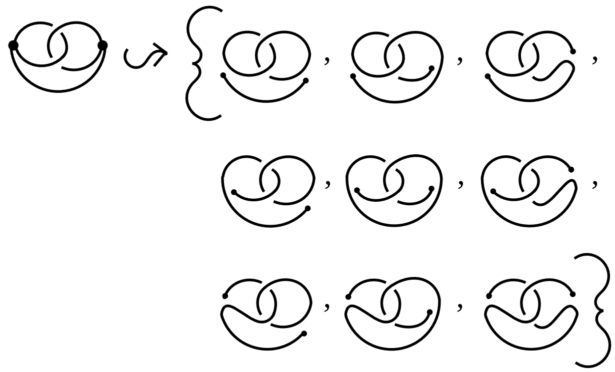


Due to contradictory orientations, there are only four resolutions for an oriented 4-valent vertex, and these are depicted below:

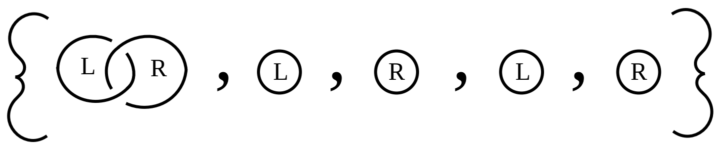


Remark 1. Notice that the orientations for the strands of the directed 4-valent vertex are only one of a few possible local orientations. In any case, we only consider those smoothings that do not contradict the original orientations of the strands. Similarly, all of our local pictures for directed 4-valent vertices should be viewed as one of a few possible orientations.

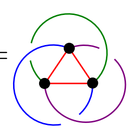
Example 2. To exemplify the entire process for a spatial graph diagram, we consider the unoriented trivalent graph diagram D_1 illustrated below, and apply all possible replacements at the two vertices in D_1:

$D_1 =$ 

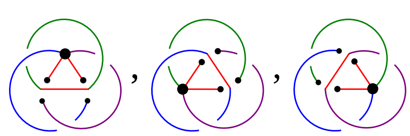
We discard all of the free arcs in the diagrams that we obtained and arrive at the following multiset $\mathcal{C}(D_1)$, where we use the letters L and R to denote the left circle and the right circle, respectively, in the original diagram D_1.

$\mathcal{C}(D_1) =$ 

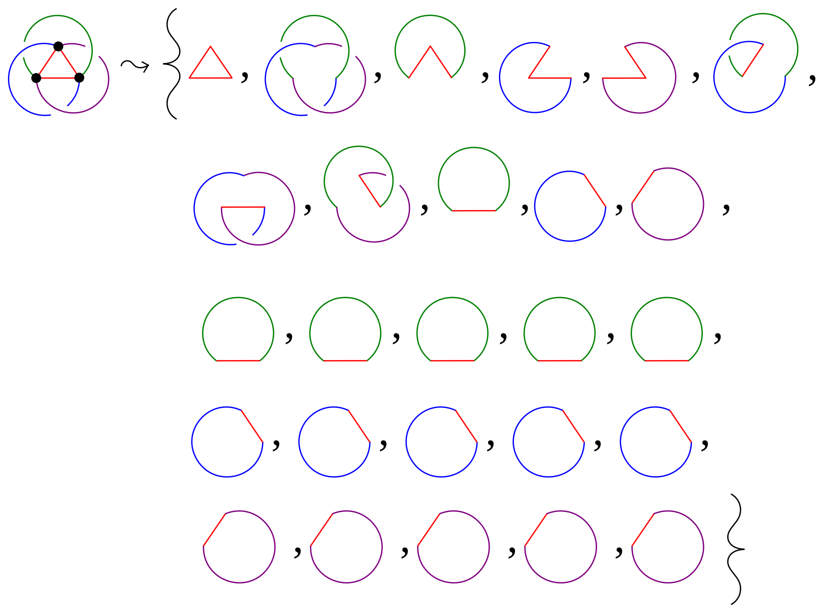
Example 3. We consider now another example, this time with a 4-valent graph to show how large the associated collections can become.

$D_2 =$ 

We have to perform the replacements at each of the three vertices in D_2. Notice that it is possible to arrive at a trivial knot or link by using a particular replacement at two of the vertices in D_2. Therefore, regardless of how we resolve the last vertex, we will always have the unknot. This situation occurs with the following graph diagrams obtained from replacements in D_2 at only two of the vertices:



So, for each of the three diagrams above, when we resolve the third vertex, there will be six diagrams containing that unknot diagram in the collection. After we discard the free arcs, we obtain the following collection $\mathcal{C}(D_2)$:



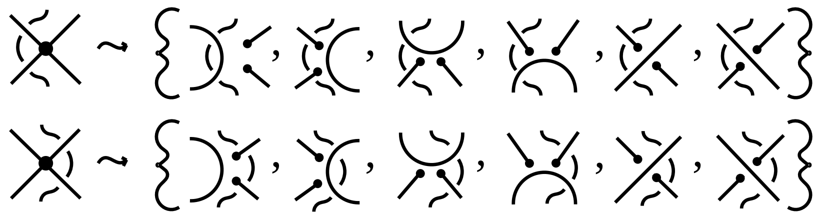
Remark 4. We want to emphasize that our collection is a multiset, unlike the collection obtained by Kauffman in *Invariants on graphs in three-space* [7]. Specifically, for the spatial graph diagram D_1 in Example 2, the associated collection of knot and link diagrams as defined in Kauffman's work retains the diagram of the Hopf link and the unknot; that is, it retains the knot or link type and not the number of times a knot or link type appears in the resulting collection. Our collection retains the Hopf link as well as four diagrams of the unknot, two from the left circle that we denoted with an L and two from the right circle that we denoted with an R. Also, oriented spatial graphs were not considered in *Invariants on graphs in three-space* [7].

Definition 5. A *cycle* in a graph Γ is a sequence of adjacent vertices such that only the first and last vertex are the same. Denote such a cycle by C. Let $f\colon \Gamma \to \mathbb{R}^3$ be an embedding of Γ into $\mathbb{R}^3$ with $f(\Gamma) = G$ the corresponding spatial graph. The image of a cycle C in Γ under the embedding f is called a *constituent knot* of the spatial graph G. A disjoint union of constituent knots in G is called a *constituent link* of G (see Kawauchi's *A survey of Knot Theory* [10] for additional details).

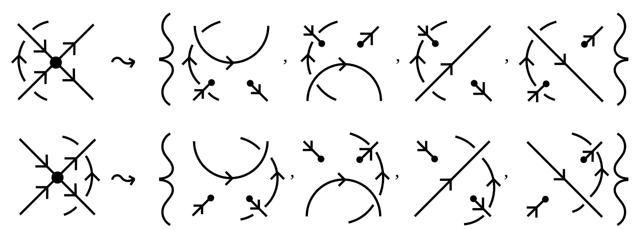
Remark 6. A careful analysis reveals that for a spatial graph G with diagram D, the multiset $\mathcal{C}(D)$ contains diagrams of the constituent knots and links of G, some of which appear with a certain multiplicity.

Proposition 7. *Let G be any spatial graph and D a diagram of G. When regarded up to Reidemeister moves I-III, the collection $\mathcal{C}(D)$ is an ambient isotopy invariant for G.*

Proof. The proof follows similarly to the proof outlined in Kauffman's *Invariants on graphs in three-space* [7]. We observe first that since we consider the collection $\mathcal{C}(D)$ up to Reidemeister moves I-III, we only need to show that $\mathcal{C}(D)$ is invariant under the moves of types IV and V. Notice that to prove the move IV, we will only show the equivalence of sliding a strand under the vertex, but the proof for sliding a strand over the vertex is similar. We will show in detail that this holds for 4-valent vertices, both unoriented and oriented. For vertices in D of other valency, the proof follows similarly. For unoriented 4-valent vertices we have:

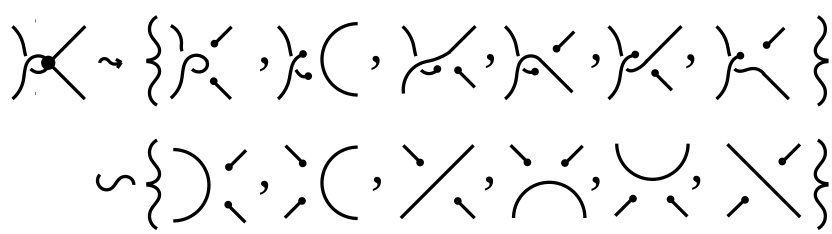


Removing the free arcs, the two collections above are the same, up to planar isotopy and the Reidemeister move II. For an oriented 4-valent vertex we have:

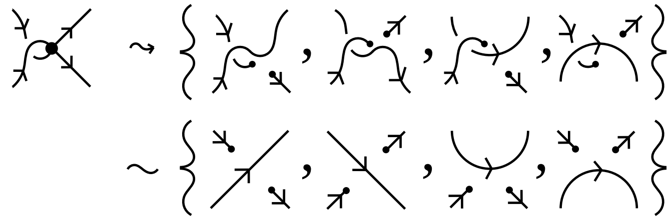


Once again, the above two collections with the free arcs removed are the same, up to planar isotopy. As in Remark 1, this is just one of a few possible local orientations for the directed vertex. However, since the other orientations produce similar sets up to rotations of the resolutions and orientations of the strands, we only consider one local orientation for this proof. Therefore, the invariance under the move of type IV is upheld.

Given an unoriented 4-valent vertex with a twist inserted, we associate the collection that we obtain by performing the vertex replacements. Applying a Reidemeister I move and planar isotopy, we obtain the collection corresponding to resolving a 4-valent vertex without a twist, as shown below:



For an oriented 4-valent vertex with a twist, we have:



Note that the collection in the second row above contains the untwisted vertex replacements. Similar to our example for an oriented vertex for type IV, this is only one of a few possible orientations. As before, we only consider one orientation for this proof because all the other cases will resolve into similar sets with different orientations. Therefore, in both unoriented and oriented cases, the invariance under the move of type V is upheld, at least for 4-valent vertices.

Now in general, we need to show that the statement holds for an n-valent vertex in D, where $n > 3$. For the generalized Reidemeister move of type IV to hold, we need to have the collection from a diagram with a strand to the left of the vertex to be the same as the collection from a diagram with a strand to the right of the vertex. This move is upheld because to have matching diagrams between our two collections, we simply have to allow a strand to slide over or under any given arc and, if the arc is not a free arc, we can apply a Reidemeister move II. If the arc is free the situation is even simpler since the free arcs are discarded regardless of their locations in a diagram.

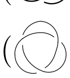
The generalized Reidemeister move of type V is upheld if the collection associated to a vertex is the same as the collection associated to a vertex with a twist inserted. We noticed that by only joining two strands at a time at any vertex in D, we eliminate the possibility of creating an additional crossing, as any crossing is removed when we discard the free arcs or by applying a Reidemeister move I. Also, every possible way to resolve the vertex will be present in our collection with the twist inserted, since we do not allow for two strands to be joined multiple times and our strand orientation does not change when a twist is performed. Each strand will be joined to every other strand that it is joined to in the original graph, and the matches between collections will all be made after the arcs are discarded and a Reidemeister move I have been performed on one of the resulting diagrams. This completes the proof. ☐

We use the multiset $\mathcal{C}(D)$ to construct a more practical set, which we now define.

Definition 8. The *weighted state set* of a spatial graph diagram D, denoted $\mathcal{C}^{\mathcal{W}}(D)$, is the set of ordered pairs (S, m_S) where S is a state of D and m_S is the multiplicity of S in the multiset $\mathcal{C}(D)$ considered up to Reidemeister moves I-III.

To exemplify this new set, we return to the spatial graph diagrams D_1 and D_2 in Examples 2 and 3, respectively. The corresponding weighted state sets are given below:

$\mathcal{C}^{\mathcal{W}}(D_1) = \{($  $, 1), (\bigcirc, 4)\}$

$\mathcal{C}^{\mathcal{W}}(D_2) = \{($  $, 1), (\bigcirc, 25)\}.$

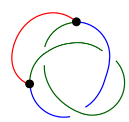
The following statement follows immediately from Proposition 7.

Proposition 9. *The weighted state set is an ambient isotopy invariant of spatial graphs.*

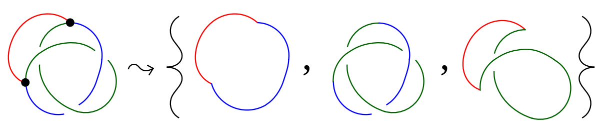
Notice that we could have defined the invariant $\mathcal{C}^{\mathcal{W}}(D)$ by using instead of the multiplicity m_S, the probability p_S that a knot type with diagram S appears in $\mathcal{C}(D)$ (in a similar fashion as in *The theory of pseudoknots* [4] for the case of pseudoknots), but that does not give us any advantages over the current definition.

Remark 10. We note that both of our invariants, $\mathcal{C}(D)$ and $\mathcal{C}^{\mathcal{W}}(D)$, are more powerful than the similar invariant introduced in Kauffman's *Invariants on graphs in three-space* [7], in the sense that they do better at distinguishing non-equivalent spatial graphs. Under Kauffman's invariant, if two distinct spatial graphs produce the same knots when resolved the invariant will not distinguish them. Under our new invariants, if the two spatial graphs produce the same knots but not the same number of each type of knot, our invariants will tell us that the two are not equivalent spatial graphs.

Example 11. Consider the spatial graph diagram D_3 given below:

$D_3 =$ 

The multiset $\mathcal{C}(D_3)$ that we obtain is

 $= \mathcal{C}(D_3)$,

and therefore, the weighted state set of D_3 is $\mathcal{C}^{\mathcal{W}}(D_3) = \{(0_1, 2), (3_1, 1)\}$, where 3_1 stands for the trefoil diagram and 0_1 for the unknot diagram.

Recall the spatial graph diagram D_2 in Example 3. The weighted state set of D_2 is $\mathcal{C}^{\mathcal{W}}(D_2) = \{(0_1, 25), (3_1, 1)\}$. Thus, the two sets of ordered pairs are distinct, and therefore D_2 and D_3 do not represent equivalent spatial graphs, which was already clear since the two graphs have different number of vertices and the vertices have different valency. However, the invariant used in Kauffman's *Invariants on graphs in three-space* [7] does not distinguish them, as it associates the unknot and the trefoil to both diagrams D_2 and D_3.

Definition 12. Let G be a spatial graph with diagram D, and let $\mathcal{C}^{\mathcal{W}}(D) = \{(S_i, m_{S_i})\}$ be the weighted state set of D. Let P be a known polynomial invariant for knots and links. Define a polynomial $\mathcal{P}$ associated with the diagram D, as given below:

$$\mathcal{P}(D) = \sum_i m_{S_i} P(S_i),$$

where the sum is over all ordered pairs in $\mathcal{C}^{\mathcal{W}}(D)$ and $P(S_i)$ is the polynomial invariant P computed from the knot/link diagram S_i.

Remark 13. As examples of polynomial invariants P for knots and links that can be employed in the previous definition we list the following:

- If G is a directed graph, the states S_i are oriented constituent knots or links of G. In this case, the polynomial P can be the Alexander polynomial which is defined in Alexander's *Topological invariants of knots and links* [2], or the Jones polynomial which can be found in Jones' *A polynomial invariant for links via von Neumann algebras* [3], or the HOMFLY-PT polynomial which is discussed in both *A new polynomial invariant of knots and links* by Freyd, et. al. [5] and *Invariants of links of Conway type* by Przytycki and Traczyk [11].

- If G is an undirected graph, the states S_i are unoriented constituent knots or links of G. In this case, we can employ the Kauffman bracket polynomial which can be found in *State models and the Jones polynomial* [6], or the two-variable Kauffman polynomial which is in *An invariant of regular isotopy* [8] to play the role for the knot/link polynomial P.

Theorem 14. *The polynomial $\mathcal{P}(D)$ is independent on the diagram D for the spatial graph G. That is, $\mathcal{P}$ is an invariant for spatial graphs.*

Proof. Let D_1 and D_2 be diagrams of a spatial graph G. Then D_1 and D_2 differ by a finite sequence of the generalized Reidemeister moves. By Proposition 7, we have that the sets $\mathcal{C}(D_1)$ and $\mathcal{C}(D_2)$ are the same up to the Reidemeister moves I-III. Moreover, by applying a polynomial invariant for knots and links to the diagrams in the two multisets above, we obtain collections of polynomials that are the same. Taking the sum of all of the polynomials of the diagrams in $\mathcal{C}(D_1)$ and separately in $\mathcal{C}(D_2)$, results in equal sums. Therefore, $\mathcal{P}(D_1) = \mathcal{P}(D_2)$. □

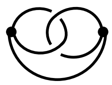
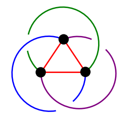
3 Second approach and a deletion-contraction recurrence relation

For a spatial graph G containing vertices of large valency, there are many constituent knot and links in G and therefore many states in the collection $\mathcal{C}(D)$, where D is a diagram of G. To find all of the corresponding states for the diagram D may be cumbersome, and thus we wanted to know whether the invariants for spatial graphs that we have defined so far satisfy a contraction-deletion recurrence relation. We have arrived at a positive answer for the case of undirected graphs, if we modify our approach to the collection of knot and link diagrams associated to D, as we now explain.

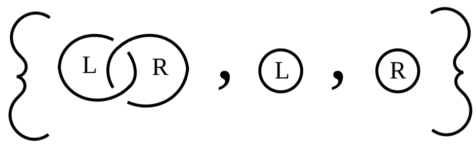
Recall that the multiset $\mathcal{C}(D)$ contains all of the states associated to D, where each state is a diagram of a constituent knot or link of the spatial graph G with diagram D. In addition, a constituent knot or link diagram may appear more than once in $\mathcal{C}(D)$. It turns out that if we consider a modified collection, $\overline{\mathcal{C}}(D)$, containing each constituent knot or link diagram exactly once, we obtain a contraction-deletion recurrence formula for the new collection $\overline{\mathcal{C}}(D)$ (which is an invariant for G), and consequently, for the corresponding polynomial invariants for G. We refer to the collection $\overline{\mathcal{C}}(D)$ as the *constituent link diagram set* of D.

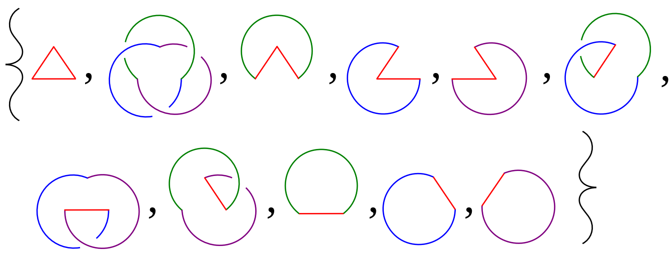
It is important to observe that $\overline{\mathcal{C}}(D)$ may still be a multiset, since distinct constituent knots/links of G may be equivalent knots/links.

Example 15. Consider again the diagrams D_1 and D_2 from Examples 2 and 3, respectively:

$D_1 =$  $D_2 =$ 

Their corresponding constituent link diagram sets are as follows:

$\overline{\mathcal{C}}(D_1) =$ 

$\overline{\mathcal{C}}(D_2) =$ 

Proposition 16. *If D and D' are diagrams of ambient isotopic spatial graphs, then the sets $\overline{\mathcal{C}}(D)$ and $\overline{\mathcal{C}}(D')$ are the same, up to Reidemeister moves I-III. That is, when regarded up to Reidemeister moves I-III, the collection $\overline{\mathcal{C}}(\,\cdot\,)$ is an ambient isotopy invariant for (unoriented and oriented) spatial graphs.*

Proof. The proof is similar to that for Proposition 7. □

Similar to our approach in Section 2, we can use the constituent link diagram set $\overline{\mathcal{C}}(D)$ to construct various polynomials associated to the original spatial graph diagram D by applying known knot/link polynomial invariants to the elements in the collection $\overline{\mathcal{C}}(D)$.

Definition 17. Let G be a spatial graph with diagram D, and let $\overline{\mathcal{C}}(D)$ be the constituent link diagram set of D with elements D_i. Let P be a polynomial invariant for knots and links. Define a polynomial $\overline{\mathcal{P}}$ associated with the diagram D and given by the following formula:

$$\overline{\mathcal{P}}(D) = \sum_i P(D_i),$$

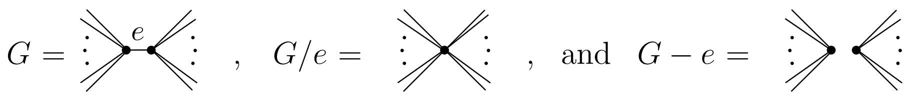
where the sum is over all elements D_i in $\overline{\mathcal{C}}(D)$ and $P(D_i)$ is the polynomial P associated to the knot or link diagram D_i.

Theorem 18. *If D and D' are diagrams of isotopic spatial graphs, then $\overline{\mathcal{P}}(D) = \overline{\mathcal{P}}(D')$. That is, $\overline{\mathcal{P}}$ is an isotopy invariant for spatial graphs.*

Proof. The proof follows similarly as that for Theorem 14, by replacing the collection $\mathcal{C}(\,\cdot\,)$ with $\overline{\mathcal{C}}(\,\cdot\,)$ and the polynomial $\mathcal{P}(\,\cdot\,)$ with $\overline{\mathcal{P}}(\,\cdot\,)$. □

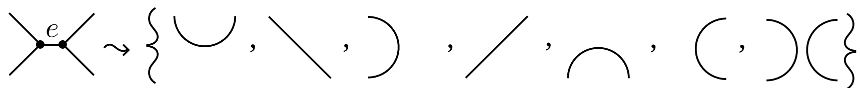
Remark 19. If the polynomial P is an ambient (or regular) isotopy invariant for knots and links then our polynomials $\mathcal{P}$ and $\overline{\mathcal{P}}$ are ambient (or regular) isotopy invariants for spatial graphs.

From here on we restrict to undirected spatial graphs. Let G be an undirected spatial graph and e an edge of G. Let G/e be the graph obtained from G by contracting the edge e, and $G-e$ be the graph obtained from G by deleting the edge e, as shown below:

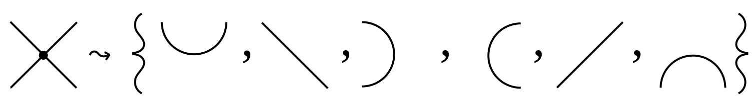


The three diagrams above are parts of spatial graph diagrams that are the same except in the neighborhood where they differ as shown. Note that the two vertices incident with the edge e in G do not need to be of the same valency; we represent them as having the same valency for purely aesthetic reasons. For simplicity, we will denote here both a spatial graph and its diagram by the symbol G.

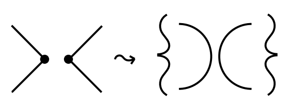
Using the second approach where we only consider the distinct constituent knots and links of a spatial graph, we found a relation among the collections $\overline{\mathcal{C}}(G)$, $\overline{\mathcal{C}}(G/e)$ and $\overline{\mathcal{C}}(G-e)$. To exemplify this relation, we start by looking at the case when we have vertices of small valency. For example, if the two vertices incident to the edge e in G are trivalent, the corresponding collection $\overline{\mathcal{C}}(G)$ is as shown below:



where all of the resulting free arcs have been discarded. For the vertex in G/e obtained by contracting the edge e, we have the following associated collection $\overline{\mathcal{C}}(G/e)$:

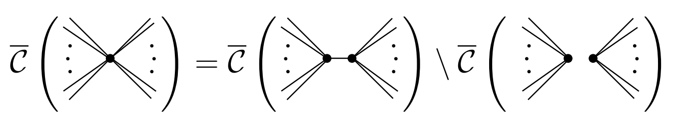


Finally, the corresponding collection $\overline{\mathcal{C}}(G-e)$ in this case is as follows:



since vertices of valency two are ignored. Looking at the three collections above, we see that we have arrived at the relation $\overline{\mathcal{C}}(G/e) = \overline{\mathcal{C}}(G) \setminus \overline{\mathcal{C}}(G-e)$, at least for the particular case considered so far. We will show now that this relation holds in general.

Theorem 20. *The following contraction-deletion recurrence relation holds:*



where the three diagrams are parts of spatial graph diagrams that are the same except in the neighborhood where they differ as shown. The equality can also be written as

$$\overline{\mathcal{C}}(G/e) = \overline{\mathcal{C}}(G) \setminus \overline{\mathcal{C}}(G-e),$$

where G is a spatial graph diagram and e is an edge of G.

Proof. When we consider the distinct cycles that can be obtained by resolving the two vertices incident with the edge e in G, we find that we can both join strands on the left and right of the edge e separately, as well as join a strand on the left with a strand on the right of e. Thus, in a small neighborhood of the edge e, the constituent knot and link diagrams of G include diagrams that contain one strand, as well as diagrams that contain two strands. When we resolve the two vertices in $G-e$ that result after deleting the edge e, we can only join the left and right strands separately. Therefore, in a small neighborhood of these two vertices in $G-e$, all of the constituent knot and link diagrams in $G-e$ will have two strands. Moreover, when we resolve the vertex in G/e obtained from G by contracting the edge e, we can only have one strand (in a small neighborhood of this vertex) in each of the constituent knot or link diagram in G. Furthermore, we observe that the diagrams in $\overline{\mathcal{C}}(G)$ with two strands are the same as those in $\overline{\mathcal{C}}(G-e)$, and that the diagrams with one strand in $\overline{\mathcal{C}}(G)$ are exactly the same as those in $\overline{\mathcal{C}}(G/e)$. Therefore, the statement holds. □

Remark 21. An equivalent statement of the relation can be written as

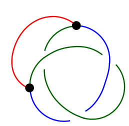
$$\overline{\mathcal{C}}(G) = \overline{\mathcal{C}}(G/e) \cup \overline{\mathcal{C}}(G-e).$$

In the proof of Theorem 20, we showed that $\overline{\mathcal{C}}(G/e) = \overline{\mathcal{C}}(G) \setminus \overline{\mathcal{C}}(G-e)$, which tells us that the new statement is a disjoint union. Then we can also write

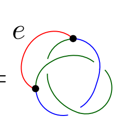
$$|\overline{\mathcal{C}}(G)| = |\overline{\mathcal{C}}(G/e)| + |\overline{\mathcal{C}}(G-e)|.$$

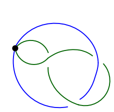
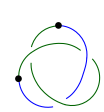
This equality gives us a method of finding the number of elements in each set, if desired.

Example 22. Consider again the graph diagram D_3 from Example 11.

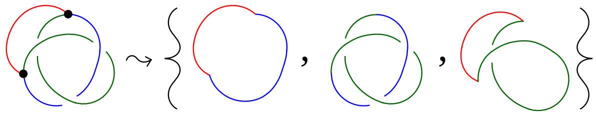
D_3 = 

We will assign the label e to the red edge that joins the two vertices in the left hand corner of the diagram. Then D_3/e and $D_3 - e$ are the graph diagrams obtained by contracting the edge e and deleting the edge e, respectively. All three diagrams are shown below.

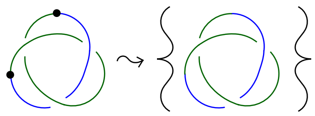
$D_3 =$ 

, $D_3/e =$ 
, and $D_3 - e =$ 

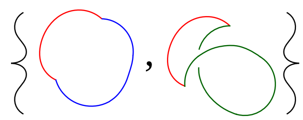
We know from Example 11 that the multiset $\overline{\mathcal{C}}(D_3)$ that we obtain is

 $= \overline{\mathcal{C}}(D_3)$.

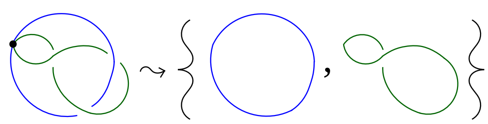
Since $D_3 - e$ has only bivalent vertices, we have only a single diagram in the collection $\overline{\mathcal{C}}(D_3 - e)$. Thus we have that

 $= \overline{\mathcal{C}}(D_3 - e)$,

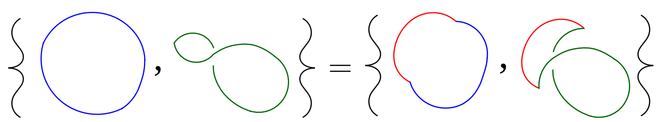
Then we have that

$\overline{\mathcal{C}}(D_3) \setminus \overline{\mathcal{C}}(D_3 - e) =$ 

Now, let's consider $\overline{\mathcal{C}}(D_3/e)$. After discarding the free arcs, we have that

 $= \overline{\mathcal{C}}(D_3/e)$,

Notice that



which demonstrates that

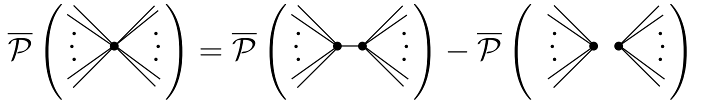
$$\overline{\mathcal{C}}(D_3/e) = \overline{\mathcal{C}}(D_3) \setminus \overline{\mathcal{C}}(D_3 - e).$$

We want to emphasize that the contraction-deletion recurrence relation in Theorem 20 is relevant because it can be used to compute the polynomial $\overline{\mathcal{P}}$ of a spatial graph in terms of polynomials of other spatial graphs with vertices of smaller valency, as explained in the following statement.

Corollary 23. *Let G be a spatial graph and e an edge of G. Then*

$$\overline{\mathcal{P}}(G/e) = \overline{\mathcal{P}}(G) - \overline{\mathcal{P}}(G - e),$$

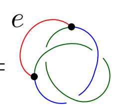
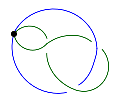
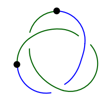
or equivalently,



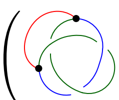
where the three diagrams are parts of spatial graph diagrams that are the same except in the neighborhood where they differ as shown.

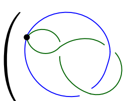
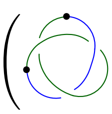
Proof. This follows at once from Theorem 20 and the definition of the polynomial $\overline{\mathcal{P}}$. □

Example 24. Recall the diagrams from Example 22.

$D_3 =$  , $D_3/e =$  , and $D_3 - e =$ 

To demonstrate Corollary 23, we will be implementing the Kauffman bracket for unoriented knots. It is well known that the Kauffman bracket for the trivial diagram of the unknot is 1 and for the diagram of the unknot with one twist is $-A^3$ or $-A^{-3}$, depending on the sign of the twist. Moreover, the Kauffman bracket for the trefoil is $A^{-7} - A^{-3} - A^5$. Thus we have that

$\overline{\mathcal{P}}\Big($  $\Big) = 1 + (A^{-7} - A^{-3} - A^5) + (-A^3),$

$\overline{\mathcal{P}}\Big($  $\Big) = 1 + (-A^3),$ $\overline{\mathcal{P}}\Big($  $\Big) = A^{-7} - A^{-3} - A^5.$

Therefore, we have the following equality

$$1 - A^3 = (1 + A^{-7} - A^{-3} - A^5 - A^3) - (A^{-7} - A^{-3} - A^5)$$

which implies that

$$\overline{\mathcal{P}}(D_3/e) = \overline{\mathcal{P}}(D_3) - \overline{\mathcal{P}}(D_3 - e).$$

4 Open Questions

We close with a few open questions:

1. Does there exist a contraction-deletion recurrence relation for oriented spatial graphs?

2. We only were able to create a contraction-deletion recurrence relation by considering the collection $\overline{\mathcal{C}}$ of a graph diagram. Does there exist a modified contraction-deletion recurrence relation by considering the collection $\mathcal{C}$ instead?

3. Is there a recurrence relation among the collections of two graphs and their connected sum?

4. Can we utilize the concept of the minor of a graph to describe our collections $\overline{\mathcal{C}}$ and $\mathcal{C}$?

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