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Spontaneous synchrony on graphs and the emergence of order from disorder

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From pulsars to pedestrians and bacteria to brain cells, objects that exhibit cyclical behavior, called oscillators, are found in a variety of different settings. When oscillators adjust their behavior in response to nearby oscillators, they often achieve a state of synchrony, in which they all have the same phase and frequency. Here, we explore the Kuramoto model, a simple and general model which describes oscillators as dynamical systems on a graph and has been used to study synchronization in systems ranging from firefly swarms to the power grid. We discuss analytical and numerical methods used to investigate the governing system of differential equations and the conditions that lead to synchronization, and demonstrate that perfect synchronization occurs only under strict conditions and for specific graph structures. We also present results from an experiment with coupled metronomes in which spontaneous emergence of synchronization, consistent with the mathematical theory, can be observed in a real-world setting.

I. BACKGROUND

Many real world systems can be modeled as networks of oscillators, objects that exhibit cyclical patterns of behavior. When two or more oscillators in a system operate with the same phase and frequency they are said to be synchronized. In nature, oscillators often interact with each other and adjust their frequencies in response to their neighbors. These interactions can cause them to operate in unison and to remain in a state of synchrony indefinitely. This phenomenon is observed to occur in many types of oscillators, including a wide variety of natural and engineered systems. For instance, fireflies in Southeast Asia spontaneously sychronize their flashing lights [7].

Often synchronization is desirable. Generators on the power grid must be in phase in order to avoid blackouts [4]. However, there are also times when synchronization can have unintended consequences and needs to be avoided. For example, synchronized marching on a bridge can cause dangerous oscillations [8]. Therefore, it is important to understand how synchronization arises and under what conditions it is likely to occur. In this project we investigated a mathematical model developed by Yoshiki Kuramoto that has been used to describe synchronization in many real world scenarios. Research thus far has focused on systems with a large (approaching infinite) number of oscillators, yet many real world systems consist of a small number of oscillators. Thus, we studied synchronization on specific graphs with a few (less than five) oscillators. In doing so, we used numerical methods to investigate the dynamics of coupled oscillators for various initial conditions and then used analytical methods to identify equilibrium solutions and assess their stability.

II. MODEL

For our investigation we used the Kuramoto-Sakaguchi model, (see equation (1)) which describes the behavior of coupled oscillators with a system of first-order nonlinear differential equations, where ψ is the phase of an oscillator, ω is its natural frequency, α represents the phase lag, which behaves like a delay in passing information between the oscillators[4], and K_{ij} is a matrix of coupling strengths for the oscillators.

$$\frac{d\psi_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\sin(\psi_j - \psi_i - \alpha) \tag{1}$$

Coupling Matrix

The coupling matrix is a square matrix with each row and each column corresponding to an oscillator on the graph. Entry K_{ij} is the coupling strength between oscillators i and j, which determines how strongly the oscillators interact. The greater the value of K_{ij}, the stronger the interaction, with zero indicating that the oscillators are uncoupled. A coupling matrix can be visually depicted as a graph with vertices representing oscillators and edges with different weights representing connections of various strengths.

The Dynamics of the Kuramoto Model

In the Kuramoto Model, when $\alpha = 0$ oscillators attempt to synchronize with their neighbors by adjusting their phase velocities. If the phase of an oscillator is slightly less than the phases of the other oscillators the sine term is positive and the phase velocity of the oscillator increases until it reaches the same phase as the others. Similarly, if the phase of an oscillator is slightly greater than the phases of the others the sine term is negative and the phase velocity of the oscillator decreases.

Assumptions and Limitations

For our investigation we assumed that all pairs of oscillators were either coupled ($K_{ij} = 1$) or uncoupled ($K_{ij} = 0$), with no other variation in coupling strength. We also assumed all couplings were bidirectional, meaning that coupled oscillators influence each other equally ($K_{ij} = K_{ji}$), that there was no self-coupling ($K_{ii} = 0$), and that the graph is "connected,", meaning no oscillators or groups of oscillators are isolated from the others. We began by considering cases with four identical oscillators, meaning all have the same natural frequency ($\omega_i = \omega$), and with a small positive phase lag ($\alpha = 0.1$). We used these limited cases as a starting point from which to gain some insight into the behavior of coupled oscillators before expanding to more general cases.

Graph Theory

We first analyzed possible configurations of oscillators that would satisfy these conditions to identify non-isomorphic arrangements (table I). An arrangement is nonisomorphic if it could not be relabeled or otherwise manipulated by stretching, twisting or flipping without adding or removing edges to be equivalent to an already established arrangement. We found these six graphs (figure 1 below) to be the only nonisomorphic cases to satisfy all conditions.

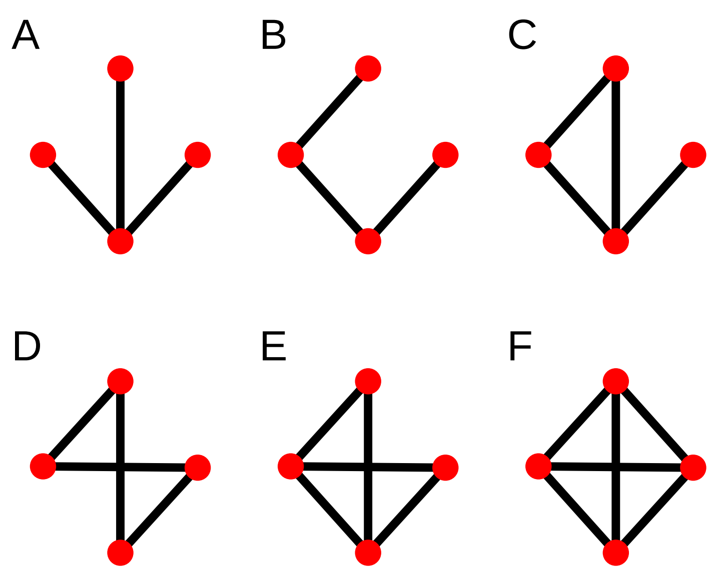

FIG. 1. The six nonisomorphic graphs. Each vertex (red dot) represents an oscillator, and each edge (line) is a connection.

III. ANALYSIS

In order to explore the behavior on these graphs and identify which subgraphs of oscillators are likely to synchronize we used the Runge-Kutta method via MATLAB to numerically solve the differential equations for each of the six cases, each under several different initial conditions including fully random initial phases for all oscillators and

TABLE I. Possible Combinations

Edges	Graphs	Connected Graphs	Non-isomorphic Graphs
6	1	1	1
5	6	6	1
4	15	15	2
3	20	16	2
2	15	0	0
1	6	0	0
0	1	0	0
Total	64	38	6

combinations of partially synchronized initial states. We noticed that oscillators that were interchangeable, meaning there were graph automorphisms that mapped them onto each other, would synchronize, while oscillators that were not interchangeable would not synchronize spontaneously, an observation previously reported in other studies[1].

Next we solved the systems of differential equations analytically from a rotating frame of reference and, using educated guesses informed by the results from the simulations and from the layouts of the graphs, found several equilibrium solutions for each case. We then computed the Jacobian matrices for the systems of differential equations at each equilibrium solution of each case and calculated the eigenvalues to determine the stability of the equilibrium solutions. The results of these computations are summarized in tables III - V. They indicate the existence of stable solutions with all oscillators which are interchangeable by automorphisms synchronized, although there are also unstable solutions with all such oscillators synchronized. They also indicate complete and spontaneous synchronization only in test cases where all oscillators have the same number of connections, and that for these test cases only synchronized solutions were stable.

We then returned to MATLAB simulation and perturbed the unstable solutions along the unstable eigenvectors by setting the initial conditions to values close to, but not exactly at, each unstable solution, in order to see what values the systems would go to from each. In agreement with our analytical results, these simulations usually showed the system approaching one of the stable equilibria. However, we also observed in some cases that some of the oscillators never reached a steady state, and seem to drift indefinitely. These behaviors are consistent with chimera state behavior other studies have observed in unequally coupled oscillators [4].

In simulations we observed that only cases D and F, in which all oscillators have the same number of connections to other oscillators, were able to spontaneously attain perfect synchronization. We also noticed that the only stable equilibrium solutions were ones in which all oscillators in each cluster were synchronized. Oscillators belong to the same cluster if they are interchangable via automorphism. From this we concluded that in order for all of the oscillators to synchronize spontaneously, they had to be a part of the same cluster, which is only possible if the sums of the coupling strengths to each oscillator are the same.

A. Proof of Synchronized Equilibrium and Stability

In order to generalize the above results beyond the initial conditions we tested, we prove the following result

Theorem 1. *A phase and frequency locked solution, where $\psi_i = \psi$ and $\frac{d\psi}{dt} = \Omega$, is an equilibrium solution if and only if $D_i = \frac{N(\omega_i - \Omega)}{\sin\alpha}$*

We begin with the Kuramoto model and the conditions for a phase and frequency locked solution:

$$\frac{d\psi_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\sin(\psi_j - \psi_i - \alpha),$$

$$\psi_i = \psi, \quad \frac{d\psi}{dt} = \Omega.$$

We then substitute the required conditions into the equation, yielding

$$\Omega = \omega_i - \frac{\sin\alpha}{N}\sum_{j=1}^{N} K_{ij}. \tag{2}$$

This can be written in terms of the degree of each oscillator, which is the sum of the coupling strengths to it

$$D_i = \sum_{j=1}^{N} K_{ij}$$

$$\Omega = \omega_i - D_i\frac{\sin\alpha}{N}.$$

The expression can be rearranged to isolate D_i

$$D_i = \frac{N(\omega_i - \Omega)}{\sin\alpha}. \tag{3}$$

This shows that in order for a phase and frequency locked equilibrium solution to occur, the degree of each oscillator must have a specific relationship to the natural frequency of that oscillator. Specifically, if the oscillators are identical ($\omega_i = \omega$), then the degrees of all the oscillators must be the same. We are also able to reverse this result to conclude that if the degrees of the oscillators have this specific relationship with the natural frequencies, then a phase and frequency locked solution is an equilibrium solution.

We now determine when this solution is stable

Theorem 2. *A phase and frequency locked equilibrium solution is a neutrally stable solution if* $\cos\alpha$ *is positive.*

$$J = \frac{\cos\alpha}{N}\begin{bmatrix} -D_1 & K_{12} & K_{13} & K_{14} \\ K_{21} & -D_2 & K_{23} & K_{24} \\ K_{31} & K_{32} & -D_3 & K_{34} \\ K_{41} & K_{42} & K_{43} & -D_4 \end{bmatrix} \tag{4}$$

We constructed the Jacobian matrix for the system of differential equations. According to the Gershgorin Circle Theorem, all the eigenvalues of a matrix must fall within one of the circles in the complex plane centered at the diagonal entries and with radii which are the sum of the off-diagonal entries. Because the sum of the off-diagonal entries in this matrix is the sum of the connection strengths, which is just the degree, and the diagonal entries are the degrees, all the circles, called Gershgorin discs, are located to the left of the imaginary axis and intersect the origin. Therefore, the real components of the eigenvalues are at most zero, and the solution is at least neutrally stable, as long as $\cos\alpha$ is positive.

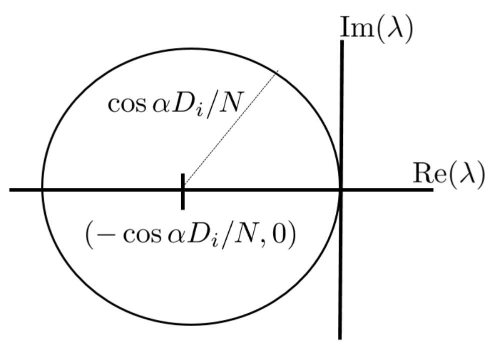

FIG. 2. Gershgorin Discs. All eigenvalues must lie inside discs centered at $(-\cos\alpha D_i/N, 0)$ and with radius $R_i = \cos\alpha D_i/N$.

Alternatively, the stability of the solution can be established by recognizing the relationship between the Jacobion matrix of the system (equation 4) and the graph Laplacian.

$$L = \begin{bmatrix} D_1 & -K_{12} & -K_{13} & -K_{14} \\ -K_{21} & D_2 & -K_{23} & -K_{24} \\ -K_{31} & -K_{32} & D_3 & -K_{34} \\ -K_{41} & -K_{42} & -K_{43} & D_4 \end{bmatrix} \tag{5}$$

The Jacobian matrix can therefore be written as the graph Laplacian (equation 5) multiplied by $-\cos\alpha/N$.

$$J = \frac{-cos\alpha}{N} L \tag{6}$$

It is known [2] that the graph Laplacian has exactly one eigenvalue with a value of zero for each connected "piece" of the graph, with the others having positive real components. Since we are only considering cases with only one "piece", and the Laplacian is multiplied by a negative to get the Jacobian, we know that the Jacobian matrix has one eigenvalue with a value of zero and that the rest are negative, so the solution is neutrally stable. The single zero eigenvalue corresponds to shifting all the oscillators such that the phase and frequency locking is maintained, so the solution is effectively stable.

Generalizations

This proof holds true for any number of oscillators and for coupling strengths other than $K_{ij} = 1$ and $K_{ij} = 0$, so long as the sums of the coupling strengths still satisfy the relationship in theorem 1. These results therefore can be generalized to any number of oscillators, as well as to non-identical oscillators [6], which may not have the same natural frequencies, although in such a case the degrees will be slightly different.

IV. THE EXPERIMENT

In order to see if these results hold under real-world conditions, we carried out an experiment involving coupled metronomes. Our goal was to reproduce the results seen in simulations and to extend them to other scenarios including different graph topologies. Inspired by recent experiments in which coupled metronomes were observed to synchronize [3, 5], we placed metronomes on a moving base that was free to translate in the horizontal direction. Considering the metronomes, the board, and the 'frictionless' rollers as a closed system, its center of mass can only move through the application of an external force. However, the only forces that are present are internal ones. Thus, if we start the experiment at rest, as the pendulum arm of the metronome moves to one side, the board will move in opposite direction to keep the center of mass in the same position. As the board moves, it will influence the movement of the metronomes' pendulum arms and this will again exert a force on the board. Over time, the common input force supplied by the board will cause the metronomes to synchronize.

The experimental setup consists of a wood platform 24.7 cm length by 13 cm wide, four metronomes (musiclily mini mechanical model LM-03), four microphones (angelia piezo contact microphone), and two PVC rollers 3/4" in diameter. It can be seen in figure 3.

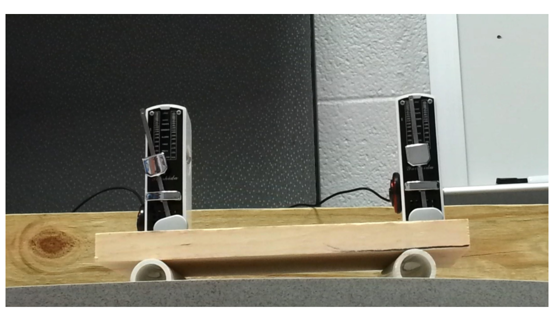

FIG. 3. The experimental setup. Links between the oscillators

A. Calibrating the metronomes

Metronomes are used by musicians to indicate the rhythm through a periodic ticking sound. Their frequency is adjustable by moving a weight along the different marked positions in its arm. The frequency ranges from 40 to 208 beats per minute. However, although the metronomes have the same design, they do present different frequencies. This is due the fact that even when set to the same nominal frequency, slight structural variations from the manufacturing process can lead to slightly different actual frequencies. Moreover, energy is lost due to friction at slightly different rates so the frequency must not be perfectly constant in the long run. Knowing that, we measured the average frequency over one minute when set to 200 bpm for various trials. The table below summarizes the results.

Table II reveals that the frequencies are similar; however there is some variations in the mean frequency. It is known

	Metronome 1	Metronome 2	Metronome 3	Metronome 4
Minimum value	200.1	200.7	200.2	200.9
Maximum value	203.7	202.7	201.1	201.9
Mean value	202.7	202.1	200.6	201.4
Standard deviation value	1.1	0.8	0.4	0.4

TABLE II. Min, Max, Mean and Standard deviation value for metronomes' frequency

from previous work that too much variation will prevent the metronomes from synchronizing. In light of this, we manually adjusted the frequencies of the metronomes in an attempt to minimize the difference in the frequencies between them.

B. Analyzing the data from the experiment

We first collected data from two metronomes using one microphone. To detect the time when the click sounds occured, we used MATLAB to look for when the amplitude of the sound wave would exceed a particular threshold. In order to avoid double counting, the threshold was increased everytime we detected a click and then let it decrease with time. This threshold was given by an envelope that approximated the shape of an isolated click. This envelope was an exponential decay function of the form $Ae^{-B(t-t_0)} + C$, where the values of A,B,C were determined through trial and error. Based on that we tried to detect the clicks. This worked well when the metronomes were not synchronized. However, when clicks occurred in quick succession, we were unable to find an envelope that enabled us to correctly detect the clicks as well as distinguishing between them.

Therefore, we implemented an alternative approach. We used a separate microphone for each metronome and recorded the sound as separate channels. This allowed us to isolate the signals from each metronome and to detect all the clicks, even when they were very close together, using a constant threshold. We did this by first normalizing the signals, setting the maximum value of the amplitude equal one. Then, the threshold was chosen to mark any jump greater than 0.6 in the vertical axis (amplitude). After detecting a click, we imposed a minimum wait time of 0.2 seconds before detecting a new click. Figure 4 is an example of the resulting data. Once we detected the click times, we fitted the graph with a sine curve representing the phase. With our available equipment, we were able to collect data from up to four metronomes.

C. Experimental data results

In order to measure the degree of synchrony in the experiment, we used the complex order parameter Z,

$$Z = Re^{i\Psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\psi_j}$$

The magnitude R measures how close to synchronization the metronomes are. If the metronomes are perfectly synchronized, then $R = 1$; if they are completely out of phase, $R = 0$. For partially synchronized states, $0 < R < 1$. After adjusting the frequencies, we noticed that with two metronomes on the board they synchronize most of the time. This occured independently of the initial phases. Even when the metronomes were disturbed, they would achieve a synchronized state over time. Figure 5 below shows in detail one particular trial of the experiment. The results of the experiment demonstrate spontaneous synchronization after a disturbance.

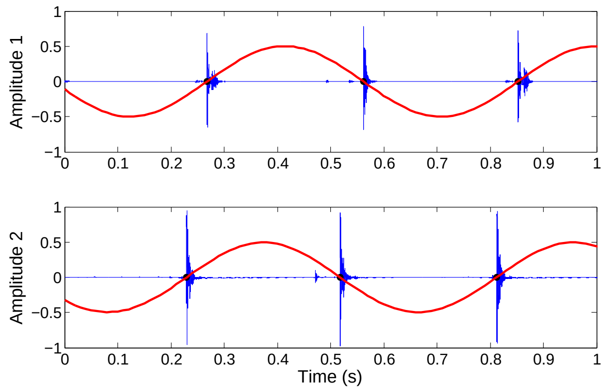

FIG. 4. The figure shows the process of analyzing the experimental data through MATLAB. Marked in blue, is the sound wave file recorded from metronome 1, metronome 2, and a sine function was fitted along all the clicks. If the metronomes were perfectly synchronized, the two sine functions would look exactly the same.

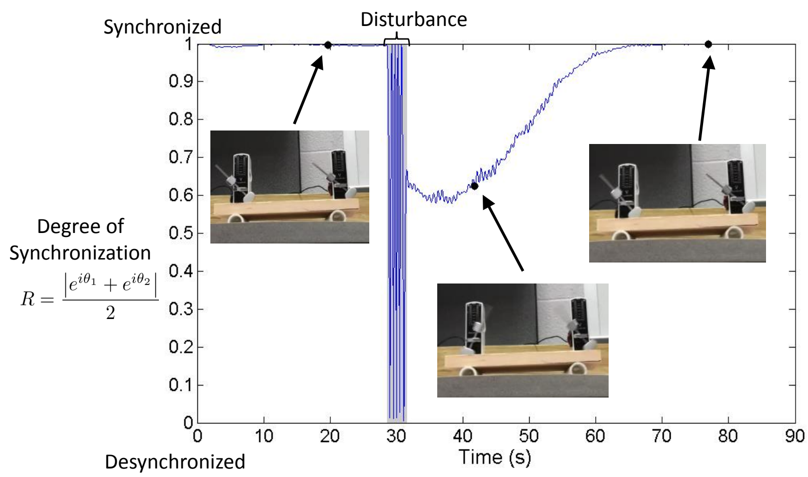

FIG. 5. Data from one trial with two metronomes. This figure highlights the fact that even when there is a disturbance and the metronomes become desynchronized, they eventually synchronize spontaneously. In this particular case, the disturbance was because one of the metronomes was held in place and then released.

In addition, we repeated the experiment with four metronomes. The data from this experiment is found in figure 6. As discussed above, the metronomes' frequencies are not exactly the same. Thus, with four metronomes, these differences prevented them from reaching a completely synchronized state, reflected in the value of around 0.9 for the order parameter magnitude R. Due to time constraints, we were unable to complete as extensive an experiment as we would have liked. In the future, it would be useful to perform the experiment with two platforms connected by springs to model more general graph topologies.

V. CONCLUSION

We were able to demonstrate with our experiment that complete synchronization can and does occur spontaneously under favorable conditions: when the frequencies are sufficiently similar. We were also able to show both numerically and analytically that perfect phase synchronization can only occur if the degrees of the oscillators have a specific relationship to the natural frequencies of the oscillators, and if the oscillators are identical then their degrees must be the same. It is important to note that this is a necessary condition, not a sufficient one, meaning that the degrees must all be the same in order for synchronization to occur, but synchronization will not necessarily occur if the degrees are all the same, as there are some initial conditions where synchronization still does not occur. Additionally, we were able to show that synchronized equilibrium solutions are stable for a large number of possible parameter values.

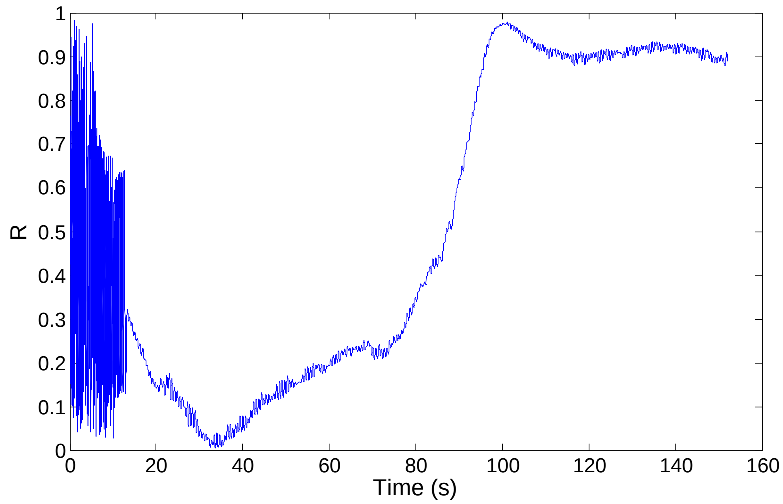

FIG. 6. Data from four metronomes. This figure shows how the order parameter R changes with time when we have four metronomes coupled together on the board. During the first 20 seconds, the data fluctuations are due to setting up the metronomes to begin. After that, we see that even after the metronomes get completely desyncrhonized (around 35 seconds), they will get close to synchronization and stabilize.

During the course of our investigation we encountered a number of possible courses of future study. As mentioned previously, our numerical investigation revealed a number of cases consistent with chimera state [4] behavior, which would merit further investigation. We would also like to further investigate graphs with unequal coupling strengths as well as implement such a system experimentally using multiple platforms connected with springs.

ACKNOWLEDGMENTS

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Appendix A: Tables

Graph	Guess	Phase Lag (α)	Solution	Eigenvalues
A	$\theta_1 = \theta_2 = \theta_3 = 0$; $\theta_4 = \theta$	$\alpha = 0.1$	$\theta = -0.050$	$\lambda_1 = 0$ $\lambda_2 = -0.247$ $\lambda_3 = -0.247$ $\lambda_4 = -0.996$
A	$\theta_1 = \theta_2 = \theta_3 = 0$; $\theta_4 = \theta$	$\alpha = 0.1$	$\theta = 3.091$	$\lambda_1 = 0.996$ $\lambda_2 = 0.247$ $\lambda_3 = 0.247$ $\lambda_4 = 0$
A	$\theta_1 = \theta_2 = \theta_3 = 0$; $\theta_4 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = -0.741$	$\lambda_1 = 0.060$ $\lambda_2 = 0.060$ $\lambda_3 = 0$ $\lambda_4 = -0.650$
A	$\theta_1 = \theta_2 = \theta_3 = 0$; $\theta_4 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 2.400$	$\lambda_1 = 0.650$ $\lambda_2 = 0$ $\lambda_3 = -0.060$ $\lambda_4 = -0.060$
B	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = 0.1$	$\theta = -0.050$	$\lambda_1 = 0$ $\lambda_2 = -0.145$ $\lambda_3 = -0.497$ $\lambda_4 = -0.850$
B	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = 0.1$	$\theta = 3.192$	$\lambda_1 = 0.497$ $\lambda_2 = 0.352$ $\lambda_3 = 0$ $\lambda_4 = -0.353$
B	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = -1.156$	$\lambda_1 = 0.087$ $\lambda_2 = 0$ $\lambda_3 = -0.0967$ $\lambda_4 = -0.423$
B	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 4.298$	$\lambda_1 = 0.183$ $\lambda_2 = -0.326$ $\lambda_3 = 0.097$ $\lambda_4 = 0$
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = 0.1$	$\theta = -5.998$; $\phi = -0.970$	$\lambda_1 = 0$ $\lambda_2 = -0.068$ $\lambda_3 = -0.529$ $\lambda_4 = -0.618$
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = 0.1$	$\theta = 2.737$; $\phi = -0.051$	$\lambda_1 = 0.366$ $\lambda_2 = 0$ $\lambda_3 = -0.646$ $\lambda_4 = -0.745$
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = 0.1$	$\theta = 2.942$; $\phi = 3.092$	$\lambda_1 = 0.640$ $\lambda_2 = 0$ $\lambda_3 = -0.250$ $\lambda_4 = -0.385$
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 17.211$; $\phi = 4.320$	$\lambda_1 = 0.107$ $\lambda_2 = 0.009$ $\lambda_3 = 0$ $\lambda_4 = -0.399$

TABLE III. Equilibrium Solutions

Graph	Guess	Phase Lag (α)	Solution	Eigenvalues
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = \frac{pi}{2} - 0.5$	$\theta = -8.741$; $\phi = -4.386$	$\lambda_1 = 0.203$ $\lambda_2 = 0.203$ $\lambda_3 = 0$ $\lambda_4 = 0$
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = -40.157$; $\phi = 1.897$	$\lambda_1 = 0.203 + 0.257i$ $\lambda_2 = 0.203 - 0.257i$ $\lambda_3 = 0$ $\lambda_4 = -0.409$
C	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta$; $\theta_4 = \phi$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 4.644$; $\phi = 4.320$	$\lambda_1 = 0.107$ $\lambda_2 = 0.00885$ $\lambda_3 = 0$ $\lambda_4 = -0.399$
D	$\theta_1 = \theta_3 = 0$; $\theta_2 = \theta_4 = \theta$	$\alpha = 0.1$	$\theta = 0$	$\lambda_1 = 0$ $\lambda_2 = -0.498$ $\lambda_3 = -0.498$ $\lambda_4 = -0.995$
D	$\theta_1 = \theta_3 = 0$; $\theta_2 = \theta_4 = \theta$	$\alpha = 0.1$	$\theta = 3.142$	$\lambda_1 = 0.995$ $\lambda_2 = 0.498$ $\lambda_3 = 0.498$ $\lambda_4 = 0$
D	$\theta_1 = \theta_3 = 0$; $\theta_2 = \theta_4 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 0$	$\lambda_1 = 0$ $\lambda_2 = -0.240$ $\lambda_3 = -0.240$ $\lambda_4 = -0.479$
D	$\theta_1 = \theta_3 = 0$; $\theta_2 = \theta_2 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = \pi$	$\lambda_1 = 0.479$ $\lambda_2 = 0.240$ $\lambda_3 = 0.240$ $\lambda_4 = 0$
E	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = 0.1$	$\theta = 0.025$	$\lambda_1 = 0$ $\lambda_2 = -0.496$ $\lambda_3 = -0.995$ $\lambda_4 = -0.996$
E	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = 0.1$	$\theta = 3.117$	$\lambda_1 = 0.995$ $\lambda_2 = 0.499$ $\lambda_3 = 0$ $\lambda_4 = -0.001$
E	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 0.475$	$\lambda_1 = 0$ $\lambda_2 = -0.012$ $\lambda_3 = -0.426$ $\lambda_4 = -0.654$
E	$\theta_1 = \theta_4 = 0$; $\theta_2 = \theta_3 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 2.666$	$\lambda_1 = 0.426$ $\lambda_2 = 0.414$ $\lambda_3 = 0$ $\lambda_4 = -0.227$

TABLE IV. Equilibrium Solutions, Continued

Graph	Guess	Phase Lag (α)	Solution	Eigenvalues
F	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta_4 = \theta$	$\alpha = 0.1$	$\theta = 0$	$\lambda_1 = -0.995$ $\lambda_2 = 0$ $\lambda_3 = -0.995$ $\lambda_4 = -0.995$
F	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta_4 = \theta$	$\alpha = 0.1$	$\theta = \pi$	$\lambda_1 = 0$ $\lambda_2 = 0.995$ $\lambda_3 = 0$ $\lambda_4 = 0$
F	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta_4 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = 0$	$\lambda_1 = -0.479$ $\lambda_2 = 0$ $\lambda_3 = -0.479$ $\lambda_4 = -0.479$
F	$\theta_1 = \theta_2 = 0$; $\theta_3 = \theta_4 = \theta$	$\alpha = \frac{\pi}{2} - 0.5$	$\theta = \pi$	$\lambda_1 = 0$ $\lambda_2 = 0.479$ $\lambda_3 = 0$ $\lambda_4 = 0$

TABLE V. Equilibrium Solutions, Continued