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Uniqueness for a Boundary Identification Problem in Thermal Imaging

Kurt M. Bryan
Rose-Hulman Institute of Technology, bryan@rose-hulman.edu

Lester F. Caudill
University of Richmond

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Kurt Bryan
Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

Lester F. Caudill, Jr.
Department of Mathematics and Computer Science
University of Richmond
Richmond, VA 23173

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Abstract

An inverse problem for a parabolic initial-boundary value problem is considered. The goal is to determine an unknown portion of the boundary of a region in $\mathbb{R}^n$ from measurements of Dirichlet data on a known portion of the boundary. It is shown that under reasonable hypotheses uniqueness results hold.

Key words. inverse problems, non-destructive testing, thermal imaging.

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1 Introduction

The goal of nondestructive evaluation is to gather information about the interior or other inaccessible portion of some material object from exterior measurements. Thermal imaging is one approach to this problem; a prescribed heat flux is applied to a portion of the surface of the object and the resulting surface temperature response is measured. From this information one attempts to determine the internal thermal properties of the object, or the shape of some unknown, inaccessible portion of the boundary. Thermal imaging holds promise as a tool for corrosion detection in aircraft, and has found utility in industrial applications. The interested reader is referred to [2], and the references therein, for a discussion of other applications of thermal imaging.

We are interested in the use of thermal imaging for the detection of so-called "back surface" corrosion and damage. The most elementary model of such a process is simple material loss which leads to a change in the surface profile of the object's boundary. This is the model we have chosen for this paper. Our focus in this work is on the issue of uniqueness—under what conditions do the proposed data measurements provide sufficient information from which to determine the shape of the "back surface?" This problem may be formulated mathematically as an inverse problem for the heat equation. More precisely, let $\Omega \subseteq \mathbb{R}^n$ represent the object to be imaged. We assume that the surface $\partial\Omega$ of Ω is piecewise C^2. We use Γ to denote the "known", accessible portion of $\partial\Omega$, and we assume that both Γ and $\partial\Omega \setminus \Gamma$ have nonzero surface measure as subsets of $\partial\Omega$. Let S_0 denote some open portion of Γ

with positive measure and let the applied heat flux $g(t, x)$ be defined for each $(t, x) \in \mathbb{R}^+ \times \partial\Omega$ with support in S_0. With some rescaling we model the propagation of heat through Ω with an initial-boundary value problem for the heat equation,

$$\begin{aligned} u_t(t,x) - \Delta_x u(t,x) &= 0\,, \quad \text{for } t \in \mathbb{R}^+,\ x \in \Omega\,, &(1.1)\\ \frac{\partial u}{\partial \eta}(t,x) &= g(t,x)\,, \quad \text{for } t \in \mathbb{R}^+,\ x \in \partial\Omega\,, &(1.2)\\ u(0,x) &= u_0(x)\,, \quad \text{for } x \in \Omega. &(1.3) \end{aligned}$$

Here $u(t, x)$ denotes the temperature in the domain Ω at the point x at time t, u_t is the derivative of u with respect to t, and η an outward unit normal vector field on $\partial\Omega$. Throughout this paper, we will refer to (1.1)-(1.3) collectively as (IBVP). Let $S_1 \subset \Gamma$ denote the portion of the boundary on which we take temperature measurements. We consider the following inverse problem: Does knowledge of $u(t, x)$ on S_1 for some time period $t_0 < t < t_1$ uniquely determine $\partial\Omega \setminus \Gamma$? Specifically, suppose $\Omega_1 \subseteq \mathbb{R}^n$ and $\Omega_2 \subseteq \mathbb{R}^n$ with Γ contained in $\partial\Omega_1 \cap \partial\Omega_2$. For $j = 1, 2$, let $u_j(x, t)$ be the solution of (1.1)-(1.3) with Ω replaced by Ω_j. If, $u_1 = u_2$ on $(t_0, t_1) \times S_1$, must it be true that $\Gamma_1 = \Gamma_2$?

Remark. Implicit in the formulation of (IBVP) is the assumption that on the unknown part of the boundary the condition $\frac{\partial u}{\partial \eta} = 0$ holds, so that the back surface acts as a perfect insulator. This is only a first approximation in most situations. In Section 5 we discuss other boundary conditions in which the back surface loses heat to the ambient environment.

The answer to the uniqueness problem posed in the present paper will be seen to depend on certain properties of the domain Ω, the initial condition u_0, and the flux g. We will show that uniqueness holds for constant u_0 and any non-zero flux g. For nonconstant u_0 one can impose reasonable conditions on the flux g to ensure uniqueness, provided Ω is bounded. The case in which the flux g is time-periodic was analyzed in [2].

This paper is organized as follows. The case of constant initial condition and nonconstant flux is analyzed in §2, where uniqueness is proved. In §3 we derive a useful eigenfunction representation and associated estimates for solutions of (IBVP), which are used in §4 to prove a uniqueness result for bounded domains. In §5, we extend our results to include other possibilities for the boundary conditions on Γ.

The fact that uniqueness for the inverse problem fails without additional hypotheses on the ingredients in (1.1)-(1.3) may be illustrated by a simple example in $\mathbb{R}^2$. Let Ω_1 be the rectangle defined by $0 < x < 2\pi,\ 0 < y < \pi$. Let Ω_2 be Ω_1 minus the rectangle $\frac{2}{3}\pi < x < \frac{4}{3}\pi,\ 0 < y < \frac{2}{3}\pi$, so that Ω_1 and Ω_2 share the "known" top boundary $\Gamma = \{(x,y) : 0 < x < 2\pi, y = \pi\}$. Let $u(t,x,y)$ be the function

$$u(t,x,y) = e^{-\frac{9}{2}t}\cos(\frac{3}{2}x)\cos(\frac{3}{2}y)\,,$$

set $u_1 \equiv u|_{\Omega_1}$, and $u_2 \equiv u|_{\Omega_2}$. One may verify directly that, for $j = 1, 2$, $\dfrac{\partial u_j}{\partial \eta} = e^{-\frac{9}{2}t}\cos(\dfrac{3}{2}x)$ on Γ while $\dfrac{\partial u_j}{\partial \eta} = 0$ everywhere else on $\partial\Omega_j$. Both u_1 and u_2 satisfy (1.1) with the same initial condition, with the same Cauchy data on Γ, and so in this case uniqueness fails. Analogous counter-examples can be constructed in other dimensions.

2 Constant Initial Condition

In this section we will develop a uniqueness result for the case in which the initial condition $u_0(x)$ is constant. The only condition on the applied flux $g(t,x)$ is that it be regular enough for IBVP to possess a unique solution, e.g., $g \in C(\mathbb{R}; L^2(\partial\Omega))$.

In what follows, we will require the following lemma.

Lemma 2.1 *Let (u_1, Ω_1) and (u_2, Ω_2) each satisfy (1.1)-(1.3). If $u_1 = u_2$ on $(0,T) \times S_1$ for some time $T > 0$, then $u_1 = u_2$ on $(0,T) \times (\Omega_1 \cap \Omega_2)$.*

In proving this lemma, we will make use of the following unique continuation result for parabolic equations. Its proof is based on the derivation of inequalities of Carleman type, and is omitted here. The interested reader is referred to the work of Saut and Scheurer [6].

Lemma 2.2 *Let Ω be a connected open set in $\mathbb{R}^n$ and $Q = (-T,T) \times \Omega$. Let $u \in L^2\left((-T,T);\, H^2_{loc}(\Omega)\right)$ be a solution of $u_t - \Delta u = 0$ which vanishes in some open subset $\mathcal{O}$ of Q. Then u vanishes in the horizontal component of $\mathcal{O}$.*

Note: Following Nirenberg [5], we define the *horizontal component* $\mathcal{O}_h$ of $\mathcal{O}$ to be the union of all open hyperplanes of the form $t =$ *constant* in Q which have nonempty intersection with $\mathcal{O}$.

Proof of Lemma 2.1. Set $\Omega' \equiv \Omega_1 \cap \Omega_2$ and set $w \equiv u_1 - u_2$. The function w satisfies the parabolic equation

$$
\begin{aligned}
w_t - \Delta w &= 0, \quad \text{on } (0,T) \times \Omega', \\
w = \frac{\partial w}{\partial \eta} &= 0, \quad \text{on } (0,T) \times S_1, \\
w(x,0) &= 0, \quad \text{on } \Omega'.
\end{aligned}
$$

We can choose some open connected subset I with $\bar{I} \subset S_1$ and open ball $B \subset \mathbb{R}^n$ such that $B \cap \partial\Omega' = I$. Let $\Omega_B = B \setminus \Omega'$ set $\tilde{\Omega} \equiv \Omega' \cup \Omega_B$. Define the function

$$
\tilde{w} \equiv \begin{cases} w, & (0,T) \times \Omega'; \\ 0, & (0,T) \times \Omega_B; \\ 0, & (-T,0] \times \tilde{\Omega}. \end{cases}
$$

For a smooth test function ϕ,

$$
\int_0^T \int_{\tilde{\Omega}} \tilde{w}\,[\phi_t + \Delta\phi]\,dxdt = 0\,,
$$

so that $\tilde{w}$ satisfies (1.1) on $\tilde{\Omega}$. Using standard parabolic regularity arguments (see, e.g., [4]), one can show that $\tilde{w} \in H^1\left((-T,T);\, H^2\left(\tilde{\Omega}\right)\right)$.

Make the identifications $Q \equiv (-T,T) \times \tilde{\Omega}$ and $\mathcal{O} \equiv (-T,T) \times \operatorname{int}(\Omega_\epsilon)$ (in this case, the horizontal component of $\mathcal{O}$ is Q) to see that $\tilde{w}$ satisfies the hypotheses of Lemma 2.2. We conclude that $\tilde{w}$ vanishes on Q and so $u_1 = u_2$ on $(0,T) \times \Omega'$. $\square$

We now present the main result of this section.

Theorem 2.1 *Let (u_1, Ω_1) and (u_2, Ω_2) be solutions of (1.1)-(1.3), with $(S_0 \cup S_1) \subseteq (\partial\Omega_1 \cap \partial\Omega_2)$. Suppose $u_0(x) = u_0$, a constant, and suppose that there is some time $T > 0$ for which the applied flux $g(t,x)$ is not identically zero* *on* $(0,T) \times S_0$. *If* $u_1 = u_2$ *on* $(0,T) \times S_1$ *then* $\Omega_1 = \Omega_2$ *and* $u_1 = u_2$ *on* $\Omega_1 = \Omega_2$.

Proof. By replacing u_j with $u_j - u_0$, for $j = 1, 2$, if necessary, it suffices to consider the case $u_0 = 0$. Suppose that $\Omega_1 \neq \Omega_2$. Then there exists some nonempty connected component D of either $\Omega_1 \backslash \Omega_2$ or $\Omega_2 \backslash \Omega_1$. Let us suppose the latter, so that u_2 is defined and satisfies (1.1) on D. The boundary ∂D of D is comprised of a portion Γ_1 of $\partial\Omega_1$ and Γ_2 of $\partial\Omega_2$. On Γ_2 we know that the normal derivative of u_2 is identically zero; on Γ_1, we know that the normal derivative (from inside Ω_1) of u_1 is zero, and since $u_2 \equiv u_1$ on $\Omega' = \Omega_1 \cap \Omega_2$ (by Lemma 2.1) and u_2 is smooth across Γ_1, we conclude that the normal derivative of u_2 vanishes on the boundary of D. Since u_2 satisfies equation (1.1) with zero initial data on D, this forces $u_2 \equiv 0$ on $(0,T) \times D$. Finally, by extending u_2 to be zero on $(-T, 0] \times (\Omega' \cup D)$, we may appeal to Lemma 2.2 to conclude that $u_2 \equiv 0$ on $(0,T) \times (\Omega' \cup D)$. This in turn implies that the flux g is identically zero on $(0,T) \times S_0$, a contradiction, and we must conclude that $\Omega_1 = \Omega_2$, as asserted. $\square$

3 Eigenfunction Expansion

In this section we derive a useful eigenfunction expansion for the function $u(t,x)$ which satisfies the parabolic initial-boundary value problem (IBVP) (1.1)-(1.3). Although the technique is very well known (see, for example, [7], or virtually any text on classical PDE), we will give a brief derivation and some estimates tailored to our needs.

We assume that the function u_0 belongs to $L^2(\Omega)$ and that for all $t > 0$

the function $g(t,x)$ belongs to $C^1((0,T);L^2(\partial\Omega))$, the space of continuously differentiable functions from $(0,T)$ to $L^2(\partial\Omega)$. We seek a solution $u(t,x)$ to (IBVP) in the space $C((0,T);L^2(\Omega))$; for such a solution the derivatives of u with respect to t and x are not well-defined, and so we must cast (IBVP) into a weak form. Multiply equation (1.1) by a smooth test function $\phi(t,x)$ with $\phi(T,x) \equiv 0$ and $\frac{\partial\phi}{\partial\eta} = 0$ on $\partial\Omega$, and then integrate over $(0,T)\times\Omega$. Integrate the term involving ϕu_t by parts in t use Green's second identity on the term involving $\phi \triangle u$ to obtain

$$\int_\Omega u_0(x)\phi(0,x)\,dx + \int_0^T \int_\Omega u\,(\phi_t + \Delta\phi)\;dx\,dt + \int_0^T \int_{\partial\Omega} \phi g\,dS_x\,dt = 0. \quad (3.1)$$

The restriction of the $L^2(\Omega)$ function u to $\partial\Omega$ is not well-defined, but since $\frac{\partial\phi}{\partial\eta} = 0$ on $\partial\Omega$ the boundary integral involving $u\frac{\partial\phi}{\partial\eta}$ vanishes. This is the weak form of (1.1) - (1.3).

In preparation for the eigenfunction expansion, let $\{\lambda_k,\, \psi_k(x)\}$, $k = 0,1,\ldots$ be an eigensystem for $-\Delta$ on Ω with homogeneous Neumann boundary conditions, so that

$$\begin{aligned} \Delta\psi_k + \lambda_k\psi_k &= 0 \text{ in } \Omega, \\ \frac{\partial\psi_k}{\partial\eta} &= 0\,. \end{aligned}$$

The eigenvalues λ_k are non-negative; order them by magnitude, so $\lambda_k \le \lambda_{k+1}$. With the boundary condition $\frac{\partial\psi_k}{\partial\eta} = 0$, the first eigenvalue $\lambda_0 = 0$, is simple, and has a constant eigenfunction. We normalize the eigenfunctions so that $\|\psi_k\|_{L^2(\Omega)} = 1$ for all k, and so obtain an orthonormal basis for $L^2(\Omega)$. The function $\psi_0(x)$ is constant and $\psi_0(x) = 1/\sqrt{|\Omega|}$. Orthogonality of the eigenfunctions then implies that

$$\int_{\Omega} \psi_k(x)\, dx = 0\,, \quad k \geq 1\,.$$

In this and later sections we will make use of the following standard estimate for solutions to Laplace's equation with Neumann boundary conditions.

Lemma 3.1 *Let $f_1 \in L^2(\Omega)$, $f_2 \in L^2(\partial\Omega)$, and let $\psi(x) \in H^1(\Omega)$ satisfy*

$$\begin{aligned} \Delta\psi &= f_1 \ \textit{in}\ \Omega, \\ \frac{\partial\psi}{\partial\eta} &= f_2 \ \textit{on}\ \partial\Omega, \\ \int_{\Omega} \psi(x)\, dx &= 0, \end{aligned} \tag{3.2}$$

Then

$$\|\psi\|_{H^1(\Omega)} \leq C(\|f_2\|_{L^2(\partial\Omega)} + \|f_1\|_{L^2(\Omega)})$$

where C depends on the domain Ω.

Proof. For all $\phi \in H^1(\Omega)$, ψ obeys

$$\int_{\Omega} \nabla\phi \cdot \nabla\psi\, dx = \int_{S_0} \phi f_2\, dS_x - \int_{\Omega} \phi f_1\, dx\,.$$

Set $\phi = \psi$ to obtain

$$\int_{\Omega} |\nabla\psi|^2\, dx = \int_{S_0} \psi f_2\, dS_x - \int_{\Omega} \psi f_1\, dx$$

so that

$$\|\nabla\psi\|^2_{L^2(\partial\Omega)} \leq \|\psi\|_{L^2(\partial\Omega)}\|f_2\|_{L^2(\partial\Omega)} + \|\psi\|_{L^2(\Omega)}\|f_1\|_{L^2(\Omega)}. \tag{3.3}$$

A standard trace inequality ([1]) yields

$$\|\psi\|_{L^2(\partial\Omega)} \leq C_1\|\psi\|_{H^1(\Omega)} \leq C_1\left(\|\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)}\right)\,. \tag{3.4}$$

Since $\int_\Omega \psi \, dx = 0$, we have a Poincaré inequality of the form

$$\|\psi\|_{L^2(\Omega)} \leq C_2 \|\nabla \psi\|_{L^2(\Omega)} , \tag{3.5}$$

where C_1 depends on Ω. The inequalities (3.5) and (3.4), in conjunction with (3.3), yield

$$\|\nabla \psi\|_{L^2(\Omega)} \leq C_1(C_2 + 1)\|f_2\|_{L^2(\partial\Omega)} + C_2\|f_1\|_{L^2(\Omega)} ,$$

which, combined with (3.5), yields the bound

$$\|\psi\|_{H^1(\Omega)} \leq C(\|f_2\|_{L^2(\partial\Omega)} + \|f_1\|_{L^2(\Omega)})$$

for an appropriate constant C. □

The main result of this section is

Lemma 3.2 *The solution $u(t,x)$ to (3.1) is unique in $C((0,T);L^2(\Omega))$, and can be expanded as*

$$u(t,x) = v(t,x) + \frac{d_0}{\sqrt{|\Omega|}} + \frac{1}{|\Omega|}\int_0^t G(s)\, ds + \sum_{k=1}^{\infty} T_k(t)\psi_k(x) \tag{3.6}$$

where $v(t,x)$ defined on $(0,T)\times\Omega$ denotes the unique function which satisfies the family of elliptic problems (indexed by t)

$$\begin{aligned} \triangle_x v &= \frac{1}{|\Omega|}G(t) \;\; \text{in } \Omega, \\ \frac{\partial v}{\partial \eta} &= g(t,x) \;\; \text{on } \partial\Omega, \\ \int_\Omega v(t,x)\, dx &= 0, \end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
G(t) &= \int_{\partial\Omega} g(t,x)\, dS_x, && (3.8)\\
d_k &= \int_\Omega (u_0(x) - v(0,x))\psi_k(x)\, dx && (3.9)\\
c_k(t) &= -\int_\Omega v_t \psi_k(x)\, dx\,, \quad k > 0\,, && (3.10)\\
c_0(t) &= \frac{G(t)}{\sqrt{|\Omega|}} && (3.11)\\
T_k(t) &= d_k e^{-\lambda_k t} + \int_0^t c_k(s) e^{-\lambda_k (t-s)}\, ds, \ \ k > 0 && (3.12)
\end{aligned}
$$

where $|\Omega|$ *denotes the measure of* Ω*,* dS_x *denotes surface measure on* $\partial\Omega$*, and* v_t *is the derivative of* $v(t,x)$ *with respect to* t *(which exists with the given hypotheses). The function* $G(t)$ *is the net rate at which heat enters* Ω *at time* t*. We also have the estimate*

$$
\sum_{k=1}^{\infty} T_k^2(t) \leq 2\left(e^{-2\lambda_1 t}\|u_0\|_{L^2(\Omega)} + \frac{t\|g_t(t,\cdot)\|_{L^2(\partial\Omega)}}{2\lambda_1}\right). \tag{3.13}
$$

Proof.

It is straightforward to show that any function $u(t,x) \in C([0,T] \times L^2(\Omega))$ which satisfies (3.1) for all smooth test functions $\phi(t,x)$ with $\phi(T,x) = 0$ and $\frac{\partial\phi}{\partial\eta} = 0$ is necessarily unique. To see this, set $g \equiv 0$ and $u_0 \equiv 0$. Then equation (3.1) becomes

$$
\int_0^T \int_\Omega u\,(\phi_t + \Delta\phi)\; dx\, dt = 0. \tag{3.14}
$$

Let $f(t,x)$ be any smooth function defined on $[0,T] \times \Omega$, and let $\phi(t,x)$ be a classical smooth solution to the parabolic problem

$$
\begin{aligned}
\phi_t + \triangle\phi &= f \text{ in } \Omega \times (0,T) \\
\frac{\partial \phi}{\partial \eta} &= 0 \text{ on } \partial\Omega \times (0,T), \\
\phi(T,x) &= 0.
\end{aligned}
$$

Note that this is well-posed, since the "initial" condition is specified at time $t = T$ and we solve backwards to $t = 0$. With this ϕ as the test function, equation (3.14) becomes

$$
\int_0^T \int_\Omega u(t,x) f(t,x)\, dx\, dt = 0,
$$

for any smooth f. We conclude that if u is integrable (for example, if $u(t,x) \in C(\mathbb{R}; L^2(\Omega))$) then $u \equiv 0$. By linearity, the solution for any u_0 and g is unique.

It is simple to verify that $u(t,x)$ defined by equation (3.6) represents a formal solution to (3.1) or to (1.1)-(1.3). Conversely, the representation can be derived by expanding the function $w(t,x) = u(t,x) - v(t,x)$ in terms of eigenfunctions, as

$$
w(t,x) = \sum_{k=0}^{\infty} T_k(t)\psi_k(x).
$$

The function w satisfies an initial-boundary value problem similar to (1.1)-(1.3) but with homogeneous boundary conditions. The initial conditions and other inhomogeneous terms, and the properties of the eigenfunctions, lead directly to the expansion (3.6).

We will now show that the function $u(t,x)$ defined by (3.6) lives in the space $C(\mathbb{R}; L^2(\Omega))$ and satisfies equation (3.1). First, the function $v(t,x)$

satisfying the boundary value problem (3.7) satisfies $v(t,\cdot) \in H^1(\Omega)$ (see [3]). Also, under the assumptions on g one can show that the function v_t exists, that $v_t(t,\cdot) \in H^1(\Omega)$, and satisfies

$$\begin{aligned}
\triangle_x v_t &= \frac{1}{|\Omega|} G'(t) \ \text{ in } \Omega, \\
\frac{\partial v_t}{\partial \eta} &= g_t(t,x) \ \text{ on } \partial\Omega, \\
\int_\Omega v_t(t,x)\, dx &= 0.
\end{aligned} \tag{3.15}$$

The above assertion can be proved by approximating v_t with difference quotients: let $v_h(t,x) = (v(t+h,x) - v(t,x))/h$; the function $v_h(t,x)$ satisfies a boundary value like (3.2), with $f_1 = (G(t+h) - G(t))/h$ and $f_2 = (g(t+h,x) - g(t,x))/h$ on the right. The bounds in Lemma 3.1 show that v_h approaches (in $H^1(\Omega)$) some function v_t which satisfies the boundary value problem (3.15).

We claim that the right side of equation (3.6) represents a function in the class $C(\mathbb{R}; L^2(\Omega))$. To prove this we first note that the first three terms on the right-hand side of (3.6) belong to $C(\mathbb{R}; L^2(\Omega))$: The second term is constant in both t and x, while the third is clearly continuous in t and constant in x. In regards to the first term $v(t,x)$, the same elliptic estimates from above allow us to bound the $L^2(\Omega)$ norm of $v(t+h,x) - v(t,x)$ in terms of the $L^2(\partial\Omega)$ norm of $g(t+h,x) - g(t,x)$; if $g(t,x)$ varies continuously as a function of t into $L^2(\partial\Omega)$ then clearly $v(t,x) \in C((0,T) \times L^2(\Omega))$.

We claim that the infinite sum is also in this space. First, for each t the sequence $\{T_k(t)\}$ is in ℓ^2, the Hilbert space of square-summable sequences. To see this note that the sequence $d_k e^{-\lambda_k t}$ is in ℓ^2 for any $t > 0$, since d_k

is square summable. Given the form of $T_k(t)$ it is clear that the sequence $\{T_k(t)\}$ will be in ℓ^2 if and only if the sequence

$$\int_0^t c_k(s)e^{-\lambda_k(t-s)}\,ds$$

is also ℓ^2 for $t > 0$. From equation (3.10) we have

$$\sum_{k=1}^{\infty} c_k^2(t) = \|v_t(t,\cdot)\|^2_{L^2(\Omega)} < \infty.$$

Also, since v_t depends continuously on t, the mapping $t \to \{c_k(t)\}$ is continuous from $\mathbb{R}$ to ℓ^2 and so for $0 < t < T$ we have $\sum_{k=1}^{\infty} c_k^2(t) \le M$ for some constant M; in fact, Lemma 3.1 shows that $\|v_t\|_{L^2(\Omega)} \le C\|g_t\|_{L^2(\partial\Omega)}$ where C depends on Ω. We can then estimate

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(\int_0^t c_k(s)e^{-\lambda_k(t-s)}\,ds\right)^2 &\le \sum_{k=1}^{\infty}\left(\int_0^t c_k^2(s)\,ds\right)\left(\int_0^t e^{-2\lambda_k(t-s)}\,ds\right) && (3.16)\\
&= \sum_{k=1}^{\infty}\left(\int_0^t c_k^2(s)\,ds\right)\left(\frac{1-e^{-2\lambda_k t}}{2\lambda_k}\right)\\
&\le \frac{1}{2\lambda_1}\sum_{k=1}^{\infty}\int_0^t c_k^2(s)\,ds\\
&= \frac{1}{2\lambda_1}\int_0^t \sum_{k=1}^{\infty} c_k^2(s)\,ds && (3.17)\\
&\le \frac{1}{2\lambda_1}\int_0^t M ds = \frac{t}{2\lambda_1}\sup_{0<t<T}\|g_t\|_{L^2(\partial\Omega)}
\end{aligned}$$

where in (3.16) we have used Hölder's inequality, and in (3.17) we have interchanged the integral and summation (permissible since the series converges). We conclude that $\{T_k(t)\}$ is square-summable and so $u(t,x)$ defined by equation (3.6) is in $L^2(\Omega)$ for each fixed t. The continuity of the map $t \to \{c_k(t)\}$ and, as shown above, boundedness of the linear map

$c_k(t) \to \int_0^t c_k(s)e^{-\lambda_k(t-s)}\,ds$ imply that the composition $t \to T_k(t)$ is continuous from $\mathbb{R}$ to ℓ^2, and so $u(t,x) \in C((0,T); L^2(\Omega))$.

We now show that $u(t,x))$ actually represents a solution to (3.1). Let $u_n(t,x)$ denote the series solution (3.6) truncated after n terms. Clearly $u_n(t,\cdot)$ converges to $u(t,\cdot)$ in $L^2(\Omega)$. In fact, $u_n \to u$ in $C((0,T); L^2(\Omega))$, for

$$\|u(t,\cdot) - u_n(t,\cdot)\|_{L^2(\Omega)} = \sum_{k=n+1}^{\infty} T_k^2(t),$$

and so the functions $f_n(t) = \sum_{k=n+1}^{\infty} T_k^2(t)$ are monotonically decreasing to zero, hence converge uniformly to zero as $n \to \infty$. We conclude that

$$\sup_{0<t<T} \|u(t,\cdot) - u_n(t,\cdot)\|_{L^2(\Omega)} \to 0$$

as $n \to \infty$.

Define

$$\begin{aligned} L(f,\phi) &= \int_\Omega u_0(x)\phi(0,x)\,dx + \int_0^T \int_{\partial\Omega} g(t,x)\phi(t,x)\,dS_x\,dt \\ &\quad + \int_0^T \int_\Omega f(t,x)\,(\phi_t + \Delta\phi)\,dx\,dt \end{aligned}$$

where $\phi(t,x)$ is a smooth function on $[0,T]\times\Omega$ with $\phi(T,x) = 0$ and $\frac{\partial\phi}{\partial\eta} = 0$ on $\partial\Omega$. Integrating by parts shows that

$$L(u_n,\phi) = \int_\Omega \phi(0,x)\left(u_0(x) - \sum_{k=1}^{n} d_k\psi_k(x)\right)dx \tag{3.18}$$

$$- \int_0^T \int_\Omega \phi(t,x)\left(v_t + \sum_{k=1}^{n} c_k(t)\psi_k(x)\right)dx\,dt \tag{3.19}$$

We claim that $L(u_n,\phi) \to 0$ as $n \to \infty$. To see this note that the first integral above can be bounded as

$$\int_\Omega \phi(0,x)\left(u_0(x) - \sum_{k=1}^{n} d_k\psi_k(x)\right)dx \le \|\phi(0,x)\|_{L^2(\Omega)}\|u_0 - \sum_{k=1}^{n} d_k\psi_k\|_{L^2(\Omega)}$$

which clearly converges to zero as $n \to \infty$, since the d_k are the Fourier coefficients of $u_0(x)$. The second integral in (3.19) can be bounded as

$$\begin{aligned}\int_0^T \int_\Omega \phi(t,x)(v_t \quad &+ \quad \sum_{k=1}^n c_k(t)\psi_k(x))\,dx\,dt \\ &\le \int_0^T \left(\|\phi(t,\cdot)\|_{L^2(\Omega)} \|v_t + \sum_{k=1}^n c_k(t)\psi_k(x)\|_{L^2(\Omega)} \right) dt \\ &\le \left(\int_0^T \|\phi(t,\cdot)\|_{L^2(\Omega)}\,dt \right) \left(\sup_{0<t<T} \sum_{k=n+1}^{\infty} c_k^2(t) \right) \qquad (3.20)\end{aligned}$$

Again, the sequence of functions $f_n(t) = \sum_{k=n+1}^{\infty} c_k^2(t)$ is monotonically decreasing and converges pointwise to zero, hence $f_n(t)$ converges uniformly to zero on $(0,T)$. We conclude that the right side of equation (3.20) converges to zero. This shows that $L(u_n, \phi)$ converges to zero. But since since $u_n \to u$ in $C((0,T); L^2(\Omega))$ it is easy to check that $L(u_n,\phi) \to L(u,\phi)$, and so we conclude that $L(u,\phi) = 0$. It follows that $u(t,x)$ defined by equation (3.6) represents the unique solution to equation (3.1). $\square$

4 Uniqueness for Bounded Regions

We now consider the more general case in which the initial condition u_0 need not be constant. Here we will assume that Ω is a bounded region. The essential idea in this section is simple. We note from the proof of Theorem 2.1 that if uniqueness fails then there must be some "insulated" region D. Within such a region, heat neither enters nor leaves, so that the average temperature of D cannot increase with time. This is the basis of the argument that follows: intuitively, if the applied flux g pumps enough heat into Ω over a long enough period then no region D can remain at the same average temperature, and so a uniqueness result must hold. We make this physical argument precise below.

Theorem 4.1 *Let $g(t,x)$ denote a flux in the class $C^1(\mathbb{R}; L^2(\partial\Omega))$ supported for $x \in S_0$ with $\|g(t,\cdot)\|_{L^2(\partial\Omega)} \leq M_0$ for all $t > 0$ and $\|g_t(t,\cdot)\|_{L^2(\partial\Omega)} \leq M_1$ for all $t > 0$. Suppose also that $G(t)$ defined by equation (3.8) satisfies $G(t) \geq G_0 > 0$ for all t. Let $u(t,x)$ be the solution to (1.1)-(1.3) or its weak form (3.1) (the initial condition u_0 is not considered known). Then knowledge of $u(t,x)$ for $0 < t < \infty$ and $x \in S_1$ uniquely determines the region Ω and the initial condition u_0.*

Proof of Theorem 4.1. Suppose that u_1 and u_2 are solutions to the weak form (3.1) of (IBVP) on domains Ω_1 and Ω_2, respectively, with initial conditions $u_1(0,x) = u_0(x)$ and $u_2(0,x) = \tilde{u}_0(x)$. We assume that temperature measurements are taken on an open subset $S_1 \subset (\partial\Omega_1 \cap \partial\Omega_2)$ and the same flux $g(t,x)$ applied on an open subset $S_0 \subset (\partial\Omega_1 \cap \partial\Omega_2)$. We will show that there is some time $T > 0$ such that measurements of u_1 and u_2 on $(0,T) \times S_1$ must differ.

Assume that $u_1 \equiv u_2$ on $(0,\infty) \times S_1$, and set $\Omega' \equiv \Omega_1 \cap \Omega_2$. Let $w = u_1 - u_2$. The function w then satisfies

$$\frac{\partial w}{\partial t} - \Delta w = 0\,, \text{ in } \Omega' \times (0,\infty)\,, \tag{4.1}$$

with $w = \frac{\partial w}{\partial \eta} = 0$ on $S_1 \times (0,\infty)$. Let p be a point in S_1 and B a ball centered at p such that $B \cap \partial\Omega' \subset S_1$. Let B_0 denote that portion of B which lies outside Ω'. Define $\tilde{w}(t,x)$ on $\Omega' \cup B_0$ as

$$\tilde{w}(t,x) = \begin{cases} w(t,x), & x \in \Omega' \\ 0 & x \in B_0 \end{cases}$$

Standard regularity results (see [4]) show that $w \in L^2((0,T); H^2(\Omega'))$ for any $T > 0$. Since $w = \frac{\partial w}{\partial \eta} \equiv 0$ on S_1, it is easy to check that $\tilde{w} \in L^2((0,T); H^2(\Omega' \cup B_0))$. The function $\tilde{w}$ vanishes on $B_0 \times (0,\infty)$ and we conclude from Lemma 2.2 (with the minor alteration $-T \to 0$) that $\tilde{w}$ vanishes on $\Omega' \times (0,\infty)$. This shows that $u_1 \equiv u_2$ on $\Omega' \times (0,\infty)$. Also, since (4.1) has a unique solution for given initial and boundary conditions, we conclude that $u_0 = \tilde{u}_0$ on Ω'.

Suppose that $\Omega_1 \neq \Omega_2$. Then either $\Omega_1 \backslash \Omega_2$ or $\Omega_2 \backslash \Omega_1$ contains a nonempty connected component D; we assume the latter, so $D \subset (\Omega_2 \setminus \Omega_1)$. The boundary of D consists of portions of $\partial\Omega_1 \setminus (S_0 \cup S_1)$ and $\partial\Omega_2 \setminus (S_0 \cup S_1)$. On these portions of the boundary the applied flux g is identically zero. Standard regularity results then show that u_2 is a classical solution to the heat equation and smooth on $\bar{D}$, and we therefore have $\frac{\partial u_2}{\partial \eta} \equiv 0$ on ∂D. Since D is bounded and u_2 is smooth,

$$\frac{d}{dt}\int_D u_2(t,x)\,dx = \int_D \frac{\partial u_2}{\partial t}\,dx = \int_D \triangle u_2\,dx = \int_{\partial D} \frac{\partial u_2}{\partial \eta}\,dS_x = 0.$$

The integral $\int_D u_2(t,x)\,dx$ on the left is just the total thermal energy inside D, and since D is insulated this integral must be constant. We will now show that this is impossible for an applied flux $g(t,x)$ of the form specified in the statement of the theorem.

Let $u_2(t,x)$ be expressed via an eigenfunction expansion as in equation (3.6). Integrating over D shows that

$$\int_D u_2(t,x)\,dx = \int_D v(t,x)\,dx + \frac{d_0|D|}{\sqrt{|\Omega_2|}} + \frac{|D|}{|\Omega_2|}\int_0^t G(s)\,ds + \int_D \sum_{k=1}^{\infty} T_k(t)\psi_k(x)\,dx \tag{4.2}$$

where $T_k(t)$ is defined by equation (3.12), d_0 by equation (3.9), and v satisfies (3.7) with Ω replaced by Ω_2. Since $G(t) \geq G_0$ for all t, that integral $\int_0^t G(s)\,ds$ grows at least as fast as $G_0 t$; however, the other terms in the equation can be shown to be $o(t)$ as $t \to \infty$, and this will show that $\int_D u_2(t,x)\,dx$ cannot be constant.

To see that the first integral on the right side of equation (4.2) is bounded in t, note that

$$\begin{aligned} \left|\int_D v(t,x)\,dx\right| &\leq \sqrt{|D|}\|v(t,\cdot)\|_{L^2(D)} \\ &\leq \sqrt{|D|}\|v(t,\cdot)\|_{L^2(\Omega)}, \end{aligned}$$

and apply Lemma 3.1 with the fact that $\|g(t,\cdot)\|_{L^2(\partial\Omega)} \leq M_0$.

The second term in equation (4.2) is constant and, therefore, bounded in t. The last term can be estimated by noting that

$$\begin{aligned} \left|\int_D \sum_{k=1}^{\infty} T_k(t)\psi_k(x)\,dx\right| &\leq \sqrt{|D|}\left\|\sum_{k=1}^{\infty} T_k(t)\psi_k(x)\,dx\right\|_{L^2(D)}, \\ &\leq \sqrt{|D|}\left\|\sum_{k=1}^{\infty} T_k(t)\psi_k(x)\,dx\right\|_{L^2(\Omega)}, \\ &= \sqrt{|D|}\sqrt{\sum_{k=1}^{\infty} T_k^2(t)}. \end{aligned} \tag{4.3}$$

From equation (3.12) and the estimates (3.16)-(3.17) we can bound

$$\begin{aligned} \sum_{k=1}^{\infty} T_k^2(t) &\leq 2\left(\sum_{k=1}^{\infty}(d_k e^{-\lambda_k t})^2 + \left(e^{-\lambda_k t}\int_0^t c_k(s)e^{\lambda_k s}\,ds\right)^2\right) \\ &\leq 2\left(e^{-2\lambda_1 t}\|u_0\|_{L^2(\Omega)}^2 + \frac{CM_1 t}{2\lambda_1}\right) \end{aligned}$$

Combining this with (4.3) shows that

$$\int_D \sum_{k=1}^{\infty} T_k(t)\psi_k(x)\,dx \leq 2\sqrt{|D|}\left(e^{-2\lambda_1 t}\|u_0\|_{L^2(\Omega)}^2 + \frac{M_1 t}{2\lambda_1}\right)^{1/2}. \tag{4.4}$$

The quantity on the right side of (4.4) is clearly $o(t)$, and so grows more slowly than $\int_0^t G(s)\,ds$. Equation (4.2) then shows that for sufficiently large t the integral $\int_D u_2(t,x)\,dx$ must increase, a contradiction that proves Theorem 4.1. $\square$

If in addition to the conditions above g is analytic in t (for example, if g is independent of t, so $g = g(x)$) then we can do better. Suppose that $g(t,x) \in C^\omega((0,T); L^2(\partial\Omega))$, i.e. for each $t_0 > 0$ there is some $\delta > 0$ such that $g(t,x)$ can be written as

$$g(t,x) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} g_k(x)$$

for all t with $|t - t_0| < \delta$, where $g_k \in L^2(\partial\Omega)$. In this case the solution to (1.1)-(1.3) is analytic in t, i.e.,

$$u(t,x) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} u_k(x)$$

where $u_k \in L^2(\Omega)$. Suppose that two domains Ω_1 and Ω_2 give rise to the same temperature measurements on $(t_1,t_2) \times S_1$ with $t_1 < t_2$. Arguing as in the proof of Theorem 4.1 we find that $u_1 \equiv u_2$ on $(t_1,t_2) \times (\Omega_1 \cap \Omega_2)$, but since u_1 and u_2 are analytic in t we have $u_1 \equiv u_2$ on $(0,\infty) \times (\Omega_1 \cap \Omega_2)$. The rest of the proof of Theorem 4.1 remains unchanged and we have

Theorem 4.2 *Let $g(t,x)$ denote a flux in the class $C^\omega(\mathbb{R}; L^2(\partial\Omega))$ supported for $x \in S_0$ with $\|g(t,\cdot)\|_{L^2(\partial\Omega)} \le M_0$ for all $t > 0$ and $\|g_t(t,\cdot)\|_{L^2(\partial\Omega)} \le M_1$ for $t > 0$. Suppose also that $G(t)$ defined by equation (3.8) satisfies $G(t) \ge G_0 > 0$ for all t. Let $u(t,x)$ be the solution to the IBVP (1.1)-(1.3) or its weak form (3.1) (the initial condition u_0 is not considered known). Then* *knowledge of $u(t,x)$ for any open time interval $0 < t_1 < t < t_2$ and $x \in S_1$ uniquely determines the region Ω and the initial condition u_0.*

5 Other Boundary Conditions

The results of the previous section show that we can uniquely identify the unknown portion of the surface of Ω if we pump in enough heat for a long enough time. However, under such conditions the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ becomes less realistic from a physical standpoint. Those portions of the boundary on which a nonzero flux is not applied will tend to lose heat to the surrounding environment. In this section we consider uniqueness results under boundary conditions which model this heat loss. The proofs are quite similar to those of the previous section.

Suppose that $u(t,x)$ satisfies the initial-boundary value problem

$$\frac{\partial u}{\partial t} - \triangle u = 0 \text{ on } \mathbb{R}^+ \times \Omega \tag{5.1}$$

$$\frac{\partial u}{\partial n} + \alpha u = g(t,x), \text{ on } \mathbb{R}^+ \times \partial\Omega \tag{5.2}$$

$$u(0,x) = u_0(x) \text{ on } \Omega \tag{5.3}$$

with $\alpha > 0$ and $S_0 \subset \partial\Omega$. The Robin boundary condition $\frac{\partial u}{\partial n} + \alpha u = 0$ corresponds to a Newton-cooling type of heat loss on the boundary with ambient temperature scaled to zero; note that we have assumed that the loss term $-\alpha u$ applies even on S_0, where the flux g is applied.

The solution u to the initial-boundary value problem (5.1)-(5.3) can be represented with an eigenfunction expansion, as

$$u(t,x) = v(t,x) + \sum_{k=0}^{\infty} T_k(t)\psi_k(x) \tag{5.4}$$

where $v(t,x)$ satisfies the family of elliptic problems (indexed by t)

$$\begin{aligned} \Delta_x v &= 0 \text{ in } \Omega, && (5.5)\\ \frac{\partial v}{\partial n} + \alpha v &= g \text{ on } \partial\Omega, && (5.6) \end{aligned}$$

$T_k(t)$ is defined by

$$T_k(t) = d_k e^{-\lambda_k t} + e^{-\lambda_k t}\int_0^t c_k(s) e^{\lambda_k s}\, ds, \tag{5.7}$$

and

$$\begin{aligned} c_k(t) &= -\int_\Omega v_t \psi_k(x)\, dx, && (5.8)\\ d_k &= \int_\Omega (u_0(x) - v(0,x))\psi_k(x)\, dx && (5.9) \end{aligned}$$

and finally, $\{\lambda_k, \psi_k(x)\}$ is an orthonormal eigensystem for $-\Delta$, i.e., for each k,

$$\begin{aligned} \Delta\psi_k + \lambda_k\psi_k &= 0 \text{ in } \Omega,\\ \frac{\partial \psi_k}{\partial n} + \alpha\psi_k &= 0, \text{ on } \partial\Omega\,. \end{aligned}$$

We order the eigenvalues by magnitude. It is easy to check that all eigenvalues are strictly positive.

As before, we assume that we have measurements of $u(t,x)$ for $x \in S_1 \subset \partial\Omega$.

Theorem 5.1 *Let (u_1, Ω_1) and (u_2, Ω_2) be solutions of (5.1)-(5.3) with $(S_0 \cup S_1) \subseteq (\partial\Omega_1 \cap \partial\Omega_2)$. Suppose that the applied flux $g(t,x) \in C^1(\mathbb{R}; L^2(\partial\Omega))$ and is supported in S_0 for each t. Also, assume that*

1. *$g(t,x)$ is not identically zero.*

2. $g(t,x), \frac{\partial g}{\partial t}(t,x) \geq 0$ *for all* x *and* t.

3. $\left\| \frac{\partial g}{\partial t}(t,\cdot) \right\|_{L^2(\partial\Omega)} \to 0$ *as* $t \to \infty$.

Then $u_1 \equiv u_2$ *on* $\mathbb{R}^+ \times S_1$ *implies that* $\Omega_1 = \Omega_2$.

Proof. Our proof proceeds by contradiction. Suppose $\Omega_1 \neq \Omega_2$. The same reasoning as in the proof of Theorem 2.1 shows that we must have some nonempty region $D \subset \Omega$ (where Ω is Ω_1 or Ω_2) on which

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 \text{ on } \mathbb{R}^+ \times D \\ \frac{\partial u}{\partial n} + \alpha u &= 0, \text{ on } \mathbb{R}^+ \times \partial D \\ u(0,x) &= u_0(x) \text{ on } D \end{aligned}$$

(where u is either u_1 or u_2). We first observe that the solution $u(t,x)$ on D must tend exponentially rapidly to zero as $t \to \infty$. To see this, note that $u(t,x)$ can be expanded on D in terms of eigenfunctions

$$u(t,x) = \sum_{k=0}^{\infty} d_k e^{-\tilde{\lambda}_k t} \tilde{\psi}_k(x) \tag{5.10}$$

with

$$d_k = \int_D u_0(x) \tilde{\psi}_k(x)\, dx.$$

and $\{\tilde{\lambda}_k, \tilde{\psi}_k(x)\}$ is an eigensystem for $-\Delta$ on D with boundary conditions $\frac{\partial \tilde{\psi}_k}{\partial n} + \alpha \tilde{\psi}_k = 0$ on ∂D. Again, the eigenvalues are strictly positive. From the representation (5.10)

$$\begin{aligned} \left| \int_D u(t,x)\, dx \right| &\leq \sqrt{|D|} \|u\|_{L^2(D)} \\ &= \sqrt{|D|} \left(\sum_{k=0}^{\infty} (d_k e^{-\tilde{\lambda}_0 t})^2 \right)^{1/2} \\ &= O(e^{-\tilde{\lambda}_0 t}) \end{aligned} \tag{5.11}$$

where $\tilde{\lambda}_0 > 0$ is the smallest eigenvalue for the above eigensystem.

We shall complete the proof of Theorem 5.1 by contradicting relation (5.11) in the following way: We shall show that, under the hypotheses on g,

(1) $\int_D u(t,x)\,dx \longrightarrow \int_D v(t,x)\,dx$ as $t \to \infty$.

(2) $\int_D v(t,x)\,dx$ is bounded away from zero, uniformly in t.

From these two facts, it is clear that $\int_D u(t,x)\,dx$ must be bounded away from zero, uniformly in t, the desired contradiction to (5.11).

To establish (1), we first show that $\|u-v\|_{L^2(\Omega)} \to 0$ as $t \to \infty$. Note that (5.4) and (5.7) imply

$$\begin{aligned}
\|u-v\|^2_{L^2(\Omega)} &= \sum_{k=0}^{\infty} T_k^2(t) \\
&\leq 2\sum_{k=0}^{\infty}\left(\int_0^t c_k(s)e^{-\lambda_k(t-s)}ds\right)^2 + 2\sum_{k=0}^{\infty} d_k^2 e^{-2\lambda_k t} \\
&= 2\sum_{k=0}^{\infty}\left(\int_0^t c_k(s)e^{-\lambda_k(t-s)}ds\right)^2 + o(1) \qquad (5.12)
\end{aligned}$$

where the last equality follows from the fact that $\lambda_k > 0$ for each k. The integral appearing under the sum on the right can be bounded as

$$\begin{aligned}
\left(\int_0^t c_k(s)e^{-\lambda_k(t-s)}ds\right)^2 &= \left(\int_0^{t/2} c_k(s)e^{-\lambda_k(t-s)}ds + \int_{t/2}^t c_k(s)e^{-\lambda_k(t-s)}ds\right)^2 \\
&\leq 2\left(\int_0^{t/2} c_k(s)e^{-\lambda_k(t-s)}ds\right)^2 \\
&\quad +2\left(\int_{t/2}^t c_k(s)e^{-\lambda_k(t-s)}ds\right)^2 \\
&\leq 2\left(\int_0^{t/2} c_k^2(s)\,ds\right)\left(\int_0^{t/2} e^{-2\lambda_k(t-s)}\,dx\right)
\end{aligned}$$

$$
\begin{aligned}
& +2\left(\int_{t/2}^{t} c_k^2(s)\,ds\right)\left(\int_{t/2}^{t} e^{-2\lambda_k(t-s)}\,dx\right) \\
= \;& 2\left(\frac{e^{-\lambda_k t}-e^{-2\lambda_k t}}{\lambda_k}\right)\int_0^{t/2} c_k^2(s)\,ds \\
& +2\left(\frac{1-e^{-\lambda_k t}}{\lambda_k}\right)\int_{t/2}^{t} c_k^2(s)\,ds \qquad (5.13)
\end{aligned}
$$

From the bounds (5.12) and (5.13) we conclude that

$$
\begin{aligned}
\|u-v\|_{L^2(\Omega)}^2 \;&\leq\; C\sum_{k=0}^{\infty}\left(\frac{e^{-\lambda_k t}-e^{-2\lambda_k t}}{\lambda_k}\int_0^{t/2} c_k^2(s)\,ds+\frac{1-e^{-\lambda_k t}}{\lambda_k}\int_{t/2}^{t} c_k^2(s)\,ds\right) \\
&\leq\; C((e^{-\lambda_0 t}-e^{-2\lambda_0 t})\int_0^{t/2}\sum_{k=0}^{\infty} c_k^2(s)\,ds \\
&\quad +(1-e^{-\lambda_0 t})\int_{t/2}^{t}\sum_{k=0}^{\infty} c_k^2(s)\,ds) \qquad (5.14)
\end{aligned}
$$

for some constant C, where we have interchanged the summation and integral for the convergent series and used that fact that $\lambda_0 < \lambda_k$ for $k > 0$.

Next, note that from equation (5.8) we have $\sum_{k=0}^{\infty} c_k(t)^2 = \|v_t(t,\cdot)\|_{L^2(\Omega)}^2$, where v_t satisfies

$$
\begin{aligned}
\triangle v_t \;&=\; 0 \text{ in } \Omega, \qquad (5.15) \\
\frac{\partial v_t}{\partial \eta}+\alpha v_t \;&=\; g_t \text{ on } \partial\Omega \qquad (5.16)
\end{aligned}
$$

Standard estimates similar to Lemma 3.1 show that we can bound

$$
\|v_t\|_{L^2(\Omega)} \leq C\|g_t\|_{L^2(\partial\Omega)}. \qquad (5.17)
$$

It then follows from (5.14) and (5.17) that

$$
\begin{aligned}
\|u-v\|_{L^2(\Omega)}^2 \;&\leq\; \frac{Ct}{2}((e^{-\lambda_0 t}-e^{-2\lambda_0 t})\sup_{0<s<t/2}\|g_t(s,\cdot)\|_{L^2(\partial\Omega)} \\
&\quad +(1-e^{-\lambda_0 t})\sup_{t/2<s<t}\|g_t(s,\cdot)\|_{L^2(\partial\Omega)}) \qquad (5.18)
\end{aligned}
$$

Since $\|g_t\|_{L^2(\partial\Omega)}$ tends to zero as t increases, it is clear that the right side of (5.18) tends to zero, and so $\|u - v\|_{L^2(\Omega)} \to 0$ as $t \to \infty$. In fact, $\|u - v\|_{L^2(D)} \to 0$ for any $D \subset \Omega$, so that

$$\left| \int_D (u - v)\, dx \right| \leq \|u - v\|_{L^1(D)} \leq \sqrt{|D|}\|u - v\|_{L^2(D)} \to 0 \,,$$

from which we conclude

$$\int_D u(t,x)\, dx \to \int_D v(t,x)\, dx \tag{5.19}$$

as $t \to \infty$, which is (1).

It remains only to establish (2). To this end, let us first consider the case in which $g(t,x) \in C^1(\mathbb{R}; C^2(\partial\Omega))$. Then the function $v(t,\cdot) \in C^2(\bar{\Omega})$ for all t. We have by the maximum principle that the minimum value of $v(t,x)$ on $\bar{\Omega}$ occurs at a point on $\partial\Omega$ at which $\frac{\partial v}{\partial \eta} \leq 0$. At such a point, $\alpha v = g - \frac{\partial v}{\partial \eta} \geq 0$, from which we conclude that $v(t,x) \geq 0$ for $x \in \Omega$. In particular, for any $D \subset \Omega$ we have $\int_D v(t,x)\, dx \geq 0$, with equality if and only if $v(t,x) \equiv 0$. Since $v \equiv 0$ if and only if $g \equiv 0$ (from (5.15)-(5.16)), the hypotheses on g imply that $\displaystyle\int_D v(t,x)\, dx > 0$ for each t. (In particular, $\displaystyle\int_D v(0,x)\, dx > 0$.) Furthermore, since $g_t \geq 0$, the same reasoning shows that $\int_D v_t(t,x)\, dx \geq 0$ for any $D \subset \Omega$ and all $t > 0$. Consequently, for each $t > 0$,

$$\begin{aligned}
\int_D v(t,x)\, dx &= \int_D v(0,x)\, dx + \int_0^t \frac{\partial}{\partial s} \int_D v(s,x)\, dx ds \\
&= \int_D v(0,x)\, dx + \int_0^t \int_D v_t(s,x)\, dx ds \\
&\geq \int_D v(0,x)\, dx > 0 \,,
\end{aligned}$$

where we have used the fact that D is bounded and smooth enough to interchange the order of integration and differentiation. This establishes (2) for $g(t,x) \in C^1(\mathbb{R}; C^2(\partial\Omega))$.

Finally, we note that the same conclusion holds if $g(t,\cdot)$ is merely $L^2(\partial\Omega)$, rather than $C^2(\partial\Omega)$, for we can approximate any non-negative $g \in L^2(\partial\Omega)$ arbitrarily closely (in $L^2(\partial\Omega)$) with a non-negative function $\tilde{g}(t,\cdot) \in C^2(\partial\Omega)$. From the standard estimate

$$\|v - \tilde{v}\|_{L^2(\Omega)} \leq C\|g - \tilde{g}\|_{L^2(\partial\Omega)}$$

where $\tilde{v}$ satisfies the boundary value problem (5.5)-(5.6) with g replaced by $\tilde{g}$, we conclude that $\|v - \tilde{v}\|_{L^2(D)}$ can be made arbitrarily small. Since

$$\left|\int_D v(t,x)\,dx - \int_D \tilde{v}(t,x)\,dx\right| \leq \sqrt{|D|}\|v - \tilde{v}\|_{L^2(D)} \leq C\|g - \tilde{g}\|_{L^2(\partial\Omega)}\,,$$

and since $\int_D \tilde{v}(t,x)\,dx > 0$ uniformly in t, we conclude that $\int_D v(t,x)\,dx > 0$ uniformly in t also. This establishes (2), and completes the proof. $\square$

6 Concluding Remarks

We have examined a variety of settings in which the Cauchy data for the heat equation uniquely determines the shape of the region on which the heat equation is defined. Specifically, if the initial temperature is constant over the region of interest then the Cauchy data—temperature and heat flux—on any open portion of the boundary of the region over any time interval determines the shape of the region. In the case that initial conditions are not constant the Cauchy data on the time interval $(0,\infty)$ uniquely determines the shape of the region, provided that the flux satisfies certain reasonable conditions. For insulate boundary conditions $\frac{\partial u}{\partial \eta} = 0$ the flux $g(t,x)$ must provide a net positive heat flux at all times, bounded away from zero, and g_t must be bounded. For the Robin boundary condition $\frac{\partial u}{\partial \eta} + \alpha u = 0$ the flux should be positive at all points and times.

While a uniqueness result holds, this inverse problem is most certainly ill-posed; the shape of the region will not be a continuous function of the measured data in any reasonable norm. A next logical step is to examine and quantify the nature of the ill-posedness and the kinds of features of the boundary that can be stably estimated from the Cauchy data. This should give insight into useful reconstruction algorithms. Such an algorithm might be based on the ideas in [2]—linearize the forward problem and examine the linearized map from the "boundary shape" space to the measured temperature data. The forward map will be given as an integral operator with smooth kernel, and will of course have an unbounded inverse. We are currently studying such an approach to gain an understanding of stability and reconstruction possibilities.

References

[1] R.A. Adams, *Sobolev Spaces.* New York: Academic Press, 1975.

[2] K. Bryan, L.F. Caudill, Jr. An inverse problem in thermal imaging. *SIAM J. Appl. Math.* **56** (1996), 715–735.

[3] D. Gilbarg, N. Trudinger. *Elliptic Partial Differential Equations of Second Order* (Second Edition). Berlin: Springer-Verlag, 1983.

[4] O. Ladyzhenskaya, V. Solonnikov, N. Uraltseva. *Linear and Quasilinear Equations of Parabolic Type* (Translations of Mathematical Monographs, v. 23). Providence: American Mathematical Society, 1968.

[5] L. Nirenberg. Uniqueness in Cauchy problems for differential equations with constant leading coefficients. *Comm. Pure Appl. Math.* **10** (1957), 89-105.

[6] J.C. Saut, B. Scheurer. Unique continuation for some evolution equations. *J. Differential Equations* **66** (1987), 118-139.

[7] I. Stakgold. *Green's Functions and Boundary Value Problems.* New York: John Wiley & Sons, 1979.