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Abstract

We study the effective behaviour of a periodic array of microscopic cracks inside a homogeneous conductor. Special emphasis in placed on a rigorous study of the case in which the corresponding effective conductivity becomes nearly singular, due to the fact that adjacent cracks nearly touch. It is heuristically shown how thin clusters of such extremely close cracks may macroscopically appear as a single crack. The results have implications for our earlier work on impedance imaging.

0 Introduction

Impedance imaging is a technique that has shown promise as a means for noninvasive testing. The applications range from medical imaging to nondestructive testing of aircraft parts. In practice this technique involves imposing an electric current (or a voltage potential) on the exterior boundary of the object and then measuring the induced voltage potential (or the associated electric current) on the same boundary. From this information one attempts to determine the interior electrical conductivity of the object and thereby to recover information about the object's internal condition.

The usual mathematical formulation of impedance imaging results in an inverse problem, specifically, the recovery of an unknown coefficient in an elliptic partial differential equation from information about the Cauchy data for one or more solutions. This inverse problem is ill-posed; small amounts of noise in the boundary data (electric currents and voltage potentials) can lead to large variations in the estimated conductivity. Frequently, however, one has a certain amount of a priori information about the nature of the conductivity, and the incorporation of this information into the inversion process can serve to stabilize the recovery.

One such example concerns the recovery of a collection of perfectly insulating or perfectly conducting cracks inside an otherwise uniform and isotropic two-dimensional electrical conductor (*cf.*, [1], [3], [6] and [10]). By a "crack" we in general mean an open C^1 curve which does not intersect itself. However, some of the work on continuous dependence estimates and most of the computational work has concentrated on the even more stable situation when the cracks are assumed to be linear (*cf.*, [2], [7], [10], and [12]). One question which naturally arises out of these investigations is how well one can distinguish closely spaced cracks, *i.e.*, to what extent do closely spaced cracks appear as a single crack, and what is the quantitative relation between the crack spacing and the resulting perturbations in the boundary data?

In this paper we provide an answer to this question. We determine the effective behavior of a periodic cluster of small linear cracks and therefore implicitly quantify the effect on the boundary data in terms of the orientation and spacing of the cracks. Our results show that the continuous dependence results for recovering closely spaced cracks from boundary data should be remarkably good; even small gaps between the cracks make their presence strongly felt in the boundary data. Our results are almost entirely rigorous convergence estimates; there is only one point at which we refer to heuristics – it is our firm belief that this passage could also be made rigorous, however, to make the presentation of reasonable length we have abstained from this here.

The organization of the paper is as follows. In the first section we consider a homogeneous electric conductor with a periodic array of linear cracks, and we determine its electrical properties as the array spacing tends to zero. Our analysis makes use of the techniques of *homogenization* and *two-scale* convergence. We find that in the limit the material behaves effectively like an anisotropic conductor whose precise form depends on the orientation and relative spacing of the cracks. The determination of the exact dependence of this effective conductor on the orientation and the relative spacing of the cracks requires some detailed analysis and estimates concerning the so-called "cell" problem. Our main results concerning this depedence are stated as Theorems 2.1 and 2.2, the completely rigorous analysis leading to these results is the subject of sections 3 and 4 and an appendix (dealing with conformal mappings). In section 5 we consider the behavior of a "thin" cluster of cracks inside a larger homogeneous isotropic conductor, and we determine to what extent such a cluster manifests itself as a single conductive or insulating crack. It is at this point in the paper we refer to heuristics, essentially concerning the "locality of homogenization". This section also contains a brief discussion of the implications of our results for the stability of recovery of cracks by impedance imaging.

In this section we will examine the behavior of an electric potential in a material which has an array of periodically spaced insulating cracks. In particular we wish to review the derivation of the effective equations which govern the potential as the array spacing approaches zero. To this end we follow the work of Attouch and Murat ([5]) and that of Allaire ([4]). Let Ω be a bounded region in $\mathbb{R}^2$ with C^2 boundary, and let Y denote $[0,1]^2$, the unit square in $\mathbb{R}^2$. $C^k_\#(Y)$ denotes the space of k times differentiable functions on $\mathbb{R}^2$ which are periodic with period cell Y, while $L^2_\#(Y)$ and $H^1_\#(Y)$ denote the spaces obtained by completing $C^\infty_\#(Y)$ with respect to the usual $L^2(Y)$ and $H^1(Y)$ norms, respectively.

Let σ be a linear crack in Y as illustrated in Figure 1, and let α denote the angle of σ relative to the y_1 axis. We suppose that the cell Y has apriori been centered so that the crack, σ, extends from $y_1 = s$ to $y_1 = 1 - s$ through the point $(1/2, 1/2)$. We will use Y^* to denote the region $Y \setminus \sigma$. We use Σ to denote the set of all periodic integer translates of σ, i.e., all the points in $\mathbb{R}^2$ of the form $(y_1 + j, y_2 + k)$ with $(y_1, y_2) \in \sigma$ and j and k integers.

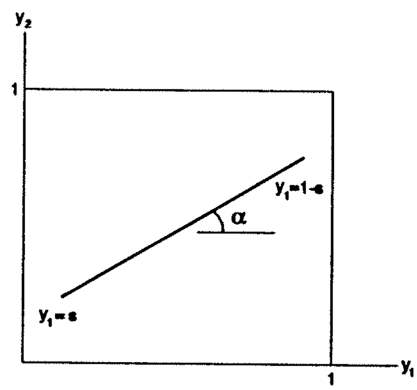

Figure 1: The region $Y^* = [0,1]^2 \setminus \sigma$.

Let Σ_ϵ denote the set of points of the form ϵy, $y \in \Sigma$, with the further proviso that they fall inside a rescaled and translated period cell $(j\epsilon, k\epsilon) + \epsilon Y$ fully contained in Ω. Cracks for which the corresponding ϵ-sized period cells intersect $\partial\Omega$ are excluded, to avoid any problems with the boundary conditions. We define the domain $\Omega_\epsilon = \Omega \setminus \Sigma_\epsilon$ as illustrated in Figure 2:

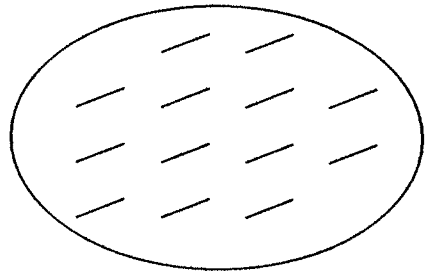

Figure 2: The domain Ω_ϵ.

We assume that the material part of Ω_ϵ has a constant isotropic electrical conductivity equal to one. The cracks are insulating and so block the flow of electrical current. If an input current $g \in L^2(\partial\Omega)$ is applied on the outer boundary of Ω_ϵ then the induced electrostatic voltage potential $u_\epsilon(x)$ will satisfy the elliptic boundary value problem

$$\Delta u_\epsilon = 0 \quad \text{in } \Omega_\epsilon, \tag{1.1}$$

$$\frac{\partial u_\epsilon}{\partial n} = 0 \quad \text{on } \partial\Omega_\epsilon \setminus \partial\Omega, \tag{1.2}$$

$$\frac{\partial u_\epsilon}{\partial n} = g \quad \text{on } \partial\Omega, \tag{1.3}$$

where n denotes an unit normal vector field on $\partial\Omega_\epsilon$, which points outward on $\partial\Omega$; it doesn't matter which direction n points on $\partial\Omega_\epsilon \setminus \partial\Omega$ (the cracks). Since all the cracks have the same orientation we shall indeed take the vectorfield n to be constant on $\partial\Omega_\epsilon \setminus \partial\Omega$.

The function $u_\epsilon(x)$ is unique if we also require the normalization

$$\int_{\Omega_\epsilon} u_\epsilon(x)\, dx = \int_\Omega u_\epsilon(x)\, dx = 0 \quad .$$

The weak formulation of equations (1.1)-(1.3) is that $u_\epsilon \in H^1(\Omega_\epsilon)$ satisfies

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla\phi\, dx = \int_{\partial\Omega} g\phi\, dS_x \tag{1.4}$$

for all functions $\phi \in H^1(\Omega_\epsilon)$. Here dS_x denotes surface measure on $\partial\Omega$.

We are interested in determining the effective electrical properties of the conducting region Ω_ϵ as the period cell size becomes infinitely small. This means examining the behavior of u_ϵ in the limit as ϵ tends to zero. Such an analysis can be carried out following exactly the same technique as that used by Murat and Attouch for an almost identical problem ([5]). One different aspect of the problem studied by Murat and Attouch owes to the fact that they impose a unilateral constraint on the jump of the potential across the cracks. Such a unilateral constraints for the normal displacement is very natural when modeling cracks in elastic media – it is not relevant when u represents a voltage potential. Another difference is that they impose homogeneous Dirichlet boundary conditions on $\partial\Omega$ and a non-zero external force on the right hand side of the equation. None of these differences are essential. The first step following the analysis of Attouch and Murat is to prove that

$$\int_{\Omega}(u_\epsilon)^2 dx \leq C\int_{\Omega_\epsilon}|\nabla u_\epsilon|^2 dx \ . \tag{1.5}$$

The proof of this proceeds exactly as in ([5]) using the same restriction-extension operator as constructed there (a natural generalization of the extension operator used earlier by Cioranescu and Saint Jean Paulin ([8]). The only difference for the present problem is that we must use a Poincaré inequality with $\int_\Omega v\,dx = 0$, as opposed to that with $v = 0$ on $\partial\Omega$. As an almost immediate biproduct of the analysis which leads to (1.5) we also get that

$$\int_{\partial\Omega}(u_\epsilon)^2 dS_x \leq C\int_{\Omega_\epsilon}|\nabla u_\epsilon|^2 dx \ . \tag{1.6}$$

These estimates guarantee the existence of appropriately convergent subsequences. The (unique) limit associated with these subsequences satisfies what amounts to the "standard" homogenized boundary value problem. In order to describe this we need some notation. Let $\chi_k(y) \in H^1_\#(Y^*)$ denote the periodic solution to the "cell" problem

$$\begin{aligned} \Delta_y\chi_k &= 0 \quad \text{in } \mathbb{R}^2\setminus\Sigma \\ \frac{\partial\chi_k}{\partial n} &= -n_k \quad \text{on } \Sigma\ , \end{aligned} \tag{1.7}$$

the weak formulation of which is

$$\int_{Y^*}\nabla\chi_k\nabla w\ dy = -\int_{Y^*}\nabla y_k\nabla w\ dy \quad \forall w\in H^1_\#(Y^*)\ . \tag{1.8}$$

The function χ_k is uniquely determined up to a constant. Let γ denote the symmetric, positive definite 2×2 matrix with entries

$$\begin{aligned} \gamma_{ij} &= \int_{Y^*}\nabla(y_i+\chi_i)\cdot\nabla(y_j+\chi_j)\,dy \\ &= \int_{Y^*}\left(\delta^i_j + \frac{\partial\chi_j}{\partial y_i}\right)\,dy. \end{aligned}$$

Let $u \in H^1(\Omega)$ denote the unique solution to

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega \qquad \gamma \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \,, \tag{1.9}$$

with the normalization $\int_\Omega u \, dx = 0$, and define $u_1 \in L^2(\Omega; H^1_\#(Y^*))$ by

$$u_1 = \frac{\partial u}{\partial x_1}(x)\chi_1(y) + \frac{\partial u}{\partial x_2}(x)\chi_2(y) \,. \tag{1.10}$$

In order to describe the convergence of the solutions u_ϵ and their derivatives we find it convenient to use the notion of two-scale convergence as developed by Nguetseng ([13]) and Allaire ([4]), even though the corresponding result is in a certain sense slightly weaker than that in ([5]). We recall that a sequence of functions $v_\epsilon(x)$ in $L^2(\Omega)$ is said to *two-scale* converge to a function $v_0(x,y) \in L^2(\Omega \times Y)$ if

$$\lim_{\epsilon \to 0} \int_\Omega v_\epsilon(x)\psi(x, x/\epsilon)\, dx = \int_\Omega \int_Y v_0(x,y)\psi(x,y)\, dy\, dx$$

for any function $\psi \in C^\infty(\Omega; C^\infty_\#(Y))$. Here we have deviated a little from the definition given in ([4]), in that we do not require ψ to be compactly supported in Ω – since we only work with sequences, $\{v_\epsilon\}$, that are uniformly bounded in $L^2(\Omega)$ this makes no difference.

Let $\tilde{u}_\epsilon$ and $\tilde{\nabla} u_\epsilon$ denote the extensions of u_ϵ and ∇u_ϵ (by zero, say) onto the entire domain Ω. Note that since Σ_ϵ is a set of measure 0 it is really quite immaterial how these extensions are done. We will use $\eta(y)$ to denote the characteristic function of Y^*. The main theorem concerning the convergence of the u_ϵ is thus

Theorem 1.1 *The sequences $\tilde{u}_\epsilon$ and $\tilde{\nabla} u_\epsilon$ two-scale converge to $u(x)$ and $\nabla u(x) + \eta(y)\nabla_y u_1(x,y)$, respectively, where the pair $(u, u_1) \in H^1(\Omega) \times L^2(\Omega; H^1_\#(Y^*))$ is as defined by (1.9) and (1.10).*

Proof: The weak form of the boundary value problem for u_ϵ, (1.4), together with the estimate (1.6) immediately implies that the $\tilde{\nabla} u_\epsilon$ are uniformly bounded in $L^2(\Omega)$. The estimate (1.5) now guarantees that the same holds for the $\tilde{u}_\epsilon$.

Having established these uniform bounds in $L^2(\Omega)$, the rest of the argument proceeds along the lines of the proof of Theorem 2.9 in [4]. Indeed, an argument identical to that shows that (after possible extraction of subsequences) the sequences $\tilde{u}_\epsilon$ and $\tilde{\nabla} u_\epsilon$ two-scale converge to functions $u_0(x,y) \in L^2(\Omega \times Y)$ and $v(x,y) \in L^2(\Omega \times Y)$, respectively. These functions are of the form

$$u_0(x,y) = u(x), \tag{1.11}$$

$$v(x,y) = \nabla u(x) + \eta(y)\nabla_y u_1(x,y), \tag{1.12}$$

for some function $u(x) \in H^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y^*))$. One small difference is that since our functions u_ϵ do not vanish on $\partial\Omega$, the limit function $u(x)$ in this case is merely in $H^1(\Omega)$, not in $H^1_0(\Omega)$.

With the representations (1.11) and (1.12) (and in particular knowing that $u(x) \in H^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y^*))$) it is not very difficult to find a homogenized system satisfied by u and u_1. Consider the weak form of the ϵ-dependent problem given by equation (1.4), and insert $\phi(x) = \phi_0(x) + \epsilon\phi_1(x, x/\epsilon)$ where $\phi_0 \in C^\infty(\Omega)$ and $\phi_1 \in C^\infty(\Omega; C^\infty_\#(Y))$. This yields

$$\int_\Omega \tilde{\nabla} u_\epsilon \cdot \nabla\phi_0 \, dx + \epsilon \int_\Omega \tilde{\nabla} u_\epsilon \cdot (\nabla_x \phi_1)(x, x/\epsilon)\, dx + \int_\Omega \tilde{\nabla} u_\epsilon \cdot (\nabla_y \phi_1)(x, x/\epsilon)\, dx$$
$$= \int_{\partial\Omega} g(x)\phi_0(x)\, dS_x + \epsilon \int_{\partial\Omega} g(x)\phi_1(x, x/\epsilon)\, dS_x \,.$$

Let now $\epsilon \to 0$ and use the fact that $\tilde{u}_\epsilon$ and $\tilde{\nabla} u_\epsilon$ two-scale converge to $u(x)$ and $\nabla u(x) + \eta(y)\nabla_y u_1(x,y)$ respectively to obtain

$$\int_\Omega \int_{Y*} (\nabla u + \nabla_y u_1) \cdot (\nabla\phi_0 + \nabla_y \phi_1)\, dy\, dx = \int_{\partial\Omega} \phi_0 g \, dS_x \,. \tag{1.13}$$

This represents a natural weak formulation of the following system of equations

$$\nabla_x \cdot \left(\int_{Y*} (\nabla u + \nabla_y u_1)\, dy \right) = 0, \quad \text{in } \Omega \tag{1.14}$$

$$\nabla_y \cdot (\nabla u + \nabla_y u_1) = 0 \quad \text{in } \Omega \times Y^*, \tag{1.15}$$

$$\left(\int_{Y*} (\nabla u + \nabla_y u_1)\, dy \right) \cdot n_x = g \quad \text{on } \partial\Omega, \tag{1.16}$$

$$(\nabla u + \nabla_y u_1) \cdot n_y = 0 \quad \text{on } \Omega \times \sigma. \tag{1.17}$$

From equation (1.15) we find that $\Delta_y u_1 = 0$ in $\mathbb{R}^2 \setminus \Sigma$, since u does not depend on y. Also, from equation (1.17) we have

$$\frac{\partial u_1}{\partial n_y}(x,y)|_{y\in\Sigma} = -\nabla_x u(x) \cdot n_y = -\left(\frac{\partial u}{\partial x_1}(x) n_{y,1} + \frac{\partial u}{\partial x_2}(x) n_{y,2} \right) \quad \text{a.e.} \quad x \in \Omega \,,$$

where $n_{y,k}$ denotes the kth component of the unit normal vector n_y. Linearity now implies that (modulo a function of x)

$$u_1 = \frac{\partial u}{\partial x_1}\chi_1 + \frac{\partial u}{\partial x_2}\chi_2 ,$$

where the χ_k denote the periodic cell-functions defined earlier. It then follows that

$$\nabla_y u_1 = \frac{\partial u}{\partial x_1}\nabla_y\chi_1 + \frac{\partial u}{\partial x_2}\nabla_y\chi_2 .$$

When the above identity is substituted into (1.13) with $\phi_1 = 0$, then we obtain exactly the weak form of the homogenized boundary value problem (1.9) for u. The fact that $\int_\Omega u\, dx = 0$ follows directly from the fact that $\int_\Omega \tilde{u}_\epsilon\, dx = 0$ and the fact that $\tilde{u}_\epsilon$ converge to u weakly in $L^2(\Omega)$, (an immediate consequence of the two-scale convergence). This completes the outline of the proof of the theorem. □

Remark: The statement of the corresponding convergence result following the lines of Attouch and Murat would involve the introduction of the restriction-extension operator, Q^ϵ, constructed in that paper. While this makes the statement of the theorem somewhat more complicated (the reason we didn't follow that approach here) – it also renders it slightly more general. For the present problem we want to mention two consequences of the slightly stronger statement: 1. From the point of view of impedance imaging a slightly more natural normalization (rather than $\int_{\Omega_\epsilon} u_\epsilon\, dx = 0$) would be $\int_{\partial\Omega} u_\epsilon\, dS_x = 0$. The fact that the corresponding homogenized solution, u, obeys the normalization $\int_{\partial\Omega} u\, dS_x = 0$ would follow immediately from the facts that $Q^\epsilon u_\epsilon = u_\epsilon$ on $\partial\Omega$ and that $Q^\epsilon u_\epsilon$ converges to u weakly in $H^1(\Omega)$. 2. Using the operator Q^ϵ it follows fairly immediately that the quadratic form $\int_{\Omega_\epsilon} |\nabla u_\epsilon|^2\, dx$ converges to $\int_\Omega \gamma |\nabla u|^2\, dx$. This is relevant for impedance imaging since the boundary measurements in a certain sense represent measurements of exactly this quadratic form. □

For our subsequent analysis of the properties of the matrix γ it convenient to write it in terms of a single cell-function. Let χ denote the Y-periodic solution to

$$\begin{aligned} \Delta_y \chi &= 0 \quad \text{in } \mathbb{R}^2 \setminus \Sigma , \\ \frac{\partial \chi}{\partial n} &= -1 \quad \text{on } \Sigma . \end{aligned} \tag{1.18}$$

We recall that n is a constant vector on Σ (and as a consequence it also has the same direction on the two "sides" of each crack). We shall pick $n = (n_1, n_2) = (-\sin\alpha, \cos\alpha)$ where α is the angle shown in Figure 1. Based on the above equation and equation (1.7) it is easy to see that $\chi_1 = n_1\chi$ and $\chi_2 = n_2\chi$, so that

$$\begin{aligned}\gamma_{11} &= 1 + n_1\int_{Y^*}\frac{\partial\chi}{\partial y_1}\,dy, \quad \gamma_{12} = n_2\int_{Y^*}\frac{\partial\chi}{\partial y_1}\,dy\ , \\ \gamma_{21} &= n_1\int_{Y^*}\frac{\partial\chi}{\partial y_2}\,dy, \quad \gamma_{22} = 1 + n_2\int_{Y^*}\frac{\partial\chi}{\partial y_2}\,dy\ .\end{aligned}$$

Stated in a variational form χ is the minimizer of the expression

$$E(w) = \int_{Y^*} |\nabla(w + n_k y_k)|^2\,dy \tag{1.19}$$

among all functions $w \in H^1_\#(Y^*)$. Using the Y-periodicity of χ it is not hard to check that

$$\int_{Y^*}\frac{\partial\chi}{\partial y_k}\,dy = -n_k\int_\sigma [\chi](y)\,dy\ \ ,$$

where $[\chi]$ denotes the jump across the crack σ in the direction of the normal vector n (that is, $[\chi](x) = \chi_+(x) - \chi_-(x)$, with $\chi_+(x)$ denoting the limiting value of χ as one approaches x from the side which n points into and with $\chi_-(x)$ denoting the limiting value of χ as one approaches x from the side which n points away from). If we define

$$R = \int_\sigma [\chi](y)\,dS_y\,,$$

then the matrix γ can be written

$$\gamma = \begin{bmatrix} 1 - Rn_1^2 & -Rn_1n_2 \\ -Rn_1n_2 & 1 - Rn_2^2 \end{bmatrix} = I - R\begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix}\ . \tag{1.20}$$

The matrix (other than the identity) appearing in the last expression simply represents the projection onto the one-dimesional subspace of $\mathbb{R}^2$ spanned by n.

The quantity R also depends on the crack separation parameter s and the angle α, and we signify this dependence by writing $R(s,\alpha)$. Fairly simple manipulations show that

$$R(s,\alpha) = \int_\sigma [\chi](y)\,dS_y = \int_{Y\setminus\sigma} |\nabla\chi|^2\,dy = 1 - E(\chi)\ \ , \tag{1.21}$$

where $E(\chi)$ is the energy expression introduced in (1.19).

When $s > 0$ then the set $\mathbb{R}^2 \setminus \Sigma$ is connected and hence the periodic function χ cannot have $\nabla\chi = -n$. This implies that $E(\chi) > 0$. Since χ is also not identically zero it now follows from the last two expressions in (1.21) that $0 < R(s,\alpha) < 1$ for $s > 0$. Through the expression (1.20) this just reaffirms the statement made earlier that γ is positive definite for $s > 0$.

When $s = 0$ but $\alpha \neq k\pi/4$ the set $\mathbb{R}^2 \setminus \Sigma$ remains connected, and we conclude that $0 < R(0,\alpha) < 1$. The matrix γ thus remains positive definite, and the proof of theorem about homogenization convergence still carries through.

For $s = 0$ and $\alpha = k\pi/4$ the cell problem still makes sense and we get $R(0, k\pi/4) = 1$. The matrix γ therefore becomes degenerate. It is given by

$$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

for $k = 0$ and $k = 1$ respectively. Formulas for other values of k follow by periodicity and obvious symmetries. The previous proof of the homogenization theorem no longer carries through in these cases. It does become very interesting to study the behaviour of γ (or equivalently of $R(s,\alpha)$) as (s,α) approaches these points of degeneracy.

A very detailed study of the behaviour of $R(s,\alpha)$ is the focus of the following three sections, and is at the same time the main result of this paper.

In this section we state our main results, which concern the behavious of $R(s,\alpha)$. We are not interested in the case when the cracks become small relative to the cell ($s \to 1/2$) rather we are interested in the case when they reach (or nearly reach) from one side of the cell to the other. We may thus without lack of generality suppose that $0 \le s \le 1/4$. By obvious symmetries it suffices to consider angles in the interval $0 \le \alpha \le \pi/4$. As noted in the last section $0 < R(s,\alpha) < 1$ at all points $(s,\alpha) \in [0,1/4] \times [0,\pi/4] \setminus \{(0,0),\ (0,\pi/4)\}$. Near such points the behaviour of R is very regular. Our first result proves that R is Lipschitz there. The proof of this fact is very simple, as seen in the following section. We are convinced that R is indeed C^∞ at these points – but this fact would be of no relevance for our primary application: the study of the mechanism by which clusters of densely spaced microscopic cracks may become electrostatically equivalent to a single macroscopic crack.

Theorem 2.1 *Suppose that* $(s,\alpha) \to (s_0,\alpha_0)$ *with*

$$0 \le s_0 \le 1/4 \quad \textit{and} \quad \alpha_0 \in [0,\pi/4] \setminus \{(0,0),\ (0,\pi/4)\}\ .$$

Then $R(s,\alpha) \to R(s_0,\alpha_0) \in (0,1)$, *and furthermore one has the estimate*

$$|R(s,\alpha) - R(s_0,\alpha_0)| \le C(|s - s_0| + |\alpha - \alpha_0|)\ , \tag{2.22}$$

with C *independent of* (s,α) *but dependent on* (s_0,α_0).

The second theorem, which is of most relevance for our primary application, concerns the behaviour of $R(s,\alpha)$ near the singular points $(0,k\pi/4)$ (at which points R takes the value 1).

Theorem 2.2 *Suppose that* $0 \le s \le 1/4$ *and* $\alpha \in [0,\pi/4] \setminus \{(0,0),\ (0,\pi/4)\}$, *and suppose that* $(s,\alpha) \to (0,k_0\pi/4)$ *for* $k_0 = 0$ *or* $k_0 = 1$. *Then* $R(s,\alpha) \to 1$, *and furthermore it has the asymptotic form*

$$R(s,\alpha) = 1 + \frac{\pi}{2\ln(s + |\alpha - k_0\pi/4|)} + o\left(\frac{1}{\ln(s + |\alpha - k_0\pi/4|)}\right)\ , \tag{2.23}$$

where $o(\delta)$ *denotes a term which is asymptotically smaller than* $|\delta|$ *as* $\delta \to 0$.

It is quite clear from this theorem that R is continuous at the singular points, but not Hölder continuous of any order. Let us briefly consider the asymptotic behaviour of the matrix γ for (s, α) close to one of the singular points, say $(0,0)$. From (1.20) we get

$$\gamma = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-\pi}{2\ln(s+|\alpha|)} \end{bmatrix} + o\left(\frac{1}{\ln(s+|\alpha|)}\right) , \tag{2.24}$$

for (s, α) sufficiently close to $(0,0)$. Similar formulas may be obtained corresponding to the other singular points. Later, when we discuss the mechanism by which clusters of densely spaced microscopic cracks may become equivalent to a single macroscopic crack, it is exactly this asymptotic formula we shall depend upon.

3 Proof of Theorem 2.1

For the proof of this theorem as well as Theorem 2.2 it is convenient to work with a period cell which is a translate of $Y = [0,1]^2$ by a half unit in the y_1 direction. We define $Q = Y - (1/2, 0)$, and we define

$$Q^*_{s,\alpha} = Q \setminus \Sigma_{s,\alpha}$$

to replace Y^*. To make the dependence on s and α explicit we have added these as indices. The set $Q^*_{s,\alpha}$ is shown in Figure 3.

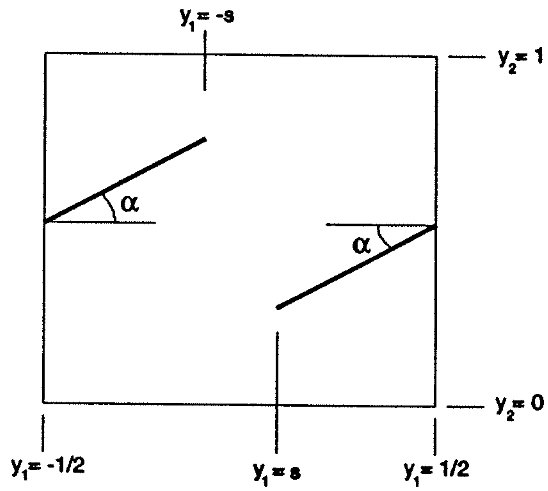

Figure 3

It is not difficult to see that when (s_0, α_0) is in $[0,1/4] \times [0,\pi/4] \setminus \{(0,0),\ (0,\pi/4)\}$ and when (s,α) is in a neighborhood of (s_0,α_0), then the the endpoints of any two succesive cracks remain bounded away from each other. It is therefore not difficult to see that for any (s,α) sufficiently close to $(s_0,\alpha_0) \in [0,1/4] \times [0,\pi/4] \setminus \{(0,0),\ (0,\pi/4)\}$ it is possible to construct a mapping $\Psi_{s,\alpha}$ with the properties that

a) $\Psi_{s,\alpha}$ is a $W^{1,\infty}$ diffeomorphism of $\overline{Q}$ onto $\overline{Q}$, *i.e.*, $\Psi_{s,\alpha}$ along with its inverse are one-to-one $W^{1,\infty}$ mappings of $\overline{Q}$ onto $\overline{Q}$.

b) $\Psi_{s,\alpha}$ maps $Q^*_{s_0,\alpha_0}$ onto $Q^*_{s,\alpha}$.

c) $\Psi_{s,\alpha}(y) = y \quad \forall y \in \partial Q$.

d) There exists a constant C independent of (s,α) (but dependent on (s_0,α_0)) such that

$$\|\Psi_{s,\alpha} - I\|_{W^{1,\infty}(\overline{Q})} \le C(|s - s_0| + |\alpha - \alpha_0|) \,. \tag{3.25}$$

To prove the estimate (2.22) we shall use the formula for R which results from the two identities (1.19) and (1.21), namely

$$\begin{aligned} R(s,\alpha) &= 1 - \min_{w \in H^1_\#(Q^*_{s,\alpha})} \int_{Q^*_{s,\alpha}} |\nabla(w + n_k y_k)|^2 \, dy \\ &= 1 - \min_{v - n_k y_k \in H^1_\#(Q^*_{s,\alpha})} \int_{Q^*_{s,\alpha}} |\nabla v|^2 \, dy \ . \end{aligned} \tag{3.26}$$

Let v_{s_0,α_0} denote the minimizer of the last expression in (3.26), corresponding to the choice of parameters (s_0, α_0), and define the function $\tilde{v}_{s,\alpha}$ by

$$\tilde{v}_{s,\alpha}(\Psi_{s,\alpha}(y)) = v_{s_0,\alpha_0}(y) \quad , y \in Q^*_{s_0,\alpha_0} \ .$$

If we use the notation $\tilde{y} = \Psi_{s,\alpha}(y)$ for coordinates in $Q^*_{s,\alpha}$, then this definition of $\tilde{v}_{s,\alpha}$ leads to the identities

$$\begin{aligned} &\int_{Q^*_{s_0,\alpha_0}} |\nabla_y v_{s_0,\alpha_0}|^2(y) \, dy \qquad (3.27) \\ &= \int_{Q^*_{s,\alpha}} (\nabla_{\tilde{y}} \tilde{v}_{s,\alpha}(\tilde{y}))^t D\Psi_{s,\alpha}(y) D\Psi^t_{s,\alpha}(y) \nabla_y \tilde{v}_{s,\alpha}(\tilde{y}) \frac{1}{|det\,(D\Psi_{s,\alpha})\,|(y)} \, d\tilde{y} \\ &= \int_{Q^*_{s,\alpha}} |\nabla_{\tilde{y}} \tilde{v}_{s,\alpha}(\tilde{y})|^2 \, d\tilde{y} \left[1 + O(|s - s_0| + |\alpha - \alpha_0|)\right] \end{aligned}$$

where, due to the estimate (3.25), the term $O(|s - s_0| + |\alpha - \alpha_0|)$ satisfies

$$|O(|s - s_0| + |\alpha - \alpha_0|)| \leq C(|s - s_0| + |\alpha - \alpha_0|) \ ,$$

with a constant C that is independent of s and α (but dependent on s_0 and α_0). Because of the property **c)** of $\Psi_{s,\alpha}$ immediately follows that $\tilde{v}_{s,\alpha} - n_k y_k \in H^1_\#(Q^*_{s,\alpha})$. Using the variational definition (3.26) of R we now get

$$\begin{aligned} R(s_0,\alpha_0) &= 1 - \int_{Q^*_{s_0,\alpha_0}} |\nabla v_{s_0,\alpha_0}|^2 \, dy \\ &= 1 - \int_{Q^*_{s,\alpha}} |\nabla_{\tilde{y}} \tilde{v}_{s,\alpha}(\tilde{y})|^2 \, d\tilde{y} \left[1 + O(|s - s_0| + |\alpha - \alpha_0|)\right] \\ &\leq 1 - \min_{v - n_k y_k \in H^1_\#(Q^*_{s,\alpha})} \int_{Q^*_{s,\alpha}} |\nabla v|^2 \, dy \left[1 + O(|s - s_0| + |\alpha - \alpha_0|)\right] \\ &= R(s,\alpha) + O(|s - s_0| + |\alpha - \alpha_0|) \ . \end{aligned}$$

In summary this proves that

$$R(s_0,\alpha_0) \leq R(s,\alpha) + C(|s - s_0| + |\alpha - \alpha_0|) \ .$$

By completely similar means we can also prove that

$$R(s,\alpha) \leq R(s_0,\alpha_0) + C(|s-s_0| + |\alpha-\alpha_0|) \quad ,$$

with C independent of s and α, but dependent on s_0 and α_0. A combination of these two inequalities leads to the desired estimate.

4 Proof of Theorem 2.2 for horizontal cracks

In this section we give a proof of Theorem 2.2 in the case when the cracks are horizontal, *i.e.*, when $\alpha = 0$. To be quite explicit we prove

Theorem 4.1 *Suppose that $s \in (0, 1/4]$, and suppose that $s \to 0$. Then $R(s, 0) \to 1$ and furthermore it has the asymptotic form*

$$R(s,0) = 1 + \frac{\pi}{2\ln s} + o\left(\frac{1}{\ln s}\right) , \tag{4.28}$$

where $o(\delta)$ denotes a term which is asymptotically smaller than $|\delta|$ as $\delta \to 0$.

The proof of this theorem consists of three parts. **1)** By use of symmetries we first express $R(s, 0)$ in terms of a function u_s which satisfies a boundary value problem where the original, insulating crack is replaced by an infinitely conducting crack shrinking to zero. **2)** By a conformal transformation u_s may be expressed in terms of the solution, v_s, to a similar problem in which the small crack is replaced by a small disk. **3)** The third, and final step, then consists in a careful analysis of the function v_s, using polar coordinates and separation of variables.

The geometric situation is as illustrated in Figure 4 below. The cracks are the darker horizontal lines; a period cell is as seen at the center of this figure. As in section 3 we place the coordinate system so that this particular translate is $Q = [-1/2, -1/2] \times [0, 1]$. The lines L_1-L_4 in addition to being cell-boundaries, are also lines of symmetry as will be addressed shortly.

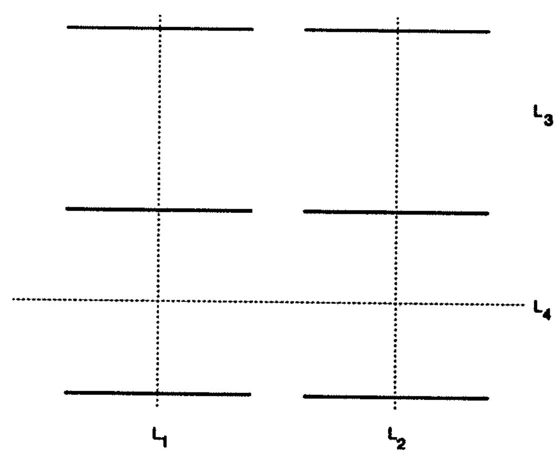

Figure 4

From (1.21) we know that $R(s,0)$ is given by

$$R(s,0) = \int_\sigma [\chi](y)\, dS_y,$$

where σ is a single crack, and χ is the solution to

$$\begin{aligned} \Delta\chi &= 0 \quad \text{on } \mathbb{R}\backslash\{\text{cracks}\}, \\ \frac{\partial\chi}{\partial n} &= -1 \quad \text{on the cracks.} \end{aligned}$$

with $y \to \chi(y)$ periodic, with period 1. If on the cracks we select the constant field n to be $n = (0,1)$, then the boundary condition on the cracks becomes the condition

$$\frac{\partial\chi}{\partial y_2} = -1 \quad \text{on the cracks,}$$

and the jump $[\chi]$ that appears in the above formula for $R(s,0)$ is formed by taking a value above the crack and subtracting the corresponding value below the crack. It is easy to see that the solution $\chi(y)$ is even with respect to the lines L_1 and L_2 and that after subtraction of an appropriate constant it is odd with respect to L_3 and L_4. With this observation we may replace the periodic boundary conditions on the cell Q, and instead characterize χ (which before was only unique up to a constant) as the unique solution to

$$\begin{aligned} \Delta\chi &= 0 \quad \text{on } Q\backslash\{\text{cracks}\}, \\ \chi &= 0 \quad \text{on the top and bottom of } Q, \\ \frac{\partial\chi}{\partial y_2} &= -1 \quad \text{on the cracks,} \\ \frac{\partial\chi}{\partial y_1} &= 0 \quad \text{on the vertical sides of } Q. \end{aligned}$$

The domain $Q\backslash\{\text{cracks}\}$ is shown in Figure 5.

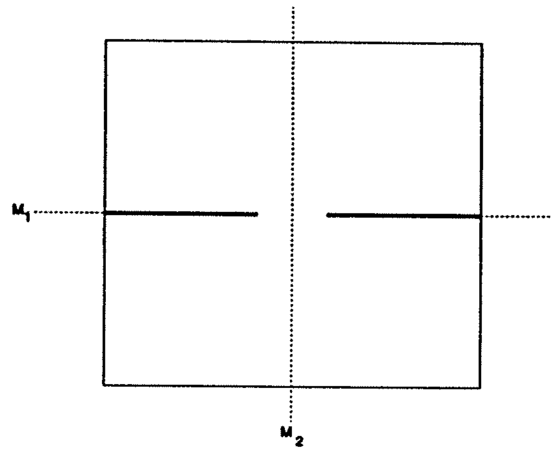

Figure 5

Again, it is easy to see that the solution χ is odd with respect to the line M_1. Let $\tilde{Q}$ denote the rectangle $[-\frac{1}{2},\frac{1}{2}]\times[\frac{1}{2},1]$ (the upper half of the period cell). We are then led to the following characterization of χ

$$\begin{aligned} \Delta\chi &= 0 \text{ in } \tilde{Q}, \\ \chi &= 0 \text{ on } y_2 = 1, \\ \chi &= 0 \text{ on } y_2 = \tfrac{1}{2},\ |y_1| < s, \\ \frac{\partial\chi}{\partial y_2} &= -1 \text{ on } y_2 = \tfrac{1}{2},\ s < |y_1| < \tfrac{1}{2}, \\ \frac{\partial\chi}{\partial y_1} &= 0 \text{ on the sides } y_1 = \pm\tfrac{1}{2}\,. \end{aligned}$$

Our original formula for $R(s,0)$ translates into

$$R(s,0) = \int_\sigma [\chi](y)dS_y = 2\int_{s<|y_1|<1/2} \chi(y_1,\frac{1}{2})dy_1\ , \tag{4.29}$$

due to the fact that χ is odd with respect to the line M_1. Define the function $v(x) = \chi(x) + (y_2 - \frac{1}{2})$; the function v satisfies

$$\begin{aligned} \Delta v &= 0 \text{ in } \tilde{Q}, \\ v &= \frac{1}{2} \text{ on } y_2 = 1, \\ v &= 0 \text{ on } y_2 = \tfrac{1}{2},\ |y_1| < s, \\ \frac{\partial v}{\partial y_2} &= 0 \text{ on } y_2 = \tfrac{1}{2},\ s < |y_1| < \tfrac{1}{2}, \\ \frac{\partial v}{\partial y_1} &= 0 \text{ on the sides } y_1 = \pm\tfrac{1}{2}. \end{aligned}$$

Also,

$$2\int_{s<|y_1|<1/2} \chi(y_1,\frac{1}{2})\,dy_1 = 2\int_{s<|y_1|<1/2} v(y_1,\frac{1}{2})\,dy_1 \ .$$

Let n denote the unit outward normal to $\tilde{Q}$. Integration by parts shows that

$$\begin{aligned} 0=\int_{\tilde{Q}}(y_2-\frac{1}{2})\,\Delta\, v\,dy &= \int_{\partial(\tilde{Q})}(y_2-\frac{1}{2})\frac{\partial v}{\partial n}\,dS_y - \int_{\partial(\tilde{Q})} v\frac{\partial}{\partial n}(y_2-\frac{1}{2})\,dS_y \\ &= \frac{1}{2}\int_{-1/2}^{1/2}\frac{\partial v}{\partial y_2}(y_1,1)\,dy_1 \ -\frac{1}{2}+\int_{s<|y_1|<1/2} v(y_1,\frac{1}{2})\,dy_1 \ , \end{aligned}$$

from which we conclude that

$$2\int_{s<|y_1|<1/2} v(y_1,\frac{1}{2})\,dy_1 = 1-\int_{-1/2}^{1/2}\frac{\partial v}{\partial y_2}(y_1,1)\,dy_1 \ . \tag{4.30}$$

Let $\tilde{P}\subset\tilde{Q}$ denote the rectangle $\tilde{P}=[-1/2,1/2]\times[3/4,1]$, with unit outward normal denoted by n. Integration by parts shows that

$$\begin{aligned} 0=\int_{\tilde{P}}(y_2-\frac{3}{4})\,\Delta\, v\,dy &= \int_{\partial(\tilde{P})}(y_2-\frac{3}{4})\frac{\partial v}{\partial n}\,dS_y - \int_{\partial(\tilde{P})} v\frac{\partial}{\partial n}(y_2-\frac{3}{4})\,dS_y \\ &= \frac{1}{4}\int_{-1/2}^{1/2}\frac{\partial v}{\partial y_2}(y_1,1)\,dy_1 \ -\frac{1}{2}+\int_{-1/2}^{1/2} v(y_1,\frac{3}{4})\,dy_1, \end{aligned}$$

from which we conclude that

$$1-\int_{-1/2}^{1/2}\frac{\partial v}{\partial y_2}(y_1,1)\,dy_1 = 4\int_{-1/2}^{1/2} v(y_1,\frac{3}{4})\,dy_1 \ -1\ . \tag{4.31}$$

A combination of (4.29), (4.30) and (4.31) now gives

$$R(s,0)=2\int_{s<|y_1|<1/2} v(y_1,\frac{1}{2})\,dy_1 = 4\int_{-1/2}^{1/2} v(y_1,\frac{3}{4})\,dy_1 \ -1\ . \tag{4.32}$$

It is not essential here that we have chosen the value 3/4, but for our later applications it is convenient that the last integral be taken on a line with $y_2=c$, with c strictly between 1/2 and 1.

Let u_s denote the function obtained by extending v (defined on $\tilde{Q}$) as an even function across the line $y_2=1/2$. Let $\tilde{\sigma}$ denote the set $\{|y_1|<s,\ y_2=1/2,\}$ in Q. The function u_s is defined on Q, and due to the fact that $\frac{\partial v}{\partial y_2}=0$ along the two line segments $\{s<y_1<1/2,\ y_2=1/2\}$ and $\{-1/2<y_1<s,\ y_2=1/2\}$, we find that u_s satisfies

$$\Delta u_s \ = \ 0 \ \text{ in } Q\setminus\tilde{\sigma},$$

$$
\begin{aligned}
u_s &= \frac{1}{2} \quad \text{on } y_2 = 0 \text{ and } y_2 = 1, \\
u_s &= 0 \quad \text{on } \tilde{\sigma} = \{|y_1| < s,\ y_2 = \frac{1}{2}\}\ , \\
\frac{\partial u_s}{\partial y_1} &= 0 \quad \text{on the sides } y_1 = \pm\tfrac{1}{2}.
\end{aligned}
\tag{4.33}
$$

We note that the strong version of the maximum principle (Hopf's Lemma) asserts that $\partial v/\partial y_2(y_1, 1/2) > 0$ for $|y_1| < s$, $y_2 = 1/2$; the quantity $\partial u_s/\partial y_2$ will thus have a jump across $\tilde{\sigma}$, so that u_s will not be harmonic across $\tilde{\sigma}$. The set $Q \setminus \tilde{\sigma}$ is pictured in Figure 6 below:

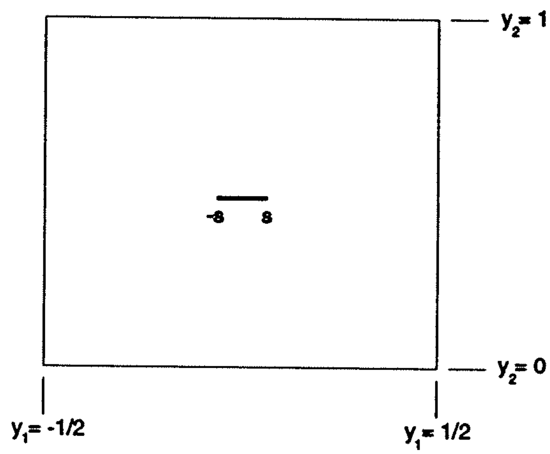

Figure 6

At this point we have completed the first step of the proof of Theorem 4.1, in that we have verified

Lemma 4.1 *Let u_s denote the solution to the boundary value problem (4.33) on $Q\setminus\tilde{\sigma}$. Then*

$$R(s,0) = 4\int_{-1/2}^{1/2} u_s(y_1, \frac{3}{4})\, dy_1 \ - 1\ .$$

We now proceed with the second step, to calculate the quantity $\int_{-1/2}^{1/2} u_s(y_1, \frac{3}{4})\ dy_1$ by means of a conformal mapping.

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ by choosing complex coordinates $z = y_1 + iy_2$; the crack $\tilde{\sigma}$ then corresponds to the the line segment between $z = -s + \frac{1}{2}i$ and $z = s + \frac{1}{2}i$. We will conformally map the region $Q \setminus \tilde{\sigma}$ to a new region in which the crack appears as a circular hole. This geometry allows us to use polar coordinates and separation of variables to examine the behavior of the correponding electrostatic potential.

Let $\phi_1(z) = \frac{z-s}{z+s}$; this conformal map takes $\mathbb{C} \setminus [-s,s]$ to $\mathbb{C} \setminus \{(-\infty,0] \cup \{1\}\}$ and the real interval $(-s,s)$ to the negative real axis. Define $\phi_2(z) = \sqrt{z}$ with the branch cut along the negative real axis. This maps the set $\mathbb{C} \setminus \{(-\infty,0] \cup \{1\}\}$ to the set $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0, z \neq 1\}$, the right half plane minus the point $z = 1$. Finally, define $\phi_3(z) = \frac{1}{2}s\frac{1+z}{1-z}$, which maps this set to the *exterior* of the closed ball $B_0(s/2)$. Let $\phi_s(z) = \phi_3(\phi_2(\phi_1(z - \frac{1}{2}i))) + \frac{1}{2}i$. Returning to real coordinates the conformal mapping $\phi_s(\cdot)$ maps $\mathbb{R}^2 \setminus \tilde{\sigma}$ conformally onto $\mathbb{R}^2 \setminus B_{(0,\frac{1}{2})}(s/2)$. It's easy to check that ϕ_1, ϕ_2 and ϕ_3 are each injective, and so each is invertible on its range. The mapping $\phi_s(\cdot)$ is thus invertible on its range. It's also easy to check that $\phi_s(y) \to y$ as $s \to 0$ for y uniformly away from the point $(0, \frac{1}{2})$, i.e., ϕ_s approaches the identity map away from the point $(0, \frac{1}{2})$.

Let Γ_1 denote the union of the top and bottom of Q and let Γ_2 denote the union of the vertical sides of Q. Define $\Gamma_{i,s} = \phi_s(\Gamma_i)$, $i = 1,2$ and let D_s denote the bounded domain enclosed by $\Gamma_{1,s} \cup \Gamma_{2,s}$. With this definition we get $\phi_s(Q \setminus \tilde{\sigma}) = D_s \setminus B_{(0,\frac{1}{2})}(s/2)$. The function, v_s, given by $v_s(x) = u_s(\phi_s^{-1}(x))$ therefore satisfies the boundary value problem

$$\begin{aligned} \triangle v_s &= 0 \text{ in } D_s \setminus B_{(0,\frac{1}{2})}(s/2) \;, \\ v_s &= \frac{1}{2} \text{ on } \Gamma_{1,s} \;, \\ \frac{\partial v_s}{\partial n} &= 0 \text{ on } \Gamma_{2,s} \;, \\ v_s &= 0 \text{ on } \partial B_{(0,\frac{1}{2})}(s/2) \;. \end{aligned} \tag{4.34}$$

In order to study the asymptotics of v_s (and therefore of u_s) it is convenient to introduce a particular Green's function. For any $x \in D_s$ let $M_s(x,y)$ denote the solution to

$$\begin{aligned} \triangle_y M_s(x,y) &= \delta_x \quad y \in D_s \;, \\ M_s(x,y) &= 0 \quad y \in \Gamma_{1,s} \;, \\ \frac{\partial M_s}{\partial n_y}(x,y) &= 0 \quad y \in \Gamma_{2,s} \;. \end{aligned} \tag{4.35}$$

Later in this section we prove that

Lemma 4.2 *Let v_s denote the solution to (4.34), and let M_s denote the Green's function defined by (4.35). Then*

$$v_s(y) = \frac{1}{2} - M_s((0,\frac{1}{2}),y)\frac{\pi}{\ln s} + O((\ln s)^{-3/2}(y)) \quad ,$$

where for any fixed $\epsilon_0 > 0$ the term $O((\ln s)^{-3/2}(y))$ satisfies

$$|O((\ln s)^{-3/2})(y)| \leq C|\ln s|^{-3/2} \ ,$$

uniformly in $y \in D_s \setminus B_{(0,\frac{1}{2})}(\epsilon_0)$, for s sufficiently small.

Before we give a proof of this lemma we briefly show how, in combination with Lemma 4.1, this immediately leads to a

Proof of Theorem 4.1

Due to the definition of v_s and Lemma 4.2, the function u_s has the form

$$u_s(y) = \frac{1}{2} - M_s((0,\frac{1}{2}),\phi_s(y))\frac{\pi}{\ln s} + O((\ln s)^{-3/2}(y)) \quad , \tag{4.36}$$

where for any fixed $\epsilon_0 > 0$ the term $O((\ln s)^{-3/2}(y))$ satifies

$$|O((\ln s)^{-3/2})(y)| \leq C|\ln s|^{-3/2} \ ,$$

uniformly in $y \in Q \setminus B_{(0,\frac{1}{2})}(\epsilon_0)$, for s sufficiently small. It is quite easy to check that the function $M_s((0,\frac{1}{2}),\phi_s(y))$ is bounded on the line segment $\{y = (y_1,y_2) : -1/2 < y_1 < 1/2 \ , \ y_2 = 3/4\}$ uniformly in s. It is equally easy to see that

$$M_s((0,\frac{1}{2}),\phi_s(y)) \to M_0((0,\frac{1}{2}),y) \quad \text{as } s \to 0$$

on the same line segment $\{y = (y_1,y_2) : -1/2 < y_1 < 1/2 \ , \ y_2 = 3/4\}$. Combining this with Lemma 4.1 and (4.36) we get that

$$\begin{aligned} R(s,0) &= 4\int_{-1/2}^{1/2} u_s(y_1,\frac{3}{4})\,dy_1 \ -1 \\ &= 1 - \frac{4\pi}{\ln s}\int_{-1/2}^{1/2} M_s((0,\frac{1}{2}),\phi_s(y_1,\frac{3}{4}))\,dy_1 + O((\ln s)^{-3/2}) \qquad (4.37) \\ &= 1 - \frac{4\pi}{\ln s}\int_{-1/2}^{1/2} M_0((0,\frac{1}{2}),(y_1,\frac{3}{4}))\,dy_1 + o\left(\frac{1}{\ln s}\right) \ . \end{aligned}$$

Integration by parts yields

$$
\begin{aligned}
1 &= \int_Q \triangle M_0((0,\frac{1}{2}),y)\,dy \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial y_2} M_0((0,\frac{1}{2}),(y_1,y_2))|_{y_2=1}\,dy_1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial y_2} M_0((0,\frac{1}{2}),(y_1,y_2))|_{y_2=0}\,dy_1 \\
&= 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial y_2} M_0((0,\frac{1}{2}),(y_1,y_2))|_{y_2=1}\,dy_1 \ ,
\end{aligned}
$$

where we have used the symmetry of $M_0((0,\frac{1}{2}),y)$ about $y_2=\frac{1}{2}$. Let $\tilde{P}\subset Q$ denote the rectangle $\tilde{P}=\{(y_1,y_2): -\frac{1}{2}<y_1<\frac{1}{2}\ ,\ \ \frac{3}{4}<y_2<1\}$. Integration by parts also yields

$$
\begin{aligned}
0 &= \int_{\tilde{P}} \triangle M_0((0,\frac{1}{2}),y)(y_2-\frac{3}{4})\,dy \\
&= \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial y_2} M_0((0,\frac{1}{2}),(y_1,y_2))|_{y_2=1}\,dy_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} M_0((0,\frac{1}{2}),(y_1,\frac{3}{4}))\,dy_1 \ .
\end{aligned}
$$

A combination of these two formulas gives

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} M_0((0,\frac{1}{2}),(y_1,\frac{3}{4}))\,dy_1 = -\frac{1}{8}\ ,
$$

which after insertion into the last expression of (4.37) yields

$$
R(s,0) = 1 + \frac{\pi}{2\ln s} + o\left(\frac{1}{\ln s}\right)\ ,
$$

exactly as stated in Theorem 4.1. □

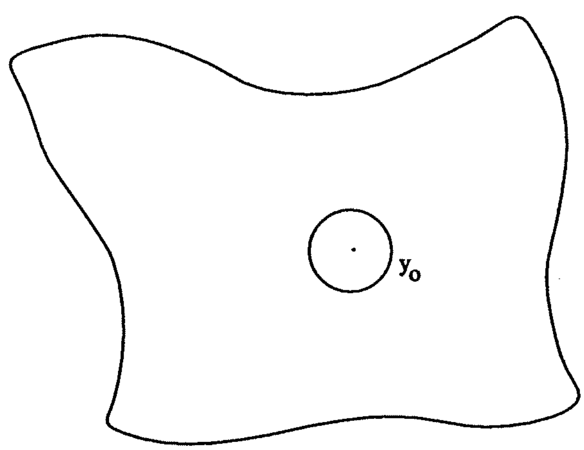

Figure 7

It now only remains to provide a proof of Lemma 4.2. This is the focus of the remainder of this section. We use the notation y_0 for the point $y_0 = (0, \frac{1}{2})$. The domain $D_s \setminus B_{y_0}(s/2)$, with exterior boundary parts $\Gamma_{1,s}$ and $\Gamma_{2,s}$ is schematically shown in Figure 7 above. As before v_s denotes the solution to the boundary value problem (4.34). We shall need a bound of the L^2-norm of the gradient of v_s.

Lemma 4.3 *For s sufficiently small one has the estimate*

$$\int_{D_s\setminus B_{y_0}(s/2)} |\nabla v_s|^2\, dy \le \frac{\pi}{|\ln(s/2)|} .$$

Proof: Let $w_s(y)$ be defined by

$$w_s(y) = \begin{cases} \frac{1}{2} & \text{outside } B_{y_0}(\sqrt{s/2}) \\ -\frac{\ln(2r/s)}{\ln(s/2)} & \text{for } s/2 \le r \le \sqrt{s/2} \end{cases}$$

where $r = |y - y_0|$. It is simple to check that, for sufficiently small s, $w_s(y)$ satisfies $w_s \in H^1(D_s \setminus B_{y_0}(s/2))$, $w_s = 0$ on $\partial B_{y_0}(s/2)$, and $w_s = \frac{1}{2}$ on $\Gamma_{1,s}$. The function v_s, being the solution to the boundary value problem (4.34), is variationally characterized as the minimizer of the expression

$$\int_{D_s\setminus B_{y_0}(s/2)} |\nabla v|^2\, dy$$

in the set $H^1(D_s \setminus B_{y_0}(s/2)) \cap \{v = 0 \text{ on } \partial B_{y_0}(s/2),\ \ v = \frac{1}{2} \text{ on } \Gamma_{1,s}\}$. As a consequence

$$\int_{D_s\setminus B_{y_0}(s/2)} |\nabla v_s|^2\, dy \le \int_{D_s\setminus B_{y_0}(s/2)} |\nabla w_s|^2\, dy . \tag{4.38}$$

It is an easy computation to show that

$$\begin{aligned} \int_{D_s\setminus B_{y_0}(s/2)} |\nabla w_s|^2\, dy &= 2\pi \int_{B_{y_0}(\sqrt{s/2})\setminus B_{y_0}(s/2)} \left|\frac{\partial w_s}{\partial r}\right|^2 r\, dr \\ &= \frac{2\pi}{|\ln(s/2)|^2} \int_{s/2}^{\sqrt{s/2}} \frac{dr}{r} \\ &= \frac{\pi}{|\ln(s/2)|} . \end{aligned} \tag{4.39}$$

A combination of the inequality (4.38) with the formula (4.39) establishes the lemma. □

We shall also need some results concerning the behaviour of v_s on the boundary of $B_{y_0}(s/2)$, as $s \to 0$.

Lemma 4.4 *The following estimates hold for s sufficiently small:*

$$\int_{\partial B_{y_0(s/2)}} \frac{\partial v_s}{\partial r}\, dS = -\frac{\pi}{\ln s} + O(|\ln s|^{-3/2}), \quad \textit{and}$$
$$\int_{\partial B_{y_0(s/2)}} \left|\frac{\partial v_s}{\partial r}\right| dS \le \frac{C}{|\ln s|}.$$

Proof: Since $\Gamma_{1,s}$ is non-trivial, we can use a Poincaré inequality to assert that there exists a constant C such that

$$\int_{D_s} |w|^2\, dy \le C \int_{D_s} |\nabla w|^2\, dy$$

for any $H^1(D_s)$-function, w, which vanishes on $\Gamma_{1,s}$. Since $\Gamma_{1,s}$ varies smoothly with s it is not difficult to see that the constant C may indeed be chosen independently of s. Let $\tilde{v}_s$ denote the function defined by $\tilde{v}_s = v_s - \frac{1}{2}$ in $D_s \setminus B_{y_0}(s/2)$ and $\tilde{v}_s = -\frac{1}{2}$ on $B_{y_0}(s/2)$. The function $\tilde{v}_s$ is in $H^1(D_s)$ (since $v_s \equiv 0$ on $\partial B_{y_0}(s)$), it also vanishes on $\Gamma_{1,s}$, so that the above estimate yields

$$\int_{D_s} |\tilde{v}_s|^2\, dy \le C \int_{D_s} |\nabla \tilde{v}_s|^2\, dy,$$

for some constant C, independent of s. For the remainder of the proof of this lemma it is convenient to introduce the notation $s' = s/2$. Select a fixed $s_0 < 1$ so small that $B_{y_0}(s_0) \subset D_s$ for all s sufficiently close to zero. Since $\nabla \tilde{v}_s \equiv 0$ on $B_{y_0}(s')$, we obtain

$$\begin{aligned}\int_{B_{y_0}(s_0)\setminus B_{y_0}(s')} |v_s - \frac{1}{2}|^2\, dy &\le \int_{D_s\setminus B_{y_0}(s')} |v_s - \frac{1}{2}|^2\, dy \le \int_{D_s} |\tilde{v}_s|^2\, dy \\ &\le C\int_{D_s} |\nabla \tilde{v}_s|^2\, dy = C \int_{D_s\setminus B_{y_0}(s')} |\nabla v_s|^2\, dy, \end{aligned} \tag{4.40}$$

with C independent of s. The point is that the L^2 norm of $v_s - \frac{1}{2}$ on any domain $B_{x_0}(s_0) \setminus B_{y_0}(s')$ can be bounded in terms of the L^2 norm of ∇v_s on $D_s \setminus B_{y_0}(s')$, which is in turn bounded by $\frac{\pi}{|\ln s'|}$ according Lemma 4.3.

The next step is to express the harmonic function $\tilde{v}_s = v_s - \frac{1}{2}$ in an infinite series on the domain $B_{y_0}(s_0) \setminus B_{y_0}(s')$. In polar coordinates around y_0

$$\tilde{v}_s(r,\theta) = (c_0^s - \frac{1}{2}) + d_0^s \ln r + \sum_{k=1}^{\infty}(c_k^s r^k + d_k^s r^{-k})\cos k\theta + \sum_{k=1}^{\infty}(\tilde{c}_k^s r^k + \tilde{d}_k^s r^{-k})\sin k\theta$$

where the $-\frac{1}{2}$ has been explicitly accounted for in the constant term $c_0^s - \frac{1}{2}$. The coefficients depend on s, as indicated by the superscripts. The condition $\tilde{v}_s \equiv -\frac{1}{2}$ on the circle $r = s'(=$

$s/2$) forces

$$\begin{aligned} d_0^s &= -\frac{c_0^s}{\ln s'}, \\ d_k^s &= -c_k^s(s')^{2k} \quad \text{and} \quad \tilde{d}_k^s = -\tilde{c}_k^s(s')^{2k} \quad \text{for} \quad k > 0 . \end{aligned}$$

The expansion for $\tilde{v}_s$ then becomes

$$\begin{aligned} \tilde{v}_s(r,\theta) = \left(c_0^s - \frac{1}{2} - c_0^s\frac{\ln r}{\ln s'}\right) &+ \sum_{k=1}^{\infty} c_k^s(r^k - (s')^{2k}r^{-k})\cos k\theta \\ &+ \sum_{k=1}^{\infty} \tilde{c}_k^s(r^k - (s')^{2k}r^{-k})\sin k\theta. \end{aligned} \tag{4.41}$$

Using this representation we may now integrate $|\tilde{v}_s|^2$ over $B_{y_0}(s_0) \setminus B_{y_0}(s')$ to obtain

$$\begin{aligned} \int_{B_{y_0}(s_0)\setminus B_{y_0}(s')} |\tilde{v}_s|^2\, dy &= \int_0^{2\pi}\int_{s'}^{s_0} |\tilde{v}_s(r,\theta)|^2\, r\, dr\, d\theta, \\ &= 2\pi\int_{s'}^{s_0}\left(c_0^s - \frac{1}{2} - c_0^s\frac{\ln r}{\ln s'}\right)^2 r\, dr \\ &+ \pi\sum_{k=1}^{\infty}((c_k^s)^2 + (\tilde{c}_k^s)^2)\int_{s'}^{s_0}(r^k - (s')^{2k}r^{-k})^2\, r\, dr \end{aligned} \tag{4.42}$$

Here we have used that the sines and cosines form an orthogonal basis in $L^2(0,2\pi)$. Upon deletion of the last terms in equation (4.42) it is clear that

$$2\pi\int_{s'}^{s_0}\left(c_0^s - \frac{1}{2} - c_0^s\frac{\ln r}{\ln s'}\right)^2 r\, dr \le \int_{B_{y_0(s_0)}\setminus B_{y_0(s')}} |\tilde{v}_s|^2\, dy.$$

The estimate (4.40) and Lemma 4.3 now show that

$$2\pi\int_{s'}^{s_0}\left(c_0^s - \frac{1}{2} - c_0^s\frac{\ln r}{\ln s'}\right)^2 r\, dr \le C\int_{D_s\setminus B_{y_0}(s')} |\nabla v_s|^2\, dy \le \frac{\pi C}{|\ln s'|}.$$

A little rearrangement of the leftmost integral in the last inequality yields

$$\int_{s'}^{s_0}\left[\left(c_0^s - \frac{1}{2}\right)\left(1 - \frac{\ln r}{\ln s'}\right) - \frac{1}{2}\frac{\ln r}{\ln s'}\right]^2 r\, dr \le \frac{C}{2|\ln s'|}.$$

As s approaches zero the integral $\int\left[\frac{1}{2}\frac{\ln r}{\ln s'}\right]^2 r\, dr$ is negligible with respect to $\frac{C}{2|\ln s'|}$ and as a consequence we conclude that

$$\int_{s'}^{s_0}\left[\left(c_0^s - \frac{1}{2}\right)\left(1 - \frac{\ln r}{\ln s'}\right)\right]^2 r\, dr \le \frac{C}{2|\ln s'|}.$$