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Abstract

It is shown under very general conditions that the solutions of equations depend continuously on the coefficients or parameters of the equations. The standard examples are solutions of monic polynomial equations and the eigenvalues of a matrix. However, the proof methods apply to any finite map $T : \mathbb{C}^n \to \mathbb{C}^n$.

Contents

1 Introduction

Suppose that we are trying to solve an equation or system of equations that depends continuously on coefficients or other parameters. Two examples that

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come immediately to mind are finding the complex solutions of a standard monic polynomial equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{1}$$

or the eigenvalue equation

$$\det(xI - A) = 0. \tag{2}$$

Examples involving a system of equations are finding the real intersection of a fixed sphere and a general plane

$$\begin{aligned} s^2 + t^2 + u^2 &= 1 \\ as + bt + cu &= d \end{aligned} \tag{3}$$

or the real or complex intersection points of two generic conic sections

$$\begin{aligned} au^2 + buv + cv^2 + du + ev + f &= 0 \\ \alpha u^2 + \beta uv + \gamma y^2 + \delta uv + \epsilon v + \zeta &= 0. \end{aligned} \tag{4}$$

We would certainly hope that the solutions of these equations depend continuously on the coefficients a_i, matrix entries $a_{i,j}$, or the coefficient vectors (a, b, c, d) and $(a, b, \ldots, \epsilon, \zeta)$. The key case for continuity of solutions is case (1), though we formulate and prove continuity theorems in a more general setting. The third and fourth cases are more subtle. In case (3) solutions can have dimension 0 or 1 or vanish completely. In case (4) solutions can escape to infinity, or become a one dimensional curve. We discuss further examples throughout the paper and in Section 4.

For most of the paper, we use methods of elementary algebra, topology and complex analysis. For the work on finite maps in Section 3 we need to use some concepts from algebraic topology.

1.1 Problem setup

To formally solve the continuity problem we will use the concept of *Hausdorff distance,* discussed later in the section, and the following formulation of solving an equation with parameters. Let $f : X \times W \to Y$ be a continuous map. Solving an equation with parameters may be stated as: Given $f : X \times W \to Y$, $w \in W$ (the parameter space) and a fixed $y_0 \in Y$ solve

$$f(x, w) = y_0 \tag{5}$$

i.e., find the solution set

$$S_w = \{x \in X : f(x, w) = y_0\}. \tag{6}$$

So, for example, in the sphere and plane case (3)

$$\begin{aligned} &X = \mathbb{R}^3, Y = \mathbb{R}^2, W = \mathbb{R}^4, \\ &x = (s, t, u), w = (a, b, c, d), y_0 = (1, 0), \\ &f(x, w) = (s^2 + t^2 + u^2, as + bt + cu - d). \end{aligned}$$

The question we which to answer is the following. What are the continuity properties of the map

$$w \to S_w. \tag{7}$$

An alternate and convenient formulation is that of fibres of a continuous map. Set

$$X^W = \{(x, w) : f(x, w) = y_0\} \tag{8}$$

and

$$T : X^W \to W, \ T(x, w) = w, \tag{9}$$

so that

$$T^{-1}(w) = S_w \times w. \tag{10}$$

The map from W to solutions is then given by $\pi_1 \circ T^{-1} : w \to S_w$, and so we can alternatively study

$$w \to T^{-1}(w) \tag{11}$$

for any map $T : X \to W$, where we have dropped the W from X^W or convenience of notation. The interpretation of X should be obvious from context.

Assumption 1 *Throughout this note we are going to make the following assumptions.*

1. *The spaces X, Y, and W are locally compact metric spaces.*
2. *The metrics on X and W satisfy the property that there is an $\epsilon_0 > 0$ such that $\overline{B_\epsilon(x)}$ is compact ($\overline{B_\epsilon(w)}$ is compact) for every $\epsilon \leq \epsilon_0$ and $x \in X$ ($w \in W$).*

These conditions will certainly hold for the cases of interest. Also note that X^W satisfies the same properties.

In the remainder of this section we set up a technical framework that allows us to rigorously define continuous dependence of solutions. In Section 2 we consider the continuity problem in a general topological framework. In Section 3 we consider the case of systems with finitely many algebraic solutions over $\mathbb{R}$ or $\mathbb{C}$, and in Section 4 we consider some further examples.

Acknowledgement This paper was inspired by a talk given at the Indiana MAA section meeting by Vania Mascioni entitled “On the homeomorphism between polynomials and their roots” [Mas].

1.2 Hausdorff distance

We are going use the Hausdorff metric to quantify the notion that two sets are close. Let (X, d) be a metric space, let $A \subseteq X$ be a subset, and let $\epsilon > 0$. Then

$$\begin{aligned} A_\epsilon &= \{x \in X : \exists y \in A, d(x, y) < \epsilon\} \\ &= \bigcup_{x \in A} B_\epsilon(x), \end{aligned}$$

where $B_\epsilon(x)$ the open ϵ-ball about x. Observe that

$$\overline{A} = \bigcap_{\epsilon>0} A_\epsilon.$$

If A and B are two compact subsets of X then the Hausdorff distance between A and B is defined by

$$\begin{aligned} d_H(A, B) &= \inf\{\epsilon : A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon \\ &= \max\left(\max_{x\in A} d(x, B), \max_{y\in B} d(y, A)\right). \end{aligned}$$

So two sets A, B are ϵ close if every point of A is within ϵ of a point B and vice versa. The following is well known.

Proposition 2 *Let X be a metric space and X_H the set of compact, non-empty subsets of X. Then (X_H, d_H) is a metric space. If X is complete then so is (X_H, d_H).*

1.3 Definition of continuous dependence

Definition 3 *Let $f : X \times W \to Y$, respectively $T : X \to W$, be a continuous map. The solution set to equation (5), respectively $T(x) = w$, depends continuously on the parameter $w \in W$ if and only if the map $W \to X_H$ map given by $w \to S_w$, respectively $w \to T^{-1}(w)$, is a well-defined and continuous map.*

Remark 4 *Though our primary interest is in equations with a finite number of solutions the Hausdorff distance allows for an infinite though compact set of solutions. See Example 29.*

2 The topological case

2.1 Sufficient conditions for continuous dependence

The following two examples show how continuity can fail.

Example 5 *Consider finding the real solutions to*

$$\varepsilon x^2 - 2x + 1 = 0$$

with $-\infty < \varepsilon < \infty$. The solutions are

$$\begin{aligned} \frac{1+\sqrt{1-\epsilon}}{\epsilon} &= \frac{2}{\epsilon} - \frac{1}{2} - \frac{1}{8}\varepsilon + O(\varepsilon^2), \\ \frac{1-\sqrt{1-\epsilon}}{\epsilon} &= \frac{1}{2} + \frac{1}{8}\varepsilon + O(\varepsilon^2). \end{aligned}$$

As $\varepsilon \to 0$ we lose the first solution (escapes to infinity) and for $\varepsilon > 1$ we lose both solutions. The second deficiency disappears if we look for complex solutions but there is no way to repair the situation in the following example.

Example 6 *Let* $T : X \to \mathbb{C}^2$ *be the map obtained by blowing up the origin.* $X = \{(w_1, w_2, (x : y)) \in \mathbb{C}^2 \times \mathbb{P}^1 : w_1 y = w_2 x\}$*. The map* T *is given by*

$$T : (w_1, w_2, (x : y)) \to (w_1, w_2).$$

Now $T^{-1}((0,0)) = \{(0,0)\} \times \mathbb{P}^1$ *but otherwise* $T^{-1}(w_1, w_2) = \{(w_1, w_2, (w_1 : w_2))\}$*. The exceptional fibre has infinitely many points but all other nearby fibres are singletons.*

These two examples illustrate the failure modes for continuity of S_w or $T^{-1}(w)$. Pick $w_0 \in W$ and consider $w \to w_0$.

1. The solution set S_w, respectively $T^{-1}(w)$, may have points that are not close to any points of S_{w_0}, respectively $T^{-1}(w_0)$, i.e., solutions escape to infinity.

2. The solution set S_{w_0}, respectively $T^{-1}(w_0)$, may have points that are not close to any points of S_w, respectively $T^{-1}(w)$, i.e., solutions in S_{w_0}, respectively $T^{-1}(w_0)$, vanish upon perturbation.

We can eliminate these failure modes by imposing the following conditions on the map T.

Definition 7 *A map* $T : X \to W$ *is called proper above* $w_0 \in W$ *if there is a compact neighbourhood* V *of* w_0 *such that* $T^{-1}(V)$ *is compact. The map* T *is proper if* $T^{-1}(K)$ *is compact for every compact* $K \subseteq W$*.*

Definition 8 *A map* $T : X \to W$ *is open above* $w_0 \in W$ *if there is an open neighbourhood* U *of* W *such that* $T : T^{-1}(U) \to U$ *is an open map.*

Throughout the remainder of this section we frame our discussion on solutions in terms of the fibres of a map $T : X \to W$.

Theorem 9 *Suppose that* $T : X \to W$ *is a map satisfying.*

1. T is a proper map,

2. T is an open map,

3. $T(X)$ is dense in W (if not take $W = \overline{T(X)}$).

Then, the map $T^{-1} : W \to X_H$ given by $w \to T^{-1}(w)$ is continuous.

Proof. First we show that T is surjective and has compact non-empty fibres. This implies that the map $w \to T^{-1}(w) \in X_H$ is well defined. Let $w_0 \in W$. By the local compactness of W there is a decreasing sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$, $\epsilon_n \to 0$, and the closed ball is $\overline{B_{\epsilon_n}(w_0)}$ compact. Now

$$T^{-1}(w_0) = \bigcap_{n>0} T^{-1}(\overline{B_{\epsilon_n}(w_0)})$$

since every x on the right hand intersection must satisfy $d(T(x), w_0) \leq \epsilon_n$ for all n. Since $T(X)$ is dense in W and T is proper then $T^{-1}(\overline{B_{\epsilon_n}(w_0)})$ is a non-empty compact set and hence the intersection on the right is a non-empty compact set. Thus $T^{-1} : W \to X_H$ is well defined.

Now we prove that T^{-1} is continuous. Seeking a contradiction, suppose that T^{-1} is not continuous. Then we have a sequence $w_n \to w_0$ with $d_H(T^{-1}(w_n), T^{-1}(w_0)) \geq \epsilon$. This means that for all n we have either

$$\exists x_n \in T^{-1}(w_n) \text{ such that } d(x_n, T^{-1}(w_0)) \geq \epsilon \tag{12}$$

or

$$\exists x'_n \in T^{-1}(w_0) \text{ such that } d(x'_n, T^{-1}(w_n)) \geq \epsilon, \tag{13}$$

corresponding to the two failure modes. Passing to a subsequence we may assume that either condition 12 is valid for all n or condition 13 is valid for all n. Suppose condition 12 is valid for all n. By dropping a finite number of terms we may assume that the sequence $\{x_n\} \subseteq T^{-1}(U)$ for some compact neighbourhood U of w_0. The sequence then has a convergent subsequence and by passing to a subsequence we may assume that we have a limit $x_n \to x_0$, so $d(x_n, x_0) \to 0$. Since $T(x_n) = w_n$ then $x_0 \in T^{-1}(w_0)$ by continuity. But

$$\liminf d(x_n, x_0) \geq \liminf d(x_n, T^{-1}(w_0)) \geq \epsilon > 0$$

a contradiction.

Now Suppose condition 13 is valid for all n. Then by a similar argument we have $x'_n \to x_0 \in T^{-1}(w_0)$, and by the triangle inequality

$$d(x_0, T^{-1}(w_n)) \geq d(x'_n, T^{-1}(w_n)) - d(x'_n, x_0)$$

Since $d(x'_n, x_0) < \epsilon/2$ for all large n we get $d(x_0, T^{-1}(w_n)) \geq \epsilon/2$ for all large n. But T is open and so the image $T(B_{\epsilon/4}(x_0))$ is an open set V containing w_0. It follows there are infinitely many $w_n \in V$ and hence that $T^{-1}(w_n) \cap B_{\epsilon/4}(x_0)$ is non-empty. This contradicts $d(x_0, T^{-1}(w_n)) \geq \epsilon/2$ for all large n. ■

Remark 10 *The proof actually shows that the properness of T implies that it is surjective.*

Remark 11 *The property of being proper guarantees that we do not lose solutions at $w_n \to w_0$.. i.e., failure mode 1 above. The property of being open guarantees that we do not lose solutions when we consider a perturbation w near w_0, namely failure mode 2. The next three examples show how both properties are necessary.*

Example 12 *Consider the set $X \subseteq \mathbb{C}^2 = \{(x, w) : x(xw - 1) = 0\}$. Let T be the map $(x, w) \to w$. For $w \neq 0$, $T^{-1}(w) = \{0, 1/w\}$ and $T^{-1}(0) = \{0\}$. The map T is open but not proper and we lose solutions as $w \to 0$.*

Example 13 *Consider example 6 again. The map T is proper but fails to be open at the points on the exceptional fibre $T^{-1}((0, 0))$.*

Example 14 *Here is a variant of the previous example which has finite fibres generically and one infinite compact fibre. The total space X is a Möbius band with an even number of twists. Let $\Delta \subseteq \mathbb{C}$ be the closed unit disc and let $X \subseteq \partial\Delta \times \Delta$ be the set*

$$X = \{(x, w) \in \partial\Delta \times \Delta : \operatorname{Im}(x^s \overline{w}) = 0\},$$

where s is some integer, and let $T : X \to \Delta$ be the map $(x, w) \to w$. Now $T(x, w) = w$ if and only if w is a real multiple of x^s. If $w \neq 0$ then $x^s = \pm w/|w|$ and there are $2|s|$ solutions. If $w = 0$ then $T^{-1}(w) = \partial\Delta \times \{0\}$. The map T is clearly continuous and proper but not open.

Theorem 9 is actually an equivalence under weak conditions as we show in the next proposition.

Proposition 15 *Suppose that X, Y satisfy the conditions given in Assumption 1. Suppose also that $T : X \to W$ is surjective map with compact fibres, Then the map $T^{-1} : W \to X_H$ given by $w \to T^{-1}(w)$ is continuous if and only if T is a proper, open map.*

Proof. One direction has already been shown. Suppose that the T^{-1} is continuous. First we show that T is proper. It suffices to show that for every $w_0 \in W$ that there is compact neighbourhood V of w_0 such that $T^{-1}(V)$ is compact. Let $\epsilon > 0$ be such that $\overline{B_\epsilon(x)}$ is compact for every $x \in X$. By compactness, there is a finite set $\{x_1, \ldots, x_k\} \subseteq T^{-1}(w_0)$ such that

$$T^{-1}(w_0) \subseteq \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i).$$

This implies that the closure of the $\epsilon/2$ neighbourhood $\overline{(T^{-1}(w_0))_{\epsilon/2}}$ is compact. For, if $x \in \left(T^{-1}(w_0)\right)_{\epsilon/2}$ then there is an $x' \in T^{-1}(w_0)$ such that $d(x, x') < \epsilon/2$, and in turn there is an x_i such that $d(x', x_i) < \epsilon/2$. By the triangle inequality $d(x, x_i) < \epsilon$ and $x \in \bigcup_{i=1}^{n} \overline{B_\epsilon(x_i)}$ which is a compact set. Now, by continuity of T^{-1}, there is a $\delta > 0$ such that $\overline{B_\delta(w_0)}$ is compact and $d(w, w_0) < \delta$ implies that $d_H(T^{-1}(w), T^{-1}(w_0)) < \epsilon/2$. From the definition of d_H we have $T^{-1}(w) \subseteq \left(T^{-1}(w_0)\right)_{\epsilon/2} \subseteq \overline{(T^{-1}(w_0))_{\epsilon/2}}$. Therefore every point of $T^{-1}\left(\overline{B_{\delta/2}(w_0)}\right)$ lies in $\overline{(T^{-1}(w_0))_{\epsilon/2}}$ a compact set. It follows that T is proper

Next we show that T is open. It suffices to show that for every $x_0 \in X$ and $\delta > 0$ that $T(B_\delta(x_0))$ contains some ball $B_\epsilon(w_0)$ about $w_0 = T(x_0)$. By continuity there is an $\epsilon > 0$ such that for $w \in B_\epsilon(w_0)$, $d_H(T^{-1}(w), T^{-1}(w_0)) < \delta$. This implies that there is a point $x' \in T^{-1}(w)$ such that $d(x', x_0) < \delta$. But then $w = T(x')$ lies in the image $T(B_\delta(x_0))$ and hence $T(B_\delta(x_0))$ contains $B_\epsilon(w_0)$, showing that T is open. ■

Remark 16 *The proof shows that if $T : X \to W$ is proper and open above $w_0 \in W$ then $w \to T^{-1}(w)$ is continuous at w_0.*

2.2 Continuous dependence in terms of quotient maps

Let $T : X \to W$ be any map. We can define an equivalence relation $\backsim$ on X by $x \backsim y$ if and only if $T(x) = T(y)$. The quotient space X/T is the standard quotient space $X/\backsim$ whose points are the equivalence classes of $\backsim$, i.e., the fibres of T. Let $q : X \to X/T$ be the quotient map. Let us also assume that T is continuous and surjective, and has compact fibres. There are two ways to topologize X/T.

1. The standard quotient topology in which open sets are those sets $U \subseteq X/T$ such that the union of equivalence classes $\bigcup_{u \in U} u$ is open in X, i.e., $q^{-1}(U)$ is open in X.
2. The topology induced by the Hausdorff metric on compact subsets of X, called the Hausdorff topology. This only makes sense if T has compact fibres.

We have a diagram of maps

$$\begin{array}{ccc} X & & \\ \downarrow q & \searrow & \\ X/T & \xrightarrow{p} & W \end{array}$$

where, $q : x \to \overline{x} = T^{-1}(T(x))$ is the canonical projection and $p(u) = T(x)$ for any $x \in u$. Note that p is a bijection and identifies X/T with W. The map q is automatically continuous for the quotient topology on X/T and the quotient topology is the finest topology on X/T for which q is continuous, it need not be continuous for the Hausdorff topology. The map p is continuous for both topologies. For, if $U \subseteq W$ is open then $p^{-1}(U) = q(T^{-1}(U))$ which is open in quotient topology. On the other hand if $u_n \to u_0$ in X/T in the Hausdorff topology then there are $x_n \in u_n$ and $x_0 \in u_0$ such that $x_n \to x_0$ and $\lim_{n\to\infty} q(u_n) = \lim_{n\to\infty} T(x_n) = T(x_0) = q(u_0)$.

We would like to know conditions under which p is a homeomorphism, for the two different topologies. This would give us an alternative formulation of continuous dependence of preimages and solutions. We have already answered the questions for the Hausdorff topology. The complete conditions are given in the following proposition.

Proposition 17 *Let X and W be locally compact metric spaces and let $T : X \to W$ be a continuous surjective map with compact fibres. Let X/T be the quotient space defined above and $p : X/T \to W$ the canonical bijection. Then*

1. *If T is an open map, then p is a homeomorphism with respect to the standard quotient topology on X/T.*
2. *If T is a proper, open map then then p is a homeomorphism with respect to the Hausdorff topology on X/T.*

3. *If T is a proper map, then the Hausdorff topology of X/T is finer than the quotient topology of X/T.*

4. *If T is a proper, open map, then the quotient topology of X/T and the Hausdorff topology of X/T are the same.*

Proof. Statement 1. We need to show that for each open set $U \subseteq X/T$ that $p(U)$ is open in W. But by definition $q^{-1}(U)$ is open in X and $p(U) = T(q^{-1}(U))$ which is open if T is an open map.

Statement 2. Assume that T is a proper, open map. It was shown in the proof of Theorem 9 that $T^{-1} : W \to X_H$, $w \to T^{-1}(w)$ was continuous. This is equivalent to p^{-1} being continuous for the Hausdorff topology. Since p is continuous, then p is a homeomorphism.

Statement 3. A set $U \subseteq X/T$ is open in the quotient topology if and only if $q^{-1}(U) = T^{-1}(p(U))$ is open. A set $U \subseteq X/T$, open in Hausdorff topology, has the following characterization. For each $u_0 \in X/T$ and $\epsilon > 0$ let $w_0 = p(u_0)$ and set

$$\begin{aligned} B_{X/T}(u_0, \epsilon) &= \{u \in X/T : d_H(u, u_0) < \varepsilon\} \\ &= p^{-1}\left\{w \in W : d_H(T^{-1}(w), T^{-1}(w_0)) < \epsilon\right\}. \end{aligned}$$

A set is open in the Hausdorff topology if and only if is a union of $B_{X/T}(u_0, \epsilon)$ for various u_0 and ϵ.

To show that the Hausdorff topology of X/T is finer than the quotient topology of X/T, when T is proper, we proceed as follows. Select $U \subseteq X/T$, open in the quotient topology, we must show that it is open in the Hausdorff topology, i.e., for each $u_0 \in U$ we must find an $\epsilon > 0$ such that $B_{X/T}(u_0, \epsilon) \subseteq U$. As U is open in the quotient topology then $T^{-1}(p(U))$ is open, hence $V = T^{-1}(p(U))$ is an open subset of X containing $T^{-1}(w_0)$. We are first going to show that V contains $T^{-1}(B_\eta(w_0))$ for some small η, from which it follows that $B_{X/T}(u_0, \epsilon) \subseteq U$. Let $\{\eta_n\}$ be a decreasing sequence of positive numbers with limit 0. We may suppose that $T^{-1}(\overline{B_{\eta_n}(w_0)})$ is compact for all n. If none of the $T^{-1}(\overline{B_{\eta_n}(w_0)})$ lie in V then there is $x_n \in (X - V) \cap T^{-1}(\overline{B_{\eta_n}(w_0)})$ for every n. Since $T^{-1}(\overline{B_{\eta_1}(w_0)})$ is compact and the sequence $\{x_n\}$ lies in the compact set $(X - V) \cap T^{-1}(\overline{B_{\eta_1}(w_0)})$, then there is a convergent subsequence of $\{x_n\}$. Passing to the subsequence, we may assume that $\{x_n\}$ is convergent with limit x_0 in $(X - V) \cap T^{-1}(\overline{B_{\eta_1}(w_0)})$. But $d(T(x_n), w_0) \leq \eta_n$ so $T(x_0) = \lim_{n\to\infty} T(x_n) = w_0$. Thus $x_0 \in T^{-1}(w_0) \subseteq V$, a contradiction.

Now we show that $q^{-1}(B_{X/T}(u_0, \epsilon)) \subseteq T^{-1}(B_\eta(w_0) \subseteq V$, for some ϵ and hence $B_{X/T}(u_0, \epsilon) \subseteq U$ which is what we require. Seeking a contradiction, suppose that $q^{-1}(B_{X/T}(u_0, \epsilon))) \nsubseteq T^{-1}(B_\eta(w_0))$ for all ϵ. Then we may find a sequence u_n with $d_H(u_n, u_0) \to 0$ and such that $w_n = p(u_n)$ satisfies $d(w_n, w_0) \geq \eta$ and $d_H(T^{-1}(w_n), T^{-1}(w_0)) \to 0$. By the definition of the Hausdorff metric, it follows that there are $y_n \in T^{-1}(w_n)$ and $z_n \in T^{-1}(w_0)$ such that $d(y_n, z_n) < d_H(T^{-1}(w_n), T^{-1}(w_0))$. Since $T^{-1}(w_0)$ is compact we may assume by passing to a subsequence that $z_n \to z_0 \in T^{-1}(w_0)$. Likewise $y_n \to z_0$ as $d(y_n, z_n) \to 0$.

But, we get $\lim_{n\to\infty} w_n = \lim_{n\to\infty} T(y_n) = T(z_0) = w_0$, a contradiction. This completes the proof of Statement 3.

Statement 4. Assume that T is a proper, open map. By statements 1 and 2, p is a homeomorphism for both topologies. Therefore the topologies have to be the same. ∎

3 The algebraic case

3.1 The general polynomial case

We now consider the polynomial case (1). We have the following theorem.

Theorem 18 *The set of roots of a polynomial equation*

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{14}$$

depend continuously on the vector of coefficients $w = (a_1, \ldots, a_n)$ in the sense of Definition 3.

Proof. Let $X = \mathbb{C}$, $W = \mathbb{C}^n$, then we just have to show that the induced map $T : X^W \to W$ given by equations (8) and (9) is proper and open. Let us begin by showing that T is proper. It suffices to show that if $M > 0$ is any constant and $|a_i| \leq M$ for all i, then there is a constant N, dependent on M, such that roots $(x_1, \ldots, x_n)$ must satisfy $|x_i| \leq N$ for all i. For, if K is any compact set then for some M, $K \subseteq \overline{B_M} \subseteq W$, the closed ball of radius M in the maximum norm. It follows that that $T^{-1}(K) \subseteq (B_N \times B_M) \cap (X \times W)$ and hence is compact. To find N, we argue as follows. From (14) we get for $x \neq 0$,

$$x^n \left(1 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right) = 0$$

so $\left(1 + \frac{a_1}{x} x^{n-1} + \cdots + \frac{a_n}{x^n}\right) = 0$. Pick x and b so that $|x| \geq b > 1$. Then

$$\begin{aligned} 1 &\leq \left|1 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right| + \left|-\frac{a_1}{x} x^{n-1} - \cdots - \frac{a_n}{x^n}\right| \\ &= \left|\frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right|; \end{aligned}$$

and, so,

$$\begin{aligned} 1 &\leq \left|\frac{a_1}{x} + \cdots + \frac{a_n}{x^n}\right| \leq \left|\frac{a_1}{x}\right| + \cdots + \left|\frac{a_n}{x^n}\right| \\ &\leq \frac{M}{b} + \cdots + \frac{M}{b^n} \leq \frac{nM}{b}. \end{aligned}$$

If $b > nM$, then we get a contradiction and so $|x| \leq b$. The largest possible value b is nM so we may take $N = nM$. Thus T is proper.

Next we show that T is open. It suffices to show that for arbitrary $(x_0, w_0) \in X^W$ and every open neighbourhood $(x_0, w_0) \in U \subseteq X^W$, $T(U)$ contains an open neighbourhood of W. We may take U to be of the form $B_\epsilon(x_0) \times B_\eta(w_0) \cap X^W$ where $B_\epsilon(x_0) \subset X$, and $B_\eta(w_0) \subset W$ will be appropriately chosen. Now set $f(x,w) = x^n + a_1 x^{n-1} + \cdots + a_n$ so that in factored form

$$f(x,w) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{i=1}^{s} (x - x_i)^{e_i},$$

where $x_0 = x_i$ for some i. We are going to use contour integrals to recover the number of zeros near x_0. Using logarithmic differentiation with respect to x we get:

$$\frac{f'(x,w)}{f(x,w)} = \sum_{i-1}^{s} \frac{e_i}{x - x_i}.$$

Let γ_0 be a small circular loop enclosing x_0 but no other roots. From basic complex analysis, the contour integrals

$$\frac{1}{2\pi i} \oint_{\gamma_0} \frac{dx}{x - x_i} dx = 1,$$

if x_i is enclosed by γ_0, and

$$\frac{1}{2\pi i} \oint_{\gamma_0} \frac{dx}{x - x_i} dx = 0$$

otherwise. Thus

$$\frac{1}{2\pi i} \oint_{\gamma_0} \frac{f'(x,w)}{f(x,w)} dx = \sum_{i-1}^{s} \oint_{\gamma_0} \frac{e_i dx}{x - x_i} = e_0$$

Now if we vary the coefficients w_0 to a nearby w, then $\frac{1}{2\pi i} \oint_{\gamma_0} \frac{f'(x,w)}{f(x,w)} dx$ varies continuously as long as no zero of $f(x,w)$ crosses γ_0. Using a similar argument, $\frac{1}{2\pi i} \oint_{\gamma_0} \frac{f'(x,w)}{f(x,w)} dx$ counts the number of zeros (with multiplicity) of $f(x,w) = 0$ contained within γ_0. Picking $B_\eta(w_0)$ sufficiently small so that the above argument is holds and $B_\epsilon(x_0)$ to be the interior of γ_0, we obtain for each $w \in B_\eta(w_0)$ an $x \in B_\epsilon(x_0)$ such that $f(x,w) = 0$. Hence $T(B_\epsilon(x_0) \times B_\eta(w_0) \cap X^W)$ contains $B_\eta(w_0)$ and it follows that T is open. All is now proven. ■

3.2 Finite maps

Let $K = \mathbb{C}$ or $\mathbb{R}$, and topologize K^n with the maximum norm. Let $T : K^n \to K^n$ be a polynomial map which is integral in the following sense. For each coordinate function $\xi_i(x) = x_i$ there is an equation of integral dependence

$$\xi_i^{n_i}(x) + a_{i,1}(T(x))\xi_i^{n_i - 1}(x) + \cdots + a_{i,n_i}(T(x)) = 0. \tag{15}$$

for some polynomial functions $a_{i,j}(w)$ on K^n.

Definition 19 *A polynomial map $T : K^n \to K^n$ is called* finite *if each of the coordinate functions ξ_i satisfies an equation of integral dependence as given in equation (15).*

Example 20 *Let $T : K^2 \to K^2$ be defined by $(z, w) = T(x, y) = (x + y, xy)$. The equations of integral dependence can be written as*

$$\begin{aligned} x^2 + zx + w &= 0, \\ y^2 + zy + w &= 0. \end{aligned}$$

Example 21 *More generally let $T : K^2 \to K^2$ be defined by*

$$T(x_1, \ldots, x_n) \to (\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_n(x_1, \ldots, x_n)),$$

where the σ_k are the elementary symmetric functions

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_k} x_{i_1} \cdot \cdots \cdot x_{i_k}$$

taken over k-subsets $\{i_1, \ldots, i_k\}$ of distinct indices. Then, it is well known that for each x_i

$$x_i^n + \sum_{k=1}^{n} (-1)^n \sigma_k x_i^{n-j} = 0.$$

Proposition 22 *A finite map has finite fibres.*

Proof. Fixing $w_0 = T(x_0)$ the coordinates of $\xi_i(x_0)$ must satisfy an equation as in (15). Thus there are only finitely many values for $\xi_i(x_0)$. It follows that the are only finitely many possibilities for x_0. ∎

Proposition 23 *If $T : K^n \to K^n$ is finite, then T is proper.*

Proof. The proof is essentially the same as the proof of properness given in the proof of Theorem 18. If the values of $w = T(x)$ are bounded then the solutions $\xi_i(x)$ in equation (15) are also bounded by the argument of the proof of Theorem 18. This implies that $T^{-1}(V)$ is compact whenever V is compact. ∎

To get results on openness we need to define the notion of multiplicity of a zero of a system of complex equations, see [Mil], [Or], [Pal], [Sto]. For this discussion we are going to use the standard Euclidean metric in $\mathbb{C}^n$:

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i \overline{x_i}}.$$

Suppose that $w_0 = T(x_0)$, there are no nearby solutions and so for a small ball $B_\epsilon(x_0)$ the map $M_{x_0} : \partial B_\epsilon(x_0) \to \mathbb{S}^{2n-1}$ defined by

$$M_{x_0}(T - w_0) = \frac{T(x) - w_0}{\|T(x) - w_0\|}$$

is well defined and continuous. The induced map

$$(M_{x_0})_* : H_{2n-1}(\partial B_\epsilon(x_0); \mathbb{Z}) \to H_{2n-1}(\mathbb{S}^{2n-1}; \mathbb{Z})$$

is multiplication by an integer and called the multiplicity of the zero of $T(x) = w_0$ at x_0. We denote this by $\mu_{x_0}(T - w_0)$.The multiplicity $\mu_{x_0}(T - w_0)$ is a positive integer assuming there is a zero at x_0 and is independent of ϵ for all sufficiently small ϵ. If w' is a point near w_0

$$\mu_{x_0}(T - w_0) = \sum_{x'} \mu_{x'}(T - w') \tag{16}$$

where x' ranges over all roots $T(x') = w'$ contained in $B_\epsilon(x_0)$. The point w' must be chosen in a small ball $B_\eta(w_0)$ such that $T - w'$ never vanishes on $\partial B_\epsilon(x_0)$. This can be arranged because of the continuity of $\|T(x) - w'\|$ in x and w', and the compactness of $\partial B_\epsilon(x_0)$.

Proposition 24 *If $T : \mathbb{C}^n \to \mathbb{C}^n$ is a finite map then T is open.*

Proof. The proof is similar to the proof of openness for Theorem 18. It suffices to show that for every solution x_0 of $T(x) = w_0$ there are nearby solutions to $T(x) = w'$ for w' close to w_0. This is guaranteed by the multiplicity equation (16). ■

We now have.

Theorem 25 *Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a finite map then the preimages $T^{-1}(w)$ depend continuously $w \in \mathbb{C}^n$ in the sense of Definition 3.*

Example 26 *Following example 20, consider the equation $x^2 + w_1 x + w_2 = 0$ solved over the reals, and the associated map $T(x_1, x_2) = (-x_1 - x_2, x_1 x_2)$. The image of the roots in the parameter plane is given by $w_1^2 - 4w_2 \geq 0$ This is a closed region bounded by a parabola. The associated map T fails to be open above the boundary locus $w_1^2 - 4w_2$ which is where the loss of solutions occurs. Of course, the problem disappears when we consider complex coefficients and solutions.*

4 Examples

Example 27 *Consider the problem of finding the eigenvalues of a matrix. By expanding the determinant in equation (2) we may use Theorem 18 to conclude that the eigenvalues depend continuously on the matrix entries.*

Example 28 *Are their interesting examples where the solution sets are not finite sets of points? Consider the problem of finding eigenvectors of a matrix, i.e., Solve*

$$A\mathbf{x} = \lambda\mathbf{x},\ \lambda \in \mathbb{C},\ \mathbf{x} \in \mathbb{C}^n.$$

Since the solution set in $\mathbf{x}$ *is never compact, then we need to projectivize. So let* $\mathbb{P}^{n-1}$ *be the projective space obtained from* $\mathbb{C}^n$ *and observe that for* A *and* λ *fixed the equation* $A\mathbf{x} = \lambda\mathbf{x}$ *makes sense in projective space. It will be a closed subset of projective space and hence a compact set. We may formulate our parametric solution problem* $T : X \to W$ *as follows:*

$$\begin{aligned} X &= \left\{(\lambda, \mathbf{x}, A) \in \mathbb{C} \times \mathbb{P}^{n-1} \times M_n(\mathbb{C}) : A\mathbf{x} = \lambda\mathbf{x}\right\}, \\ W &= M_n(\mathbb{C}),\ T(\lambda, \mathbf{x}, A) = A. \end{aligned}$$

If A *is a matrix with distinct eigenvalues then there will be* n *points of the form* $(\lambda_i, \mathbf{x}_i)$ *lying above* A*. If there is a multiple eigenvalue, say a double eigenvalue with geometric multiplicity 2, then there* $n-2$ *points of the form* $(\lambda_i, \mathbf{x}_i)$ *and a projective line* $\lambda\times\ \mathbb{P}^1$*, the latter corresponding to the double eigenvalue.*

Example 29 *An interesting example of polynomial maps* $T : \mathbb{C}^n \to \mathbb{C}^n$ *are gradient maps defined by*

$$T(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right),$$

where $f : \mathbb{C}^n \to \mathbb{C}$ *is a polynomial. To be specific, pick*

$$f(x, y, z) = x^4 + y^4 + z^4 + xyz,$$

then,

$$T(x, y, z) = (4x^3 + yz, 4y^3 + xz, 4z^3 + xy).$$

The topological properties of this gradient map have consequences for the variation of topology of the family of surfaces defined by

$$f(x, y, z) + ax + by + cz = d.$$

See [Br] for more details. By Bezout's Theorem the typical fibre has 27 points. Using Maple's Groebner basis package one can show that x*,* y *and* z *all satisfy degree 27 polynomials with coefficients that are polynomials* $w_1 = \frac{\partial f}{\partial x}$*,* $w_2 = \frac{\partial f}{\partial y}, w_3 = \frac{\partial f}{\partial z}$*, and so we conclude that* ∇f *is finite.*

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